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DESCRIPTION OF CHAOS-ORDER TRANSITION WITH RANDOM MATRICES WITHIN THE MAXIMUM ENTROPY PRINCIPLE

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Abstract

The Deformed Gaussian Orthogonal Ensemble introduced earlier is here developed for large dimensional matrices. Both the spacing and eigenvector distributions are studied and compared to other ones suggested for the chaos-order transition problem. The concept of a universal lower entropy with respect to the Gaussian Orthogonal Ensemble entropy is proved very useful.

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It is expected that for systems whose classical motion is neither regular nor fully chaotic, the statistical behaviour is intermediate between the Poisson and the Gaussian Orthogonal Ensemble (GOE) limits[1−16]. Several authors have suggested empirical functional forms for the level spacing distributions. We cite here the Brody distribution[11], the Berry-Robnik distribution[5] and the Robnik distribution [6]. What one is usually seeking are intermediate distributions that exhibit a degree of universality close to that of their Poisson and Wigner (GOE) limits[17]. Further, the distribution of eigenvectors of a system that is fully chaotic (GOE) is known to be of the Porter-Thomas form [18]. It is therefore hoped that in the intermediate case the spacing distribution alluded to above dictates to some extent the form of the eigenvector distribution. This way one would have a fully universal description of systems intermediate between Chaos and order.

Several of the above questions have been discussed in the past. In particular we cite the work of Alhassid and Collaborators [19−22], Lenz and Haake[7−8], and Guhr and Weidenmüller [23]. Our aim in the present work is to develop a general framework through which all of the above questions can be addressed. We shall shown that it is possible to derive a joint distribution for the spacings and eigenvectors valid in the intermediate regime. From this distribution, the spacing distribution is obtained by interating out the eigenvectors and similarly for the eigenvector distribution. The cases of 2×2 and 3×3 matrices have already been worked out analytically [24−25]. Here we present a thorough numerical study for large dimentional matrices.

Our theory is based on the maximum entropy principle, which we briefly outline in the following.

We define the entropy associated with the distribution $P(H)$ of the Hamitonian ensemble H

$$S = -\int dH\, P(H) \ln P(H) \tag{1}$$

g We now maximize S subject to the usual constraints of the GOE.

$$\langle \mathrm{Tr}\, H^2 \rangle \equiv \int dH\, P(H) \mathrm{Tr}\, H^2 = \mu \tag{2}$$

$$\langle 1 \rangle = 1 \quad , \tag{3}$$

and obtain

$$P_{\mathrm{GOE}}(H) = \exp[-\lambda_0 - 1 - \alpha_0 \,\mathrm{Tr}\, H^2] \tag{4}$$

$$\alpha_0 = \frac{N(N+1)}{4\mu} \quad , \quad \exp(-\lambda_0 - 1) = 2^{-\frac{N}{2}} \left(\frac{\pi}{2\alpha_0}\right)^{-\frac{N(N+1)}{4}} \tag{5}$$

Denoting the eigenvalues by $E_1, E_2, \ldots E_N$ and amplitudes $C_1, C_2, \ldots, C_N$, one can easily obtain the joint distribution function

$$P\,(E_1, E_2, \ldots E_N; C_1, C_2, \ldots C_N) \equiv P\,(E_1, \ldots E_N)\; P\,(C_1, \ldots C_N) \tag{6}$$

from which the spacing distribution $P(s)$ and amplitude (eigenvector) distribution $P(c)$ can be derived. When $N \to \infty$, we obtain the Wigner distribution

$$P_W(s) = \frac{\pi}{2D^2}\, s \,\exp\left[-\frac{\pi}{4}\frac{s^2}{D^2}\right] \tag{7}$$

and the Porter-Thomas distribution

$$P_{\mathrm{PT}}(c) = \left(\frac{N}{2\pi}\right)^{1/2} \exp\left(-\frac{N}{2}\, C^2\right) \tag{8}$$

We should emphasize that the joint distribution function Eq.(6), implies no correlations between the E - and C - distributions. This is a consequence of the GOE, namely P(H) is invariant under arbitrary rotation of the basis.

Before turning our attention to the intermediate case, it is helpful to mention that constraint (2) gives rise to the Gaussian distribution (4) with the second moment

$$\langle H_{ij}\, H_{kl} \rangle = (\delta_{ik}\, \delta_{jl} + \delta_{jk}\, \delta_{jk})\, \frac{1}{4\alpha_0} \quad , \tag{9}$$

with α_0 independent of the label. Further, the GOE entropy can be straightforwardly derived from (1), (4) and (5). We find for the entropy per degree of freedom (there are $N(N+1)/2$ degrees of freedom for our symmetric real matrices)

$$s_{\mathrm{GOE}} = \frac{1}{2}\left(1 + \ln\frac{\pi}{2\alpha_0}\right) + (N+1)^{-1} \ln 2 \tag{10}$$

Thus simple universal features of the intermediate distribution we are seeking are 1) a second moment that depends on the label, and 2) an entropy per degree of freedom that is *smaller* than s_{GOE}, Eq.(10).

Within the maximum entropy principle, an intermediate distribution can be defined through the addition of more constraints. Here we use the simplest possible one that allows α_0 to depend on the label. If we devide the random matrix H into four blocks and introduce the following notation (see Fig. 1)

$$H = PHP + QHQ + PHQ + QHP$$
$$P \equiv \textstyle\sum_{i=1}^{M} |i><i| \tag{11}$$

with $P + Q = 1, P^2 = P, Q^2 = Q, PQ = QP = 0$ then the desired constraint reads

$$\langle \mathrm{Tr}\, PHQHP \rangle = \nu \tag{12}$$

We now maximize S subject to the GOE constraints (2) and (3) and the new one (12), to obtain the intermediate distribution. By fixing the value of $\langle \mathrm{Tr}\,(PHQHP) \rangle$, we are deforming the GOE. Of course the system still maintains full axial symmetry about the P-"direction" the new ensemble, which we called the Deformed Gaussian Orthogonal Ensemble (DGOE) in Ref.[20] is invariant under transformation that leaves vectors in P unchanged. Further understanding of the ensemble can be gained by spelling out the second moment

$$\langle H_{ij}\, H_{k\ell} \rangle = (\delta_{ik}\, \delta_{j\ell} + \delta_{i\ell}\, \delta_{jk})\, \frac{1}{4\alpha + 2\beta\, |\delta_{ki} - \delta_{\ell i}|} \quad , \tag{13}$$

clearly showing the label dependence mentioned above.

The DGOE distributions we obtain has the general form[24)]

$$P_{\rm DGOE}(H) = P_{\rm GOE}(H)\, \exp[-\beta\, {\rm Tr}(PHQHP)] \left[1+\frac{\beta}{2\alpha}\right]^{\frac{M(N-M)}{2}} \qquad (14)$$

The entropy per degree of freedom of the DGOE is easily obtained (more precisely one speaks of the information content, I, of DGOE relative to the GOE[26–27)]

$$s_{\rm DGOE} = s_{\rm GOE} - \frac{M(N-M)}{N(N+1)} \ln\left(1+\frac{\beta}{2\alpha}\right) \qquad (15)$$

or

$$I \equiv s_{\rm GOE} - s_{\rm DGOE} = \frac{M(N-M)}{N(N+1)} \ln\left(1+\frac{\beta}{2\alpha}\right) \qquad (15')$$

Eq.(15) clearly shows that a system described by the DGOE is less chaotic, since the difference $s_{\rm DGOE} - s_{\rm GOE} < 0$. Said differently, the information content I is positive. Further, the degree of order in the DGOE is measured by both M, the dimention of the symmetric non-diagonal block matrix, and β. For very large matrices ($N \to \infty$), the DGOE is not much different from the GOE if M is taken to be small. For M comparable to N, namely $N \to \infty \Rightarrow \frac{M}{N} \equiv n < 1$ finite, we obtain a saturation limit for I,

$$I = n(n-1) \ln\left(1+\frac{\beta}{2\alpha}\right) \qquad (16)$$

Our detailed numerical calculation described below corroborates our discussion above, namely that for fixed value of β/α, the amplitude and the spacing distributions saturate with respect to the dimension of the matrix. This is the universal feature we are seeking.

Before presenting the numerical results we mention that in Ref.24) we have worked out fully analyticatly the cases of 2×2 and 3×3 matrices. The 2×2 case coincides with the

results of Ref.12b) and with Alhassid and Levine[20)]. It differs, however, from that of Lenz and Haake[8)] as these authors devise an interpolating formula; it is guaranteed that the spacing distribution is Poissonian when $\beta = \infty$ and Wigner when $\beta = 0$. There is no way, however, to trace this transition to special characteristcs of the ensemble of matrices. The 3×3 case ($M = 1, I = \frac{1}{6}\ln(1+\frac{\beta}{2\pi})$) involves a complicated double integral (see Ref.24) for details). However two limiting cases are worth citing here. The $\beta \to \infty$ limit

$$P(s) = 2\sqrt{\frac{38}{8\pi}} \left\{1+\sqrt{\frac{\pi\alpha}{6}}\, s\, e^{\alpha s^2/6}\, {\rm efr}\left[\sqrt{\frac{\alpha}{6}}\, 5\right]\right\} \exp\left(-\frac{2\pi}{3}s^2\right) \qquad (17)$$

$$P(c) = \frac{1}{4}\left[3\delta(c)+\delta(c-1)+\delta(c+1)\right] \qquad (18)$$

and the small spacing limit $s \to 0$

$$P(s) = \left[0.86\left[1+\frac{\beta}{2\pi}\right]^{1/2} + 0.64\right] s \qquad (19)$$

The $\beta \to \infty$ limit corresponds to the fully regular case, and Eq.(17) teaches us that for small dimension matrices, the regular case, though obviously with no level repulsion, is far from being Poissonian (exponential). The amplitude distribution, Eq.(18) contains three delta functions ($N = 3$), and is clearly *not* a Porter-Thomas (Gaussian). All of the above is expected for small matrices. The interesting conclusion from this exercise rests on Eq.(19), which clearly shows that for increasing β, $P(s)$ goes as $0.86\frac{\beta}{2\pi}^{1/2}$ and so there is *always* level repulsion (see, however Berry and Robnik [5)] and in accordance with us, Robnik[6)]). The level repulsion goes away when β is rigorously set equal to ∞. This, when generalized to larger matrices indicates that the $\beta \to \infty$ limit must correspond to two *decoupled* GOES and $\beta \to 0$ to the case of two fully mixed GOE's (and thus a single doubly larger GOE). In fact it can be shown that our DGOE can be reformulated in such a way that the quantity $\frac{1}{\left(1+\frac{\beta}{2\alpha}\right)^{1/2}} \equiv \lambda$ acts as a coupling constant in a description involving the following Hamiltonian

$$H(\lambda) = (PH_GP + QH_GP) + \lambda(PH_GQ + QH_GP) \quad , \tag{20}$$

$$\equiv H_0 + \lambda V \tag{21}$$

λ taking the values from $0(\beta = \infty)$, which is the regular case, to 1 ($\beta = 0$), which is the fully chaotic case ($H(\lambda = 1) = H_G$). Note that PH_GP, QH_GQ, PH_GQ and QH_GP are all random matrices.

Several authors have addressed the problem of chaos-order transition using the decomposition (21) for H. In particular we mention Guhr and Weidenmüller[23], who treat the problem of isospin mixing in compound nuclear reactions. There is also the work of Lenz and Haake[7-8] who consider a more general case of H_0 and V belonging to different ensembles (e.g. H_0: GOE, V: GUE). Alhassid and Levine[16] considered the same problem using Dyson's random walk formulation and they obtained the s- and c-distributions for the 2×2 matrices case. The way we formulate the chaos-order transition, through the DGOE, Eq. (14), allows a realistic large dimensional numerical study of *both* the spacing and the eigenvector distributions. Before proceeding we mention that for large matrix Hamiltonians H_0 and V, the DGOE information content, I, is given by (see Eq. (15))

$$I \equiv s_{H_G} - s_H = 2n(1-n)\ln(1/\lambda) \quad . \tag{22}$$

For $\lambda = 0$, I is infinitely positive. Eq. (22) is an interesting way to quantitatively measure how much more information is contained in $H(\lambda)$ with respect to H_G.

We have considered ensemble of matrices of varying dimensions pertaining to the DGOE as described before. The spacing distributions for $\beta = 10^4\alpha$ is shown in figure 2 for $N = 100, 400$ and 800. We took $\frac{M}{N} = \frac{1}{2}$. We see clearly that as N increases the distribution becomes independent on N as our entropy argument based on Eq. (16) tell us. We next turn to the variation of the distribution for large enough N ($N = 800$) with

β. This is shown in Fig. 3. Also shown in the figure are the corresponding Dyson-Mehta Δ_3 distributions[28]. The gradual shift of the maximum from the Wigner one towards small s is gradual. At very large values of β (small λ), the distribution is not exactly Poissonian simply because though repulsion is gone, there is still level repulsion in the diagonal PHP, QHQ block matrix distributions. In this respect, we are in complete agreement with Guhr and Weidenmuller [23]. Note that the Δ_3 behaviour saturates at large β below the Poisson value.

We next turn to the eigenvector distribution eigenvector distribution. We plot in fig. 3c $P(y)$ vs. $\ln(y/\langle y\rangle); y \equiv c^2$. We show the histogram of our numerical diagonalization, the Porter-Thomas distribution (dashed curve) and the χ^2-distribution suggested by Alhassid and Levine [19],

$$P_\nu(y) = \left[\frac{\nu}{2\langle y\rangle}\right]^{\nu/2} \frac{y^{\frac{\nu}{2}-1}\exp[-\nu y/2\langle y\rangle]}{\Gamma(\nu/2)} \qquad y \equiv c^2 \tag{23}$$

where ν is a parameter that measures the number of degrees of freedom. When $\nu = 1$, one recovers the PT distribution. From the figure, we see clearly, that when constraining both the spacing and the amplitude distribution through the DGOE, the resulting $P(y)$ deviates appreciably from Eq. (23), for large values of β (small λ).

The value of ν that best fits the "data" is determined by inverting the equation[19]

$$\langle \ln y/\langle y\rangle\rangle = \psi(\nu/2) - \ln(\nu/2)\ , \tag{24}$$

Where $\psi(x)$ is the digamma function. The quantity $\langle \ln y\rangle$ which is constructed from the "data" is plotted in Fig. (4) for several values of $N(M = \frac{N}{2})$ vs n/N where n denotes the label of the eigenvector coefficient $|E_k\rangle = \sum_n C_n^k|n\rangle$. We see a great of amount of fluctuation that is smoothed out when averaging over an ensemble of matrices (the curve for $N = 400$). We verified that the ensemble average is close to the n-average. The larger dimentional cases ($N = 600$ and 800) are seen to fluctuate a lot and their average, needed to obtain ν above, was found by the simple n-average mentioned above. We notice from the figure a certain degree of saturation attained for $N = 600$. The average for $N = 600$ is close to that for $N = 800$.

We should mention that the amplitude distribution for very large β shown in figure 3c which deviates appreciably from the χ^2-distribution, Eq. (23), corresponds to a situation of two almost completely decoupled GOE'S (PHP and QHQ), since $\lambda \sim 0.0$. For such a case one would expect to better account for the "data" (histogram) by using a sum of two χ^2-distributions, one for PHP and the other for QHQ, namely

$$P_{\nu,\mu}(y) = a\,P_\nu(y) + b\,P_\mu(y) \quad ; \qquad a + b = 1 \tag{25}$$

The surprisal procedure used by Alhassid and Levine can be applied for this more general case to find a, b, ν and μ. The result of our calculation is shown in Fig. 5. For $\beta = 10^4\,\alpha$ (Fig. 5a), we find $\alpha = 0.25$, $b = 0.75$, $\nu = 0.96$ and $\mu = 0.72$, whereas for $\beta = 5\times 10^4\,\alpha$ (Fig. 5b), we find $a = 0.23$, $b = 0.77$, $\nu = 0.98$ and $\mu = 0.54$. The values of these parameter satisfy the general surprisal consistancy condition $\langle \ln y/\langle y\rangle\rangle = a\left(\psi(\frac{\nu}{2}) - \ln\frac{\nu}{2}\right) + b\left(\psi(\frac{\mu}{2}) - \ln\frac{\mu}{2}\right)$.

In conclusion, we have presented a detailed numerical study of the Deformed Gaussian Orthogonal Ensemble[24)] for large matrices. Our guiding principle has been the dimension-independence of the reduction in the entropy with respect to that of the GOE. The general conclusions drawn from our study is that it is possible to discuss both the spacing distribution and amplitude distribution using the same ensemble appropriates for intermediate situation between chaos and order. Our theory should be useful for the study of nuclear statistics, symmetry breaking, as well as the general question of chaos-order transition. We are presently applying our theory to the isospin mixing problem considered in Ref. 23.

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FIGURE CAPTIONS

Fig. 1. The block structure of the Hamiltonian. See text for details.

Fig. 2. The spacing distribution of the DGOE with $\beta = 10^4\alpha$ for different dimensions of PHP is $\frac{1}{2}N$.

Fig. 3. The spacing distribution $P(s)$, Fig. 3a, spectral rigidity $\Delta_3(L)$, Fig. 3b, and the intensity distribution $P(y)$, Fig. 3c for several values of β. The dimension of the Hamiltonian matrix is 700 and that of PHP is 350.

Fig. 4. The surprisal, $\langle \ln y/\langle y\rangle\rangle$ v.s. n/N, where n represents the label of the eigenvector coefficient (see text for details). The dashed curve, $N = 100$, the full curve, $N = 400$ and the dotted curve $N = 800$. The dashed-dotted curve represents the Porter-Thomas result.

Fig. 5. The intensity distribution for $N = 700$ and $\beta = 10^4\alpha$ (5a) and $5 \times 10^4\alpha$ (5b). The full curve is obtained from Eq. 25, see text for details.

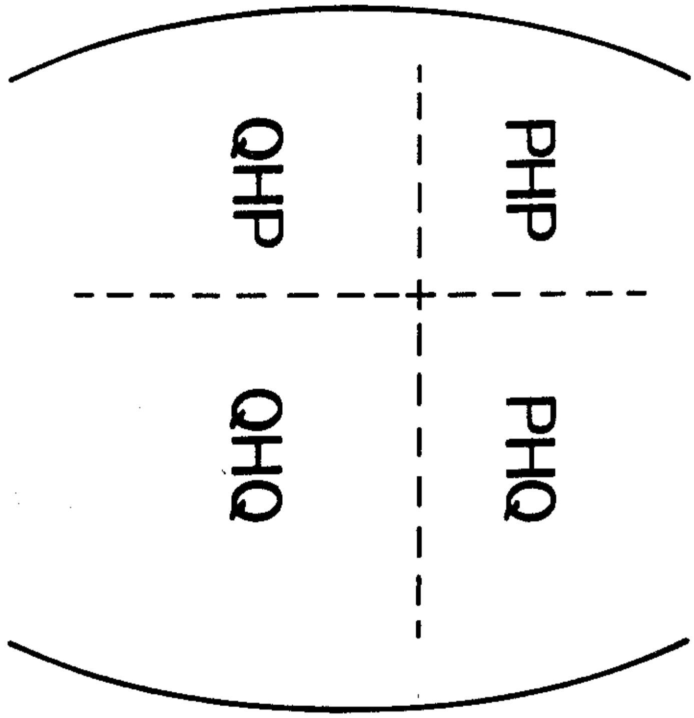

PHP
PHQ
QHP
QHQ

Fig. 1

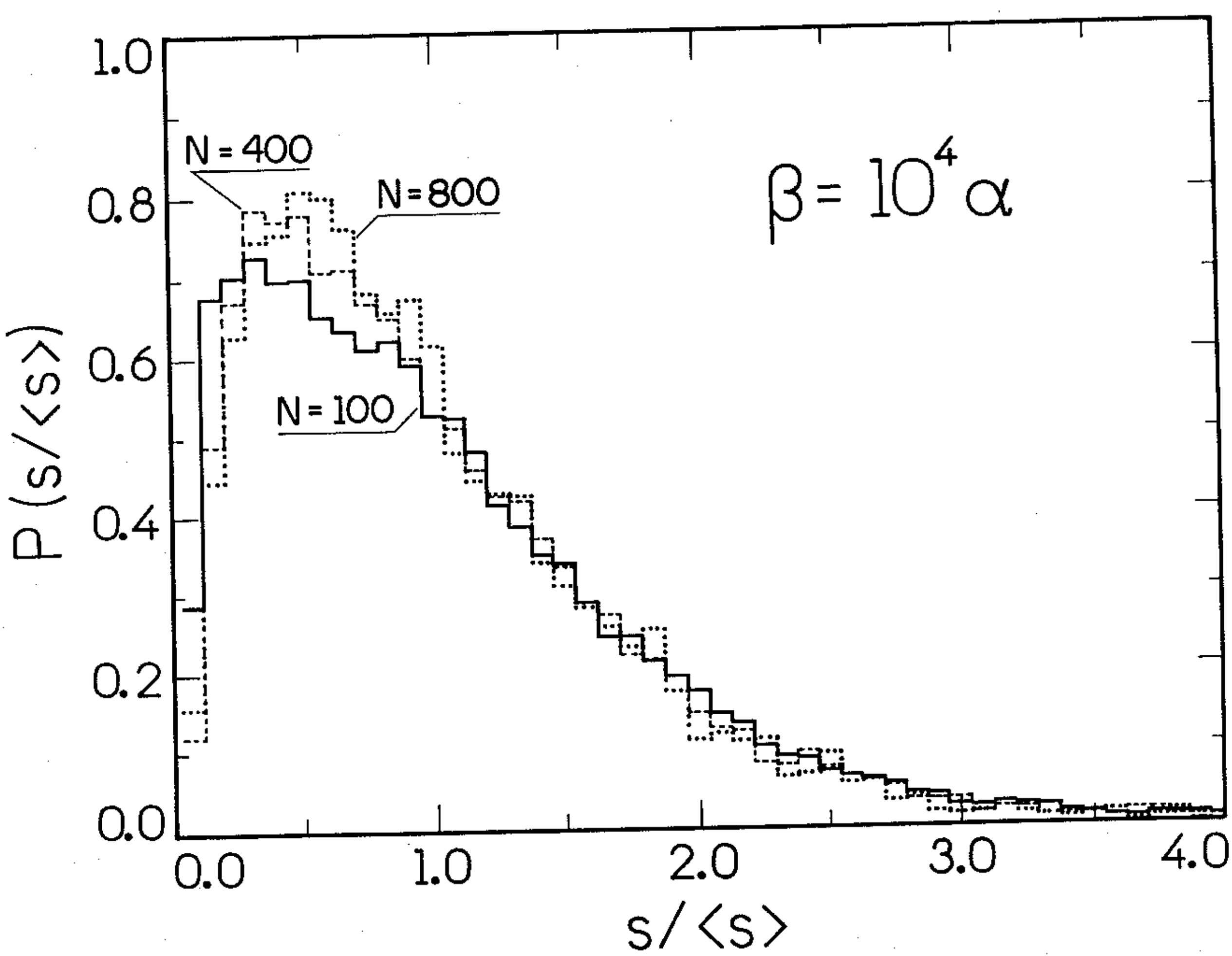
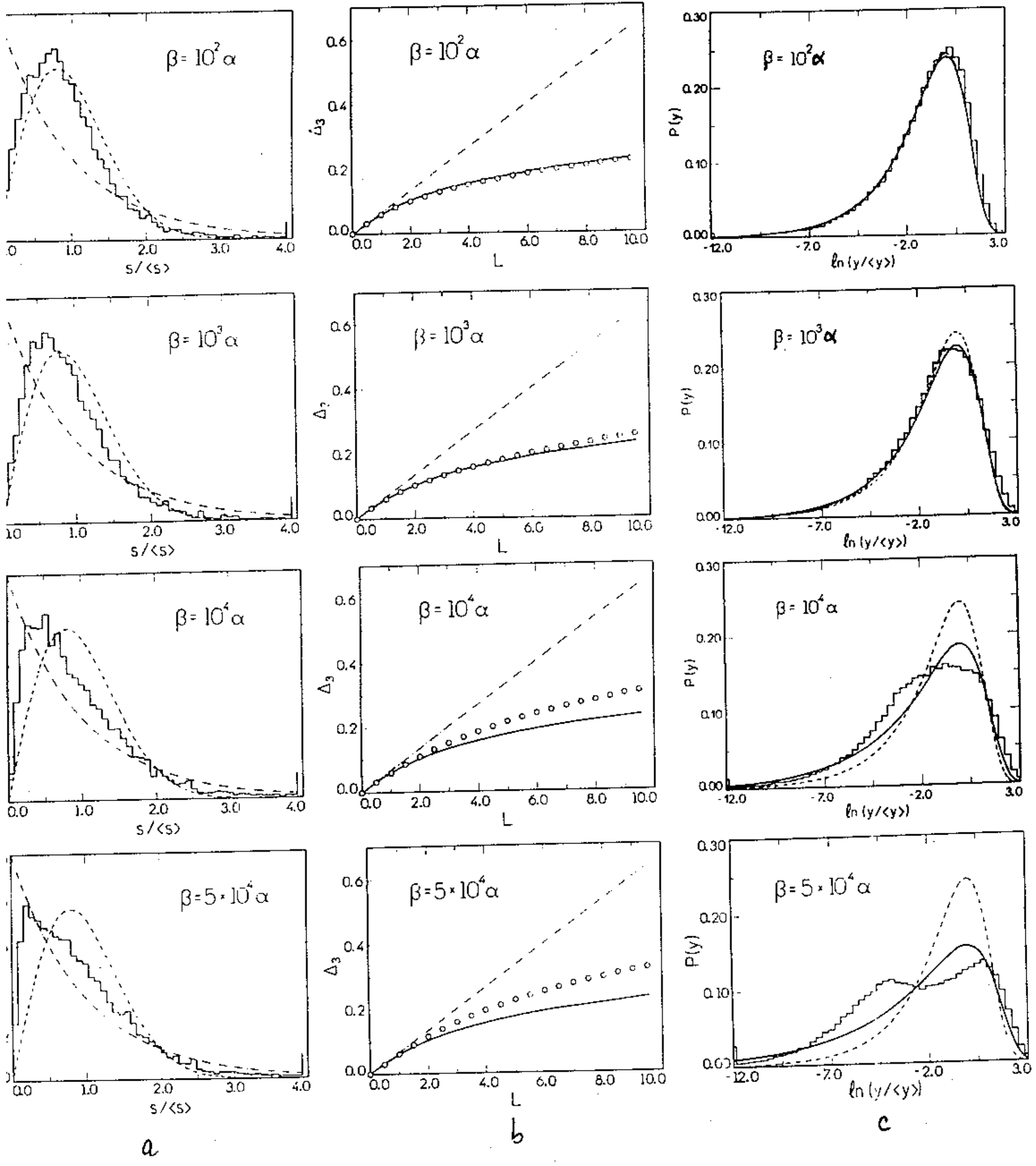
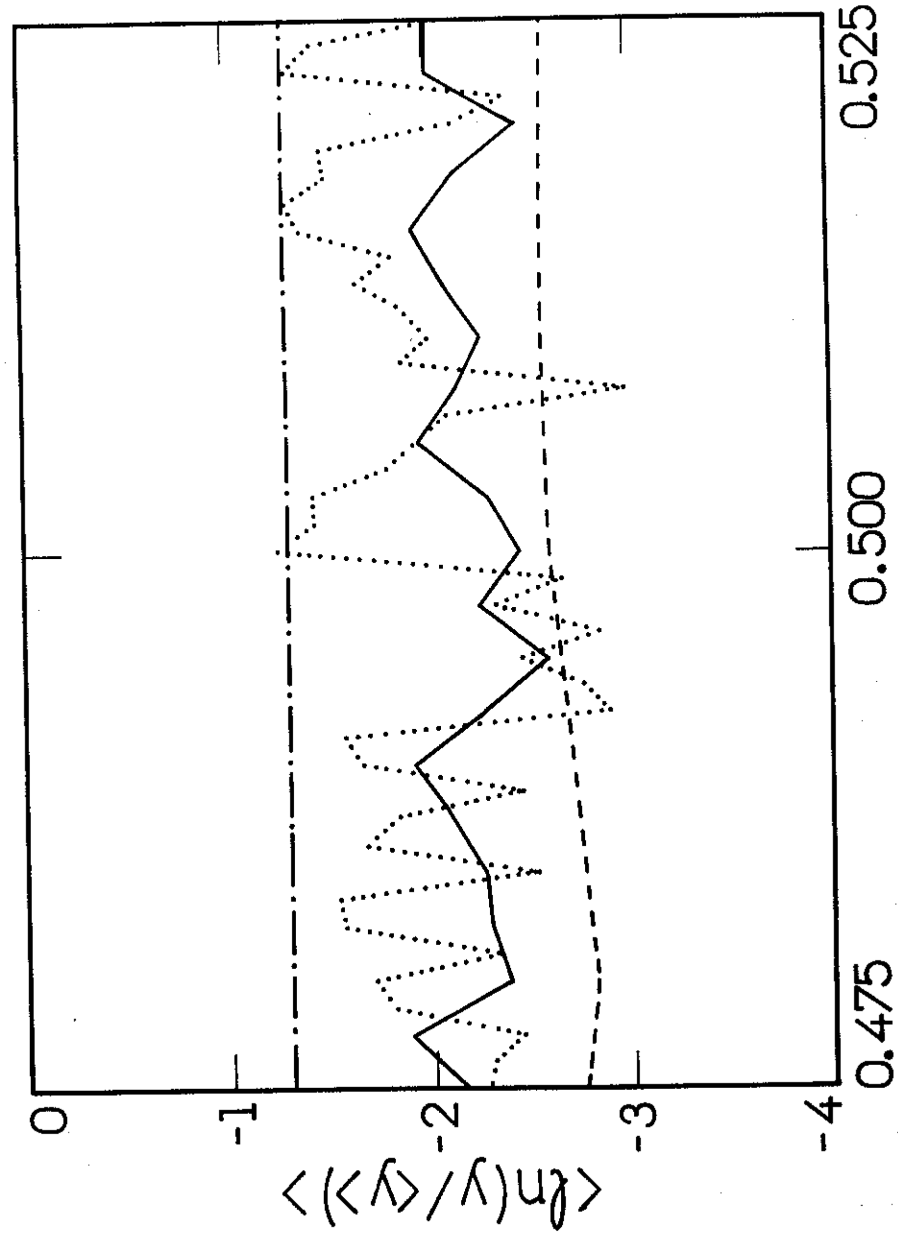

N = 400
N = 800
N = 100
β = 10^4 α
P (s/<s>)
s/<s>
0.0
0.2
0.4
0.6
0.8
1.0
0.0
1.0
2.0
3.0
4.0

Fig. 2



β = 10²α
β = 10³α
β = 10⁴α
β = 5×10⁴α
s/⟨s⟩
L
Δ3
P(y)
ln(y/⟨y⟩)
a
b
c

Fig. 3



0.525
0.500
0.475
n/N
0
-1
-2
-3
-4
⟨ln(y/⟨y⟩)⟩

Fig. 4

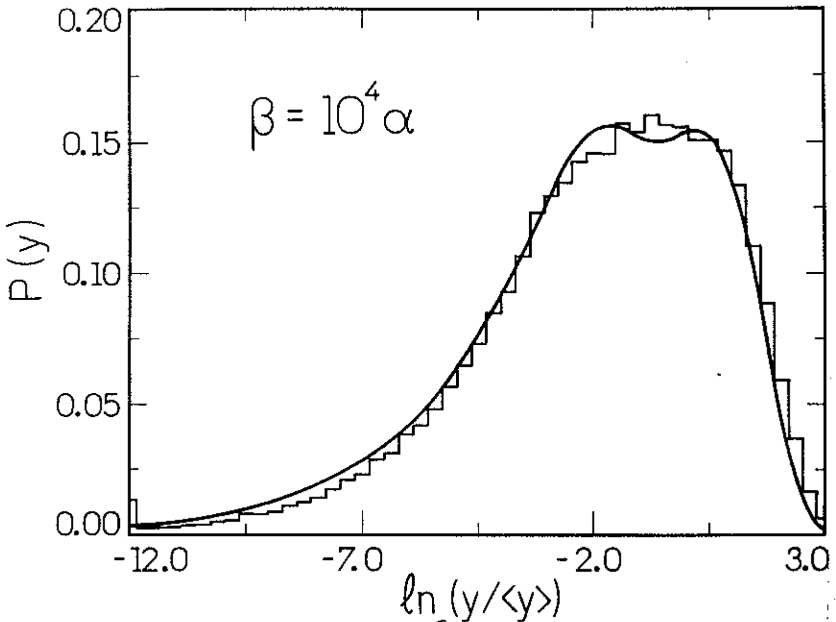

0.20
0.15
0.10
0.05
0.00
P(y)
β = 10⁴α
-12.0
-7.0
-2.0
3.0
ℓn (y/⟨y⟩)

Fig. 5a

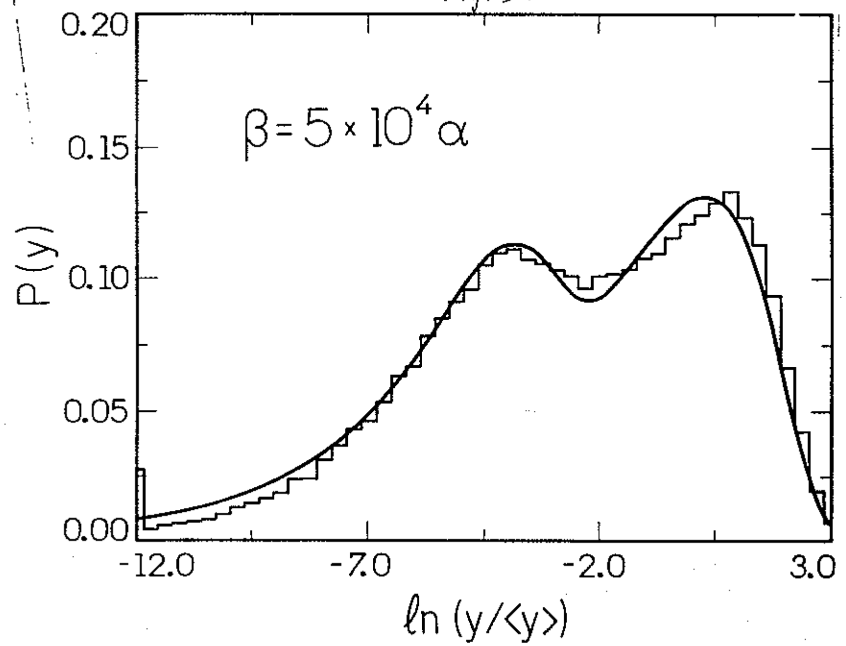

0.20
0.15
0.10
0.05
0.00
P(y)
β = 5 × 10⁴α
-12.0
-7.0
-2.0
3.0
ℓn (y/⟨y⟩)

Fig. 5b