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PSEUDOCLASSICAL MODEL OF SPINNING PARTICLE WITH ANOMALOUS MAGNETIC MOMENTUM

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Pseudoclassical Model of Spinning Particle with Anomalous Magnetic Momentum

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A generalization of the pseudoclassical action of a spinning particle in the presence of an anomalous magnetic momentum is given. The action is written in reparametrization and supergauge invariant form. The Dirac quantization, based on the Hamiltonian analyses of the model, leads to the Dirac-Pauli equation for a particle with an anomalous magnetic momentum in an external electromagnetic field. Due to the structure of first-class constraints in that case, one has to solve an operators ordering problem, to make the quantization consistent.

11.15.Kc, 11.10.Ef

In recent years numerous classical models of relativistic particles and superparticles were discussed intensively. First, the interest to such models was initiated by the close relation with the string theory, but now it is also clear that the problem itself has an important meaning for the deeper understanding of the structure of quantum theory. One of the basic, in the above mentioned set of classical models, is the pseudoclassical model of Fermi particle with spin 1/2, proposed first in the works [1, 2], investigated and quantized in the works [1–9]. The model can be formulated in gauge invariant (reparametrization and supersymmetric) form. The action of the model in an external electromagnetic field has the form [9]:

$$S = \int_0^1 \left[-\frac{\dot{x}^2}{2e} - e\frac{m^2}{2} - g\dot{x}^\alpha A_\alpha + igeF_{\alpha\beta}\psi^\alpha\psi^\beta + i\left(\frac{\dot{x}_\alpha\psi^\alpha}{e} - m\psi^5\right)\chi - i\psi_n\dot{\psi}^n\right] d\tau\,, \tag{1}$$

where x^α, e are even and ψ^n, χ are odd variables dependent on τ, which plays the role of the time in this theory, $A_\alpha(x)$ is an external electromagnetic field potential, $F_{\alpha\beta}(x)$ is the Maxwell strength tensor, and g the electrical charge. Greek indices run over $\overline{1,3}$ and Latin indices run over $\overline{1,3}, 5$. The metric tensors: $\eta^{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$ and $\eta^{mn} = \mathrm{diag}(1,-1,-1,-1,-1)$. There are two gauge transformation in the theory with the action (1), reparametrizations,

$$\delta x = \dot{x}\xi\,, \quad \delta e = \frac{d}{dt}(e\xi)\,, \quad \delta\psi^n = \dot{\psi}^n\xi\,, \quad \delta\chi = \frac{d}{dt}(\chi\xi)\,, \tag{2}$$

and supertransformations,

$$\delta x = i\psi\epsilon\,, \quad \delta e = i\chi\epsilon\,, \quad \delta\chi = \dot{\epsilon}\,, \quad \delta\psi^\alpha = \frac{1}{2e}(\dot{x}^\alpha + i\chi\psi^\alpha)\epsilon\,,$$
$$\delta\psi^5 = \left[\frac{m}{2} - \frac{i}{me}\psi^5\left(\dot{\psi}^5 - \frac{m}{2}\chi\right)\right]\epsilon\,, \tag{3}$$

where ξ are even and ϵ odd τ-dependent parameters. The spinning degrees of freedom in such a model are described by Grassmannian variables, that's why the model is called pseudoclassical. The quantization of the model in different ways leads to the quantum mechanics of the Dirac particle, is very instructive and creates many useful analogies with problems of quantization of gauge field theories.

In this letter we present a generalization of the model when an anomalous magnetic momentum of the particle is present. The relativistic quantum theory of spinning particle, which has both the "normal" magnetic momentum $e/2m$ and an "anomalous" magnetic momentum μ, was formulated by Pauli [10]. In this case he generalized the Dirac equation to the following form:

$$\left(\hat{\mathcal{P}}_\nu \gamma^\nu - m + \frac{\mu}{2}\sigma^{\alpha\beta}F_{\alpha\beta}\right)\Psi(x) = 0, \tag{4}$$

where $\hat{\mathcal{P}}_\nu = i\partial_\nu - gA_\nu(x)$, $\sigma^{\alpha\beta} = \frac{i}{2}\left[\gamma^\alpha, \gamma^\beta\right]_-$.

Our aim is to write an analog of the action (1), whose quantization gives the Dirac-Pauli theory.

We propose the following pseudoclassical action for spinning particle with anomalous momentum:

$$S = \int_0^1 \left[-\frac{\dot{x}^2}{2e} - e\frac{M^2}{2} - \dot{x}^\alpha\left(gA_\alpha - 4i\mu\psi^5 F_{\alpha\beta}\psi^\beta\right) + igeF_{\alpha\beta}\psi^\alpha\psi^\beta \right.$$
$$\left. + i\left(\frac{\dot{x}_\alpha\psi^\alpha}{e} - M^*\psi^5\right)\chi - i\psi_n\dot{\psi}^n\right]d\tau, \tag{5}$$

where $M = m + 2i\mu F_{\alpha\beta}\psi^\alpha\psi^\beta$, and $M^* = m - 2i\mu F_{\alpha\beta}\psi^\alpha\psi^\beta$.

One can check that there are also two types of gauge transformations, under which the actions is invariant. The first one, which is the reparametrization, has the same form as (2). The second one is a supertransformation,

$$\delta x = i\psi\epsilon, \quad \delta e = i\chi\epsilon, \quad \delta\chi = \dot{\epsilon}, \quad \delta\psi^\alpha = \frac{1}{2e}(\dot{x}^\alpha + i\chi\psi^\alpha)\epsilon,$$
$$\delta\psi^5 = \left[\frac{M}{2} - \frac{i}{me}\psi^5\left(\dot{\psi}^5 + 2\mu F_{\alpha\beta}\dot{x}^\alpha\psi^\beta - \frac{M^*}{2}\chi\right)\right]\epsilon, \tag{6}$$

The form of the transformation (6) depends on an external field and anomalous momentum and generalizes the transformation (3).

Going over to the Hamiltonian formalism, we introduce the canonical momenta:

$$p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha} = -\frac{1}{e}(\dot{x}_\alpha - i\psi_\alpha\chi) - gA_\alpha + 4i\mu\psi^5 F_{\alpha\beta}\psi^\beta,$$
$$P_e = \frac{\partial L}{\partial \dot{e}} = 0, \quad P_\chi = \frac{\partial_r L}{\partial\dot{\chi}} = 0, \quad P_n = \frac{\partial_r L}{\partial\dot{\psi}^n} = -i\psi_n\ . \tag{7}$$

It follows from the equation (7) that there exist primary constraints $\Phi_a^{(1)} = 0$,

$$\Phi_a^{(1)} = \begin{cases} \Phi_1^{(1)} = P_\chi\ , \\ \Phi_2^{(1)} = P_e\ , \\ \Phi_{3n}^{(1)} = P_n + i\psi_n\ . \end{cases} \tag{8}$$

We construct the Hamiltonian $H^{(1)}$, according to the standard procedure (we use the notations of the book [12]),

$$H^{(1)} = H + \lambda_a\Phi_a^{(1)}, \ \text{where}\ H = (P\dot{q} - L)|_{\frac{\partial_r L}{\partial\dot{q}} = P}\,, \quad q = (x, e, \chi, \psi^n)\ , \tag{9}$$

and get for H:

$$H = -\frac{e}{2}\left(\left(\mathcal{P}_\alpha + 4i\mu\psi^5 F_{\alpha\beta}\psi^\beta\right)^2 + 2igF_{\alpha\beta}\psi^\alpha\psi^\beta - M^2\right) - i\left(\mathcal{P}_\alpha\psi^\alpha - M\psi^5\right)\chi, \tag{10}$$

where $\mathcal{P}_\alpha = -p_\alpha - gA_\alpha$.

From the conditions of the conservation of the primary constraints $\Phi_{1,2}^{(1)}$ in time τ, $\dot{\Phi}_{1,2}^{(1)} = \left\{\Phi_{1,2}^{(1)}, H^{(1)}\right\} = 0$, we find the secondary constraints $\Phi_{1,2}^{(2)} = 0$,

$$\Phi_1^{(2)} = \mathcal{P}_\alpha\psi^\alpha - M\psi^5 = 0, \tag{11}$$
$$\Phi_2^{(2)} = \mathcal{P}^2 + 8i\mu\psi^5 F_{\alpha\beta}\mathcal{P}^\alpha\psi^\beta + 2igF_{\alpha\beta}\psi^\alpha\psi^\beta - M^2 = 0, \tag{12}$$

and the same conditions for the constraints $\Phi_{3n}^{(1)}$ give equations for the determination of λ_{3n}. Thus, the Hamiltonian H is proportional to constraints, as one can expect in the case of a reparametrization invariant theory,

$$H = i\chi\Phi_1^{(2)} - \frac{e}{2}\Phi_2^{(2)}. \tag{13}$$

No more secondary constraints arise from the Dirac procedure, and the Lagrange's multipliers λ_1 and λ_2 remain undetermined, in perfect correspondence with the fact that the number of gauge transformations parameters equals two for the theory in question [12]. Now we are going over from the initial set of constraints $\left(\Phi^{(1)}, \Phi^{(2)}\right)$ to equivalent one $\left(\Phi^{(1)}, T\right)$, where

$$T = \Phi^{(2)} + \frac{i}{2}\frac{\partial_r \Phi^{(2)}}{\partial \psi^n}\Phi_{3n}^{(1)}. \tag{14}$$

The new set of constraints has an advantage because of it can be divided in a set of first-class constraints, which is $\left(\Phi_{1,2}^{(1)}, T\right)$ and second-class ones, which is $\Phi_{3n}^{(1)}$. In our case we perform only a partial gauge fixing, by imposing the supplementary gauge conditions $\Phi_{1,2}^{G} = 0$ to the primary first-class constraints $\Phi_{1,2}^{(1)}$,

$$\Phi_1^G = \chi = 0, \quad \Phi_2^G = e = 0. \tag{15}$$

One can check that the conditions of the conservation in time of the supplementary constraints (15) give equations for determination of the multipliers λ_1 and λ_2. Thus, on this stage we reduced our Hamiltonian theory to one with the first-class constraints T and second-class ones $\varphi = \left(\Phi^{(1)}, \Phi^{G}\right)$.

For the quantization we will use the so called Dirac method for systems with first-class constraints [11], which, being generalized to the presence of second-class constraints, can be formulated as follow: the commutation relations between operators are calculated according to the Dirac brackets with respect to the second-class constraints only; second-class constraints operators equal zero; first-class constraints as operators are not zero, but, are considered in sense of restrictions on state vectors. All the operators equations have to be realized in some Hilbert space.

In our case, the sub-set of second-class constraints $\left(\Phi_{1,2}^{(1)}, \Phi^{G}\right)$ has a special form [12], so that one can use it for eliminating of the variables e, P_e, χ, P_χ, from the consideration, then for the rest of the variables x, p, ψ^n, the Dirac brackets with respect to the constraints φ reduce to ones with respect to the constraints $\Phi_{3n}^{(1)}$ only and can be easy calculated,

$$\{x^\alpha, p_\beta\}_{D(\Phi_{3n}^{(1)})} = \delta_\beta^\alpha, \quad \{\psi^n, \psi^m\}_{D(\Phi_{3n}^{(1)})} = \frac{i}{2}\eta^{nm}, \tag{16}$$

while others Dirac brackets vanish. Thus, the commutation relations for the operators $\hat{x}, \hat{p}, \hat{\psi}^n$, which correspond to the variables x, p, ψ^n respectively, are

$$[\hat{x}^\alpha, \hat{p}_\beta]_- = i\{x^\alpha, p_\beta\}_{D(\Phi_{3n}^{(1)})} = \delta_\beta^\alpha, \quad \left[\hat{\psi}^m, \hat{\psi}^n\right]_+ = i\{\psi^m, \psi^n\}_{D(\Phi_{3n}^{(1)})} = -\frac{1}{2}\eta^{mn}. \tag{17}$$

Besides, the operator equations hold:

$$\hat{\Phi}_{3n}^{(1)} = \hat{P}_n + i\hat{\psi}_n = 0. \tag{18}$$

The commutation relations (17) and the equations (18) can be realized in the space of the four columns $\Psi(x)$ dependent on x^α, as: $\hat{x}^\alpha$ are operators of multiplication, $\hat{p}_\alpha = -i\partial_\alpha$, $\hat{\psi}^\alpha = \frac{i}{2}\gamma^5\gamma^\alpha$, and $\hat{\psi}^5 = \frac{i}{2}\gamma^5$, where γ^n are the γ-matrices $(\gamma^\alpha, \gamma^5)$, $[\gamma^m, \gamma^n]_+ = 2\eta^{mn}$. The first-class constraints $\hat{T}$ as operators have to annihilate the physical vectors; in virtue of (18), (14) these conditions are reduced to the equations:

$$\hat{\Phi}_{1,2}^{(2)}\Psi(x) = 0, \tag{19}$$

where $\hat{\Phi}_{1,2}^{(2)}$ are operators, which correspond to the constraints (11), (12). There is no ambiguity in construction of an operator $\hat{\Phi}_1^{(2)}$, according to the classical function $\Phi_1^{(2)}$ from (11). Thus, taking into account the realizations of the commutation relations (17), one easily can see that the first equation (19) reproduces the Dirac-Pauli equation (4). As to the construction oh the operator $\hat{\Phi}_2^{(2)}$, according to the classical function $\Phi_2^{(2)}$ from (12), we meet here an ordering problem since the constraint $\Phi_2^{(2)}$ contains terms with products of the momenta and functions of the coordinates, namely terms of the form $p_\alpha A^\alpha$, $p_\alpha F^{\alpha\beta}$. For such terms we choose the symmetrized form of the corresponding operators,

$$p_\alpha A^\alpha \rightarrow \frac{1}{2}\left[\hat{p}_\alpha, A^\alpha(\hat{x})\right]_+ , \quad p_\alpha F^{\alpha\beta} \rightarrow \frac{1}{2}\left[\hat{p}_\alpha, F^{\alpha\beta}(\hat{x})\right]_+ , \tag{20}$$

which, in particular, provides the hermiticity of the operator $\hat{\Phi}_2^{(2)}$. But the main reason is, the correspondence rule (20) provides the consistency of the two equations (19). Indeed, in this case we have

$$\hat{\Phi}_2^{(2)} = \left(\hat{\Phi}_1^{(2)}\right)^2 , \tag{21}$$

and the second equation (19) appears to be merely the consequence of the first equation (19), i.e. of the Dirac-Pauli equation (4). To verify the validity of (21), one needs only to take into account that the operator, which corresponds to the term $8i\mu\psi^5 F_{\alpha\beta}\mathcal{P}^\alpha\psi^\beta$ in the constraint $\Phi_2^{(2)}$ (12), in virtue of the structure of the γ-matrices, can be written in the form:

$$\begin{aligned} &8i\mu\psi^5 F_{\alpha\beta}\mathcal{P}^\alpha\psi^\beta \rightarrow i\mu\left[F_{\alpha\beta}(\hat{x}), \hat{\mathcal{P}}^\alpha\right]_+ \gamma^\beta = \\ &2i\mu F_{\alpha\beta}(\hat{x})\hat{\mathcal{P}}^\alpha\gamma^\beta - \mu\partial^\alpha F_{\alpha\beta}(\hat{x})\gamma^\beta = \left[\hat{\mathcal{P}}_\alpha\gamma^\alpha, \frac{\mu}{2}\sigma^{\alpha\beta}F_{\alpha\beta}(\hat{x})\right]_+ \end{aligned} \tag{22}$$

To complete the quantization, one has to present an inner product in the space of realization of commutation relations. The general method of its construction in the frame of the Dirac method we used, is unfortunately still unknown. Nevertheless, in this concrete case, the space of physical vectors, obeying the condition (19), can be transformed into a Hilbert space, if one takes for the inner product ordinary scalar product of solutions of the Dirac equation, which does not depend on x^0, in spite of

the integration is fulfilled over x^i only. It is not difficult to verify that the introduced operators, obey of natural properties of hermiticity, which are known from the Dirac relativistic mechanics. In particular, the operator $\hat{p}_0$, which has to be considered on the same foot with $\hat{p}_i$, is also hermitian on the solutions of the Dirac-Pauli equation (4), in virtue of the above mentioned conservation of the scalar product in time x^0.

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