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VACUUM FLUCTUATIONS, RADIATION REACTION AND SUB-HEISENBERG STATES

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ABSTRACT

The effects of the vacuum electromagnetic fluctuations and the radiation reaction fields on the time development of a simple microscopic system are identified. This is done by studying a charged mechanical oscillator (frequency ω_0) within the realm of stochastic electrodynamics, where the "vacuum" plays the role of a large energy reservoir. We show how the Liouville equation is transformed into a Schrödinger's like stochastic equation with an *arbitrary* "quantum" of action $\hbar'$. The role of the physical Planck's constant $\hbar$ is introduced through the zero-point vacuum electromagnetic fields. The perturbative and the exact solutions of the stochastic Schrödinger equation are presented. The non-perturbative solutions appear in the form of sub-Heisenberg states for which the initial *classical uncertainty relation* takes the form $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = (\hbar'/2)^2$, which includes the limit of zero indeterminacy ($\hbar' = 0$). We show how the radiation reaction and the vacuum fields govern the evolution of these sub-Heisenberg states in phase space guaranteeing that they decay to the stationary state with average energy $\hbar\omega_0/2$. Environmental and thermal effects are briefly indicated and the connection with similar works within the realm of quantum electrodynamics is also presented.

I. INTRODUCTION

Even in its ground state, a microscopic system possesses fluctuations which are associated to the zero-point (or zero temperature) energy which exists in empty space. The most striking example are the electric and magnetic vacuum fields which can be indirectly observed[1,2]. The spectral distribution $\rho_0(\omega)$ of these electromagnetic fields is well known and it is related to the correlation function of these fluctuating fields through the ensemble average[1,3]:

$$\frac{3}{4\pi}\,\langle E_{VF}(t)\,E_{VF}(0)\rangle = \int_0^\infty d\omega\,\rho_0(\omega)\,\cos(\omega t) \quad . \tag{1.1}$$

Here $E_{VF}(t)$ is the x component of the vacuum electric field at time t and at the origin of the coordinate system.

Within the realm of stochastic electrodynamics (SED) these E_{VF} fields are classical random fields with zero mean[1,3]. However, within quantum electrodynamics (QED) these electromagnetic fields are considered quantized, that is, they have a dual (wave-particle) nature. Nevertheless in both theories (SED and QED) the spectral distribution is such that[1–3]

$$\rho_0(\omega) = \hbar\omega^3/2\pi^2c^3 \quad , \tag{1.2}$$

where $\hbar$ is the Planck's constant and c the velocity of light.

In this paper we shall study a charged oscillator by considering the stochastic electrodynamics approach for two reasons. The historical reason, which is inspired in the attempts of Planck[4], Einstein and Stern[5] and Nernst[6] "to return to continuous changes in energy". The philosophical, or unifying reason, which is the SED attempt to bring classical and quantum theories to a closer (and maybe nonconflicting) relation[7].

The classical equation of motion of a charged oscillator (charge e and mass m) is given by

$$\ddot{x} = -\omega_0^2 x + \frac{e}{m}\left[E_{VF}(t) + E_{RR}(t)\right] \quad , \tag{1.3}$$

where $e\,E_{RR}(t)$ is the radiation reaction force. The total electric field acting on the particle, that is, $E(t) \equiv E_{VF} + E_{RR}$ will be considered only a function of time within our nonrelativistic approximation.

According to this classical view, the probability distribution in phase space x and $p \equiv m\dot{x}$ (kinetical momentum) will be denoted by $W(x,p,t)$, and will evolve in time according to the Liouville equation, namely:

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x}(\dot{x}\,W) + \frac{\partial}{\partial p}(\dot{p}\,W) = 0 \quad . \tag{1.4}$$

Since $\dot{p}/m = \ddot{x}$ is related to the stochastic "vacuum" field $E_{VF}(t)$ (see (1.3)), the above equation (1.4) can be transformed into a Fokker-Planck equation in a standard manner[8]. Here, however, we prefer to transform the Liouville equation directly into a Schrödinger's like stochastic equation. The main reasons for this choice will be:

a) to show that the Schrödinger equation for an oscillator is entirely equivalent to the classical Liouville equation. Therefore the so called "quantum" calculations using the Schrödinger equation are in fact a classical description of the oscillator.

b) We also show that the wave functions and the energy levels obtained using the Schrödinger equation have no physical meaning being only mathematical tools, useful for calculation purposes, but not strictly necessary for a physical description of the system.

We shall present the above results in a new form through the introduction of an *arbitrary* constant $\hbar'$ with the dimension of action.

II. LIOUVILLE EQUATION IN SCHRÖDINGER'S FORM

The transformation of the classical equation (1.4) into another (equivalent) equation which looks like the Schrödinger equation is known since the famous work by Wigner[9] which was published 60 years ago. However, we shall present the derivation in a new form which will be very convenient for our purpose, that is, to obtain sub-Heisenberg states and to discuss the effects of E_{VF} and E_{RR} on the time evolution of any physical state in phase space.

Using a procedure similar to that introduced by Wigner[9], we shall *define* an auxiliary function $\psi(x,t)$ through the Fourier transform[10]:

$$\psi^*(x+y,t)\,\psi(x-y,t) \equiv \int_{-\infty}^{\infty} dp\, W(x,p,t)\,\exp\left(-\frac{2ipy}{\hbar'}\right) \quad , \tag{2.1}$$

where $\hbar'$ is an *arbitrary* constant with dimension of action.

We shall keep $\hbar' \neq \hbar$ ($\hbar$ is the true Planck's constant) in order to stress that $\hbar'$ has no dynamical meaning. In other words, we shall see that $\hbar$ (introduced in (1.2)) has a dynamical meaning and determines the equilibrium state of the system, whereas $\hbar'$ (introduced in (2.1)) will be related to the shape of the sub-Heisenberg states. It is worthwhile to mention that the ψ functions are not uniquely determined by (2.1). Nevertheless it is easy to show that ψ will satisfy a Schrödinger's like equation as we shall see in a while.

Having defined $\psi(x,t)$ through the Fourier transform (2.1) one can ask what will be the equation obeyed by $\psi(x,t)$ if we impose that $W(x,p,t)$ obeys the Liouville equation (1.4). The answer is very simple. The substitution of (2.1) into (1.4) leads to[9,10]

$$i\hbar'\frac{\partial \psi}{\partial t} = \left[-\frac{\hbar'^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_0^2 x^2 - ex\left(E_{VF} + E_{RR}\right)\right]\psi \quad . \tag{2.2}$$

Therefore, (2.2) may be interpreted as a *classical stochastic* Schrödinger's like equation for the auxiliary function $\psi(x,t)$. However, one must avoid to interpret $\psi(x,t)$ as "wave function". The equivalence between (1.4) and (2.2) is exact only for the harmonic oscillator[11].

Since E_{VF} is the stochastic vacuum field, the solutions of equation (2.2) will have properties that depends on the statistical properties of E_{VF}[12]. Within SED, the zero-point electromagnetic fields are Gaussian random fields.

If we look for solutions of the Liouville-Schrödinger equation (2.2) we can use the fact that the Hermite functions, namely:

$$\phi_n(x) = \left(\frac{m\,\omega_0}{\pi\hbar'}\right)^{1/4} \frac{\exp\left(-\frac{m\,\omega_0}{\hbar'}\,x^2\right)}{(2^n\,n!)^{1/2}}\, H_n\left(x\sqrt{\frac{m\omega_0}{\hbar'}}\right) \quad , \tag{2.3}$$

form an orthogonal and complete set, thus fulfilling the condition

$$\sum_{n=0}^{\infty} \phi_n^*(x)\,\phi_n(y) = \delta(x-y) \quad . \tag{2.4}$$

As a matter of fact, the functions $\psi = \phi_n(x)\,e^{-i\epsilon_n t/\hbar'}$ are solutions of (2.2) if $e = 0$. However, the set of "energies" ϵ_n

$$\epsilon_n = \hbar'\,\omega_0\left(\frac{1}{2} + n\right) \quad , \tag{2.5}$$

cannot be interpreted as the energy levels of the oscillator because $\hbar'$ is arbitrary. The true Planck's constant $\hbar$ only appears through the influence of the vacuum fields $E_{VF}(t)$ whose spectral distribution $\rho_0(\omega)$ was introduced in (1.2). Another reason which forbids us to give physical interpretation to the "states" $(\phi_n(x), \epsilon_n)$ is that they do not lead to positive definite probability distribution in phase space[13], with the exception of the "state" $(\phi_0(x),\, \epsilon_0 = \frac{\hbar'\omega_0}{2})$. However, even in this case, we cannot identify $\phi_0(x)$ with the true ground state of the oscillator because $\hbar'$ is *arbitrary*. It

is important to remark that, from the logical point of view, the "states" $\phi_n(x)$ and the "energy levels" ϵ_n are unphysical quantities even if we choose $\hbar' = 1.0545887 \times 10^{-27}$ erg sec. The reason is that (2.2) is the *classical* Liouville equation (1.4) no matter the numerical value of the auxiliary constant $\hbar'$.

Nevertheless, since the set of functions $\phi_n(x)$ is complete, we can write the solution of the stochastic Schrödinger equation (2.2) in the form

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n(t)\,\phi_n(x)\,e^{-i\,\epsilon_n\,t/\hbar'} \quad , \tag{2.6}$$

and try to find the coefficients $a_n(t)$ by substituting (2.6) into (2.2). We shall discuss the solutions of (2.2) in what follows. Let us first consider the usual approximate method.

III. PERTURBATIVE ANALYSIS

Let us introduce the *mathematical hypothesis* that for $t = 0$ the ψ function is such that

$$\psi(x,0) = \phi_\ell(x) \quad , \tag{3.1}$$

where $\phi_\ell(x)$ is given by (2.3) and ℓ is an arbitrary integer. A standard perturbation calculation will give

$$a_n(t) \simeq \delta_{\ell n} + \frac{ie}{\hbar'}\,x_{n\ell}\int_0^t dt'\,[E_{VF}(t') + E_{RR}(t')]\,e^{i\omega_{n\ell}t'} + \cdots \quad , \tag{3.2}$$

where $\hbar'\omega_{n\ell} \equiv \epsilon_n - \epsilon_\ell$ and $x_{n\ell}$ are defined as:

$$x_{n\ell} \equiv \int_{-\infty}^{\infty} dx\,x\,\phi_n^*(x)\,\phi_\ell(x) \simeq -\frac{\ddot{x}_{n\ell}}{\omega_0^2} \quad . \tag{3.3}$$

We would like to make a few remarks at this point. Firstly, the Planck's constant h contributes to a_n through E_{VF} only. Note, however, that $\langle E_{VF}\rangle = 0$. Secondly, the approximate equality in the right hand side of (3.3) is valid if the radiative forces in (1.3) may be considered less important than the harmonic force $-m\,\omega_0^2\,x$.

The average rate of exchange of energy between the charge and the *total* radiation field is such that

$$\begin{aligned}\dot{\mathcal{E}}(\ell) &\equiv \left\langle \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp\, W(x,p,t)\, e\,[E_{RR}(t) + E_{VF}(t)]\,\frac{p}{m}\right\rangle \\ &\equiv \dot{\mathcal{E}}_{RR} + \dot{\mathcal{E}}_{VF} \quad . \end{aligned} \tag{3.4}$$

Here it is understood that $W(x,p,t)$ will be calculated by using (2.1), (2.6), (3.1) and (3.2). The ensemble average indicated by $\langle\ \rangle$ has exactly the same meaning as was indicated in (1.1). Within this perturbative analysis we shall calculate $\dot{\mathcal{E}}(\ell)$ up to the order e^2.

The radiation reaction force will be approximated by

$$e\,E_{RR}(t) \simeq \frac{2}{3}\,\frac{e^2}{c^3}\,\dddot{x} \simeq -\frac{2}{3}\,\frac{e^2\omega_0^2}{c^3}\,\dot{x} \equiv -\gamma p \quad , \tag{3.5}$$

where we have introduced the notation $\gamma = 2e^2\omega_0^2/mc^3$ for the damping constant.

Therefore, up to order e^2, $\dot{\mathcal{E}}_{RR}$ will be given by

$$\dot{\mathcal{E}}_{RR} \simeq -\frac{2}{3}\,\frac{e^2}{c^3}\,\omega_0^4 \int_{-\infty}^{\infty} dx\, x^2\,|\phi_\ell(x)|^2 = -\gamma\,\epsilon_\ell \quad , \tag{3.6}$$

since $a_n \simeq \delta_{n\ell}$ (see (3.2)) in first approximation. Such a result, which is extremely simple, means that if only the radiation reaction is present, the oscillator states are all unstable[14], since the square of x (or $\ddot{x}/\omega_0^2$) has a non zero average value in such states.

A convenient form to write this simple result is

$$\dot{\mathcal{E}}_{RR} = -\frac{2}{3}\,\frac{e^2}{c^3}\sum_{n=0}^{\infty} \ddot{x}_{\ell n}\,\ddot{x}_{n\ell} \quad , \tag{3.7}$$

where we have used (2.4) and (3.3).

In order to obtain a similar expression for $\dot{\mathcal{E}}_{VF}$ we must take into account the second term which contributes to the coefficients $a_n(t)$ (formula (3.2)). Considering that $\langle E_{VF}\rangle = 0$ and that the correlation function for the vacuum electromagnetic fields is given by (1.1) we obtain from (3.4) that, in the limit $\omega_0\,t \ll 1$, $\dot{\mathcal{E}}_{VF}$ will be given by

$$\dot{\mathcal{E}}_{VF} \simeq -\frac{4\pi^2e^2}{3\omega_0^3\,\hbar'}\sum_{n=0}^{\infty} \rho_0\,(\omega_{\ell n})\;\ddot{x}_{\ell n}\,\ddot{x}_{n\ell} \quad , \tag{3.8}$$

where $\rho_0(\omega)$ is the zero-point spectral distribution introduced in (1.2). This approximate result was obtained previously within the realm of QED[14] using the particular value $\hbar' = \hbar$. The calculation is justified[15] only if we consider that the time t is such that $\tau \ll t \ll 1/\omega_0$ where τ is correlation time associated to (1.1). This time τ was assumed to be very short compared to the period of oscillation $2\pi/\omega_0$. Similar calculations where also performed within the context of SED[12].

Combining equations (3.7) and (3.8), equation (3.4) can be cast into the interesting form

$$\dot{\mathcal{E}}(\ell) \simeq -\frac{2}{3}\,\frac{e^2}{c^3}\left[\left(1+\frac{\hbar}{\hbar'}\right)\sum_{n(<\ell)}\ddot{x}_{\ell n}\,\ddot{x}_{n\ell} + \left(1-\frac{\hbar}{\hbar'}\right)\sum_{n(>\ell)}\ddot{x}_{\ell n}\,\ddot{x}_{n\ell}\right] \quad , \tag{3.9}$$

which is the main new result of this section and deserves a few comments.

The first term in (3.9) represents downward transitions (to lower energy states $\epsilon_n < \epsilon_\ell$) while the second term represents upward transitions. Therefore, since the constant $\hbar'$ is arbitrary, both type of transitions are allowed. We also conclude that

$\dot{\mathcal{E}}(\ell)$ is always non-zero. If, however, we take the less general situation in which $\hbar' = \hbar$ we obtain

$$\dot{\mathcal{E}}(\ell) \;=\; -\frac{4}{3}\,\frac{e^2}{c^3}\sum_{n(<\ell)}\ddot{x}_{\ell n}\,\ddot{x}_{n\ell} \;=\; -\frac{2}{3}\,\frac{e^2\omega_0^2}{mc^3}\,(\epsilon_\ell - \epsilon_0) \quad , \tag{3.10}$$

which is the more familiar result from QED calculations[14]. In other words, if $\hbar' = \hbar$ we are not able to see the upward transitions which are present in a more general situation which we shall present in the next section. We can also conclude that the stationary state (average energy $\epsilon_0 = \hbar\,\frac{\omega_0}{2}$) cannot be stable in the absence of the vacuum energy fluctuations which exactly balance the energy loss due to self reaction. Therefore, a striking conclusion of the above analysis is that it is the "vacuum" energy that prevents the oscillator collapse and gives us the criterium to identify the equilibrium state. It is a matter of taste if we decide to make the mathematical analysis of our system using the Liouville equation (1.4) or the Schrödinger like equation (2.2). Both descriptions will be conceptually classical.

IV. NONPERTURBATIVE ANALYSIS AND SUB-HEISENBERG STATES

Let us see how one can easily find a set exact solutions of the classical stochastic Schrödinger equation (2.2). We shall also show once more that the arbitrary constant $\hbar'$ has no dynamical role.

Using a well known procedure (see the paper by Schrödinger[16]) one should look for solutions of (2.2) with the form of "coherent"[17] states

$$\psi_c(x,t) = \phi_0(x - x_c)\,\exp\left[\frac{i}{\hbar'}\,(xp_c - g(t))\right] \quad . \tag{4.1}$$

Here we are using $\hbar' \neq \hbar$ and $\phi_0(x)$ is given by (2.3) with $n = 0$. We shall call these states as classical sub-Heisenberg states.

It is straightforward to show that $p_c(t) = m\dot{x}_c(t)$ and that $x_c(t)$ must obey the classical equation of motion (1.3). It is also not difficult to obtain a closed expression for $g(t)$ which will be given by[17]:

$$g(t) \;=\; \frac{\hbar'\omega_0\,t}{2} + \int_0^t dt'\left[\frac{p_c^2(t')}{2m} - \frac{m\,\omega_0^2\,x_c^2(t')}{2}\right] \quad . \tag{4.2}$$

The classical trajectories x_c can be easily found if we consider the approximation (3.5) for the radiation reaction force. It can be put in the form

$$x_c(t) = x_d(t) + x_f(t) \quad , \tag{4.3}$$

where $x_d(t)$ is the deterministic part which depends on x_0 (initial position) and p_0 (initial momentum) and $x_f(t)$ is the fluctuating part. More explicitly:

$$x_d = \left[x_0\,\cos\omega_1 t + \left(\frac{m\,\gamma\,x_0 + 2p_0}{2m\,\omega_1}\right)\sin\omega_1 t\right]\,e^{-\frac{\gamma t}{2}} \quad , \tag{4.4}$$

where $\omega_1^2 = \omega_0^2 - \gamma^2/4$.

The fluctuating part, x_f, depends on the vacuum field $E_{VF}(t)$ and has the simple form

$$x_f(t) \;=\; -\,\frac{e}{m\,\omega_1}\int_0^t dt'\,E_{VF}(t')\,\sin\left[\omega_1(t-t')\right]\,e^{\frac{\gamma}{2}(t-t')} \quad . \tag{4.5}$$

With the knowledge of these nonperturbative solutions $\psi_c(x,t)$, one can obtain exact expressions for the coefficients $a_n(t)$ used in the earlier expansion formula (2.6). We shall not present it here since they have been explicitly written previously[12,17] (the only difference is that here $\hbar'$ is *arbitrary*). It is also interesting to recall that for each x_0 and p_0 (see (4.4)) we have a different sub-Heisenberg state[12,17]. Therefore, one can obtain various sets of phase space distributions $W(x,p,t)$ through different ψ_c with different x_0, p_0 and $\hbar'$.

The Wigner function associated to these states $\psi_c(x,t)$ also have a simple form. If we substitute (4.1) in (2.1) we get[17]

$$W(x,p,t) = (\pi\hbar')^{-1} \exp\left[-\frac{m\omega_0}{\hbar'}(x-x_c)^2 - \frac{(p-p_c)^2}{\hbar' m\omega_0}\right] \quad , \tag{4.6}$$

which has very interesting properties.

Since $\hbar'$ is arbitrary we can consider, for instance, the particular case

$$\lim_{\hbar'\to 0} W(x,p,t=0) = \delta(x-x_0)\,\delta(p-p_0) \quad , \tag{4.7}$$

which is a typical *deterministic* initial phase space distribution.

If for instance, $x_0 = p_0 = 0$ but $\hbar' \neq 0$, we get an initial phase space distribution

$$W(x,p,t=0) = \frac{1}{\pi\hbar'} \exp\left(-\frac{m\omega_0 x^2}{\hbar'} - \frac{p^2}{\hbar' m\omega_0}\right) \equiv W_{\rm in}(x,p) \quad , \tag{4.8}$$

which corresponds to an *initial classical uncertainty* relation of the form

$$\left\langle(\Delta x)^2\right\rangle\left\langle(\Delta p)^2\right\rangle = (\hbar'/2)^2 \quad . \tag{4.9}$$

Therefore, we can analyse the phase space evolution[18,19] of our system by using the sub-Heisenberg states (4.1). Our conclusions are not restricted to the perturbative domain (see (3.9) and (3.10)), it also include arbitrary initial shape because $\hbar'$ is also arbitrary. This freedom is possible within SED but is, up to now, avoided within QED.

V. EVOLUTION IN PHASE SPACE

The classical trajectory, $x_c(t)$, and momentum, $p_c(t) = m\dot{x}_c(t)$, which appear in the Wigner function $W(x,p,t)$, obtained above (see (4.6)) are *correlated*. The various moments can be obtained from the definitions (4.4), (4.5) and also with the

use of the statistical properties of the vacuum electric field (see (1.1)). In this section we shall obtain the ensemble averaged phase space distribution, namely $\langle W(x,p,t)\rangle$, by using the transition probability[20] in phase space $Q(xpt|x'p')$. In other words, if $W_{\rm in}(x',p')$ is the initial $(t=0)$ phase space distribution, then, at later times, we get

$$\langle W(x,p,t)\rangle = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dp'\, Q(xpt|x'p')\, W_{\rm in}(x',p') \quad . \tag{5.1}$$

The transition probability Q satisfies a Fokker-Planck[20] equation which is the generalization of the Liouville equation (1.4). Within SED, this equation is well known and can be written as[8]

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p}{m}Q\right) + \frac{\partial}{\partial p}\left[-\left(\gamma p + m\omega_0^2 x\right) + D\,\frac{\partial}{\partial p}\right] Q = 0 \quad , \tag{5.2}$$

where D is the diffusion coefficient.

At zero temperature, the diffusion coefficient is given by

$$D = \gamma\left[\lim_{t\to\infty}\langle p_f^2(t)\rangle\right] = \frac{\hbar\gamma m\omega_0}{2} \quad , \tag{5.3}$$

where $p_f(t) = m\dot{x}_f(t)$, $x_f(t)$ given by (4.5).

The solution of (5.2) is also well-known and can be written in the form[20,21]

$$2\pi\,\alpha_1\,\alpha_2\sqrt{1-\xi^2}\;Q(xpt|x'p') =$$

$$= \exp\left\{-\left[\frac{(p-p_d)^2}{\alpha_1^2} + \frac{(x-x_d)^2}{\alpha_2^2} - \frac{2\xi}{\alpha_1\alpha_2}(x-x_d)(p-p_d)\right] \Big/ 2(1-\xi^2)\right\} \quad , \tag{5.4}$$

where x_d $(p_d = m\dot{x}_d)$ is the deterministic trajectory (see (4.4) and replace $x_0 \to x'$ and $p_0 \to p'$).

The function $\alpha_1(t)$ is given by[21]

$$\alpha_1^2(t) = \frac{D}{\gamma}\left\{1 - \left[1 + \frac{\gamma^2}{2\omega_1^2}\,\sin^2(\omega_1 t) - \frac{\gamma}{2\omega_1}\,\sin(2\omega_1 t)\right] e^{-\gamma t}\right\} \quad , \tag{5.5}$$

whereas $\alpha_2(t)$ is such that

$$\alpha_2^2(t) = \frac{D}{\gamma\, m^2\, \omega_0^2}\left\{1 - \left[1 + \frac{\gamma^2}{2\omega_1^2}\,\sin^2(\omega_1 t) + \frac{\gamma}{2\omega_1}\,\sin(2\omega_1 t)\right] e^{-\gamma t}\right\} \quad . \tag{5.6}$$

The correlation ξ is given by

$$\xi\,\alpha_1\,\alpha_2 = \frac{D}{m\,\omega_1^2}\,\sin^2(\omega_1 t)\, e^{-\gamma t} \quad , \tag{5.7}$$

and, since $Q(xpt|x't')$ is a *classical* transition probability within SED we also have

$$\lim_{t\to 0}\; Q(xpt|x'p') = \delta(x - x')\,\delta(p - p') \quad . \tag{5.8}$$

Let us assume that $W_{\rm in}(x'|p')$ is given by (4.8), which characterizes a simple initial sub-Heisenberg state in phase space. We now want to calculate the average energy $\mathcal{E}(t)$ as a function of time. The explicit definition is

$$\begin{aligned}\mathcal{E}(t) \;&\equiv\; \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx\, \left(\frac{p^2}{2m} + \frac{1}{2}\, m\,\omega_0^2\, x^2\right) \langle W(x,p,t)\rangle \equiv \\ &\equiv\; \frac{\overline{p^2}}{2m} + \frac{1}{2}\, m\,\omega_0^2\, \overline{x^2} \quad . \end{aligned} \tag{5.9}$$

Using (5.1) and also (5.4) we obtain for $\overline{x^2}$ the following result

$$\begin{aligned}\overline{x^2} \;=\; &\frac{\hbar'}{2m\,\omega_0}\left[\cos^2(\omega_1 t) + \left(1 + \frac{\gamma^2}{4\omega_1^2}\right)\sin^2(\omega_1 t) + \frac{\gamma}{2\omega_1}\,\sin(2\omega_1 t)\right] e^{-\gamma t} + \\ + \;&\frac{\hbar}{2m\,\omega_0}\left\{1 - \left[1 + \frac{\gamma^2}{2\omega_1^2}\,\sin^2(\omega_1 t) + \frac{\gamma}{2\omega_1}\,\sin(2\omega_1 t)\right] e^{-\gamma t}\right\} \quad , \end{aligned} \tag{5.10}$$

nd a similar expression for $\overline{p^2}$.

It is interesting to observe that for $\gamma t \gg 1$ we get $\overline{x^2} = \hbar/2m\omega_0$. Note that ere $\hbar$ is the Planck's constant whose origin is the vacuum field which appears in the tochastic Schrödinger equation (2.2).

The final result to the average energy is

$$\mathcal{E}(t) = \frac{\hbar\omega_0}{2} + \frac{(\hbar' - \hbar)\,\omega_0}{2}\, e^{-\gamma t}\left[1 + \frac{\gamma^2}{2\omega_0^2}\,\sin^2(\omega_1 t)\right] \quad , \tag{5.11}$$

here we have neglected γ^4/ω_0^4 as compared to 1. It is easy to see that $\mathcal{E}(t)$ aries from $\mathcal{E}(0) = \dfrac{\hbar'\omega_0}{2}$ to $\mathcal{E}(\infty) = \dfrac{\hbar\omega_0}{2}$ which is the average energy of the echanical oscillator in the stationary regime. The above result (5.11) is more general han our previous formula for $\dot{\mathcal{E}}(\ell)$, presented in Section III (see (3.9) and (3.10)), in wo respects. Firstly, (5.11) is valid for times in the wide interval from $0 < t < \infty$ hereas (3.9) is valid only for small t $(\omega_0 t \ll 1)$. Secondly, the initial state ψ_c is a ub-Heisenberg state with contributions from all states $\phi_\ell(x)$.

I. CONCLUSION AND DISCUSSION

Let us briefly discuss the effects of non-zero temperatures and also other interesting nvironmental effects on the evolution of the sub-Heisenberg states analyzed in this aper.

If the oscillator is inside a cavity with temperature T, then the spectral distribution f the "vacuum" electromagnetic fluctuations is given by[1,3]

$$\rho_T(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\left[\frac{1}{2} + \frac{1}{\exp\left(\dfrac{\hbar\omega}{kT}\right) - 1}\right] \quad , \tag{6.1}$$

instead of (1.2). Therefore, a simple rule, to generalize our previous results to finite temperatures, is to replace $\hbar$ by $\hbar \coth(\hbar\omega_0/2kT)$ in every place in which Planck's constant appears (for instance in the diffusion coefficient D given in (5.3)). If we want, for instance, to describe finite temperature effects (and dissipation by radiation reaction) on the oscillator, within the usual QED formalism, we should use (2.2) with $\hbar' = \hbar$. The thermal radiation effects will appear through E_{VF}, which now has the spectral distribution (6.1). However, the equilibrium distribution, that is, the "ground" state distribution at finite temperature, can be obtained directly from (2.2) by taking $e = 0$ but choosing $\hbar' = \hbar \coth(\hbar\omega_0/2kT)$. The detailed justification of this procedure (within quantum mechanics) is not trivial[22].

Another interesting environmental effect on the charged oscillator occurs when it is inside resonant cavities[23] or between two perfect plane mirrors[24, 25]. In the latter case, the emission and the absorption of the oscillator depend on the position and the orientation which the oscillator has with respect to the mirror plates. If the oscillator is oriented *parallel* to the plates, the damping constant γ (see (3.5)), is modified to another one ($\gamma_{\parallel}(\omega_0)$) which is a more complicated function of ω_0, namely[25]:

$$\gamma_{\parallel}(\omega_0) = \gamma \, \frac{3\pi c}{2\omega_0 a} \sum_{s=0}^{[\omega_0 a/\pi c]} \left[1 + \left(\frac{\pi c s}{\omega_0 a}\right)^2\right] \sin^2\left(\frac{\pi b s}{a}\right) \quad . \tag{6.2}$$

Here, a is the distance between the mirror plates and b ($b < a$) is the distance from the oscillator and one of the plates. An interesting remark is that $\gamma_{\parallel}(\omega_0) = 0$ if $\omega_0 < \frac{\pi c}{a}$. In this case the oscillator cannot loose energy and *all states are stable*. In other words, the charged oscillator behaves as an uncharged *Newtonian* oscillator. Even the true Planck's constant $\hbar$ has no role in this particular case. The reason is that the spectral distribution is also modified by the perfect mirrors, that is, $\rho_0(\omega)$ is replaced by $\rho_{\parallel}(\omega)$ such that[25]

$$\rho_{\parallel}(\omega) = \frac{\gamma_{\parallel}(\omega)}{\gamma} \, \frac{\hbar\omega^3}{2\pi^2 c^3} \quad . \tag{6.3}$$

Therefore, $\rho_{\parallel}(\omega) = 0$ for frequencies $0 \leq \omega \leq \pi c/a$. If the oscillator frequency is in this interval then the "vacuum" field E_{VF} is unable to excite the oscillator which is oriented parallel to the perfect mirror plates (the situation is completely different if the oscillator is oriented perpendicularly to the plates). It is interesting to see what happens, from the QED point of view, with the "Newtonian" oscillator if we remove the perfect mirrors, allowing radiation of all frequencies to excite the oscillator. Within the QED approach the electromagnetic fields E_{VF} and E_{RR} are operators. Therefore, we can use equation (1.3) as the Heisenberg equation for the position operator x. The solution of the Heisenberg equation can be written as before (see (4.3) to (4.5)) if we approximate the damping force as proportional to the momentum (see (3.5)). Due to the effect of the perfect mirror plates, the oscillator is initially Newtonian and we can consider x_0 and p_0 (see (4.4)) as commuting variables. Therefore, after removing the perfect mirror plates at $t = 0$ it is possible to show that the commutation relation at later times will be[29, 30]

$$[x, p] = i\hbar \left(1 - e^{-\gamma t}\right) \tag{6.4}$$

This shows that, in principle, sub-Heisenberg states are also possible within the realm of QED because $[x, p] \neq i\hbar$ for small times ($\gamma t \ll 1$). However, when the plates conductivity is finite, the discontinuities of (6.2) become smooth because there is always some residual noise at all frequencies[26], that is, $\rho_{\parallel}(\omega) \neq 0$ for $0 < \omega < \infty$. This forbids practical observations of sub-Heisenberg states by this method. However, other related interesting phenomena (as the suppression of spontaneous emission) have been observed experimentally[27] and explained theoretically within the realm of SED[7]. In this case, the corresponding QED calculation, based on the Lamb-Bethe theory, also

agrees with the experiment but, as far as we know, these quantum calculation have not been published[27].

Finally we would like to recall that Born, Heisenberg and Jordan (1926) and Dirac (1927), showed how to systematically quantize the free electromagnetic field by exploiting the representation of each field mode as a harmonic oscillator[28]. However, it is now well known that the harmonic oscillator must be considered an essentially classical system[9–12]. We have shown this fact, once more, by discussing, in section II, the equivalence between the classical Liouville equation and the Schrödinger equation for the oscillator. Therefore, we think that the famous Einstein question ("What are light quanta?") must be reconsidered within the realm of QED and SED[31].

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