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OFF CRITICAL CURRENT ALGEBRAS

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Abstract

We discuss the infinite dimensional algebras appearing in integrable perturbations of conformally invariant theories, with special emphasis in the structure of the consequent non-abelian infinite dimensional algebra generalizing W_∞ to the case of a non abelian group. We prove that the pure left-symmetry as well as the pure right-sector of the thus obtained algebra coincides with the conformally invariant case. The mixed sector is more involved, although the general structure seems to be near to be unraveled. We also find some subalgebras that correspond to Kac-Moody algebras. The constraints imposed by the algebras are very strong, and in the case of the massive deformation of a non-abelian fermionic model, the symmetry alone is enough to fix the 2- and 3-point functions of the theory.

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1. Introduction

The appearance of the Virasoro[1] and the $W_\infty(V)$ algebras[2] as off-critical symmetries of certain class of integrable models (IM) addresses the question whether all the known infinite algebras (super-Virasoro, $\widehat{G}_n$-Kac-Moody, W_N, etc $\cdots$) reappear again as symmetries of the *nonconformal* integrable models. From the way Virasoro algebra arises in the off-critical XY, Ising and Potts models one could expect to find the $\widehat{G}_n$-Kac-Moody algebra (and the larger current algebra $\widehat{G}_n \otimes \mathrm{Vir}$) studying the algebra of the conserved charges of the integrable perturbations[3] of the conformal WZW models[4,10].

The simplest example is the $k=2$, $SU(2)$-WZW model perturbed by the field $\psi^k_{\Delta,j}(z)\overline{\psi}^k_{\Delta,j}(\overline{z})$ of spin $j=1$ and $\Delta = 1/2$. This model is equivalent to the theory of massive Majorana fermion in the $O(3)$-vector representation. A more general fact is that all $k=1$, $O(n)$-WZW models represent free fermions[5], and their massive perturbation is described by the action

$$S = \int \frac{1}{2}\left(i\overline{\psi}^i \not{\partial}\psi^i + m\overline{\psi}^i\psi^i\right) d^2z \quad . \tag{1.1}$$

Our problem is to find all the symmetries of this simple fermionic system and to see *whether the $\widehat{O}(n)$-Kac-Moody algebra (or $\widehat{O}(n) \otimes \mathrm{Vir}$)- algebra take place as symmetries of (1.1).* The basic ingredients in the construction of all the conserved charges for (1.1) are the following two infinite sets of conserved tensors:

$$\begin{aligned} T^{ji}_{2s} = T^{ij}_{2s} = \psi^{(i}\partial^{2s-1}\psi^{j)} \quad &, \quad \overline{\partial}T^{ij}_{2s} = \partial\theta^{ij}_{2s-2} \\ J^{ji}_{2s-1} = -J^{ij}_{2s-1} = \frac{1}{2}\psi^{[i}\partial^{2s-2}\psi^{j]} \quad &, \quad \overline{\partial}J^{ij}_{2s-1} = \partial\widetilde{\theta}^{ij}_{2s-3} \quad ; s = 1,2,\cdots \end{aligned} \tag{1.2}$$

The conserved charges we are looking for are specific combinations of the "higher momenta" in z and $\overline{z}$ of these tensors. The first surprising fact is that the charges of the currents J^{ij}_{2s-1}, $\overline{J}^{ij}_{2s-1}$, namely

$$\begin{aligned} Q^{ij}_s &= \int J^{ij}_{2s-1}dz - \int \overline{\widetilde{\theta}}^{ij}_{2s-3}d\overline{z} \\ \overline{Q}^{ij}_s &= \int \overline{J}^{ij}_{2s-1}d\overline{z} - \int \widetilde{\theta}^{ij}_{2s-3}dz \qquad s = 1,2,\cdots \end{aligned}$$

close the (*nonchiral!*) $\widehat{O}(n)$-Kac-Moody algebra

$$[Q^{ij}_{s_1}, Q^{kl}_{s_2}] = \delta_{ik}Q^{jl}_{s_1+s_2} + \delta_{jl}Q^{ik}_{s_1+s_2} - \delta_{il}Q^{jk}_{s_1+s_2} - \delta_{jk}Q^{il}_{s_1+s_2} + \frac{p}{2}s_1(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\delta_{s_1+s_2} \tag{1.3}$$

($-\infty \le s_i \le \infty$). The *central charge is $p=1$, i.e. the same as for free massless fermions.* Constructing specific "first momenta" of $\delta_{ij}T^{ij}_{2s}$ (see eq.(3.13)) one can complete (1.3) to the larger current algebra $\mathrm{Vir} \otimes \widehat{O}(n)$. The appearance of this algebra as symmetries of (1.1) is a strong indication that we can use their degenerate representations to solve exactly the model. As we demonstrate in section 3 the simplest Ward identity of Vir $\otimes\widehat{O}(n)$ written for the fermionic two point function leads to the K_1-Bessel equation.

It is important to mention that the $\widehat{O}(n)$ $\otimes$Vir-symmetry of the action (1.1) *have nothing to do* with the *local gauge transformations* and *reparametrizations* in $2-D$ *"space-time"* $(z\bar{z})$. This is evident from the explicit higher derivatives realization (3.9) and (3.13) of the action of the generators Q_s^{ij} and V_k on $\psi^k(z,\bar{z})$. The same formulae (3.9), (3.13) written however in the p-space $(p\bar{p}=m^2)$ make transparent the following *remarkable fact*: the algebra Vir $\otimes\widehat{O}(n)$ does represent specific local gauge transformations and reparametrizations in the p-space formulation of the model (1.1). The parameters of these transformations $\omega^{ij}(p)$ and $\epsilon(p)$ have to satisfy the conditions:

$$\omega^{ij}(p)=\omega^{ij}(-p) \quad , \quad \epsilon(p)=-\epsilon(-p) \quad .$$

One should look for a better understanding of this fact in the *specific phase space geometry of the model (1.1)*.

The current algebra Vir $\otimes\widehat{O}(n)$ does not exhaust all the symmetries of the model (1.1). Together with the "first momenta" of T_{2s}^{ij} and J_{2s-1}^{ij} one can consider all the conserved "higher momenta" $\overset{(-)}{L}{}_{-k}^{ij(2s)}$ and $\overset{(-)}{Q}{}_{-p}^{ij(2s-1)}$ $(0\leq k\leq 2s-1\,,\,0\leq p\leq 2s-2)$ given by eqs.(3.17) and (3.18). The problem of deriving their algebra requires some preliminary investigation. Having in mind the essential role of the conformal W_∞ algebra[6] in the construction of the full off-critical symmetry algebra for the Ising model[2] we have first to find an appropriate *conformal current algebra analog* of W_∞. Our starting point is the conformal OPE algebra of *all the descendents* T_{2s}^{ij} of the stress-tensor T and J_{2s-1}^{ij}, of the $O(n)$-current J^{ij}. Following the standard conformal technique we have constructed in section 2 a *new class of W_∞-current algebras* $W_\infty(\widehat{G}_n)$ for $\widehat{G}_n=\widehat{O}(n)$. We have calculated the structure constants $g_r^{s_1s_2}$ (2.7) of $W_\infty(\widehat{G}_n)$ using the basis (2.3) of the quasiprimary descendents. This makes difficult the comparison with the corresponding structure constants $c_r^{s_1s_2}$ of usual W_∞[6], which are calculated in a specific nonquasiprimary basis. There are however many indications that our universal structure constants (2.7) has to coincide with these ones of $W_{1+\infty}$ [9], taken in our quasiprimary basis.

The *off-critical analog* of the conformal $W_\infty(\widehat{G}_n)$ is defined as the algebra $\widetilde{W}_\infty(\widehat{G}_n)$ of all the symmetries of (1.1). As one can see from our discussion in section 4 it has quite complicated structure: two subalgebras $\widetilde{W}_\infty^{L(R)}(\widehat{G}_n)$ of $W_\infty(\widehat{G}_n)$, which do not commute, two more (incomplete $n\geq -1$) Vir $\otimes\widehat{O}(n)$-current algebras, one $\widehat{O}(n)$-Kac-Moody algebra etc. Similarly to the simplest Vir $\otimes\widehat{O}(n)$-algebra (1.3), the generators displayed explicitly in (3.14) have a specific higher derivatives realization and *do not represent* local gauge transformations and reparametrizations. For example, the generators of one of the incomplete Vir $\otimes\widehat{O}(n)$-algebra are of the form:

$$(Q_n^{ij})_{kl}=-\frac{1}{2}(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})[\bar{z}\bar{\partial}-z\partial-1]_n\partial^n \quad , \quad n\geq 0 \quad , \tag{1.4}$$

where $[A]_n=A(A-1)\cdots(A-n+1)$. The realization (1.4) is completely different from the one of the *standard* $\widehat{O}(n)$-Kac-Moody subalgebra of the conformal $W_\infty(\widehat{G}_n)$:

$$(Q_n^{ij})_{kl}=-\frac{1}{2}(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})z^n \quad , \quad -\infty\leq n\leq\infty \quad . \tag{1.5}$$

However the symmetries generated by (1.4) are not a specific feature of the massive action (1.1) only. For the massless case where the full algebra of symmetries is the conformal $W_\infty(\widehat{G}_n)$ one can find a subalgebra spanned by

$$(Q_n^{ij})_{kl}=-\frac{1}{2}(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})(z\partial+1)_n\partial^n \tag{1.6}$$

which indeed close $\widehat{O}(n)$-Kac-Moody algebra (1.3).

All these similarities between the conformal $W_\infty(\widehat{G}_n)$ and the off-critical $\widetilde{W}_\infty(\widehat{G}_n)$ reflect on the form of the corresponding Ward identities as well. The conclusion is that in order to construct the representations of $\widetilde{W}_\infty(\widehat{G}_n)$ and to solve its Ward identities it is better to do it first for the conformal $W_\infty(\widehat{G}_n)$. In fact, the major part of the properties of the off-critical model are hidden in the $W_\infty(\widehat{G}_n)$-symmetries of the conformal model we have started with.

2. Conformal $W_\infty(\widehat{G}_n)$ algebras.

We are going to study the off-critical properties of certain integrable perturbations of the WZW models. The crucial role of the conformal W_∞ algebra in the construction of the corresponding off-critical algebra $W_\infty(V)$ for the Ising and Potts models[2] addresses the question about the *relevant conformal current algebra analog of* W_∞. In the same way W_∞ algebra[6,9] is realized as the algebra of all the descendents $T_{2s}(z)$ $(s=1,2,\ldots)$ of the stress-tensor T one expects the W_∞-current algebra we are looking for to be spanned by the descendents $T_{2s}(z)$ and $J_{2s-1}^i(z)$, (or $J_{2s-1}^{ij}=\frac{1}{2}\epsilon^{ijk}J_{2s-1}^k(z)$) of $T(z)$ and of the $SU(2)$-currents $J^i(z)$. In order to find the $W_\infty(A_1)$ algebra we have first to write the conformal OPE's of J_{2s-1}^i and T_{2s}. We have to be sure, however, that J_{2s-1}^i, T_{2s} form a basis in the space of all the conserved tensors, i.e., the product of each two of them contains the currents from the defined basis only. The problematic OPE is the one of two currents: $J_{2s-1}^i(z)J_{2p-1}^k(w)$. Its anti-symmetric part can be exhausted by terms of the type $\epsilon^{ijk}J_{2l-1}^k(z)$. The only remaining possibility for the symmetric part would be $\delta^{ij}T_{2m}(z)$. This is however not generally the case, and one can see this fact realizing T_{2s} and J_{2s-1}^i in terms of free fermions ψ^i, $(i=1,2,3)$ as

$$\begin{aligned} J_{2s-1}^i &= \epsilon^{ijk}\left(\partial^{2s-2}\psi^j\right)\psi^k \\ T_{2s} &= \psi^i\partial^{2s-1}\psi^i \quad . \end{aligned} \qquad s=1,2,\cdots \tag{2.1}$$

Using the standard OPE's for the fermions ψ^i one can easily verify that $\delta^{ij}T_{2s}$ does not exhaust the symmetric part of the aforementioned OPE's. Indeed, there are new terms of the form

$$T_{2s}^{ij}=T_{2s}^{ji}=\psi^i\partial^{2s-1}\psi^j+\psi^j\partial^{2s-1}\psi^i \quad , \quad T_{2s}=\frac{1}{2}\delta^{ij}T_{2s}^{ij} \quad .$$

The conclusion is that the algebra we have to consider is the one of T^{ij}_{2s} and J^{ij}_{2s-1}. Applying the usual conformal technique[7,4], one can easily derive the OPE's of the *quasiprimary currents* T^{ij}_{2s} and J^{ij}_{2s-1}. At this point we are going to follow closely the method described in detail in section (3.5) of ref.[8]. As a result of a simple and standard computation we obtain the following OPE's algebra:

$$\begin{aligned}
T^{ij}_{2s_1}(z_1)T^{kl}_{2s_2}(z_2) = \sum_{s=1}\sum_{r=0}\Big\{ & D^{2s_1,2s_2}_{2s}C^{2s_1,2s_2}_{2s}(r)z_{12}^{2(s-s_1-s_2)+r}\times \\
&\times\partial_2^r\left(\delta^{ik}T^{jl}_{2s}(2)+\delta^{il}T^{jk}_{2s}(2)+\delta^{jk}T^{il}_{2s}(2)+\delta^{jl}T^{ik}_{2s}(2)\right) \\
&+D^{2s_1,2s_2}_{2s-1}C^{2s_1,2s_2}_{2s-1}(r)z_{12}^{2(s-s_1-s_2)+r-1}\times \\
&\times\partial_2^r\left(\delta^{ik}J^{jl}_{2s-1}(2)+\delta^{il}J^{jk}_{2s-1}(2)+\delta^{jk}J^{il}_{2s-1}(2)+\delta^{jl}J^{ik}_{2s-1}(2)\right)\Big\} \\
&+N^{(2)}_{2s_1}z_{12}^{-4s_1}\delta_{s_1s_2}(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk})
\end{aligned}$$

$$\begin{aligned}
T^{ij}_{2s_1}(z_1)J^{kl}_{2s_2-1}(z_2) = \sum_{s=1}\sum_{r=0} & \\
\Big\{ & D^{2s_1,2s_2-1}_{2s}C^{2s_1,2s_2-1}_{2s}(r)z_{12}^{2(s-s_1-s_2)+r+1}\partial^r\left(\delta^{ik}T^{jl}_{2s}(2)-\delta^{il}T^{jk}_{2s}(2)+\delta^{jk}T^{il}_{2s}(2)-\delta^{jl}T^{ik}_{2s}(2)\right) \\
&+D^{2s_1,2s_2-1}_{2s-1}C^{2s_1,2s_2-1}_{2s-1}(r)z_{12}^{2(s-s_1-s_2)+r}\times \\
&\times\partial^r\left(\delta^{ik}J^{jl}_{2s-1}(2)-\delta^{il}J^{jk}_{2s-1}(2)+\delta^{jk}J^{il}_{2s-1}(2)-\delta^{jl}J^{ik}_{2s-1}(2)\right)\Big\}
\end{aligned}$$

$$\begin{aligned}
J^{ij}_{2s_1-1}(z_1)J^{kl}_{2s_2-1}(z_2) = \sum_{s=1}\sum_{r=0}\Big\{ & D^{2s_1-1,2s_2-1}_{2s-1}C^{2s_1-1,2s_2-1}_{2s-1}(r)z_{12}^{2(s-s_1-s_2)+r+1}\times \\
&\times\partial^r\left(-\delta^{ik}J^{jl}_{2s-1}(2)+\delta^{il}J^{jk}_{2s-1}(2)+\delta^{jk}J^{il}_{2s-1}(2)-\delta^{jl}J^{ik}_{2s-1}(2)\right) \\
&+D^{2s_1-1,2s_2-1}_{2s}C^{2s_1-1,2s_2-1}_{2s}(r)z_{12}^{2(s-s_1-s_2)+r+2}\times \\
&\times\partial^r(-\delta^{ik}T^{jl}_{2s}(2)+\delta^{il}T^{jk}_{2s}(2)+\delta^{jk}T^{il}_{2s}(2)-\delta^{jl}T^{ik}_{2s}(2))\Big\} \\
&+N^{(2)}_{2s_1}z_{12}^{-4s_1+2}\delta_{s_1s_2}(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk}) \qquad (2.2)
\end{aligned}$$

where

$$C^{nm}_s(r) = \frac{(2s-1)!\,(n-m+s+r-1)!}{r!\,(2s+r-1)!\,(n-m+s-1)!}$$

and

$$D^{nm}_s = \frac{N^{(3)}_{nms}}{N^{(2)}_s} \quad .$$

What remains is to calculate the normalization constants $N^{(2)}_s$ and $N^{(3)}_{nms}$ of the 2- and 3-point functions:

$$\begin{aligned}
\left\langle T^{ij}_{2s_1}(z_1)T^{kl}_{2s_2}(z_2)\right\rangle &= N^{(2)}_{2s_1}\left(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{kj}\right)\delta_{s_1,s_2}z_{12}^{-4s_1} \\
\left\langle J^{ij}_{2s_1-1}(z_1)J^{kl}_{2s_2-1}(z_2)\right\rangle &= N^{(2)}_{2s_1-1}\left(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{kj}\right)\delta_{s_1,s_2}z_{12}^{-4s_1+2}
\end{aligned}$$

and say,

$$\begin{aligned}
\left\langle T^{ij}_{2s_1}(z_1)T^{kl}_{2s_2}(z_2)T^{pt}_{2s_3}(z_3)\right\rangle &= N^{(3)}_{2s_1,2s_2,2s_3}\left(\delta^{ik}\delta^{jp}\delta^{lt}+\delta^{jk}\delta^{ip}\delta^{lt}+\ldots\right) \\
&\times z_{12}^{-2(s_3-s_1-s_2)}z_{13}^{-2(s_2-s_1-s_3)}z_{23}^{-2(s_1-s_2-s_3)}
\end{aligned}$$

In order to find $N^{(2)}$ and $N^{(3)}$ we can take the following free fermion realization of the *quasiprimary currents* T^{ij}_{2s} and J^{ij}_{2s-1}:

$$\begin{aligned}
T^{ij}_{2s}(1) &= \sum_{k=1}^{2s}(-1)^k\binom{2s-1}{k-1}^2 :\partial^{2s-k}\psi^i(1)\partial^{k-1}\psi^j(1): \\
J^{ij}_{2s-1}(1) &= \sum_{k=1}^{2s-1}(-1)^k\binom{2s-2}{k-1}^2 :\partial^{2s-k-1}\psi^i(1)\partial^{k-1}\psi^j(1):
\end{aligned} \qquad (2.3)$$

The explicit form of $N^{(2)}$ and $N^{(3)}$ *does depend* on the concrete construction (2.3) of T^{ij}_{2s} and J^{ij}_{2s-1} we have chosen but the ratio $D^{s_1s_2}_r$ *should not depend on it.* In words, the structure constants of the algebra are independent of the explicit realization of their generators. It is only the central term that depends on it. Then by straightforward computation, we get

$$\begin{aligned}
D^{s_1s_2}_r = & \frac{(-1)^{s_1+s_2-r}}{[2(s_1+s_2-r)-2]!}\sum_{k_1=1}^{s_1}\sum_{k_2=1}^{s_2}\sum_{k_3=1}^{s_1+s_2-r}(-1)^{s_1+k_1+k_3}2^{s_1-s_2-k_1+k_3}\times \\
&\binom{s_1-1}{k_1-1}^2\binom{s_2-1}{k_2-1}^2\binom{s_1+s_2-r-1}{k_3-1}^2(s_1-1-k_1+k_2)!\,(s_2-1-k_2+k_3)!\,(s_1+s_2-r-1-k_3+k_1)!
\end{aligned} \qquad (2.4)$$

The next step is to derive the corresponding Lie algebra, encoded in the singular terms of the OPE's (2.2). It is generated by the Laurent modes of the currents T^{ij}_{2s} and J^{kl}_{2s-1}:

$$\begin{aligned}
\mathcal{L}^{ij(2s)}_n &= \oint z^{2s+n-1}T^{ij}_{2s}(z)dz \quad , \\
\mathcal{Q}^{kl(2s-1)}_n &= \oint z^{2s+n-2}J^{kl}_{2s-1}(z)dz \quad .
\end{aligned} \qquad (2.5)$$

Then by simply integrating (2.2) we find the $W_\infty(A_1)$-current algebra in the form:

$$\begin{aligned}
\left[\mathcal{L}_{n_1}^{ij(2s_1)}, \mathcal{L}_{n_2}^{kl(2s_2)}\right] = \sum_{r=0}^{s_1+s_2-2} & \left\{ g_{2r}^{2s_1,2s_2}(n_1,n_2)\left(\delta \circ \mathcal{L}_{n_1+n_2}^{(2s_1+2s_2-2r-2)}\right)^{ijkl} \right. \\
& \left. + g_{2r-1}^{2s_1,2s_2}(n_1,n_2)\left(\delta \circ \mathcal{Q}_{n_1+n_2}^{(2s_1+2s_2-2r-1)}\right)^{ijkl} \right\} \\
& + N_{2s_1}^{(2)} \frac{(2s_1+n_1-1)!}{(4s_1-1)!(n_1-2s_1)!}\delta_{n_1+n_2}\delta_{s_1-s_2}(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}) \\
\left[\mathcal{L}_{n_1}^{ij(2s_1)}, \mathcal{Q}_{n_2}^{kl(2s_2-1)}\right] = \sum_{r=0}^{s_1+s_2-2} & \left\{ g_{2r}^{2s_1,2s_2-1}(n_1,n_2)\left(\delta \bullet \mathcal{L}_{n_1+n_2}^{2s_1+2s_2-2r-2)}\right)^{ijkl} \right. \\
& \left. + g_{2r-1}^{2s_1,2s_2-1}(n_1,n_2)\left(\delta \bullet \mathcal{Q}_{n_1+n_2}^{(2s_1+2s_2-2r-3)}\right)^{ijkl} \right\} \\
\left[\mathcal{Q}_{n_1}^{ij(2s_1-1)}, \mathcal{Q}_{n_2}^{kl(2s_2-1)}\right] = \sum_{r=0}^{s_1+s_2-2} & \left\{ g_{2r-1}^{2s_1-1,2s_2-1}(n_1,n_2)\left(\delta \star \mathcal{Q}_{n_1+n_2}^{2s_1+2s_2-2r-3)}\right)^{ijkl} \right. \\
& \left. + g_{2r}^{2s_1-1,2s_2-1}(n_1,n_2)\left(\delta \star \mathcal{L}_{n_1+n_2}^{(2s_1+2s_2-2r-4)}\right)^{ijkl} \right\} \\
& + N_{2s_1-1}^{(2)} \frac{(2s_1+n_1-2)!}{(4s_1-3)!(n_1-2s_1+1)!}\delta_{n_1+n_2}\delta_{s_1-s_2}(\delta^{ik}\delta^{jl}-\delta^{il}\delta^{jk})
\end{aligned} \tag{2.6}$$

where

$$(\delta \circ \mathcal{Y})^{ijkl} = \left(\delta^{ik}\mathcal{Y}^{jl} + \delta^{il}\mathcal{Y}^{kj} + \delta^{kj}\mathcal{Y}^{il} + \delta^{jl}\mathcal{Y}^{ik}\right)$$

$$(\delta \bullet \mathcal{Y})^{ijkl} = \left(\delta^{ik}\mathcal{Y}^{jl} - \delta^{il}\mathcal{Y}^{kj} + \delta^{kj}\mathcal{Y}^{il} - \delta^{jl}\mathcal{Y}^{ik}\right)$$

$$(\delta \star \mathcal{Y})^{ijkl} = \left(-\delta^{ik}\mathcal{Y}^{jl} + \delta^{il}\mathcal{Y}^{kj} + \delta^{kj}\mathcal{Y}^{il} - \delta^{jl}\mathcal{Y}^{ik}\right)$$

The $SU(2)$ independent part of the structure constants, $g_r^{s_1,s_2}(n_1,n_2)$ is given by:

$$\begin{aligned}
g_r^{s_1,s_2}(n_1,n_2) = D_r^{(s_1,s_2)} \sum_{l=0}^{r+1} & \frac{(-1)^l}{l!(r+1-l)!} \frac{(2s_1-r-1+l)![2(s_1+s_2-r)-1]!}{(2s_1-r-1)![2(s_1+s_2-r)-1+l]!} \times \\
& \times \frac{(s_1+n_1-1)!(s_1+s_2-r+n_1+n_2-1+l)!}{(s_1+n_1-2-r+l)!(s_1+s_2-r+n_1+n_2-1)!} ,
\end{aligned} \tag{2.7}$$

and $N_{2s}^{(2)}, N_{2s-1}^{(2)}$ are two parameters representing the central charges of the algebra (2.6).

We should mention that in the way we arrived at the algebra (2.6) we did not use any specific property of $SU(2) \sim O(3)$. The only specification we have used is that the

generators of the current algebra are antisymmetric tensors, and that in their OPE's the symmetric tensors T_{2s}^{ij} also contribute. It is evident that all $\widehat{O}(n)$-current algebras, for $n > 2$ share these properties. Therefore *the algebra (2.6) represents the universal form of the new family of W_∞ algebras: $W_\infty(\widehat{B}_n)$ and $W_\infty(\widehat{D}_n)$*. Concerning the classification of the W_∞-current algebras it is clear that one has to follow the classification of the usual current algebras. For each of them, namely $\widehat{A}_n^{(1)}$, $\widehat{A}_n^{(2)}$, $\widehat{B}_n$, $\widehat{C}_n \ldots$, etc. one has to find the OPE algebra of all the descendents. For example, the generators of $W_\infty(\widehat{A}_n)$ can be taken as having the form:

$$\begin{aligned}
J_{2s-1}^A &= \psi^\alpha \tau_{\alpha\beta}^A \partial^{2s-2}\psi^\beta \\
T_{2s}^{(AB)} &= \psi^\alpha \left\{\tau^A, \tau^B\right\}_{\alpha\beta} \partial^{2s-1}\psi^\beta
\end{aligned} \tag{2.8}$$

where $\tau_{\alpha\beta}^A$ are anti-hermitian matrices representing the $SU(n)$ algebra: $\left[\tau^A, \tau^B\right] = f_C^{AB}\tau^C$. The corresponding $W_\infty(\widehat{A}_n)$ algebra have a form similar to (2.6) with the adequate group structure in order to conform to the A_n case. It is important to note that *the structure constants $g_r^{s_1 s_2}$* given by (2.7) *are universal quantities for all the groups* $O(n), SU(n)$ etc.

3. Symmetries of the off-critical $k = 1, O(n)$ WZW models

Our problem is to construct explicitly all the conserved charges of the models given by (1.1), i.e. - $O(n)$-Majorana massive fermions $\overline{\psi}^i(z,\overline{z})\,(i = 1,\cdots,n)$. What we have to do is **(1)** find the conserved tensors T_{2s}^{ij} and J_{2s-1}^{ij} in terms of $\psi^i, \overline{\psi}^i$; **(2)** verify that their conservation laws satisfy the criterion of existence of noncommuting charges[2]; **(3)** construct these charges and compute their algebra. One could expect that the case of n massive fermions in the $O(n)$-vector representation is a straightforward generalization of the results for one massive fermion[2]. There exist however few important differences. The first is that together with $T_{2s} = \delta_{ij}T_{2s}^{ij}$ and J_{2s-1}^{ij} we have to consider all components of the symmetric conserved tensor T_{2s}^{ij}. The second very important point is that the algebra of the standard conserved charges:

$$\begin{aligned}
P_s^{ij} &= \int T_{2s}^{ij} dz - \int \theta_{2s-2}^{ij} d\overline{z} \quad , \quad \overline{P}_s^{ij} = \int \overline{T}_{2s}^{ij} d\overline{z} - \int \theta_{2s-2}^{ij} dz \\
Q_s^{ij} &= \int J_{2s-1}^{ij} dz - \int \widetilde{\theta}_{2s-3}^{ij} d\overline{z} \quad , \quad \overline{Q}_s^{ij} = \int \overline{J}_{2s-1}^{ij} d\overline{z} - \int \widetilde{\theta}_{2s-3}^{ij} dz
\end{aligned} \tag{3.1}$$

is **nonabelian**. Its abelian subalgebra is spanned by $P_s = \delta_{ij}P_s^{ij}$ and $\overline{P}_s = \delta_{ij}\overline{P}_s^{ij}$. In order to find this algebra, it is better to realize P_s^{ij}, Q_s^{ij} etc in terms of differential operators. Following the standard massive fermion technology[2] we start with the "nonquasiprimary" form of the conserved tensors:

$$\begin{aligned}
T_{2s}^{ij} &= \psi^i \partial^{2s-1}\psi^j + \psi^j \partial^{2s-1}\psi^i \quad , \quad T_2^{ij} = \frac{1}{2}(\psi^i \partial\psi^j + \psi^j \partial\psi^i) \\
J_{2s-1}^{ij} &= \frac{1}{2}\left(\psi^i \partial^{2s-2}\psi^j - \psi^j \partial^{2s-2}\psi^i\right)
\end{aligned} \tag{3.2}$$

and similar expressions for $\overline{T}^{ij}_{2s}$ and $\overline{J}^{ij}_{2s-1}$. Using the equations of motion:

$$\overline{\partial}\psi^k = m\overline{\psi}^k \quad , \quad \partial\overline{\psi}^k = -m\psi^k \tag{3.3}$$

one can show that (3.2) are indeed conserved tensors. For example the first few J^{ij}_{2s-1}-current conservation laws are in the form:

$$\begin{aligned} &\overline{\partial}J_1^{ij} + \partial\overline{J}_1^{ij} = 0 \\ &\overline{\partial}J_3^{ij} = \partial^2\widetilde{\theta}^{ij} + m^2\partial J_1^{ij} \\ &\overline{\partial}J_5^{ij} = \partial^4\widetilde{\theta}^{ij} + m^2\partial J_3^{ij} + \frac{3}{2}m^2\partial^3 J_1^{ij} \end{aligned} \tag{3.4}$$

etc., where $\widetilde{\theta}^{ij} = \frac{m}{2}(\psi^j\overline{\psi}^i - \psi^i\overline{\psi}^j)$. For the "stress-tensor's" set of conserved quantities we have:

$$\begin{aligned} &\overline{\partial}T_2^{ij} = \partial\theta^{ij} \quad , \quad \theta^{ij} = m(\overline{\psi}^i\psi^j + \overline{\psi}^j\psi^i) \\ &\overline{\partial}T_4^{ij} = \partial^3\theta^{ij} + 2m^2\partial T_2^{ij} \\ &\overline{\partial}T_6^{ij} = 2\partial^5\theta^{ij} + m^2\partial T_4^{ij} + 4m^2\partial^3 T_2^{ij} \end{aligned} \tag{3.5}$$

etc. Substituting (3.2) in (3.1) and using the equations of motion (3.3) one could exclude the time derivatives in the integrands of (3.1) and then take $t = 0$ ($z = t+x, \overline{z} = t-x, \partial_x = \partial$). The result of these computations is:

$$\begin{aligned} P_1^{ij} &= -\frac{1}{2}\int dx\left[\psi^i\partial\psi^j + \psi^j\partial\psi^i + m(\overline{\psi}^i\psi^j + \overline{\psi}^j\psi^i)\right] \\ P_2^{ij} &= -\frac{1}{2}\int dx\left[\psi^{(i}\partial^3\psi^{j)} + m\overline{\psi}^{(i}\partial^2\psi^{j)} + m^2\psi^{(i}\partial\psi^{j)} - \frac{m^2}{2}\overline{\psi}^{(i}\partial\overline{\psi}^{j)} + \frac{3}{2}m^3\overline{\psi}^{(i}\psi^{j)}\right] \\ Q_0^{ij} &= -\int dx\left[\psi^{[j}\psi^{i]} + \overline{\psi}^{[j}\overline{\psi}^{i]}\right] \\ Q_1^{ij} &= -\int dx\left[4\partial^2\psi^{[i}\psi^{j]} - 4m\partial\overline{\psi}^{[i}\psi^{j]} + m^2(\psi^{[i}\psi^{j]} + \overline{\psi}^{[i}\overline{\psi}^{j]})\right] \end{aligned} \tag{3.6}$$

etc. In order to get the momenta space form of (3.6) we have to insert the usual creation and annihilation decomposition of ψ^i and $\overline{\psi}^i$:

$$\begin{aligned} \psi^i(x,t) &= \int\frac{dp}{2\pi}\sqrt{\frac{p_0-p}{2p_0}}\left(e^{ip\cdot x}a_i^+(p) + e^{-ip\cdot x}a_i^-(p)\right) \\ \overline{\psi}^i(x,t) &= i\int\frac{dp}{2\pi}\sqrt{\frac{p_0+p}{2p_0}}\left(e^{ip\cdot x}a_i^+(p) - e^{-ip\cdot x}a_i^-(p)\right) \end{aligned} \tag{3.7}$$

where $p_0^2 - p^2 = m^2$ and $p\cdot x = p_0t + px$.

The result is the following compact form of the charges (3.1):

$$\begin{aligned} \overset{(-)}{P}{}^{ij}_{s} &= \frac{1}{4}\int\frac{dp}{2\pi}(p_0 \pm p)^{2s-1}{:}(a_i^+a_j^- + a_j^+a_i^-){:} \\ \overset{(-)}{Q}{}^{ij}_{s} &= \int\frac{dp}{2\pi}(p_0 \pm p)^{2s}{:}(a_i^+a_j^- - a_j^+a_i^-){:} \end{aligned} \tag{3.8}$$

(we denote $p \equiv p_0 - p$ and $\overline{p} \equiv p_0 + p$ from now on). The desired differential form of P^{ij}_s and Q^{ij}_s is a simple consequence of (3.7), (3.8) and the anticommutation relations $\{a_i^+(p), a_j^-(q)\} = 2\pi\delta_{ij}\delta(p-q)$ and reads:

$$\left[\overset{(-)}{P}{}^{ij}_{s}, \psi^k(z,\overline{z})\right] = (\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})\overset{(-)}{\partial}{}^{2s-1}\psi^l(z,\overline{z}) \tag{3.9a}$$

$$\left[\overset{(-)}{Q}{}^{ij}_{s}, \psi^k(z,\overline{z})\right] = (\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})\overset{(-)}{\partial}{}^{2s}\psi^l(z,\overline{z}) \quad , \quad s = 1, 2, \cdots \tag{3.9b}$$

With eqs. (3.9) at hands we are prepared now to find the algebra of the charges (3.1):

$$\left[Q^{ij}_{s_1}, Q^{kl}_{s_2}\right] = \delta^{ik}Q^{jl}_{s_1+s_2} + \delta^{jl}Q^{ik}_{s_1+s_2} - \delta^{il}Q^{jk}_{s_1+s_2} - \delta^{jk}Q^{il}_{s_1+s_2} \tag{3.10a}$$

$$\left[P^{ij}_{s_1}, Q^{kl}_{s_2}\right] = \delta^{ik}P^{jl}_{s_1+s_2} - \delta^{jl}P^{ik}_{s_1+s_2} + \delta^{il}P^{jk}_{s_1+s_2} - \delta^{jk}P^{il}_{s_1+s_2} \tag{3.10b}$$

$$\left[P^{ij}_{s_1}, P^{kl}_{s_2}\right] = -\delta^{ik}Q^{jl}_{s_1+s_2-1} - \delta^{jl}Q^{ik}_{s_1+s_2-1} - \delta^{il}Q^{jk}_{s_1+s_2-1} - \delta^{jk}Q^{il}_{s_1+s_2-1} \tag{3.10c}$$

and the same algebra for $\overline{Q}^{ij}_s$ and $\overline{P}^{ij}_s$. The mixed left-right algebra takes the form ($s_1 \leq s_2$):

$$\begin{aligned} \left[P^{ij}_{s_1}, \overline{P}^{kl}_{s_2}\right] &= -(-m^2)^{2s_1-1}(\delta^{ik}\overline{Q}^{jl}_{s_2-s_1} + \delta^{jl}\overline{Q}^{ik}_{s_2-s_1} + \delta^{il}\overline{Q}^{jk}_{s_2-s_1} + \delta^{jk}\overline{Q}^{il}_{s_2-s_1}) \\ \left[P^{ij}_{s_1}, \overline{Q}^{kl}_{s_2}\right] &= (-m^2)^{2s_1-1}(\delta^{ik}\overline{P}^{jl}_{s_2-s_1+1} - \delta^{jl}\overline{P}^{ik}_{s_2-s_1+1} + \delta^{il}\overline{P}^{jk}_{s_2-s_1+1} - \delta^{jk}\overline{P}^{il}_{s_2-s_1+1}) \\ \left[Q^{kl}_{s_1}, \overline{P}^{ij}_{s_2}\right] &= -(-m^2)^{2s_1}(\delta^{ik}\overline{P}^{jl}_{s_2-s_1} - \delta^{jl}\overline{P}^{ik}_{s_2-s_1} + \delta^{il}\overline{P}^{jk}_{s_2-s_1} - \delta^{jk}\overline{P}^{il}_{s_2-s_1}) \\ \left[Q^{ij}_{s_1}, \overline{Q}^{kl}_{s_2}\right] &= (-m^2)^{2s_1}(\delta^{ik}\overline{Q}^{jl}_{s_2-s_1} + \delta^{jl}\overline{Q}^{ik}_{s_2-s_1} - \delta^{il}\overline{Q}^{jk}_{s_2-s_1} - \delta^{jk}\overline{Q}^{il}_{s_2-s_1}) \end{aligned} \tag{3.11a, b, c, d}$$

Using the formal identity $\overline{\partial} = -m^2\partial^{-1}$ one can write $(m^2)^{-s}Q^{ij}_s \equiv \widetilde{Q}^{ij}_s$ and $(m^2)^{-s}\overline{Q}^{ij}_s \equiv \widetilde{Q}^{ij}_{-s}$ as a unique object $\widetilde{Q}^{ij}_s$ $(-\infty \leq s \leq \infty)$ which *generates the $O(n)$-Kac-Moody algebra (3.10a), (3.11d).* One can recognize the total algebra (3.10), (3.11) as a *subalgebra $GL(n,R)_{mod\,2}$ of the $\widehat{GL}(n,R)$-Kac-Moody algebra* spanned by $\widetilde{P}^{ij}_{2s-1} \equiv P^{ij}_s$ and $\widetilde{Q}^{ij}_{2s} = Q^{ij}_s$, i.e. the *closed subalgebra of symmetric generators P^{ij} with odd indices and antisymmetric generators Q^{ij} with even indices.*

One can derive the algebra (3.10), (3.11) using directly (3.8). The advantage is that in this way we calculate the value of the central charge of the $\widehat{O}(n)$-Kac-Moody algebra. Starting from (3.8) and taking care about the right normal ordering in the r.h.s. of the commutator $[Q^{ij}_s, \overline{Q}^{ij}_s]$ we find the central term in the form:

$$(-m^2)^{2s}\frac{s}{2}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \quad .$$

The central charge therefore is $p = 1$, i.e. the same as for the massless $O(n)$ -fermions. The appearance of non-zero central charge has far-reaching consequences in the application of the degenerate higher weight representations of (3.10), (3.11) for constructing the exact solutions of these models.

Having constructed the algebra of the conserved charges (3.1) we have to answer the question about its meaning. As in the case of one Majorana fermion[2], one can expect that *the charges* $\overset{(-)}{P}_s^{ij}$ *and* $\overset{(-)}{Q}_s^{ij}$ *are generators of specific new symmetries of the* $O(n)$*-massive fermions action (1.1).* It is indeed the case and the proof of this statement goes trough simple higher derivatives computations. We shall present here only one detail of this proof. Consider the commutator:

$$\left[Q_s^{kl}, \overline{\psi}^j\psi^j\right] = (\delta^{kj}\delta^{lm} - \delta^{km}\delta^{lj})\left[(\partial^{2s}\overline{\psi}^m)\psi^j - \overline{\psi}^m\partial^{2s}\psi^j\right] \tag{3.12}$$

The conclusion that the r.h.s. of (3.12) is a total derivative is based on the following identity:

$$(\partial^{2s}\overline{\psi}^m)\psi^j - \overline{\psi}^m\partial^{2s}\psi^j = \partial\left\{\sum_{l=0}^{s-1}(-1)^l[\partial^{2s-l-1}\overline{\psi}^m\partial^l\psi^j - \partial^l\overline{\psi}^m\partial^{2s-l-1}\psi^j]\right\} \quad .$$

The same is true for the commutators $[\overline{Q}_s^{kl}, \overline{\psi}^j\psi^j], [\overset{(-)}{Q}_s^{kl}, \psi^j\overline{\partial}\psi^j]$, and $[\overset{(-)}{Q}_s^{kl}, \overline{\psi}^j\partial\overline{\psi}^j]$ as well. Therefore the *action (1.1) is invariant under the infinitesimal* $\widehat{O}(n)$*-Kac-Moody transformation (3.9b):*

$$\left[\widetilde{Q}_s^{ij}, S\right] = 0 \qquad -\infty \le s \le \infty \quad .$$

It is straightforward to make an analogous conclusion concerning the P_s^{ij}-symmetries of (1.1). Then we have to complete our statement about the symmetries of (1.1): the *subalgebra (3.10), (3.11) of* $\widehat{GL}(n,R)$*-Kac-Moody algebra appears as a larger algebra of symmetries of the* $O(n)$*-massive Majorana fermions action (1.1).*

The question about the origin of such symmetries is in order. The first to be noted is that they are not related to the local $GL(n,R)$-gauge transformations in two dimensional $(z,\overline{z})$ space. However the momenta space form of the transformations (3.9):

$$\begin{aligned}
[\widetilde{P}_s^{ij}, a_k^{\pm}(p)] &= \pm i(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})p^{2s-1}a_l^{\pm}(p) \\
[\widetilde{Q}_s^{ij}, a_k^{\pm}(p)] &= \pm i(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})p^{2s}a_l^{\pm}(p) \quad , \qquad -\infty \le s \le \infty
\end{aligned} \tag{3.13}$$

is very suggestive. Remember the standard realization (1.5) of the x-space infinitesimal local gauge transformations. Therefore the specific $\widehat{GL}(n,R)_{mod\,2}$-Kac-moody algebra of symmetries of (1.1) we have found a manifest local momenta-space $GL(n,R)$ gauge invariance. The parameters $\omega^{ij}(p)$ of these transformations are restricted under the conditions:

$$\begin{aligned}
\omega^{(ij)}(p) &= -\omega^{(ij)}(-p) \\
\omega^{[ij]}(p) &= \omega^{[ij]}(-p) \quad .
\end{aligned}$$

These properties are not a specific feature of the $O(n)$-massive fermions only. One can find *two incomplete* ($s \ge 0$!) $GL(n,R)_{mod\,2}$ local (in momenta space) gauge groups of symmetries for the massless $O(n)$-fermions as well. One can further speculate that the conformal $W_\infty(\widehat{G}_n)$-algebra describes the symmetries of the phase space of the corresponding

conformal model. A part of these symmetries survives the perturbation forming the *non-conformal* $\widetilde{W}_\infty(\widehat{G}_n)$*-algebra.* The x-space symmetries are restricted now to $2-D$ Poincaré group, global G_n gauge invariance and the specific $\widetilde{W}_\infty(\widehat{G}_n)$ symmetries, for example the $GL(n,R)_{\text{mod}2}$-Kac-Moody algebra (3.10), (3.11). The latter manifests as a local gauge transformations in the p-space. In this line of arguments one can consider $\widehat{GL}(n,R)_{\text{mod}2}$ (and $\widetilde{W}_\infty(G_n)$ in general) as symmetries of the phase space of the integrable models.

Our motivation to study the full set of conservation charges for $k = 1$, $O(n)$-WZW massive models was to find the full algebra of the symmetries of the model (*presumably noncommuting*) in order to use it for the calculations of the correlation functions. We already have found one nontrivial subalgebra (3.10), (3.11), generated by the charges of the conserved tensors $\overset{(-)}{T}{}_{2s}^{ij}$ and $\overset{(-)}{J}{}_{2s-1}^{ij}$. Is this symmetry sufficient to fix all the correlation functions (without using equation of motion)? What is known from the conformal WZW models is that the conformal current algebra Vir $\otimes\widehat{G}_n$ is a powerful tool for such calculations. Therefore we have to look for more *new charges* L_n, generating the Virasoro algebra, in order to complete our $\widehat{O}(n)$ (or $\widehat{GL}(n,R)_{\text{mod}2}$) -Kac-Moody algebras (3.10), (3.11) to the larger Vir $\otimes\widehat{O}(n)$ algebra.

How to construct the Virasoro charges for one massive fermion we already know from the off-critical Ising model case. In order to generalize it for the $O(n)$-massive fermions we have to find specific combinations of the "higher momenta" of the $\overset{(-)}{T}{}_{2s}^{ij}, \overset{(-)}{J}{}_{2s-1}^{ij}, \theta^{ij}$ and $\overline{\theta}^{ij}$ to be conserved. From the explicit form (3.4) and (3.5) of these standard conservation laws one can conclude that they satisfy the criterion[2] for existence of new charges. Therefore we can construct $(4s-3)\frac{n(n+1)}{2}$ new symmetric charges $L_{-n}^{ij(2s)}, \overline{L}_{-n}^{ij(2s)}$ $(0 \le n \le 2s-1)$ and $(4s-5)\frac{n(n-1)}{2}$ antisymmetric ones $Q_{-k}^{ij(2s-1)}, \overline{Q}_{-k}^{ij(2s-1)}$ $(0 \le k \le 2s-2)$ for each $s = 2,3,\cdots$.

To begin with the first momenta, i.e. quantities linear in z and $\overline{z}$. The simplest one is the generalization of the Lorentz rotation $L_0 = \frac{1}{2}\delta^{ij}L_0^{ij}$:

$$L_0^{ij} = \int(zT_2^{ij} + \overline{z}\theta^{ij})dz - \int(\overline{z}\overline{T}_2^{ij} + z\theta^{ij})d\overline{z} \quad .$$

The next two are the off-critical analogs of the conformal first momenta of T_4^{ij} and $\overline{T}_4^{ij}$:

$$\begin{aligned}
L_{-2}^{ij(4)} &= \int(zT_4^{ij} + 2m^2\overline{z}T_2^{ij})dz - m^2\int(2zT_2^{ij} + \overline{z}\theta^{ij})d\overline{z} \\
\overline{L}_{-2}^{ij(4)} &= \int(\overline{z}\overline{T}_4^{ij} + 2m^2z\overline{T}_2^{ij})d\overline{z} - m^2\int(2\overline{z}\overline{T}_2^{ij} + z\theta^{ij})dz
\end{aligned}$$

One can go further and construct $\overset{(-)}{L}{}_{-4}^{ij(6)}$ etc. However all of them are straightforward $O(n)$-matrix generalization of the corresponding one fermion charges $\overset{(-)}{L}{}_{-2s+2}^{(2s)} = \frac{1}{2}\delta^{ij}\overset{(-)}{L}{}_{-2s+2}^{ij(2s)}$ and we can take them in the following differential form:

$$\begin{aligned}
\left[L_{-2s+2}^{ij(2s)}, \psi^k(z,\overline{z})\right] &= -i(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})\left(\overline{z}\overline{\partial} - z\partial - \frac{2s-1}{2}\right)\partial^{2s-2}\psi^l \\
\left[\overline{L}_{-2s+2}^{ij(2s)}, \psi^k(z,\overline{z})\right] &= -i(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})\left(\overline{z}\overline{\partial} - z\partial + \frac{2s-3}{2}\right)\overline{\partial}^{2s-2}\psi^l
\end{aligned} \tag{3.14}$$

The proof that they are indeed the conserved charges we are looking for is again based on the fact that they do generate new symmetries of the action (1.1), i.e.

$$[L^{ij(2s)}_{-2s+2}, S] = 0 = [\overline{L}^{ij(2s)}_{-2s+2}, S]$$

The last statement follows from specific higher derivatives identities similar to the one used in the proof of (3.12)

The question about the algebra of these new symmetries is now in order. By direct calculations, using (3.14) one can see that $L^{ij(2s)}_{-2s+2}$ and $\overline{L}^{ij(2s)}_{-2s+2}$ *does not close* an algebra. It is necessary to consider together with them the first momenta $\overset{(-)}{Q}{}^{ij(2s-1)}_{-2s+3}$ of the current J^{ij}_{2s-1}. Before doing this we should mention that the traces $\overset{(-)}{L}{}^{(2s)}_{-2s+2} = \frac{1}{2}\delta_{ij}\overset{(-)}{L}{}^{ij(2s)}_{-2s+2}$ *do close* an algebra which coincides with the off-critical Virasoro algebra V_c of the off-critical Ising model[2]. One could wonder what is then the algebra of $\widetilde{Q}^{ij}_s$ and these Virasoro generators:

$$V_k = \frac{1}{4}(-m^2)^k\delta_{ij}L^{ij(2k+2)}_{-2k} \quad , \quad V_{-k} = \frac{1}{4}(-m^2)^k\delta_{ij}\overline{L}^{ij(2k+2)}_{-2k}$$

As one could expect the result of simple computations is the larger current algebra $V_c \otimes \widehat{O}_n$:

$$\begin{aligned}
\left[V_{m_1}, V_{m_2}\right] &= (m_1 - m_2)V_{m_1+m_2} + \frac{n}{24}m_1(m_1^2-1)\delta_{m_1+m_2} \\
\left[V_{m_1}, \widetilde{Q}^{ij}_{m_2}\right] &= -\, m_2\widetilde{Q}^{ij}_{m_1+m_2} \\
\left[\widetilde{Q}^{ij}_{m_1}, \widetilde{Q}^{kl}_{m_2}\right] &= \delta^{ik}\widetilde{Q}^{jl}_{m_1+m_2} + \delta^{jl}\widetilde{Q}^{ik}_{m_1+m_2} - \delta^{il}\widetilde{Q}^{jk}_{m_1+m_2} - \delta^{jk}\widetilde{Q}^{il}_{m_1+m_2} + \\
&\quad + \frac{n}{2}m_1\delta_{m_1+m_2}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \quad .
\end{aligned} \tag{3.15}$$

We have enlarged in this way the known symmetries of the action (1.1) to the $V_c \otimes \widehat{O}(n)$-algebra.

Turning back to our problem of constructing the first momenta of the current J^{ij}_{2s-1} we start with the explicit form of the simplest two of them:

$$\begin{aligned}
Q^{ij(3)}_{-1} &= \int (zJ^{ij}_3 - 2m^2\overline{z}J^{ij}_1)dz - \int \left[2\widetilde{\theta}^{ij} - 2m^2zJ^{ij}_1 - \partial(z\widetilde{\theta}^{ij}) + 2m^2z\overline{J}^{ij}_1\right]d\overline{z} \\
Q^{ij(5)}_{-3} &= \int (zJ^{ij}_5 + m^2\overline{z}J^{ij}_3)dz - \int [m^4\overline{z}J^{ij}_1 - \frac{9}{2}m^2\partial J^{ij}_1 + \frac{3}{2}m^2\partial^2(zJ^{ij}_1) + m^2zJ^{ij}_3 \\
&\quad + m^2\partial(\overline{z}\widetilde{\theta}^{ij}) + \partial^3(z\widetilde{\theta}^{ij}) - 4\partial^2\widetilde{\theta}^{ij}]d\overline{z}
\end{aligned}$$

Following the method we used above for $\widetilde{P}^{ij}_s$ and $\widetilde{Q}^{ij}_s$ we arrive at the following general differential form for $\overset{(-)}{Q}{}^{ij(2s-1)}_{-2s+3}$:

$$\begin{aligned}
\left[Q^{ij(2s-1)}_{-2s+3}, \psi^k(z,\overline{z})\right] &= -i(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})(\overline{z}\overline{\partial} - z\partial - s + 1)\partial^{2s-3}\psi^l(z,\overline{z}) \\
\left[\overline{Q}^{ij(2s-1)}_{-2s+3}, \psi^k(z,\overline{z})\right] &= -i(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})(\overline{z}\overline{\partial} - z\partial + s - 2)\overline{\partial}^{2s-3}\psi^l(z,\overline{z})
\end{aligned} \tag{3.16}$$

Considering $\overset{(-)}{Q}{}^{ij(2s-1)}_{-2s+3}$ together with $\overset{(-)}{L}{}^{ij(2s)}_{-2s+2}$, $\overset{(-)}{Q}{}^{ij}_s \equiv \overset{(-)}{Q}{}^{ij(2s-1)}_{-2s+2}$ $\overset{(-)}{P}{}^{ij}_s \equiv \overset{(-)}{L}{}^{ij(2s)}_{-2s+1}$ we are expecting them to close an algebra. However this is not the case. One can easily check using (3.16) that the commutator $[Q^{ij(2s_1-1)}_{-2s_1+3}, Q^{kl(2s_2-1)}_{-2s_2+3}]$ contains higher momenta of J^{ij}_{2s-1} and T^{ij}_{2s} as well. For example, the simplest one has the form:

$$\begin{aligned}
\left[Q^{ij(3)}_{-1}, Q^{kl(3)}_{-1}\right] &= -\delta^{ik}(Q^{jl(5)}_{-2} - 4Q^{jl(3)}_{-2}) + \delta^{il}(Q^{jk(5)}_{-2} - 4Q^{jk(3)}_{-2}) \\
&\quad + \delta^{jk}(Q^{il(5)}_{-2} - 4Q^{il(3)}_{-2}) - \delta^{jl}(Q^{ik(5)}_{-2} - 4Q^{ik(3)}_{-2}) \quad .
\end{aligned} \tag{3.17}$$

It includes together with the "zero momenta" $Q^{il(3)}_{-2} \equiv Q^{il}_1$, the *second momenta* $Q^{ij(5)}_{-2}$ of J^{ij}_5:

$$\begin{aligned}
\left[Q^{ij(5)}_{-2}, \psi^k(z,\overline{z})\right] &= -i(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})\left[\frac{15}{4} + (\overline{z}\overline{\partial} - z\partial - 3/2)^2\right]\partial^2\psi^l \\
&= -\frac{i}{2}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk})\left[(\overline{z}\overline{\partial} - z\partial)_2 + (\overline{z}\overline{\partial} - z\partial - 4)_2\right]\partial^2\psi^l
\end{aligned} \quad , \tag{3.18}$$

where $(A)_n = A(A+1)\cdots(A+n-1)$.

All this discussion is to demonstrate that *the algebra of the first momenta* of T^{ij}_{2s} and J^{ij}_{2s-1} *is not closed.* Involving the higher momenta of T^{ij}_{2s}, J^{ij}_{2s-1} we are constructing in this way an algebra of the $W_\infty(\widehat{G}_n)$-type (2.6). Leaving the problem of the general structure of the algebra of all the symmetries of (1.1) to the next section we address here the question about its subalgebras. Up to now we have constructed two such subalgebras: $\widehat{GL}(n,R)_{\text{mod}2}$ given by (3.10), (3.11) and Vir $\otimes \widehat{O}(n)$ of eq. (3.15). Deriving the missing commutator

$$\left[V_{m_1}, \widetilde{P}^{ij}_{m_2}\right] = -(m_2 - 1/2)\widetilde{P}^{ij}_{m_1+m_2} \tag{3.19}$$

we can unify them in an unique current algebra, namely: Vir $\otimes \widehat{GL}(n,R)_{\text{mod}2}$. Are there more subalgebras of this type? As in the cases of one[2] and two[1] fermions one could expect to find two incomplete ($n \geq -1$) Virasoro subalgebras. In our case they are generated by a specific combination of $\delta^{ij}\overset{(-)}{L}{}^{ij(2k)}_{-s+1}$ and $\delta^{ij}\overset{(-)}{L}{}^{ij}_{-1} \equiv \overset{(-)}{P}_1$:

$$\overset{(-)}{\mathcal{L}}_n = \sum_{k=\left[\frac{n}{2}\right]}^{n} \beta_k \overset{(-)}{L}{}^{ij(2k)}_{-n+1}\delta_{ij} = \left[\overline{z}\overline{\partial} - z\partial \pm 1/2\right]_{n+1} \overset{(-)}{\partial}{}^n \quad . \tag{3.20}$$

Do they have an $\widehat{O}(n)$-Kac-Moody counterpart? The form of the commutator (3.17) suggests to consider Q^{ij}_0, $Q^{ij(3)}_{-1}$ and $Q^{ij(5)}_{-2} - 4Q^{ij(3)}_{-2}$ as appropriate candidates for the generators $\mathcal{Q}^{ij}_0$, $\mathcal{Q}^{ij}_1$ and $\mathcal{Q}^{ij}_2$ of the incomplete ($n \geq 0$) $\widehat{O}(n)$-Kac-Moody algebra. The differential form of these generators

$$\begin{aligned}
(\mathcal{Q}^{ij}_0)_{kl} &= -i(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \\
(\mathcal{Q}^{ij}_1)_{kl} &= (\mathcal{Q}^{ij}_0)_{kl}(\overline{z}\overline{\partial} - z\partial - 1)\partial \\
(\mathcal{Q}^{ij}_2)_{kl} &= (\mathcal{Q}^{ij}_0)_{kl}(\overline{z}\overline{\partial} - z\partial - 1)(\overline{z}\overline{\partial} - z\partial - 2)\partial^2
\end{aligned}$$

allows us to guess the general form of the $\widehat{O}(n)$ generators:

$$(Q_s^{ij})_{kl} = (Q_0^{ij})_{kl}\left[\bar{z}\bar{\partial} - z\partial - 1\right]_n \partial^n \quad , \quad n \geq 0 \quad . \tag{3.21}$$

Using the simple identity:

$$\partial^k\left[\bar{z}\bar{\partial} - z\partial - 1\right]_n = \left[\bar{z}\bar{\partial} - z\partial - 1 - k\right]_n \partial^k$$

one can easily verify that (3.21) indeed close $\widehat{O}(n)$-Kac-Moody algebra (3.10a). Similarly, the conserved charges

$$(\overline{Q}_s^{ij})_{kl} = (Q_0^{ij})_{kl}\left[\bar{z}\bar{\partial} - z\partial\right]_n \bar{\partial}^n \quad , \quad n \geq 0$$

generate one more $\widehat{O}(n)$-current algebra. These two algebras however do not mutually commute.

There are certain indications that the algebras of symmetries of (1.1) we have described up to now are sufficient for the calculation of the correlation functions. Leaving aside the problem of how to use the null-vector corresponding to (3.15), (3.19) or how to solve the infinite system of Ward identities (W.I.'s) for the charges $\widetilde{P}_s^{ij}$, $\widetilde{Q}_s^{ij}$, V_s , $\mathcal{L}_n$ and $\overset{(-)}{\mathcal{Q}}{}_n^{ij}$ we shall mention the following simple and *remarkable fact*: the $Q_{-1}^{ij(3)}$ (or $\overline{Q}_{-1}^{ij(3)}$) W.I.'s for the 2-point function

$$g^{lm}(z_1, z_2|\bar{z}_1, \bar{z}_2) = \langle \psi^l(z_1, \bar{z}_1)\psi^m(z_2, \bar{z}_2)\rangle$$

coincide with the K_1-Bessel equation.

Taking into account the Poincaré invariance (i.e., L_0, $\overset{(-)}{L}_{-1} = \frac{1}{2}\delta^{ij}\overset{(-)}{L}{}_{-1}^{ij}$) we get

$$g^{lm}(z_1, z_2|\bar{z}_1, \bar{z}_2) = \delta^{lm}\sqrt{\frac{\bar{z}_{12}}{z_{12}}}K(x) \quad , \quad x = \sqrt{-4m^2 z_{12}\bar{z}_{12}}$$

We next require the $Q_{-1}^{ij(3)}$-Ward identity:

$$\left\langle Q_{-1}^{ij(3)}\psi^l(z_1, \bar{z}_1)\psi^m(z_2, \bar{z}_2)\right\rangle = 0 \quad .$$

As a consequence of (3.16) and $Q_{-1}^{ij(3)}$-invariance of the vacua we obtain the following equation:

$$\left(\bar{z}_{12} + \frac{2}{m^2}\partial_{12} + \frac{1}{m^2}z_{12}\partial_{12}^2\right)\sqrt{\frac{\bar{z}_{12}}{z_{12}}}K(x) = 0$$

which is equivalent to the K_1-Bessel equation:

$$x^2K''(x) + xK'(x) - (x^2+1)K(x) = 0$$

We have in fact to solve an infinite system of higher order differential equations representing the remaining Ward identities:

$$\langle O_n\psi^l(z_1, \bar{z}_1)\psi^m(z_2, \bar{z}_2)\rangle = 0$$

where $O_n = \{\widetilde{P}_s^{ij}, \widetilde{Q}_s^{ij}, V_s, \mathcal{L}_n, \overset{(-)}{\mathcal{Q}}{}_n^{ij}\}$. The algebraic explanation of why all they could be solved in terms of K_1 is not known. One non-trivial check is the $L_{-2}^{ij(4)}$ and $\mathcal{L}_1$-W.I.'s which are specific third order differential equation. In the $L_{-2}^{ij(4)}$ case for example we have:

$$x^3K'''(x) + 2x^2K''(x) - x(x^2+1)K'(x) - (x^2+1)K(x) = 0$$

As it has been demonstrated in [2], this equation has $K_1(x)$ as a solution.

4. Off-critical $\widetilde{W}_\infty(\widehat{G}_n)$-algebra

Although it seems reasonable that the Vir $\otimes\widehat{GL}(n, R)_{\mathrm{mod}2}$ and the other $\widehat{O}(n)$-Kac-Moody and Virasoro algebras) are the most important part of the symmetries of (1.1) we find interesting to study the full algebra as well. To do this we have to continue with the constructions of the higher momenta of T_{2s}^{ij} and J_{2s-1}^{ij}. One such example is the second momenta of J_3^{ij}:

$$Q_0^{ij(3)} = \int \mathcal{F}_0^{ij}dz - \int \mathcal{G}_0^{ij}d\bar{z}$$

where

$$\mathcal{F}_0^{ij} = z^2J_3^{ij} - 4m^2 z\bar{z}J_1^{ij} + 2m^2z^2\overline{J}_1^{ij} - \bar{\partial}\left(\bar{z}^2\widetilde{\theta}^{ij}\right) + 4\bar{z}\widetilde{\theta}^{ij}$$

$$\mathcal{G}_0^{ij} = -\bar{z}^2\overline{J}_3^{ij} + 4m^2 z\bar{z}\overline{J}_1^{ij} - 2m^2z^2J_1^{ij} - \partial\left(z^2\widetilde{\theta}^{ij}\right) + 4z\widetilde{\theta}^{ij}$$

are components of conserved tensor:

$$\bar{\partial}\mathcal{F}_0^{ij} = \partial\mathcal{G}_0^{ij}$$

One can realize $Q_0^{ij(3)}$ as differential operator as well:

$$(Q_0^{ij(3)})_{kl} = -\frac{1}{2}(Q_0^{ij})_{kl}\left[(\bar{z}\bar{\partial} - z\partial)_2 + (\bar{z}\bar{\partial} - z\partial - 2)_2\right] \quad .$$

This formula together with (3.16) and (3.18) allows us to make a conjecture about the general form of all the conserved charges related to J_{2s-1}^{ij}

$$\begin{aligned}\left(Q_{-m}^{ij(2s-1)}\right)_{kl} &= -\frac{1}{2}\left(Q_0^{ij}\right)_{kl}\Big[(\bar{z}\bar{\partial} - z\partial + \alpha)_{2s-2-m} + \\ &\qquad + (\bar{z}\bar{\partial} - z\partial + \alpha - 2s + 2)_{2s-2-m}\Big]\partial^m \\ \left(\overline{Q}_{-m}^{ij(2s-1)}\right)_{kl} &= -\frac{1}{2}\left(Q_0^{ij}\right)_{kl}\Big[(\bar{z}\bar{\partial} - z\partial + \bar{\alpha} - 2s + m + 3)_{2s-2-m} + \\ &\qquad + (\bar{z}\bar{\partial} - z\partial + \bar{\alpha} + m + 1)_{2s-2-m}\Big]\bar{\partial}^m \\ 0 \leq m &\leq 2s - 2\end{aligned} \tag{4.2}$$

where $\alpha = 0\,, \overline{\alpha} = -1$ for ψ and $\alpha = 1\,, \overline{\alpha} = 0$ for $\overline{\psi}$. The proof of this conjecture is again indirect. By tedious higher derivatives calculus and identities similar to the one used before (for the case Q^{ij}_s) one can verify that (4.2) are indeed symmetries of (1.1), i.e. $\left[\overset{(-)}{Q}{}^{ij(2s+1)}_{-m}, S\right] = 0$.

We have next to find the differential form of the remaining part $\overset{(-)}{L}{}^{ij(2s)}_{-m}$ $(0 \le m \le 2s-1)$ of the generators. We have already mentioned their close relation with the one fermion's generators $\overset{(-)}{L}{}^{(2s)}_{-m}$ (see ref.[2]). This fact allows us to write $\overset{(-)}{L}{}^{ij(2s)}_{-m}$ using the $\overset{(-)}{L}{}^{(2s)}_{-k}$ differential operators:

$$\begin{aligned}
\left(L^{ij(2s)}_{-m}\right)_{kl} &= -\frac{1}{2}\left(I^{ij}\right)_{kl}\left[\left(\overline{z}\overline{\partial} - z\partial + \alpha\right)_{2s-1-m} + \left(\overline{z}\overline{\partial} - z\partial + \alpha - 2s + 1\right)_{2s-1-m}\right]\partial^m \\
\left(\overline{L}^{ij(2s)}_{-m}\right)_{kl} &= -\frac{1}{2}\left(I^{ij}\right)_{kl}\left[\left(\overline{z}\overline{\partial} - z\partial + \overline{\alpha} - 2s + m + 2\right)_{2s-1-m} + \left(\overline{z}\overline{\partial} - z\partial + \overline{\alpha} + m + 1\right)_{2s-1-m}\right]\overline{\partial}^m
\end{aligned} \tag{4.3}$$

where $\left(I^{ij}\right)_{kl} = \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}$.

We have exhausted in this way all the symmetries of the $O(n)$-massive free fermions. What we are going to show now is that *their algebra consists of two noncommuting (incomplete $0 \le k \le 2s-1$) $W_\infty(\widehat{G}_n)$algebras*. In words the conformal structure constants (2.7) reappear again in the massive theory as structure constants of the "left" and "right" subalgebras spanned by $L^{ij(2s)}_{-m}, Q^{ij(2s-1)}_{-m}$ and $\overline{L}^{ij(2s)}_{-m}, \overline{Q}^{ij(2s-1)}_{-m}$. The method is similar to the one used in the case of $W_\infty(V)$-algebra[2]. We start with the "conformal decomposition" of the "left" generators:

$$\begin{aligned}
L^{ij(2s)}_{-k} &= \sum_{l=0}^{2s-1-k}\binom{2s-1-k}{l}\left(\overline{z}\overline{\partial}^2 + \alpha\overline{\partial}\right)^l \mathcal{L}^{ij(2s)}_{-k-l}(-m^2)^l\,, \\
Q^{ij(2s-1)}_{-k} &= \sum_{l=0}^{2s-2-k}\binom{2s-2-k}{l}\left(\overline{z}\overline{\partial}^2 + \alpha\overline{\partial}\right)^l \mathcal{Q}^{ij(2s-1)}_{-k-l}(-m^2)^l\,.
\end{aligned} \tag{4.4}$$

In the calculation of the commutators of (4.4) we are using the conformal CR's (2.6) and the fact that the operators $(\overline{z}\overline{\partial}^2 + \alpha\overline{\partial})^l$ are commuting. In order to prove that $L^{ij(2s)}_{-m}$ and $Q^{ij(2s-1)}_{-m}$ satisfy (2.6) with the same structure constants we do need the following property of $g^{s_1 s_2}_r(n_1, n_2)$ to be satisfied:

$$\begin{aligned}
g^{s_1,s_2}_r(-k_1,-k_2)&\binom{s_1+s_2-r-k_1-k_2-1}{n} = \\
&= \sum_l \binom{s_1-k_1-1}{l}\binom{s_2-k_2-1}{n-l} g^{s_1,s_2}_r(-k_1-l, -k_2-n+l)
\end{aligned} \tag{4.5}$$

We have to remember at this point that the only structure constants we know are the ones calculated in the quasiprimary basis (2.3). Therefore we are forced to use them in the proof of (4.5). The question now is whether the identity (4.5) depends on the basis.

The answer is that (4.5) holds in all the basis. It is related to the fact that the form of the conformal decomposition (4.4) is universal. To see it we have to write all the generators (conformal and nonconformal) in the quasiprimary basis.

$$\begin{aligned}
\left(\widetilde{\mathcal{L}}^{ij(2s)}_{-m}\right)_{kl} &= \left(I^{ij}\right)_{kl}\sum_{p=1}^{2s}(-1)^p\binom{2s-1}{p-1}^2 \partial^{p-1}\left(z^{2s-m-1}\partial^{2s-p}\right) \\
\left(\widetilde{L}^{ij(2s)}_{-m}\right)_{kl} &= \left(I^{ij}\right)_{kl}\sum_{p=1}^{2s}(-1)^p\binom{2s-1}{p-1}^2 \left(\overline{z}\overline{\partial} - z\partial - p + 1\right)_{2s-m-1}\partial^m
\end{aligned} \tag{4.6}$$

etc. The crucial point is that using (4.6) we arrive again at the same "conformal decomposition" (4.4). The last step of the proof is to substitute (2.7) in (4.5) and verify that it holds. The conclusion is that the "left" off-critical algebra shares the same form and same g^{s_1,s_2}_r as the conformal $W_\infty(\widehat{G})$-algebra (note that $0 \le m \le 2s-1$). The same is true for the "right" algebra.

From the explicit form (4.6) of $\overset{(-)}{L}{}^{ij(2s)}_{-m}, \overset{(-)}{Q}{}^{ij(2s-1)}_{-m}$ one can easily see that "left" and "right" algebra do not commute. The problem of computing the structure constants $\overline{g}^{s_1,s_2}_r$ of the mixed algebra $(m_1 < m_2)$:

$$\begin{aligned}
\left[L^{ij(2s_1)}_{-m_1}, \overline{L}^{kl(2s_2)}_{m_2}\right] = (m^2)^{m_1}\sum_{r=0}^{s_1+s_2-m_1-2}&\left\{\overline{g}^{2s_1,2s_2}_{2r}\left(\delta \circ \overline{L}^{2(s_1+s_2-m_1-1-r)}_{-m_2+m_1}\right)^{ijkl}\right. \\
&\left. + \overline{g}^{2s_1,2s_2}_{2r-1}\left(\delta \circ \overline{Q}^{2(s_1+s_2-m_1-r)-1)}_{-m_2+m_1}\right)^{ijkl}\right\}
\end{aligned}$$

$$\begin{aligned}
\left[Q^{ij(2s_1-1)}_{-m_1}, \overline{Q}^{kl(2s_2-1)}_{m_2}\right] = (m^2)^{m_1}\sum_{r=0}^{s_1+s_2-m_1-2}&\left\{\overline{g}^{2s_1-1,2s_2-1}_{2r}\left(\delta \star \overline{L}^{2(s_1+s_2-m_1-r)-1}_{-m_2+m_1}\right)^{ijkl}\right. \\
&\left. + \overline{g}^{2s_1-1,2s_2-1}_{2r-1}\left(\delta \star \overline{Q}^{2(s_1+s_2-m_1-r)-1)}_{-m_2+m_1}\right)^{ijkl}\right\}
\end{aligned}$$

etc. is more complicated. One can apply in principle the method[2] used for $W_\infty(V)$-algebra in the case of $\widetilde{W}_\infty(\widehat{G}_n)$ as well, but up to now the problem of the computation of the mixed structure constants $g^{s_1 s_2}_r(n_1, n_2)$ is still open.

5. Further generalizations

The fact that the $O(n)$-massive fermionic action (1.1) has Vir $\otimes \widehat{GL}(n, R)_{\mathrm{mod}2}$ and $\widetilde{W}_\infty(\widehat{G}_n)$ as algebras of symmetries leads to a natural question: *whether one can find* *similar infinite dimensional algebras studying more general systems of free massive fields*: N fermions and M bosons taken in appropriate representations of the internal group G_n. The purely fermionic case is straightforward generalization of the n-Majorana fermions in vector representation of $O(n)$ we have described above. For example all the conserved charges of the massive fermions in the fundamental representation of $A_N = SU(N+1)$ are generated by the conserved tensors (2.8) (written now for the massive fermions $\psi^{\alpha}(z,\overline{z})$). The case of massive bosons in the vector representation of $O(n)$:

$$S = \frac{1}{2}\int d^2z\,(\partial\varphi^i\overline{\partial}\varphi^i + m^2\varphi^i\varphi^i) \tag{5.1}$$

requires certain modifications in the construction of the conserved tensors:

$$\begin{aligned} T^{ij}_{2s} &= \partial\varphi^i\partial^{2s-1}\varphi^j + \partial\varphi^j\partial^{2s-1}\varphi^i \\ J^{ij}_{2s-1} &= \varphi^i\partial^{2s-1}\varphi^j - \varphi^j\partial^{2s-1}\varphi^i \quad . \end{aligned} \tag{5.2}$$

Using the equation of motion one can easily obtain the corresponding conservation laws:

$$\begin{aligned} &\overline{\partial}T^{ij} = \partial\theta^{ij} \quad , \quad \theta^{ij} = m^2\varphi^i\varphi^j \\ &\overline{\partial}J^{ij} + \partial\overline{J}^{ij} = 0 \quad , \quad \overline{J}^{ij} = \varphi^i\overline{\partial}\varphi^j - \varphi^j\overline{\partial}\varphi^i \\ &\overline{\partial}T^{ij}_4 = \partial^3\theta^{ij} + m^2\partial T^{ij} \\ &\overline{\partial}J^{ij}_3 = -\partial^3\overline{J}^{ij} + m^2\partial J^{ij} \qquad ,\text{etc} \quad . \end{aligned} \tag{5.3}$$

Due to the specific form of the r.h.s. of (5.3) T^{ij}_{2s} and J^{ij}_{2s-1} do satisfy the criterion of ref.[2] of existence of new noncommuting charges. The constructions of the conserved charges are similar to the fermionic ones and as a consequence they span the same algebra $\widetilde{W}_{\infty}(G_n)$.

Having a system of massive fermions and bosons one could expect larger symmetries which mix the fermionic and bosonic degrees of freedom, i.e. supersymmetric generalization $\widetilde{SW}_{\infty}(G_n)$ of the $\widetilde{W}_{\infty}(G_n)$. The simplest case is of one Majorana fermion $\psi(z,\overline{z}),\overline{\psi}(z,\overline{z})$ and one boson $\varphi(z,\overline{z})$. We can take the conserved tensors in the form:

$$\begin{aligned} T_{2s} &= \psi\partial^{2s-1}\psi + \partial\varphi\partial^{2s-1}\varphi \\ G_{2s-1/2} &= \psi\partial^{2s-1}\varphi \quad , \quad \overline{G}_{2s-1/2} = \overline{\psi}\,\overline{\partial}^{2s-1}\varphi \quad . \end{aligned} \tag{5.4}$$

The corresponding conservation laws have specific form with higher derivatives in the r.h.s., which allows the construction of "higher momenta" conserved charges. For example:

$$\begin{aligned} &\overline{\partial}G_{3/2} = \partial\Theta \quad , \quad \Theta = m\overline{\psi}\varphi\,,\,\overline{\Theta} = m\psi\varphi \\ &\partial\overline{G}_{3/2} = -\overline{\partial}\,\overline{\Theta} \\ &\overline{\partial}G_{7/2} = \partial^3\Theta + m\partial^2\overline{\Theta} + m^2\partial G_{3/2} \quad , \end{aligned}$$

etc. and therefore we can construct an infinite set of supersymmetric charges $G^{(2s-1)}_{-k}$ (and $\overline{G}^{(2s-1)}_{-k}$):

$$\begin{aligned} \mathcal{G}_{-1/2} &= \int G_{3/2}dz - \int \Theta d\overline{z} \\ \mathcal{G}^{(3)}_{-3/2} &= \int G_{7/2}dz - \int \left(\partial^2\Theta + m\partial\overline{\Theta} + m^2G_{3/2}\right)d\overline{z} \\ \mathcal{G}^{(3)}_{-1/2} &= \int \left(zG_{7/2} + m^2\overline{z}G_{3/2}\right)dz - \int \left(\partial^2 z\Theta - 3\partial\Theta + m\partial(z\overline{\Theta}) - 2m\overline{\Theta} + m^2zG_{3/2}\right)d\overline{z} \end{aligned}$$

etc. Together with the conserved "momenta" $L^{(2s)}_k$ of T_{2s} they span the $N=1$ supersymmetric analog of the $W_{\infty}(V)$ algebra [2]. By similar constructions considering say three fermions and three bosons we can derive the off-critical supersymmetric analog of $W_{\infty}(A_1)$, i.e. the current superalgebra $\widetilde{SW}_{\infty}(A_1)$.

It becomes clear from this short discussion that having at hands free massive fermions and bosons one can construct large class of N-supersymmetric and supercurrent off-critical $W_{\infty}(G_n)$ algebras.

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References.

[1] H. Itoyama, H.B. Thacker, *Nucl. Phys.* **B320** (1989)541.
[2] G. Sotkov, M. Stanishkov IFT-preprint 001/93.
[3] L. Bonora, Y.Z. Zhang, M. Martellini, *Int. J. Mod. Phys.* **A6** (1991)1617.
[4] V. G. Knizhnik, A. B. Zamolodchikov, *Nucl. Phys.* **B247** (1984)83.
[5] E. Witten *Commun. Math. Phys.* **92** (1984)455.
[6] C.N. Pope, X. Shen, L.J. Romans *Nucl. Phys.* **B339** (1990)191.
[7] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov *Nucl. Phys.* **B241** (1984)333.
[8] P. Furlan, G. Sotkov, I. T. Todorov, *Rivista del Nuovo Cimento* **12** (1989)1.
[9] C.N. Pope, L.J. Romans, X. Shen, Phys. Lett. **B242** (1990)101.
[10] P. Gepner, E. Witten, Nucl. Phys. B287(1987)493.