

# Bose–Einstein Condensation and Spontaneous Symmetry Breakdown

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**Abstract**—Bose–Einstein condensation is studied as a phenomenon of spontaneous symmetry breakdown. The order parameter is the condensate wave function. We analyze the functional dependence of the free energy on the order parameter. An expression for the order parameter in terms of the Green's function of the theory is derived. A method for deriving the thermodynamics of a Bose–Einstein condensed system is provided. For the Bogoliubov condensate, we derive the London relation and an equation of state valid for very low temperatures. Finally, we discuss the loop expansion method, which provides a framework for computational calculations of the thermodynamics of Bose–Einstein condensed systems.

## 1. INTRODUCTION

The vast interest in Bose–Einstein condensation (BEC) arises from the fact that this phenomenon is composed of several physical disciplines, promoting the creation of a link between them. As a phase transition in thermodynamics or as matter waves in quantum coherent phenomena up to quantum field theory, where BEC is seen as a spontaneous breaking of symmetry, the possibility of experimental observations and tests of fundamental theories has reinvigorated the field.

When the symmetries of the physical states are smaller than the symmetries of the Hamiltonian, there occurs spontaneous symmetry breakdown [1]. Spontaneous symmetry breakdown plays an essential role in the formulation of the now standard theory of fundamental interactions. In fact, identification of the  $SU(2) \times U(1)$  gauge symmetry and its spontaneous breakdown, by using the Higgs mechanism, was the key to understanding electroweak interactions [2].

The relevance of spontaneous symmetry breakdown in condensed matter physics has been stressed mainly by Anderson [3], who has suggested that broken symmetry is one of the two most important principles of condensed matter physics. Anderson has also advanced the idea that this basic notion will be essential to understanding many physical phenomena and to establishing an alternative methodology in condensed matter physics.

When spontaneous symmetry breakdown occurs, one needs a new variable in order to specify the microscopic state in the lower symmetry phase. This new variable ("created by the breakdown of symmetry" [3]) is the order parameter. We shall assume that the order parameter can be parameterized in terms of local generalized variables  $Q_i$ :

$$Q_i = Q_i(x). \quad (1.1)$$

The above comment implies that, in order to get the hydrodynamics of the fluid or the thermodynamics of the system (in equilibrium), one has to determine the free energy dependence on these generalized variables. This means that, in order to obtain a complete description of the phenomena, one has to know the free energy functional

$$\mathcal{F} = \mathcal{F}[V, T, Q_i, \partial_\mu Q_i], \quad (1.2)$$

where the functional  $\mathcal{F}$  depends not only on  $V$  and  $T$  (as usual, when there is no spontaneous symmetry breakdown) but also on the order parameter and its derivatives.

From the generalized free energy, one can introduce generalized forces as "gradients" of the generalized coordinates:

$$F_i(x) = \frac{\delta \mathcal{F}}{\delta Q_i(x)}. \quad (1.3)$$

One can also define generalized momenta as

$$\Pi_\mu^{(i)}(x) = \frac{\delta \mathcal{F}}{\delta [\partial_\mu Q_i(x)]}. \quad (1.4)$$

As a basic principle, one follows Anderson and imposes that the generalized forces vanish. That is, the basic equation when there is a breakdown of symmetry is

$$\left. \frac{\delta \mathcal{F}}{\delta Q_i} \right|_{Q_i = Q_i^c} = 0 = F_i(x). \quad (1.5)$$

The field configurations  $Q_i^c(x)$ , for which the generalized free energy is stationary, are the actual values of the order parameter of the system. In the case of BEC,

$Q_i^c$  is the order parameter in the condensed phase. The

order parameter is associated with the condensate wave function.

We shall see in this paper that proper identification and explicit construction of the functional  $\mathcal{F}$  by using the field theoretical approach introduces an alternative methodology for dealing with the phenomenon of Bose-Einstein condensation.

When one assumes that the order parameter is  $\mathbf{x}$  and  $t$  dependent (local order parameter), the vanishing of the generalized force leads to conservation laws. In the case of macroscopic occupation of a quantum state, one gets for one of the coordinates

$$\partial^\mu \Pi_\mu^{(i)}(\mathbf{x}) = 0, \tag{1.6}$$

where  $\Pi_\mu(\mathbf{x})$  is defined in (1.4). We shall see later that (1.6) corresponds to the continuity equation in hydrodynamics. The other equation is a generalized Bernoulli equation describing isentropic potential flow.

We have also presented in this paper how the thermodynamics of the system, when BEC occurs, can be inferred. Some modifications are needed in order to deal with the thermodynamics of a condensed system. Among these modifications, we call attention to the "imaginary time" field configurations and periodic boundary conditions. In order to distinguish the hydrodynamics from thermodynamics, we work with the grand partition potential in the latter case.

This paper is organized as follow. In Section 2, we present an explicit representation of the functional dependence of the free energy on the order parameter. We show also how field theory provides a method for computing, from microscopic theory, the relevant functional.

The hydrodynamic equation follows from the generalized equilibrium conditions; so, Section 3 is dedicated to a full derivation of the ensemble average providing the most important hydrodynamic equations, which in our previously work was referred to as the Ehrenfest theorem.

In our early publication [4], we explored fluidity properties using a description of density and phase as dynamic variables. Nevertheless, that formalism can be expanded to include a thermodynamic description of the system, from which we can obtain not only hydrodynamics characteristics but also the overall thermodynamic behavior.

In Section 4, we present an approach for deriving the thermodynamics of the system in the presence of the condensate. We use the background field method in order to make explicit the dependence of the Gibbs potential with the order parameter. The Gibbs potential is the generating functional of the irreducible Green's function of the theory at finite temperatures. The order parameter is the field configuration that minimizes the generating functional.

In Section 5, we discuss the Bogoliubov condensate. The relevance of this particular type of condensate is

that it describes condensation of zero-momentum particles. It is relevant in the study of superfluidity. The thermodynamics of the system can be inferred, in this case, from the Green's functions of the theory computed at zero momentum.

Explicit and general results for the Bogoliubov condensate are presented in Section 6, where we derive an equation of state and the London relation.

In Section 7, we present a systematic method in field theory for making predicting the thermodynamics of the condensed phase. This method is loop expansion. Some results are derived for the zero-loop and one-loop approximations.

Conclusions are given in Section 8.

## 2. GENERALIZED FREE ENERGY

In this section, we will show that field theory provides a very convenient method for defining the generalized free energy and a well-defined scheme for computing it.

The partition function  $Z$  of a many-body system will be obtained, within the field theoretical approach, from the Wiegell-Jalickee representation [5]. Within this representation,  $Z$  is given as a sum over field configurations [5, 6] as

$$Z = e^{-i\mathcal{F}} = \int \mathcal{D}\psi \mathcal{D}\psi^* e^{-i \int dt dx \mathcal{L}[\psi, \partial_\mu \psi]}, \tag{2.1}$$

where  $\mathcal{F}$  in (2.1) is the free energy and  $\mathcal{L}$  is the Lagrangian density, which can be written, in the nonrelativistic case, as

$$-\mathcal{L} = i\psi^* \frac{\partial}{\partial t} \psi - \mathcal{H}[\psi^*, \psi]. \tag{2.2}$$

As usual, one splits  $\mathcal{H}$  in (2.2) into two terms. There is a kinetic term ( $K$ ) and an interaction term  $\mathcal{H}_{int}$ . That is,

$$\begin{aligned} \mathcal{H} &= K + \mathcal{H}_{int} \\ &= \frac{(\nabla\psi)^* \cdot (\nabla\psi)}{2m} + U(x)\psi^*\psi + \mathcal{H}'_{int}[\psi^*\psi], \end{aligned} \tag{2.3}$$

where the term  $U\psi^*\psi$  takes into account the interaction with external forces.

The average over the ensemble of an observable  $\hat{O}[\psi]$  is defined as

$$\langle \hat{O}[\psi] \rangle = Z^{-1} \int \mathcal{D}\psi \mathcal{D}\psi^* \hat{O}[\psi] e^{-i \int dt \int dx \mathcal{L}[\psi, \partial_\mu \psi]}. \tag{2.4}$$

One can change the integration variables in (2.1) to new variables  $\rho(x)$  and  $\phi(x)$  defined by

$$\psi(x) = \sqrt{\rho(x)} e^{i\phi(x)}. \tag{2.5}$$

In terms of the variables  $\rho$  and  $\varphi$  defined in (2.5), one can write (2.1) as

$$Z = e^{-i\mathcal{F}} = \int \mathcal{D}\rho \mathcal{D}\varphi |J| e^{-i \int dt dx \mathcal{L}[\rho, \varphi, \partial_\mu \rho, \partial_\mu \varphi]}, \quad (2.6)$$

where  $J$  is the Jacobian of the transformation (2.5) and  $\mathcal{L}[\rho, \varphi, \partial_\mu \rho, \partial_\mu \varphi]$  is given by

$$\begin{aligned} \mathcal{L}[\rho, \partial_\mu \rho, \varphi, \partial_\mu \varphi] &= +\rho \frac{\partial \varphi}{\partial t} + \mathcal{H}[\nabla \varphi, \rho, \nabla \rho] \\ &= +\rho \frac{\partial \varphi}{\partial t} + \frac{\rho}{2m} (\nabla m)^2 + \frac{1}{2m} \nabla(\sqrt{\rho}) \cdot \nabla(\sqrt{\rho}) + \mathcal{H}_{\text{int}}[\rho]. \end{aligned} \quad (2.7)$$

Spontaneous symmetry breakdown means that the average defined in (2.4) over the ensemble of the field operator  $\hat{\psi}$  is different from zero in the condensed phase. One then writes

$$\langle \hat{\psi}(x) \rangle = \sqrt{\rho_c(x)} e^{i\varphi_c(x)} \equiv \psi_c(x). \quad (2.8)$$

We shall see later that  $\psi_c(x)$  is the wave function of the condensate.

It follows from (2.8) that, since the order parameter is a complex variable, one has to use two generalized coordinates. We take these variables to be

$$\begin{aligned} Q_1(x) &= \rho_0(x), \\ Q_2(x) &= \varphi_0(x). \end{aligned} \quad (2.9)$$

In order to get the generalized free energy, all one has to do is to separate out, in the functional integration (2.1), the contribution associated with the vacuum expectation value given by (2.8). In order to get the dependence of  $\mathcal{F}$  on  $\langle \psi \rangle$ , we shall employ the background field method [7]. Suppose we want to separate out a particular set of field configurations contributing to the integral (2.6). Let  $\rho_0(x)$  and  $\varphi_0(x)$  represent this set. One writes

$$\begin{aligned} \rho(x) &= \rho_0(x) + \rho'(x), \\ \varphi(x) &= \varphi_0(x) + \varphi'(x) \end{aligned} \quad (2.10)$$

or equivalently, by using (2.1),

$$\begin{aligned} \psi(x) &= \psi_0(x) + \psi'(x), \\ \psi^*(x) &= \psi_0^*(x) + \psi^{*'}(x). \end{aligned} \quad (2.11)$$

By integrating over  $\rho'$  and  $\varphi'$  (respectively,  $\psi'$  and  $\psi^{*'}$ ) in the functional integral (2.6) (respectively, (2.11)), we get a partition function that depends on  $\rho_0$  and  $\varphi_0$  ( $\psi_0$  and  $\psi_0^*$ , respectively). In this way, one gets, by

separating out the relevant degrees of freedom, an order parameter-dependent functional,

$$\begin{aligned} Z &= Z[\rho_0, \varphi_0, \partial_\mu \rho_0, \partial_\mu \varphi_0] = e^{-i\mathcal{F}[\rho_0, \varphi_0, \partial_\mu \rho_0, \partial_\mu \varphi_0]} \\ &\equiv \int \mathcal{D}\rho' \mathcal{D}\varphi' |J| \\ &\times e^{-i \int dt dx \mathcal{L}[\rho_0 + \rho', \varphi_0 + \varphi', \partial_\mu(\rho_0 + \rho'), \partial_\mu(\varphi_0 + \varphi')]}, \end{aligned} \quad (2.12)$$

where

$$\begin{aligned} \mathcal{L}[\rho_0 + \rho', \varphi_0 + \varphi', \partial_\mu(\rho_0 + \rho'), \partial_\mu(\varphi_0 + \varphi')] \\ = (\rho_0 + \rho') \frac{\partial}{\partial t} (\varphi_0 + \varphi') + \frac{(\rho_0 + \rho')}{2m} (\nabla(\varphi_0 + \varphi'))^2 \\ + \frac{1}{2m} (\nabla \sqrt{\rho_0 + \rho'})^2 + \mathcal{H}_{\text{int}}[\rho_0 + \rho']. \end{aligned} \quad (2.13)$$

It follows from (2.12) that the free-energy functional is the free energy of the system in the presence of the background fields  $\rho_0$  and  $\varphi_0$  [7].

The actual values of the background fields that the system chooses are those for which the free-energy functional becomes stationary:

$$\left. \frac{\delta \mathcal{F}}{\delta \rho_0} [\rho_0, \varphi_0, \partial_\mu \rho_0, \partial_\mu \varphi_0] \right|_{\rho_0 = \rho_c} = 0, \quad (2.14)$$

$$\left. \frac{\delta \mathcal{F}}{\delta \varphi_0} [\rho_0, \varphi_0, \partial_\mu \rho_0, \partial_\mu \varphi_0] \right|_{\varphi_0 = \varphi_c} = 0, \quad (2.15)$$

or equivalently

$$\left. \frac{\delta \mathcal{F}}{\delta \psi_0} [\psi_0, \partial_\mu \psi_0] \right|_{\psi_0 = \psi_c} = 0, \quad (2.16)$$

$$\left. \frac{\delta \mathcal{F}}{\delta \psi_0^*} [\psi_0, \partial_\mu \psi_0] \right|_{\psi_0^* = \psi_c^*} = 0. \quad (2.17)$$

That is, those configurations for which the generalized forces vanish are the actual values of the order parameter.

### 3. HYDRODYNAMICS

In this section, we shall see that the generalized free energy, defined in (2.12), provides a framework for the study of fluid motion. This framework is equivalent to a Hamiltonian formulation for hydrodynamics. In this formulation,  $\rho$  and  $\varphi$ , defined in (2.5), play the role of canonically conjugate field variables. One of the extra results of this section is to obtain "Ehrenfest theorems" for the fluid motion. We will see that, in the sense of averages over the ensemble, the Euler equation and continuity equation hold true. For time-independent interparticle interactions and a time-independent external potential, one gets also an Ehrenfest theorem associated with the Bernoulli equation.

In dealing with fluid motion, we will be interested in defining, within the functional method, local observ-

ables, such as the fluid local density, local pressure, local fluid velocity, etc. All these relevant physical quantities can be defined as averages over the ensemble of local operators. These local operators can be defined as functional derivatives of the generalized free energy ( $\mathcal{F}$ ) or functional derivatives of averages over the ensemble of certain physical quantities. In order to stress the functional derivative definition of these quantities, we will name them local and we shall use a label  $l$  to identify them. That is, we write, for an arbitrary quantity  $\hat{F}(x)$ , the local quantity  $F^l(x)$  as

$$F^l(x) = \langle \hat{F}(x) \rangle = \frac{\delta}{\delta \alpha(x)} \langle \hat{G} \rangle, \quad (3.1)$$

with  $\alpha(x)$  and  $G$  defined accordingly. The local fluid density, defined as the vacuum expectation value of the density operator, can be written in the form of functional derivatives of the generalized free energy as

$$\rho^l(x) \equiv \langle \hat{\rho}(x) \rangle = \frac{\delta \mathcal{F}}{\delta U(x)}, \quad (3.2)$$

whereas the local current defined as

$$\mathbf{j}^l(x) \equiv \left\langle \hat{\rho}(x) \frac{\nabla \hat{\phi}(x)}{m} \right\rangle \quad (3.3)$$

is the functional derivative of the gradient of  $\phi$ . In other words, from (1.4),  $\mathbf{j}^l$  is the spatial component of the generalized momenta:

$$\mathbf{j}^l(x) = \frac{\delta \mathcal{F}}{\delta (\nabla \phi(x))}. \quad (3.4)$$

The interaction energy can be written as

$$H_{int} = \int \rho(x) e(x, \rho) dx + \int U(x) \rho(x) dx, \quad (3.5)$$

where  $U(x)$  is the external potential and  $e(x, \rho)$  is the per-particle internal energy (local internal energy).

The local external force is the force that a single particle experiences when the system is under the action of an external potential  $U(x)$ . The local force is then

$$F_{ext}^l(x) = -\nabla U(x). \quad (3.6)$$

The local internal energy and pressure are those associated with the energy and pressure of a single particle in the fluid. From (3.5), it follows that the local internal energy should be identified with the average over the ensemble of the energy per particle ( $e$ ):

$$e^l(x) \equiv \langle e(x, \rho) \rangle. \quad (3.7)$$

Following London [9], the local pressure will be defined as the functional derivative:

$$P^l(x) = \left\langle \rho^2(x) \int \frac{\delta e(x', \rho)}{\delta \rho(x)} dx' \right\rangle. \quad (3.8)$$

Finally, we define the local kinetic energy as kinetic energy where  $\rho$  is constant:

$$K^l(x) = \left\langle \frac{\delta}{\delta \rho(x)} \int \hat{K}^l(x') dx' \right\rangle = \left\langle \frac{(\nabla \phi(x))^2}{2m} \right\rangle. \quad (3.9)$$

It will be seen later that all these local quantities (local kinetic energy, local internal energy, etc.) are, within the quantum context, the operators (or vacuum expectation values) associated with the classical per-particle physical quantities.

We are now ready to write the generalized equilibrium equations (2.14) and (2.15). By using representation (2.7) for the Lagrangian density, it follows that Eqs. (2.14) and (2.15) can be written as

$$-\frac{\partial}{\partial t} \langle \phi(x) \rangle = \frac{\delta}{\delta \rho(x)} \langle \mathcal{H} \rangle, \quad (3.10)$$

$$\frac{\partial}{\partial t} \langle \rho(x) \rangle = \frac{\delta}{\delta \phi(x)} \langle \mathcal{H} \rangle. \quad (3.11)$$

Equations (3.10) and (3.11) show that  $\rho$  and  $\phi$  are a canonically conjugate pair of continuous variables.

Equation (3.11) assumes a quite familiar form if one uses (2.3). Independently of the details of the dynamics, one can write

$$\frac{\partial \langle \rho(x) \rangle}{\partial t} + \left[ \frac{\nabla \phi(x)}{m} \right] = 0, \quad (3.12)$$

which is just the continuity equation. In terms of local quantities defined in (3.2) and (3.3), one gets

$$\frac{\partial \rho^l(x)}{\partial t} + \nabla \cdot \mathbf{j}^l(x) = 0. \quad (3.13)$$

Continuity equation (3.13) can be written in terms of the generalized momentum, defined in (1.6), as

$$\partial^\mu \Pi_\mu = 0, \quad (3.14)$$

where  $\pi^0 = \frac{\delta \mathcal{F}}{\delta U} = \rho^l$  and  $\pi^k = \frac{\delta \mathcal{F}}{\delta (\partial_k \phi)} = j^k$ .

Let us turn now to Eq. (3.10). From (3.5), it follows that there will be three contributions to  $\frac{\delta H_{int}}{\delta \rho(x)}$ , which, by using (3.7) and (3.8), can be written as

$$\frac{\delta H_{int}}{\delta \rho(x)} = \left\langle \frac{P(x)}{\rho(x)} \right\rangle + \langle e(x, \rho(x)) \rangle + U(x) \quad (3.15)$$

$$\equiv h^l(x) + U(x) = \langle \hat{h}(x) \rangle + U(x),$$

where  $h^l(x) = \langle \hat{h} \rangle$  in (3.15) is the local enthalpy.

If one neglects derivatives of the density (uniform fluid), Eq. (3.10) can be written as

$$\begin{aligned} \frac{\partial \langle \hat{\phi}(x) \rangle}{\partial t} &= K'(x) + h'(x) + U(x) \\ &= \langle \hat{K}(x) \rangle + \langle \hat{h}(x) \rangle + U(x), \end{aligned} \quad (3.16)$$

where  $\hat{K}(x)$  is defined in (3.9) and  $\hat{h}(x)$  is defined in (3.15).

By defining, as usual, the momentum and velocity operators as

$$\hat{\Phi}(x) \equiv \nabla \hat{\phi}(x) \equiv m \hat{V}(x), \quad (3.17)$$

Eq. (3.16) becomes

$$-\frac{\partial}{\partial t} \langle \phi(x) \rangle = \left\langle \frac{\hat{\Phi}^2(x)}{2m} \right\rangle + \langle \hat{h}(x) \rangle + U(x). \quad (3.18)$$

In terms of the velocity operator defined in (3.17), Eq. (3.13) becomes

$$\frac{\partial \langle \rho(x) \rangle}{\partial t} + \nabla \cdot (\langle \rho(x) \hat{V}(x) \rangle) = 0. \quad (3.19)$$

If one takes the gradient of (3.18), one gets the force equation

$$\frac{\partial}{\partial t} \langle \hat{\Phi}(x) \rangle = -\nabla \cdot \left\langle \frac{\hat{\Phi}^2(x)}{2m} \right\rangle - \nabla \langle \hat{h}(x) \rangle + \mathbf{F}_{\text{ext}}(x). \quad (3.20)$$

Equations (3.19) and (3.20) correspond to Ehrenfest theorems for a fluid. Equation (3.19) is a continuity equation, whereas (3.20) expresses momentum conservation [9].

By assuming a stationary solution to (3.18), one writes

$$\phi(\mathbf{x}, t) = -Et + \phi(\mathbf{x}). \quad (3.21)$$

Equation (3.18) becomes

$$E = \left\langle \frac{\hat{\Phi}^2(\mathbf{x})}{2m} \right\rangle + \langle h(\mathbf{x}) \rangle + U(\mathbf{x}), \quad (3.22)$$

which is just the Ehrenfest theorem for the Bernoulli equation [10].

The solution associated with the fluid at rest and in equilibrium ( $\phi = \text{cte}$ ) implies

$$E = \langle h(\mathbf{x}) \rangle + U(\mathbf{x}), \quad (3.23)$$

which is the condition for equilibrium in a fluid.

For particles that interact through binary forces, that is, instead of  $\rho(x)e(x, \rho(x))$  in (3.5), we consider

$$\frac{\lambda}{2} \rho(x) \int V(x-x') \rho(x') dx';$$

Eq. (3.18) is (for uniform fluid)

$$\begin{aligned} -\frac{\partial}{\partial t} \langle \phi(x) \rangle &= \left\langle \frac{(\nabla \phi)^2(x)}{2m} \right\rangle \\ &+ \lambda \int V(x'-x) \langle \rho(x') \rangle dx' + U(x). \end{aligned} \quad (3.24)$$

The conclusion is that, from the generalized free energy, we succeeded in getting the fluid equations (3.19) and (3.20). The continuity equation can be written in the form of (1.6). One of the generalized equilibrium equations entails conservation of momentum, whereas the other equation stands for conservation of matter.

It follows from (3.17) and (3.18) that  $\phi$  is the velocity potential. In this way, the velocity potential and the fluid density constitute a pair of canonically conjugate variables.

At this point, it is worth stressing the fact that all we have shown up to now is that, by assuming a functional dependence of the free energy with regard to the density and velocity potential, we are able to obtain, from the generalized equilibrium equations, Ehrenfest theorems for the fluid. The derivation of these equations has not required associating these quantities with the vacuum expectation values of field operators.

#### 4. THERMODYNAMICS IN THE PRESENCE OF THE CONDENSATE [5, 8]

In this section, we shall analyze how the approach developed earlier can be useful in deriving the thermodynamics of a Bose-Einstein condensed system. The method employed here is again the background field method. Before starting, let us review some results on the field theoretical approach to the study of the thermodynamics of a system in equilibrium. We shall analyze the grand partition function of the system. This choice is made in order to make a distinction between the equilibrium properties and the previous case.

The thermodynamic properties of a system can be inferred from the grand partition function  $\Xi$ :

$$\Xi = \text{Tr}(e^{-\beta(H - \mu \hat{N})}), \quad (4.1)$$

where  $\beta = (T)^{-1}$ ,  $\hat{N} = \int d^3x \psi^\dagger \psi$ , and  $\mu$  is the chemical potential.

Within the functional approach, the grand partition function can be written as a sum over field configurations satisfying periodic boundary conditions [5, 11, 12]:

$$\psi(x, 0) = \psi(\beta, x). \quad (4.2)$$

A functional integration representation for Hamiltonian (2.3) has been derived by Wiegand and Jalickee [4]. One writes

$$\Xi = \int \mathcal{D}\psi \int \mathcal{D}\psi^* \exp(-\mathcal{J}[\psi]), \quad (4.3)$$

where the action  $S[\psi]$  in (4.3) is written as

$$S[\psi] = \int_0^\beta d\tau \int d^3x \left( \psi^*(\tau, \mathbf{x}) \frac{\partial \psi}{\partial \tau}(\tau, \mathbf{x}) + \frac{\nabla \psi^*(\mathbf{x}, \tau) \cdot \nabla \psi(\mathbf{x}, \tau)}{2m} - \mu \psi^*(\tau, \mathbf{x}) \psi(\tau, \mathbf{x}) \right) + \int_0^\beta d\tau \int d^3x \int d^3x' \psi^*(\mathbf{x}, \tau) \psi(\mathbf{x}, \tau) \times V(\mathbf{x} - \mathbf{x}') \psi^*(\mathbf{x}', \tau) \psi(\mathbf{x}', \tau). \tag{4.4}$$

The thermodynamic properties of the system can be inferred from the Gibbs potential  $\Omega^0$ , defined as

$$-\beta \Omega \equiv \ln \Xi. \tag{4.5}$$

That is, from  $\Omega$ , defined in (4.5), one gets the equation of state from the equations

$$P = - \left( \frac{\partial \Omega}{\partial V} \right)_{T, \mu}, \tag{4.6}$$

$$N = - \left( \frac{\partial \Omega}{\partial \mu} \right)_{T, V}, \tag{4.7}$$

whereas the internal energy ( $U$ ), free energy ( $F$ ), and entropy ( $S$ ) are obtained from

$$U = \frac{\partial}{\partial \beta} (\beta \Omega), \tag{4.8}$$

$$F = \Omega + N\mu, \tag{4.9}$$

$$S = \beta^2 \frac{\partial \Omega^0}{\partial \beta} - \mu N \beta. \tag{4.10}$$

Let us consider now the system in the presence of the condensate. The condensate can be thought of as a background field [5, 8]. The presence of the background field condensate affects the thermodynamic properties of the system. In order to derive the thermodynamics of the system in this new phase, we separate, in the functional integration, the background condensate contribution and write

$$\psi(x) = \psi'(x) + \psi_b(x). \tag{4.11}$$

By summing over  $\psi'(x)$ , we get a grand canonical partition function which is a functional of  $\psi_b(x)$ ; that is,

$$\Xi(\psi_b) = e^{-\beta \Omega(\psi_b)} = \int \mathcal{D}\psi^* \int \mathcal{D}\psi e^{-S(\psi + \psi_b)}. \tag{4.12}$$

$\Omega(\psi_b)$ , as defined in (4.3), is a functional of the background field  $\psi_b$ , and it differs from the Gibbs potential  $\Omega^0$  defined in (4.5) by a factor  $\beta$ .  $\Omega(\psi_b)$  represents in these circumstances the Gibbs potential associated with

the system in the presence of the background field  $\psi_b(\mathbf{x}, \tau)$ . The extrema of this functional, that is,

$$\left. \frac{\delta \Omega}{\delta \psi_b} \right|_{\psi_b = \psi_c} = 0, \tag{4.13}$$

is the temperature-dependent order parameter of the phase transition.

We can see that the thermodynamics of a Bose-Einstein condensed system can also be formulated as a variational problem.

It is possible to show, by using the background field method [7], that, for  $\psi_c$  defined in (4.11) and satisfying (4.12), one can write

$$\Omega(\psi_c) = \Gamma(\psi_c),$$

where

$$\Omega[\psi_c, \partial_\mu \psi_c] = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n! m!} \int dx_1 \dots dx_n dy_1 \dots dy_m \times \Gamma^{(n, m)}(x_1 \dots x_n; y_1 \dots y_m) \psi_c(x_1) \dots \times \psi_c(x_n) \psi_c^*(y_1) \dots \psi_c^*(y_m). \tag{4.14}$$

Here,  $\Gamma^{(n, m)}(x_1, \dots, x_n; y_1, \dots, y_m)$  are the finite-temperature one-particle irreducible Green's functions of the theory. That is, the generalized potential is simply the generating functional of the one-particle irreducible Green's functions of the theory. Field theory provides a scheme for computing the functional  $\mathcal{F}$ .

From (4.13), one can write

$$\Omega[\psi_c, \partial_\mu \psi_c] = \Gamma[\psi_c, \partial_\mu \psi_c], \tag{4.15}$$

where  $\Gamma[\psi_c, \partial_\mu \psi_c]$  is the effective action of the theory computed at those field configurations that satisfy (4.13).

The basic outcome of expressions (4.13) and (4.15) is that field theory provides a natural representation for the grand partition functional dependence on the order parameter [8]. We have seen that representation (2.12) is useful for deducing the hydrodynamic equations, whereas we shall see that (4.15) is useful for obtaining the thermodynamics of the condensed phase.

We shall give explicit examples in the sections that follow.

In order to illustrate how the method proposed here works, let us compute  $\Omega(\rho_0)$  explicitly in the case of free fields.

For free fields, the integral representation (4.3) is reduced to a quadratic one, namely,

$$\Xi = \int \mathcal{D}\psi^* \int \mathcal{D}\psi \times \exp \left[ - \int_0^\beta d\tau \int d^3x \left( \psi^* \frac{\partial}{\partial \tau} \psi - \mu \psi^* \psi + \frac{\nabla \psi^* \cdot \nabla \psi}{2m} \right) \right]. \tag{4.16}$$

We consider this simple exercise to be very relevant, since it illustrates the fact that the method presented here provides a clear distinction between the thermodynamic behavior of a normal system and a condensed one.

The functional integral (4.16) can be easily performed, and the result is that the Gibbs potential is given by

$$\Omega = \frac{V}{\beta} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \ln \left( 1 - e^{-\beta \left( \frac{\mathbf{p}^2}{2m} - \mu \right)} \right). \quad (4.17)$$

One gets from (4.17) the thermodynamics of an ideal Bose gas.

Let us consider the condensation of particles in the  $\mathbf{p} = 0$  state. The Bogoliubov condensate is represented by the uniform field configurations  $\psi_b$  [8, 13] (no  $(\mathbf{x}, \tau)$  dependence); that is,

$$\psi_b(\mathbf{x}, \tau) = \psi_b. \quad (4.18)$$

By making a substitution (4.11) with  $\psi_b$  given by (4.18), one gets a grand canonical partition that is dependent on the uniform background field  $\rho_b \equiv \psi_b^* \psi_b$ :

$$\Xi(\rho_b) = e^{-\beta \Omega(\rho_b)} = e^{V \beta \rho_b} \int \mathcal{D}\psi \int \mathcal{D}\psi^* \times \exp \left[ - \int_0^\beta d\tau \int d^3 \mathbf{x} \left( \psi^* \frac{\partial}{\partial \tau} \psi - \mu \psi^* \psi + \frac{\nabla \psi^* \cdot \nabla \psi}{2m} \right) \right]. \quad (4.19)$$

Performing now the quadratic integration, one gets

$$\Omega(\rho_b) \equiv V \Gamma_{\text{eff}}(\rho_b, T) = V \left( -\mu \rho_b + \frac{1}{\beta} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \times \ln \left[ 1 - \exp \left[ -\beta \left( \frac{\mathbf{p}^2}{2m} - \mu \right) \right] \right] \right). \quad (4.20)$$

The extremum condition (4.13) implies, from (4.17), that

$$\mu = 0. \quad (4.21)$$

In this case, we have two phases. In the condensed phase,  $\mu = 0$ , by imposing this condition and using (3.19)–(3.21), one gets the usual results:

$$P = - \frac{1}{\beta} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \ln \left( 1 - e^{-\beta \mathbf{p}^2 / 2m} \right), \quad (4.22)$$

$$\frac{N}{V} = \rho_c + \frac{1}{(2\pi)^3} \int d^3 \mathbf{k} \frac{1}{e^{\beta \mathbf{p}^2 / 2m} - 1}, \quad (4.23)$$

$$\frac{U}{V} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\frac{k^2}{2m}}{e^{\beta \mathbf{p}^2 / 2m} - 1}. \quad (4.24)$$

Other thermodynamic variables can be easily written out. They are found in textbooks [14] and can easily be inferred from Eqs. (3.19)–(3.23) by taking  $\mu = 0$  and  $\rho_b = 0$ .

The phase transition occurs when  $\rho_c$  is different from zero ( $\rho_c > 0$ ). From (5.20), it follows that, for free fields, the dependence of  $\rho_c$  on the temperature is

$$\rho_c = \frac{N}{V} - \frac{1}{(2\pi)^3} \int d^3 \mathbf{p} \frac{1}{e^{\beta \mathbf{p}^2 / 2m} - 1} \quad (4.25)$$

$$\equiv \frac{N}{V} - \left( \frac{m}{\pi \beta} \right)^2 \int d^3 \mathbf{k} \frac{1}{e^{k^2} - 1} \quad (4.26)$$

$$= \frac{N}{V} - \left( \frac{mT}{2\pi} \right)^{3/2} \zeta \left( \frac{3}{2} \right), \quad (4.27)$$

where  $\zeta$  in (4.25) stands for the Riemann zeta functions  $\zeta(x)$ . The condition  $\rho_c > 0$  implies that BEC occurs for  $T > T_c$ , where

$$T_c = \frac{2\pi}{m} \left( \frac{N}{V} \zeta^{-1} \left( \frac{3}{2} \right) \right)^{2/3}. \quad (4.28)$$

At the critical temperature, the order parameter is zero.

This example clearly illustrates how the approach works. It allows us to obtain the thermodynamics of the system and the temperature dependence of the order parameter.

### 5. THERMODYNAMICS OF THE BOGOLIUBOV CONDENSATE

In this section, we shall apply the approach to study a very special type of BE condensate. We shall study the condensation of zero-momentum bosons. The condensation of particles in the zero-momentum state is the key to understanding the superfluidity of helium 4 [9, 13].

In his pioneering work on Bose-Einstein condensation in an imperfect gas, Bogoliubov [15] argued that, when BEC occurs, one can treat the creation and destruction operators  $b(0)$  and  $b^+(0)$  appearing in the Fourier expansion of the field operator,

$$\hat{\psi} = V^{-1/2} \sum \hat{b}(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}} = \frac{\hat{b}_0}{\sqrt{V}} + \hat{\psi}' \equiv \hat{\psi}_0 + \hat{\psi}', \quad (5.1)$$

as classical variables. That means that one can replace these variables in zero-mode amplitudes by  $c$  numbers; i.e.,  $b_0 = b_0^+ = \sqrt{N_0}$ , with  $N_0$  being the number of particles in the condensate. The validity of treating these

variables as classical (*c*-numbers) is more reliable as the condensate becomes more and more occupied.

The fact that  $\hat{\Psi}_0$  in (5.1) can be considered a classical variable allowed him (and was later on generalized by Beliaev [16]) to propose an approximate scheme for calculations when BEC occurs. In the following, we present a very general method for dealing with the thermodynamics of condensation in the zero-momentum state.

An analog of Bogoliubov's approach in the functional approach is to associate a uniform background field ( $\mathbf{x}$  and  $\tau$  independent field configuration) with the Bogoliubov condensate. Our understanding of Bogoliubov's approach is that it enables one to identify the  $\mathbf{p} = 0$  state as the relevant degree of freedom and the variable can be obtained from a variation method.

Let us consider the thermodynamic properties of an interacting Bose gas described by Lagrangian (2.2) in the presence of a uniform background field, which we assume to be real and positive; that is, one writes

$$\Psi_b = \sqrt{\rho_b}. \quad (5.2)$$

For such a background, one can write, on general grounds,

$$\begin{aligned} \Gamma &= \Gamma(\rho_b, T) \\ &= V\beta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \tilde{\Gamma}^{(n,m)}(0, \dots, 0, 0, \dots, 0, T) \rho_b^{(n+m)/2}, \end{aligned} \quad (5.3)$$

where  $\tilde{\Gamma}^{(n,m)}$  stands for the Fourier transform of the one-particle irreducible Green's function of the theory taken at zero momenta:

$$\begin{aligned} \tilde{\Gamma}^{(n,m)}(0, \dots, 0) \\ = \tilde{\Gamma}^{(n,m)}(p_1, \dots, p_n, \dots, p_m) \Big|_{p_1 = p_2 = \dots = 0}. \end{aligned} \quad (5.4)$$

The volume term  $V$  in (5.3) comes from the  $\delta$  function involving momentum conservation, whereas  $\beta$  comes from energy conservation.

If one defines  $\Gamma_{\text{eff}}$  as

$$\begin{aligned} \Gamma_{\text{eff}}(\rho_b, T) \\ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \rho_b^{(m+n)/2} \tilde{\Gamma}^{(n,m)}(0, 0, 0, 0, \dots, 0, T), \end{aligned} \quad (5.5)$$

then, from (5.4) and (4.13), it follows that, for uniform configurations, one can write

$$\Omega(\rho_b, T) = V\Gamma_{\text{eff}}(\rho_b, T) = V\Gamma_{\text{eff}}(\rho_b, T). \quad (5.6)$$

We are now ready to establish the framework for studying zero-momentum Bose-Einstein condensation in field theory. The equation for determining the num-

ber of particles in the  $\mathbf{p} = 0$  state, for example, is equivalent to the equation

$$\left. \frac{d\Gamma_{\text{eff}}(\rho_b, T)}{d\rho_b} \right|_{\rho_b = \rho_c} = 0. \quad (5.7)$$

Solution of (5.7) leads to the determination of  $\rho_c$  as a function of  $\mu$  and  $T$ ; that is,

$$\rho_c = \rho_c(T, \mu). \quad (5.8)$$

The dependence of the condensate on  $T$  and  $\mu$  requires some care in order to determine the thermodynamic variables in the condensate phase. One needs to derive  $\Omega$  with regard to the proper variables first and then substitute the condensate configuration  $\rho_c$ .

The equation of state, for instance, in the presence of the condensate, should be inferred from the equations

$$P = -\Gamma_{\text{eff}}(\rho_c, \mu, T), \quad (5.9)$$

$$\frac{N}{V} = -\left( \frac{\partial \Gamma_{\text{eff}}(\rho_b, T)}{\partial \mu} \right)_{\rho_b = \rho_c}. \quad (5.10)$$

We say that there is BEC whenever there is a solution for Eqs. (5.7)–(5.10) satisfying the condition

$$\rho_c > 0. \quad (5.11)$$

From Eqs. (5.7) and (5.10), one should get the expression for the occupation of the  $\mathbf{p} = 0$  state as a function of  $\frac{N}{V}$  and  $T$ , that is,

$$\rho_c = \rho_c \left[ \frac{N}{V}, T \right]. \quad (5.12)$$

The system is supposed to exhibit two phases. In the noncondensed phase,

$$\rho_c = 0, \quad (5.13)$$

whereas in the condensed phase,

$$\rho_c = \rho_c \left[ \frac{N}{V}, T \right]. \quad (5.14)$$

The critical temperature is that which distinguishes the two phases, and  $\rho_c$  plays the role of the order parameter. One defines the critical temperature that for which

$$\rho_c \left[ \frac{N}{V}, T_c \right] = 0. \quad (5.15)$$

The whole thermodynamics can then be inferred from Eqs. (5.6)–(5.10) with all variables computed for the field-theoretical configuration that makes  $\Gamma$  an extremum. As far as condensation is concerned,  $\Gamma_{\text{eff}}(\rho, T)$  is the relevant potential. In terms of  $\Gamma_{\text{eff}}(\rho, T)$ , we can write, in the condensed phase,

$$P = -\Gamma_{\text{eff}}(\rho_c, T), \quad (5.16)$$



$$\frac{N}{V} = \left. \frac{\partial \Gamma_{\text{eff}}(\rho_b, T)}{\partial \mu} \right|_{\rho_b = \rho_c} \quad (5.17)$$

$$U = V \left. \frac{\partial [\beta \Gamma_{\text{eff}}(\rho_b, T)]}{\partial \beta} \right|_{\rho_b = \rho_c} \quad (5.18)$$

$$F = V \Gamma_{\text{eff}}(\rho_c, T) + \mu N, \quad (5.19)$$

$$S = \beta^2 V \left. \frac{\partial \Gamma_{\text{eff}}(\rho_b, T)}{\partial \beta} \right|_{\rho_b = \rho_c} - \mu N \beta, \quad (5.20)$$

whereas, in the noncondensed phase, the same expressions hold true with  $\rho_c = 0$ . Equations (5.16)–(5.20) are the basic set of equations describing the condensation of zero-momenta particle phenomena.

Although the field-theoretical approach has already been used to treat zero-momentum Bose–Einstein condensation, our proposal differs from the previous treatments. We have stressed the need for finding the dependence of the Gibbs potential on the order parameter and, from this, to derive the whole thermodynamics. Furthermore, our treatment is very general and explicit results depend only on our ability to determine  $\Gamma_{\text{eff}}(\rho, T)$ .

### 6. EQUATION OF STATE AND THE LONDON RELATION

Although Eqs. (5.16)–(5.20) depend on the thermodynamic potential  $\Gamma$ , which can be computed explicitly only under certain approximations (as we shall see in the next section), one can derive two relevant properties without explicit knowledge of  $\Gamma$ . These properties do not depend on the dynamics.

The first property that one expects in helium superfluid, once this phenomenon is related to BEC, is that the state of superfluid helium should be characterized by zero Gibbs energy. In fact, it follows from (5.16)–(5.20) that

$$U + PV = TS(T). \quad (6.1)$$

From (6.1), it follows that the Gibbs energy is zero in the condensed phase:

$$G = 0. \quad (6.2)$$

As a result of the third law of thermodynamics, one expects that

$$\lim_{T \rightarrow 0} TS(T) = 0. \quad (6.3)$$

By dividing (6.1) by  $N$  and then taking the zero temperature limit, we can predict the equation of state:

$$\frac{P}{\rho} = -\epsilon, \quad (6.4)$$

where  $\epsilon$  is the binding energy per atom.

The static fountain effect can be explained from the London relation [9]

$$dP = \frac{S}{V} dT. \quad (6.5)$$

We can see that this relation follows as a consequence of our basic equations describing the superfluid phase. In fact, if one derives the pressure as a function of  $T$ , one gets (independently of the details of the dynamics) from (3.19) that

$$\frac{dP}{dT} = \frac{\partial \Gamma(\rho_c, T) d\rho_c}{\partial \rho_c dT} - \left( \frac{\partial \Gamma(\rho_0, T)}{\partial T} \right)_{\rho_0 = \rho_c}. \quad (6.6)$$

From Eq. (6.1), the London relation (6.5) follows. We have shown that the London relation follows from equilibrium thermodynamics when BEC occurs. The London relation follows also from (6.1) since

$$dG = VdP - SdT. \quad (6.7)$$

### 7. LOOP EXPANSION

In this section, we present a specific method for predicting the thermodynamics of a condensed system. This method is the so-called loop expansion.

Let us consider the problem of expanding the grand partition functional in powers of  $\hbar$ . That is, one writes

$$\begin{aligned} \Omega(\psi_0(x), T) &= \Gamma(\psi_0(x), T) \\ &= \sum_{K=0}^{\infty} \Omega^{(K)}(\psi_0(x, \tau), T) \hbar^K, \end{aligned} \quad (7.1)$$

where  $\Gamma^{(K)}$  is the  $K$ th term in a series expansion of  $\Gamma$  in powers of  $\hbar$ .

It can be shown that, in field theory, the contribution to  $\Gamma^{(K)}$  comes from graphs having  $K$  loops; that is, for computing  $\Gamma^{(K)}$  in (7.1), one considers only those irreducible Green's function that have  $K$  loops. For this reason, expansion (7.1) is also referred to as loop expansion.

Loop expansion provides a systematic method for dealing with BEC. As pointed out in [7], the method is richer and more appropriate than the perturbative approach or the low-density approximation.

Let us denote by  $\Gamma^{(0)}$  the zero-loop graphs contributing to this approximation. One has, schematically,

$$\begin{aligned} \Gamma^{0(1,1)} &= \text{---} \times \text{---}, \\ \Gamma^{0(2,2)} &= \text{---} \times \times \text{---}. \end{aligned}$$

where the “vertex,” in this case, takes into account the nonlocality of the interaction.

At the zero-loop level (or classical level), one can write

$$\beta\Omega(\psi_0(\mathbf{x}, t)) = \int_0^\beta d\tau \int d^3\mathbf{x} \left( \psi_0^*(\mathbf{x}, \tau) \frac{\partial \psi_0}{\partial \tau}(\mathbf{x}, \tau) + \frac{\nabla \psi_0^*(\mathbf{x}, \tau) \cdot \nabla \psi_0(\mathbf{x}, \tau)}{2m} - \mu \psi_0^*(\mathbf{x}, \tau) \psi_0(\mathbf{x}, \tau) \right) \quad (7.2)$$

$$+ \int_0^\beta d\tau \int d^3\mathbf{x} \int d^3\mathbf{x}' \psi_0^*(\mathbf{x}, \tau) \psi_0(\mathbf{x}, \tau) \times V(\mathbf{x} - \mathbf{x}') \psi_0^*(\mathbf{x}', \tau) \psi_0(\mathbf{x}', \tau).$$

For a hard-sphere gas,

$$V(\mathbf{x} - \mathbf{x}') = \frac{\lambda}{2} \delta(\mathbf{x} - \mathbf{x}'). \quad (7.3)$$

The equation for the order parameter that follows from (4.13) and (7.2) is

$$\left( + \frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} - \mu \right) \psi_c(\mathbf{x}, \tau) = \lambda \psi_c(\mathbf{x}, \tau) (\psi_c^*(\mathbf{x}, \tau) \psi_c(\mathbf{x}, \tau)) = 0. \quad (7.4)$$

We notice at this point that Eq. (7.4) is the imaginary time version of the Gross-Pitaevskii equation and satisfies the periodic boundary condition (4.2). This is due to the finite-temperature description of a system in equilibrium.

For time-independent external electromagnetic external fields, we can search for a solution to the imaginary time Gross-Pitaevskii equation (7.4) in the form

$$\psi_c(\mathbf{x}, \tau) = e^{i\left(\frac{2\pi n}{\beta}\right)\tau} \psi_c(\mathbf{x}), \quad (7.5)$$

where  $n$  has to be an integer number ( $n = 0, 1, 2, \dots$ ) in order to meet the requirement of periodic field configuration (4.2).

The restriction to  $n = 0$  lead us to the time-independent order parameter complex field equation

$$(\nabla^2 + \mu) \psi_c(\mathbf{x}) = -\lambda \psi_c(\mathbf{x}) (\psi_c^*(\mathbf{x}) \psi_c(\mathbf{x})). \quad (7.6)$$

As a result of (7.2) and (7.5), one can see that, at the zero-loop level and for static fields, one has the following dependence of the thermodynamic potential on the order parameter:

$$\Omega^{(0)}(T, \mu) = \int d^3\mathbf{x} \left[ \nabla \psi_c^*(\mathbf{x}) \nabla \psi_c(\mathbf{x}) - \mu \psi_c^*(\mathbf{x}) \psi_c(\mathbf{x}) + \frac{\lambda}{2} (\psi_c^*(\mathbf{x}) \psi_c(\mathbf{x}))^2 \right]. \quad (7.7)$$

At the zero-loop approximation, one get, from (7.7), that

$$N = -\frac{\partial \Omega}{\partial \mu} = \int d^3\mathbf{x} \psi_c^*(\mathbf{x}) \psi_c(\mathbf{x}). \quad (7.8)$$

Defining the number of particles in the condensate by

$$N_c = \int d^3\mathbf{x} \psi_c^*(\mathbf{x}) \psi_c(\mathbf{x}), \quad (7.9)$$

one can see that, at the zero-loop level, all particles are in the condensate. That is,

$$N = N_c. \quad (7.10)$$

Let us proceed further and assume that the condensate is a Bogoliubov condensate. In this case, one can write the Gibbs potential as a function of the condensate  $\rho_b(x)$  as

$$\Omega^{(0)}(\rho_b) = V \left( -\mu \rho_b + \frac{\lambda}{2} \rho_b^2 \right). \quad (7.11)$$

The condition that  $\Omega^{(0)}$  be minimum,

$$\left. \frac{d\Omega^{(0)}}{d\rho_b} \right|_{\rho_b = \rho_c} = 0, \quad (7.12)$$

leads to the condition

$$\lambda \rho_c = \mu, \quad (7.13)$$

whereas Eqs. (3.13) and (5.5) imply

$$\rho_c = \frac{N}{V}. \quad (7.14)$$

This means that, at the zero-loop level (classical level), all particles are in the condensate. This approximation will then be useful for describing the system at zero temperature and at low densities [19].

The internal energy  $U$  is

$$U = \frac{\lambda}{2} V \left( \frac{N}{V} \right)^2, \quad (7.15)$$

whereas the pressure can be obtained from (7.11) and (7.14). We get

$$P = \frac{\lambda}{2} \left( \frac{N}{V} \right)^2. \quad (7.16)$$

One notes from (7.16) that, in order for the pressure to be positive, one has to have

$$\lambda > 0. \quad (7.17)$$

Condition (7.17) has been emphasized by Bogoliubov [15], who calls it the "condition of thermodynamic stability of a gas at absolute zero." This condition imposes a restriction on the possible types of interact-

ing forces that allow for BEC. Condition (7.17) follows also from the requirement that  $\Omega^0$  be minimum, that is,

$$\frac{d^2 \Omega^0}{d\rho^2} > 0. \tag{7.18}$$

The equation of state (7.16) was derived in [17] for a hard-sphere gas. Equations (7.13) and (7.15) can be found in some papers as well as in textbooks [18].

Let us give a general expression for the Gibbs potential within the one-loop approximation. Within this approximation, one writes

$$\Omega(\psi_0) = \Omega^{(0)}(\psi_0) + \Omega^{(1)}(\psi_0), \tag{7.19}$$

where  $\Omega^{(0)}$  is given by (7.2) and  $\Omega^{(1)}$  results in summing all contributions of the graphs containing just one loop [20, 21]. It is possible to show that

$$\begin{aligned} & e^{-\beta \Omega^{(1)}(\psi_0(x, \tau))} \\ &= \int \mathcal{D}\eta \mathcal{D}\eta^* \exp \left\{ - \int d\tau d^3x \frac{1}{2} (\eta^*, \eta) G' \begin{pmatrix} \eta \\ \eta^* \end{pmatrix} \right\} \tag{7.20} \\ & \equiv \det G'(\psi). \end{aligned}$$

Here,  $G'$  is a two-by-two matrix given by

$$\begin{aligned} & G' \tag{7.21} \\ &= \begin{bmatrix} K - \Delta E + 2\lambda \psi_0^* \psi_0(x, \tau) & \lambda \psi_0^{2*}(x, \tau) \\ \lambda \psi_0^2(x, \tau) & K - \Delta E + 2\lambda \psi_0^* \psi_0(x) \end{bmatrix}, \end{aligned}$$

where

$$K = -\frac{\nabla^2}{2m} + \frac{\partial}{\partial \tau}. \tag{7.22}$$

The final expression for the dependence of the Gibbs potential as a function of the background field  $\psi_0(x, \tau)$  is, from (7.20) and (7.21),

$$\Omega(\psi_0(x)) = \Omega^{(0)}(\psi_0(x)) + \beta^{-1} \ln \det G'(\psi_0). \tag{7.23}$$

The expression of the temperature-dependent order parameter is now obtained from the condition

$$\begin{aligned} & \left. \frac{\delta \Omega^0(\psi_0)}{\delta \psi_0(x)} \right|_{\psi_0 = \psi_c} \\ & + \beta^{-1} \frac{1}{\det G'(\psi_0)} \left. \frac{\delta (\det G'(\psi_0))}{\delta \psi_0(x)} \right|_{\psi_0 = \psi_c} = 0. \tag{7.24} \end{aligned}$$

From the above expression, one would be able to infer the temperature-dependent term

$$\psi_c(x, T). \tag{7.25}$$

The Gibbs potential is obtained by substituting (7.25) into (7.23).

We shall give explicit examples of this method in a future publication.

### 8. CONCLUSIONS

In this paper, we have developed further the notion that an understanding of many features of a Bose-Einstein condensed system can be achieved from the basic concept of spontaneous symmetry breakdown.

The central idea in this approach is that, in the condensed phase, a certain hydrodynamic potential ( $Z$ ) or a thermodynamic potential ( $\Omega$ ) is a functional of the order parameters. This idea is not new and can be traced back to the works of Ginsburg and Pitaevskii [22] and Anderson [3].

As far as the hydrodynamic equation is concerned, the functional depends on the phase and density of the condensate.

The generalized equilibrium condition

$$\left. \frac{\delta \mathcal{F}}{\delta \rho}(\rho, \partial_\mu \rho, \phi, \partial_\mu \phi) \right|_{\rho_c} = 0$$

lead to the Euler equation, whereas the generalized equilibrium condition

$$\left. \frac{\delta \mathcal{F}}{\delta \phi}(T, V, \rho, \partial_\mu \rho, \phi, \partial_\mu \phi) \right|_{\phi_c} = 0$$

gives the continuity equation.

In this paper, we have presented a method for computing this functional and made some specific predictions.

We have presented the approach in some detail for determining the thermodynamics in the presence of the condensate. In particular, we have presented the thermodynamics of the condensation of zero-momentum particles. This condensation is described by the Bogoliubov condensate.

For the Bogoliubov condensate, we can predict not only that for an increase in temperature by  $dT$  will there be an increase in the pressure  $dP$  given by the London relation [9]

$$dP = \frac{S}{V} dT$$

but also a zero-temperature equation of state

$$\frac{P}{\rho} = -\epsilon,$$

where  $\epsilon$  is the binding energy per atom.

We have presented a general method, loop expansion, for predicting the thermodynamics of a condensed system. Some specific results were obtained for the zero-loop approximation.

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