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QUANTIZATION OF PSEUDOCCLASSICAL MODEL
OF SPIN ONE RELATIVISTIC PARTICLE

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Abstract

A consistent procedure of canonical quantization of pseudoclassical model for spin one relativistic particle is considered. Two approaches to treat the quantization for the massless case are discussed, the limit of the massive case and independent quantization of a modified action. Quantum mechanics constructed for the massive case proves to be equivalent to the Proca theory and for massless case to the Maxwell theory. Results obtained are compared with ones for the case of spinning (spin one half) particle.

I. INTRODUCTION

Classical and pseudoclassical models of relativistic particles and their quantization are discussed lately in different contexts. One of the reason is on these simple examples to learn how to solve some problems which arise also in string theory, gravity and so on. On the other hand it is interesting itself to find out whether there exist classical models for any relativistic particles (with any spin), whose quantization reproduces, in a sense, the corresponding field theory or one particle sector of the corresponding quantum field theory.

A classical action of a scalar relativistic particle one can find, for example, in the Landau text book [1]. An action of spin one half relativistic particle, with spinning degrees of freedom, describing by anticommuting (grassmannian or odd) variables, was first proposed by Berezin and Marinov [2] and just after that discussed and investigated in papers [3–7]. Generalization of this model for particles with arbitrary spin was proposed in [8,9]. The actions of the models obey different kinds of gauge symmetry, in particular, of reparametrization invariance and special supertransformations. Due to the reparametrizations in all the cases Hamiltonian equal zero on the constraint surface. In the papers [10–12], devoted to the quantization of these models, they tried to avoid these difficulties, using the so called Dirac method of quantization of theories with first-class constraints [13], in which one considers the first-class constraints in the sense of restrictions on the state vectors. Unfortunately, in general case, this scheme of quantization creates many questions, e.g. with Hilbert space construction, what is Schrödinger equation and so on. A consistent, but more complicated technically way is to work in the physical sector, namely, first, on the classical level, one has to impose gauge conditions to all the first class-constraints to reduce the theory to one with second-class constraints only, and then quantize by means of the Dirac brackets (we will call such a method as canonical quantization). First canonical the quantization for a relativistic spin one half particle was done in [14]. In this paper we are going to use this approach to quantize a relativistic particle spin one. We consider a pseudoclassical model of relativistic spin one particle both massive and massless with an action, which is conventional

generalization of Berezin-Marinov action, mentioned above, with a Chern-Simons term. We impose gauge conditions to all the first class constraints, except to one first-class constraint, which is quadratic in fermionic variables. In virtue of the structure of this constraint it is difficult, and probably impossible without a reduction of the number of degrees of freedom, to impose a conjugated gauge condition, on the other hand, treating this constraint in the sense of restrictions of quantum states does not create problems with Hilbert space construction. Thus, we quantize the theory quasicanonically by means of Dirac brackets with respect to all other constraints and gauge conditions. We demonstrate that quantum mechanics constructed is equivalent to one-particle sector of the quantum theory of Proca vector field. The quantization of the massless case is considered in two ways, as the limit from the massive case and independently starting from the massless Lagrangians without the variable ψ^5 . For convenience, a comparison with spin one half case is given.

II. PSEUDOCASSICAL MODELS OF SPINNING PARTICLES.

A generalization of the pseudoclassical action of spin one-half relativistic particle to the case of arbitrary spin $N/2$ can be written in the form

$$S = \int_0^1 \left[-\frac{1}{2e} (\dot{x}^\mu - i\psi_a^\mu \chi_a)^2 - \frac{e}{2} m^2 - im\psi_a^5 \chi_a + \frac{1}{2} f_{ab} (i[\psi_{an}, \psi_b^n]_- + \kappa_{ab}) - i\psi_{an} \dot{\psi}_a^n \right] d\tau, \quad (2.1)$$

where x^μ , e and f_{ab} are even and ψ_a^n , χ_a are odd variables (f_{ab} is antisymmetric), dependent on a parameter $\tau \in [0, 1]$, which plays a role of time in this theory, $\mu = \overline{0, 3}$; $a, b = \overline{1, N}$; $n = (\mu, 5) = \overline{0, 3, 5}$; $\eta_{\mu\nu} = \text{diag}(1 - 1 - 1 - 1)$; $\eta_{mn} = \text{diag}(1 - 1 - 1 - 1)$. Spinning degrees of freedom are described by odd (grassmannian) variables ψ_a^μ and ψ_a^5 ; odd χ_a and even e play an auxiliary role to make the action reparametrization and super gauge-invariant as well as to make it possible consider both cases $m \neq 0$ and $m = 0$ on the same foot. The summand $\frac{1}{2} \kappa_{ab} \int_0^1 f_{ab} d\tau$, with even coefficients κ_{ab} plays the role of a Chern-Simons term and can be added only in case $N = 2$ without breaking of the rotational gauge symmetry [12]. Thus,

$\kappa_{ab} = \kappa \epsilon_{ab} \delta_{N,2}$ with an even constant κ and two dimensional Levi-Civita symbol ϵ_{ab} .

The are three types of gauge transformations under which the action (2.1) is invariant: reparametrizations

$$\delta x^\mu = \dot{x}^\mu \xi, \quad \delta e = \frac{d}{d\tau} (e\xi), \quad \delta f_{ab} = \frac{d}{d\tau} (f_{ab}\xi), \quad \delta \psi_a^n = \dot{\psi}_a^n \xi, \quad \delta \chi_a = \frac{d}{d\tau} (\chi_a \xi), \quad (2.2)$$

supertransformations

$$\begin{aligned} \delta x^\mu &= i\psi_a^\mu \epsilon_a, \quad \delta e = i\chi_a \epsilon_a, \quad \delta f_{ab} = 0, \quad \delta \chi_a = \dot{\epsilon}_a - f_{ab} \epsilon_b, \\ \delta \psi_a^\mu &= \frac{1}{2e} (\dot{x}^\mu - i\psi_b^\mu \chi_b) \epsilon_a, \quad \delta \psi_a^5 = \frac{m}{2} \epsilon_a, \end{aligned} \quad (2.3)$$

and $O(N)$ rotations

$$\delta x^\mu = 0, \quad \delta e = 0, \quad \delta f_{ab} = t_{ab} + t_{ac} f_{cb} - t_{bc} f_{ca}, \quad \delta \psi_a^n = t_{ab} \psi_b^n, \quad \delta \chi_a = t_{ab} \chi_b, \quad (2.4)$$

with even parameters $\xi(\tau)$, $t_{ab}(\tau) = -t_{ba}(\tau)$, and odd parameters $\epsilon_a(\tau)$.

Equations of motion have the form

$$\frac{\delta S}{\delta x_\mu} = \frac{d}{d\tau} \left[\frac{1}{e} (\dot{x}^\mu - i\psi_a^\mu \chi_a) \right] = 0, \quad (2.5)$$

$$\frac{\delta S}{\delta e} = \frac{1}{2e^2} (\dot{x}^\mu - i\psi_a^\mu \chi_a)^2 - \frac{m^2}{2} = 0, \quad (2.6)$$

$$\frac{\delta S}{\delta f_{ab}} = \frac{1}{2} (i[\psi_{na}, \psi_b^n]_- + \kappa_{ab}) = 0, \quad N \geq 2, \quad (2.7)$$

$$\frac{\delta_r S}{\delta \chi_a} = \frac{i}{e} (\dot{x}^\mu - i\psi_b^\mu \chi_b) \psi_a^\mu - im\psi_a^5 = 0, \quad (2.8)$$

$$\frac{\delta_r S}{\delta \psi_{a\mu}} = 2i (\dot{\psi}_a^\mu - f_{ab} \dot{\psi}_b^\mu) - \frac{i}{e} \chi_a (\dot{x}^\mu - i\psi_b^\mu \chi_b) = 0, \quad (2.9)$$

$$\frac{\delta_r S}{\delta \psi_{a5}} = 2i (\dot{\psi}_a^5 - f_{ab} \dot{\psi}_b^5 - \frac{m}{2} \chi_a) = 0. \quad (2.10)$$

Calculating the total angular momentum $M_{\mu\nu}$, corresponding to the action (2.1), we get

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \quad (2.11)$$

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad S_{\mu\nu} = i(\psi_{a\mu} \psi_{a\nu} - \psi_{a\nu} \psi_{a\mu}).$$

The spatial part of $S_{\mu\nu}$ forms a tree-dimensional spin vector $\mathbf{s} = (s^k)$

$$s^k = \frac{1}{2} \epsilon_{kjl} S_{lj} = \sum_a s_a^k, \quad s_a^k = i \epsilon_{kjl} \psi_{aj} \psi_{al}, \quad (2.12)$$

where ϵ_{kjl} is three-dimensional Levi-Civita symbol and there is no summation over a in the last formula (2.12). To demonstrate that this vector really behaves like a spin one should introduce an interaction with an external electromagnetic field $A_\mu^{ext}(x)$ into the model and consider the non-relativistic approximation. Unfortunately, in general case it is impossible to introduce such an interaction in the action (2.1) in the same manner as for the spin one half [4,6,7]. Namely, if one adds the terms

$$-g \dot{x}^\mu A_\mu^{ext} + i g e F_{\mu\nu}^{ext} \psi_a^\mu \psi_a^\nu \quad (2.13)$$

with arbitrary external field A_μ^{ext} , $F_{\mu\nu}^{ext} = \partial_\mu A_\nu^{ext} - \partial_\nu A_\mu^{ext}$, to the integrand (Lagrangian) of the expression (2.1), this Lagrangian becomes inadmissible, i.e. the corresponding lagrangian equations become inconsistent for $N > 1$. One can check this by straightforward calculations. However, if the external field is constant ($F_{\mu\nu} = \text{const}$), the terms (2.13) can be added to the Lagrangian; the equations of motion remain consistent, but the super gauge symmetry (2.3) of the action disappears, namely equations of motion have now only one solution for χ , $\chi = 0$. So, let us introduce the interaction with a constant magnetic field $F_{0i}^{ext} = 0$, $F_{ij}^{ext} = -\epsilon_{ijk} B^k$, where B^k are components of a magnetic field \mathbf{B} , that is enough for our purposes. Besides, we restore the velocity of light c in the equations of motion by the prescription $m \rightarrow mc$, $g \rightarrow g/c$ and impose two gauge conditions

$$\tau = x^0/c = t, \quad f_{ab} = 0, \quad (2.14)$$

to fix the gauge freedom, which corresponds to the reparametrizations and $O(N)$ rotations.

Thus, we get in the case of consideration

$$\frac{1}{e^2} \left(\frac{dx^\mu}{dt} \right)^2 - m^2 c^2 - \frac{2ig}{c} F_{\mu\nu}^{ext} \psi_a^\mu \psi_a^\nu = 0, \quad (2.15)$$

$$\frac{d}{dt} \left(\frac{1}{e} \frac{dx^\mu}{dt} \right) = \frac{g}{c} F_{\mu\nu}^{ext} \frac{dx^\nu}{dt}, \quad \dot{\psi}_{a\mu} = \frac{eg}{c} F_{\mu\nu}^{ext} \psi_a^\nu, \quad (2.16)$$

$$\frac{1}{e} \left(\frac{dx^\mu}{dt} \right) \psi_a^\mu - mc \psi_a^5 = 0, \quad \psi_a^5 = 0, \quad \chi_a = 0, \quad (2.17)$$

$$\psi_{an} \psi_b^n = \frac{i}{2} \kappa_{ab}, \quad N \geq 2. \quad (2.18)$$

In the limit $c \rightarrow \infty$ (\mathbf{B}/c - fixed) it follows from the equation (2.15) that e can be every-where replaced by $1/m$. Then we obtain from (2.16)

$$m \frac{d^2 \mathbf{x}}{dt^2} = \frac{g}{c} \left[\frac{d\mathbf{x}}{dt} \times \mathbf{B} \right], \quad \frac{d\mathbf{s}}{dt} = \frac{g}{mc} [\mathbf{s} \times \mathbf{B}]. \quad (2.19)$$

It follows from the equations (2.17) that $\psi_a^0 = \psi_a^5 = \text{const}$, and therefore the constraint (2.18) takes the form $\psi_a^i \psi_b^i = -i \kappa_{ab}/2$. Using this, one can calculate $s_a^k s_b^k = 2(\psi_a^i \psi_b^i)^2 = \kappa^2(1 - \delta_{ab})\delta_{N,2}/2$, so that

$$s^2 = \left(\sum_a s_a^k \right)^2 = \kappa^2 \delta_{N,2}. \quad (2.20)$$

Thus, one can interpret the equations (2.19) as describing the non-relativistic motion of a charged particle with the total spin momentum \mathbf{s} , ($s^2 = \kappa^2 \delta_{N,2}$), and with the total magnetic momentum $g\mathbf{s}/mc$ in a constant magnetic field.

III. HAMILTONIAN FORMULATION. CONSTRAINTS. GAUGES CONDITIONS.

Going over to the hamiltonian formulation, we introduce the canonical momenta:

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -\frac{1}{e} (\dot{x}^\mu - i \psi_{a\mu} \chi_a), \quad P_e = \frac{\partial L}{\partial \dot{e}} = 0, \\ P_{\chi_a} = \frac{\partial L}{\partial \dot{\chi}_a} = 0, \quad P_{an} = \frac{\partial L}{\partial \dot{\psi}_a^n} = -i \psi_{an}, \quad P_{fab} = \frac{\partial L}{\partial \dot{f}_{ab}} = 0. \quad (3.1)$$

It follows from (3.1) that there exist primary constraints $\Phi^{(1)} = 0$,

$$\Phi^{(1)} = \begin{cases} \Phi_{1a}^{(1)} = P_{\chi_a}, \\ \Phi_2^{(1)} = P_e, \\ \Phi_{3an}^{(1)} = P_{an} + i \psi_{an}, \\ \Phi_{4ab}^{(1)} = P_{fab}. \end{cases}$$

We construct the Hamiltonian $H^{(1)}$ according to the standard procedure (we are using the notations of the book [14]),

$$H^{(1)} = H + \lambda_A \Phi_A^{(1)}, \quad H = \left(\frac{\partial_\tau L}{\partial \dot{q}} \dot{q} - L \right) \Big|_{\frac{\partial_\tau L}{\partial \dot{q}} = p}, \quad q = (x, e, \chi, \psi, f),$$

and get for the H :

$$H = -\frac{e}{2}(p^2 - m^2) + i(p_\mu \psi_a^\mu + m\psi_a^5)\chi_a - \frac{1}{2}f_{ab} (i[\psi_{an}, \psi_b^n]_- + \kappa_{ab}). \quad (3.2)$$

From the conditions of the conservation of the primary constraints $\Phi^{(1)}$ in the time τ , $\dot{\Phi}^{(1)} = \{\Phi^{(1)}, H^{(1)}\} = 0$, we find secondary constraints $\Phi^{(2)} = 0$,

$$\Phi^{(2)} = \begin{cases} \Phi_{1a}^{(2)} = p_\mu \psi_a^\mu + m\psi_a^5, \\ \Phi_2^{(2)} = p^2 - m^2, \\ \Phi_{3ab}^{(2)} = i[\psi_{an}, \psi_b^n]_- + \kappa_{ab}, \end{cases} \quad (3.3)$$

and determine λ , which correspond to the primary constraint $\Phi_{3an}^{(1)}$. Thus, the Hamiltonian H appears to be proportional to the constraints, as one could expect in the case of a reparametrization invariant theory,

$$H = -\frac{e}{2}\Phi_2^{(2)} + i\Phi_{1a}^{(2)}\chi_a - \frac{1}{2}f_{ab}\Phi_{3ab}^{(2)}. \quad (3.4)$$

No more secondary constraints arise from the Dirac procedure, and the Lagrange's multipliers, corresponding to the primary constraints $\Phi_{1a}^{(1)}$, $\Phi_2^{(1)}$ and $\Phi_{4ab}^{(1)}$, remain undetermined. One can go over from the initial set of constraints $(\Phi^{(1)}, \Phi^{(2)})$ to the equivalent one $(\Phi^{(1)}, \tilde{\Phi}^{(2)})$, where

$$\tilde{\Phi}^{(2)} = \Phi^{(2)} \Big|_{\psi_{an} \rightarrow \tilde{\psi}_{an} = \psi_{an} + \frac{1}{2}\Phi_{3an}^{(1)}}.$$

The new set of constraints can be explicitly divided in a set of the first-class constraints, which is $(\Phi_{1,2}^{(1)}, \Phi_{4ab}^{(1)}, \tilde{\Phi}^{(2)})$, and in a set of second-class constraints, which is $\Phi_{3an}^{(1)}$. So, we are dealing with a theory with first-class constraints. Our goal is to quantize this theory. We choose the following way. We will impose supplementary gauge conditions to all the first-class constraints, excluding the constraint $\tilde{\Phi}_{3ab}^{(2)}$. As a result we will have only a set of first-class constraints, which is reduction of $\Phi_{3ab}^{(2)}$ to the rest of constraints. These constraints

we suppose to use to specify the physical states according to Dirac [13]. All other constraints will be of second-class and will be used to form Dirac brackets.

Thus, let us impose preliminary the following gauge conditions:

$$\begin{aligned} \Phi_{1a}^G = \chi_a = 0, \quad \Phi_{2ab}^G = f_{ab} = 0, \\ \Phi_3^G = x_0 - \zeta\tau = 0, \quad \Phi_{4a}^G = \psi_a^0 = 0, \end{aligned} \quad (3.5)$$

where $\zeta = -\text{sign } p_0$ (The gauge $x_0 - \zeta\tau = 0$ was first proposed in papers [14] as a conjugated gauge condition to the constraint $p^2 = m^2$ in the case of scalar and spinning particles. In contrast with the gauge $x_0 = \tau$, which together with the continuous reparametrization symmetry breaks the time reflection symmetry and therefore fixes the variables ζ , the former gauge breaks only the continuous symmetry, so that the variable ζ remains in the theory to describe states of particles $\zeta = +1$ and states of antiparticles $\zeta = -1$. Namely this circumstance allowed one to get Klein-Gordon and Dirac equations as Schrödinger ones in course of the canonical quantization. To break the supergauge symmetry the gauge condition $\psi^5 = 0$ was used in [14]. In [15] the general class of gauge conditions of the form $\alpha\psi^0 + \beta\psi^5 = 0$ was investigated in case of D -dimensional spinning particles.) The requirement of consistency of the constraint Φ_3^G , $\dot{\Phi}_3^G = 0$, gives one more gauge condition

$$\Phi_5^G = e + \zeta p_0^{-1} = 0, \quad (3.6)$$

and the same requirements for the gauge condition (3.5), (3.6) lead to the determination of the lagrangian multipliers, which correspond to the primary constraints $\Phi_1^{(1)}$, $\Phi_2^{(1)}$ and $\Phi_4^{(1)}$.

To go over to a time-independent set of constraints, we introduce the variable x'_0 , $x'_0 = x_0 - \zeta\tau$, instead of x_0 without changing the rest of the variables. This is a canonical transformation in the space of all variables with the generating function $W = x_0 p'_0 + \tau |p'_0| + W_0$, where W_0 is the generating function of the identity transformation with respect to all variables except x_0 , p_0 . The transformed Hamiltonian $H^{(1)'}$ is of the form $H^{(1)' = H^{(1)} + \partial W / \partial \tau = H + \{\Phi\}$, where $\{\Phi\}$ are terms proportional to the constraints and H is the physical Hamiltonian,

$$H = \omega = (\mathbf{p}^2 + m^2)^{1/2}, \quad \mathbf{p} = (p_k). \quad (3.7)$$

We can present all the constraints of the theory (including the gauge conditions), after the canonical transformation, in the following equivalent form: $K = 0$, $\phi = 0$, $T = 0$,

$$K = \begin{cases} \chi_a, & e - \omega^{-1}, & x'_0, & f_{ab}, & \psi_a^0, \\ P_{\chi_a}, & P_e, & |p_0| - \omega, & P_{f_{ab}}, & P_{a0}; \end{cases} \quad (3.8)$$

$$\phi = \begin{cases} p_i \psi_a^i + m \psi_a^5, \\ P_{al} + i \psi_{al}, \quad l = 1, 2, 3, 5; \end{cases} \quad (3.9)$$

$$T_{ab} = i \left([\psi_a^i, \psi_b^i]_- + [\psi_a^5, \psi_b^5]_- \right) - \kappa_{ab}. \quad (3.10)$$

The both sets of constraints K and ϕ are of second-class, only T are now first-class constraints. The set K has the so called special form [14], in this case, if we eliminate the variables χ , P_{χ_a} , e , P_e , x'_0 , $|p_0|$, f_{ab} , $P_{f_{ab}}$ and ψ_a^0 from the consideration, using these constraints, the Dirac brackets for the rest of variables with respect to all the second-class constraints (K, ϕ) reduce to ones with respect to the constraints ϕ only. Thus, we can only consider the variables x^i , p_i , ζ , ψ_a^i , P_{al} , $l = (i, 5)$ and two sets of constraints, second-class one ϕ and first-class one T . Often further we will use the transversal $\psi_a^{i\perp}$ and the longitudinal ψ_a^{\parallel} parts of ψ_a^i , because of ¹ these variables are convenient to treat both cases $m \neq 0$ and $m = 0$ on the same foot. The first constraint (3.9) is, in fact, a relation between ψ_a^{\parallel} and ψ_a^5 , $\psi_a^{\parallel} = -m \psi_a^5$, whereas $\psi_a^{i\perp}$ are not constrained. Nonzero Dirac brackets between all the variables have the form

¹Here and further we are using the following notations

$$a^{i\perp} = \Pi_j^i(\mathbf{p}) a^j, \quad a^{i\parallel} = L_j^i(\mathbf{p}) a^j, \quad a^{\parallel} = p_j a^j,$$

$$\Pi_j^i(\mathbf{p}) + L_j^i(\mathbf{p}) = \delta_j^i, \quad L_j^i(\mathbf{p}) = p^{-2} p_i p_j, \quad p = |\mathbf{p}|.$$

$$\{x^k, x^j\}_{D(\phi)} = \frac{i}{\omega^2} [\psi_a^{k\perp}, \psi_a^{j\perp}]_- + \frac{im}{\omega^2 p^2} \left(p_k [\psi_a^{j\perp}, \psi_a^5]_- - p_j [\psi_a^{k\perp}, \psi_a^5]_- \right), \quad (3.11)$$

$$\{x^i, \psi_a^{j\perp}\}_{D(\phi)} = -\frac{\psi_a^{i\perp} p_j}{p^2} + \frac{m}{p^2} \Pi_j^i \psi_a^5, \quad \{x^i, \psi_a^5\}_{D(\phi)} = -\frac{m}{\omega^2} \psi_a^{i\perp} + \frac{m^2}{\omega^2 p^2} p_i \psi_a^5,$$

$$\{\psi_a^{i\perp}, \psi_a^{j\perp}\}_{D(\phi)} = -\frac{i}{2} \delta_{ab} \Pi_j^i, \quad \{\psi_a^5, \psi_b^5\}_{D(\phi)} = -\frac{i}{2} \frac{p^2}{\omega^2} \delta_{ab}, \quad \{x^k, p_j\}_{D(\phi)} = \delta_j^k.$$

To simplify the problem of quantization one can go over to new variables, whose Dirac brackets are more simple. Namely, let us introduce θ_a^i and X^k , analogous to the case of spin one half particles [15], according to the formulas

$$X^k = x^k - \frac{i}{\omega + m} [\psi_a^{k\perp}, \psi_a^5]_-, \quad \theta_a^i = \psi_a^{i\perp} - \frac{\omega}{p^2} p_i \psi_a^5; \\ x^i = X^i - \frac{i}{\omega(\omega + m)} [\theta_a^i, \theta_a^{\parallel}]_-, \quad \psi_a^{i\perp} = \theta_a^{i\perp}, \quad \psi_a^5 = -\frac{1}{\omega} \theta_a^{\parallel}. \quad (3.12)$$

Using the brackets (3.11), one gets

$$\{X^k, p_j\}_{D(\phi)} = \delta_j^k, \quad \{X^k, X^j\}_{D(\phi)} = \{X^k, \theta_a^k\}_{D(\phi)} = 0, \\ \{\theta_a^k, \theta_b^j\}_{D(\phi)} = -\frac{i}{2} \delta_{ab} \delta_{kj}. \quad (3.13)$$

Variables X^i , p_i , ζ , θ_a^k , are independent with respect to the second-class constraints (3.9). Thus, on this stage we stay only with the first-class constraints (3.10), which being written in the new variables θ_a^k , have the form

$$T_{ab} = i [\theta_a^k, \theta_b^k]_- - \kappa_{ab}. \quad (3.14)$$

It is useful to adduce the expression for angular momentum $M_{\mu\nu}$ in terms of the independent variables,

$$M_{0j} = x_0 p_j - x_j p_0 = X_0 p_j - X_j p_0 - \frac{i p_0}{\omega(\omega + m)} [\theta_a^j, \theta_a^{\parallel}]_-, \quad x_0 = \zeta \tau, \quad p_0 = -\zeta \omega, \\ M_{kj} = x_k p_j - x_j p_k + i (\psi_a^{k\perp} \psi_a^{j\perp} - \psi_a^{j\perp} \psi_a^{k\perp}) + \frac{2im}{p^2} (p_k \psi_a^{j\perp} - p_j \psi_a^{k\perp}) \psi_a^5 \\ = X_k p_j - X_j p_k + i [\theta_a^k, \theta_a^j]_-, \quad (3.15)$$

One can check by straightforward calculations that $M_{\mu\nu}$ together with p_μ form the Poincare algebra in sense of Dirac brackets with respect to the constraints ϕ ,

$$\begin{aligned} \{M_{\mu\nu}, M_{\lambda\rho}\}_{D(\phi)} &= \eta_{\mu\lambda}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\lambda} + \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\nu\lambda}M_{\mu\rho}, \\ \{p_\mu, M_{\nu\lambda}\}_{D(\phi)} &= -\eta_{\mu\nu}p_\lambda + \eta_{\mu\lambda}p_\nu, \quad \{p_\mu, p_\nu\}_{D(\phi)} = 0. \end{aligned}$$

IV. QUANTIZATION

In the previous section we have imposed the gauge conditions to all the first-class constraints except the set of constraint (3.10). These constraints are quadratic in the fermionic variables. On the one hand, that circumstance makes it difficult to impose a conjugated gauge condition, on the other hand, imposing these constraints on states vectors does not create problems with Hilbert space construction since the corresponding operators of constraints have a discrete spectrum. Thus, we suppose to treat only the constraints T_{ab} in sense of the Dirac method. Namely, commutation relations between the operators \hat{X}^i , \hat{p}_i , $\hat{\zeta}$, $\hat{\theta}_a^k$, which are related to the corresponding classical variables, we calculate by means of Dirac brackets (3.13), so that the nonzero commutators are

$$\begin{aligned} [\hat{X}^k, \hat{p}_j]_- &= i \{X^k, p_j\}_{D(\phi)} = i\delta_j^k, \\ [\hat{\theta}_a^k, \hat{\theta}_b^j]_+ &= i \{\theta_a^k, \theta_b^j\}_{D(\phi)} = \frac{1}{2}\delta_{ab}\delta_{kj}. \end{aligned} \quad (4.1)$$

We assume also the operator $\hat{\zeta}$ to have the eigenvalues $\zeta = \pm 1$ by analogy with the classical theory, so that

$$\hat{\zeta}^2 = 1. \quad (4.2)$$

Suppose \mathcal{R} is a Hilbert space of vectors $\mathbf{f} \in \mathcal{R}$, where one can realize the relations (4.1), (4.2). Then physical vectors have to obey the conditions

$$i [\theta_a^k, \theta_b^k]_- \mathbf{f} = \kappa_{ab} \mathbf{f}. \quad (4.3)$$

Besides, they have to obey the Schrödinger equation

$$\left(i \frac{\partial}{\partial \tau} - \hat{H}\right) \mathbf{f} = 0, \quad (4.4)$$

with the quantum Hamiltonian \hat{H} constructed according to the classical physical one (3.7),

$$\hat{H} = \hat{\omega} = (\hat{\mathbf{p}}^2 + m^2)^{1/2}. \quad (4.5)$$

Going over to the physical time $x^0 = \zeta\tau$ (see [14]) one can transfer (4.4) to the form

$$\left(i \frac{\partial}{\partial x_0} - \hat{\zeta}\hat{\omega}\right) \mathbf{f} = 0. \quad (4.6)$$

Hermitian operators of angular momentum $\hat{M}_{\mu\nu}$ can be constructed according to the classical expression (3.15),

$$\begin{aligned} \hat{M}_{0j} &= \hat{X}_0\hat{p}_j - \frac{1}{2} [\hat{X}_j, \hat{p}_0]_+ - \frac{i\hat{p}_0}{\hat{\omega}(\hat{\omega} + m)} [\hat{\theta}_a^j, \hat{\theta}_a^0]_-, \\ \hat{M}_{kj} &= \hat{X}_k\hat{p}_j - \hat{X}_j\hat{p}_k + i [\hat{\theta}_a^k, \hat{\theta}_a^j]_-. \end{aligned} \quad (4.7)$$

In fact, all the formulas we adduced until this moment were written for arbitrary N . However, a realization of the relations (4.1) and (4.2) has to be considered separately for each N . In this paper we suppose to emphasize the case of spin one, which corresponds to $N = 2$. At the same time we believe that it is instructive to compare this case with the case $N = 1$, which can be quantized completely canonically [14]. Thus, below we consider construction of state spaces separately in two cases $N = 1$ and $N = 2$.

A. Spin one half

In this case $N = 1$ and the first-class constraint T_{ab} are absent. We can construct the realization of the algebra (4.1) in the Hilbert space \mathcal{R} , whose elements $\mathbf{f} \in \mathcal{R}$ are four-component columns,

$$\mathbf{f} = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{pmatrix},$$

so that $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are two components columns. We seek all the operators in the block-diagonal form, namely

$$\hat{\zeta} = \gamma^0, \quad \hat{p}_k = -i\partial_k \mathbf{I}, \quad \hat{X}^k = X^k \mathbf{I}, \quad \hat{\theta}^k = \frac{1}{2}\Sigma^k, \quad (4.8)$$

where γ^0 is the zero gamma matrix, I and \mathbf{I} are 2×2 and 4×4 unit matrices, $\Sigma^k = \text{diag}(\sigma^k, \sigma^k)$, where σ^k are Pauli matrices. We interpret $f_+(x) = f_1(x)$ as the wave function of a particle and $f_-(x) = \sigma^2 f_2^*(x)$ as that of an antiparticle and define accordingly the scalar product in \mathcal{R} ,

$$(\mathbf{f}, \mathbf{g}) = \int [f_1^+ g_1 + g_2^+ f_2] dx = \int f_\zeta^+ g_\zeta dx, \quad \zeta = \pm. \quad (4.9)$$

The operators \hat{X}^k , \hat{p}_k , $\hat{\theta}^k$, \hat{H} are self-conjugate with respect to this scalar product. It follows from (4.6) that

$$i \frac{\partial}{\partial x_0} f_\zeta = \hat{\omega} f_\zeta.$$

Thus, in this case the equations for the wave functions of a particle and antiparticle have the same form as it has to be in the absence of an external electromagnetic field.

The operators of angular momentum (4.7) and the spin operator \hat{s}^k have the following form in the realization in question

$$\begin{aligned} \hat{M}_{ij} &= \hat{X}_i \hat{p}_j - \hat{X}_j \hat{p}_i - \frac{1}{2} \epsilon_{ijk} \Sigma^k, \\ \hat{M}_{0j} &= \hat{X}_0 \hat{p}_j - \hat{X}_j \hat{p}_0 - \frac{i \hat{p}_j}{2 \hat{p}_0} + \frac{\hat{p}_0}{2 \hat{\omega} (\hat{\omega} + m)} \epsilon_{jki} \hat{p}_k \Sigma^i, \\ \hat{s}^k &= i \epsilon_{kji} \hat{\psi}^i \hat{\psi}^j = \frac{1}{2} \Sigma^k. \end{aligned} \quad (4.10)$$

As it is known, the square of the Pauli-Lubanski vector $\hat{W}^\mu = 1/2 \epsilon^{\mu\nu\lambda\sigma} \hat{M}_{\nu\lambda} \hat{p}_\sigma$ is a Casimir operator for the Poincare algebra. For this realization and in the centre mass system

$$\hat{W}^0 = 0, \quad \hat{W}^k = m \frac{\hat{p}_0}{\hat{\omega}} \hat{s}^k, \quad \hat{W}^2 = -(\hat{W}^i)^2 = -\frac{3}{4} m^2.$$

The latter confirms that the system in question has spin one half.

Now one can see that the quantum mechanics constructed is completely equivalent to the standard Dirac theory, namely it is connected with the latter by the unitary Foldy-Wouthuysen transformation [16]. Doing this transformation in the equation (4.6), we are coming to the standard Dirac equation (see [14]),

$$\mathbf{f} = \mathcal{U} \Psi, \quad \mathcal{U} = \frac{\hat{\omega} + m + \gamma \hat{\mathbf{p}}}{(2\hat{\omega}(\hat{\omega} + m))^{1/2}}, \quad (i\gamma^\mu \partial_\mu - m) \Psi = 0.$$

Besides, applying the same transformation to the operators (4.10), we get the operators of the angular momentum in the Dirac theory [15],

$$\mathcal{U}^+ \hat{M}_{\mu\nu} \mathcal{U} = \hat{X}_\mu \hat{p}_\nu - \hat{X}_\nu \hat{p}_\mu - \frac{1}{2} \sigma_{\mu\nu}, \quad \sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]_-.$$

B. Spin one

The relations (4.1) (4.2) for \hat{X}^k , \hat{p}_j and $\hat{\zeta}$ we can realize in a Hilbert space \mathcal{R}_{scal} , whose elements are two-component columns $f \in \mathcal{R}_{scal}$,

$$f = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{pmatrix}, \quad f_\zeta(\mathbf{x}) \in L_2, \quad \zeta = 1, 2,$$

in the following natural way [14]:

$$\hat{\zeta} = \sigma^3, \quad \hat{X}^k = x^k I, \quad \hat{p}_k = -i \partial_k I.$$

The scalar product in \mathcal{R}_{scal} we select in the form

$$(f, g) = \int [f_1^* g_1 + g_2^* f_2] dx. \quad (4.11)$$

The commutation relations (4.1) for θ_a^k , $a = 1, 2$, we realize in a Hilbert space \mathcal{R}_{spin} , which is a Fock space constructed by means of tree kinds of Fermi annihilation and creation operators b_k , b_k^+ , $k = 1, 2, 3$,

$$\begin{aligned} \theta_1^k &= \frac{1}{2} (b_k + b_k^+), & \theta_2^k &= \frac{i}{2} (b_k^+ - b_k), \\ [b_k, b_j^+]_{\pm} &= \delta_{kj}, & [b_k, b_j]_{\pm} &= [b_k^+, b_j^+]_{\pm} = 0. \end{aligned} \quad (4.12)$$

Due to the Fermi statistics of these operators the space \mathcal{R}_{spin} is finite-dimensional space of vectors $v \in \mathcal{R}_{spin}$, with basis vectors $v^{(0)}$, $v^{(1)}$, $v^{(2)}$, $v^{(3)}$,

$$\begin{aligned} v^{(0)} &= |0\rangle, & b_k^+ |0\rangle &= 0, \quad k = 1, 2, 3, \\ v_k^{(1)} &= b_k^+ |0\rangle, & v_k^{(2)} &= \frac{1}{2} \epsilon_{kji} b_j^+ b_i^+ |0\rangle, & v^{(3)} &= \frac{1}{6} \epsilon_{ijk} b_i^+ b_j^+ b_k^+ |0\rangle, \end{aligned} \quad (4.13)$$

which are eigen for the operator $\hat{n} = \sum_k b_k^\dagger b_k$,

$$\hat{n}v^{(n)} = nv^{(n)}, \quad n = 0, 1, 2, 3. \quad (4.14)$$

The total Hilbert space \mathcal{R} is the direct product of \mathcal{R}_{scal} and \mathcal{R}_{spin} .

Calculating the operators of angular momentum $\hat{M}_{\mu\nu}$, spin \hat{s}^k and square of Pauli-Lubanski vector in the realization, we get

$$\begin{aligned} \hat{M}_{ij} &= \hat{X}_i \hat{p}_j - \hat{X}_j \hat{p}_i + i(b_i b_j^\dagger - b_j b_i^\dagger), \\ \hat{M}_{0j} &= \hat{X}_0 \hat{p}_j - \frac{1}{2} [\hat{X}_j, \hat{p}_0]_+ + \frac{\hat{p}_0}{\hat{w}(\hat{w} + m)} (\hat{p}_k b_k b_j^\dagger - b_j \hat{p}_k b_k^\dagger), \end{aligned} \quad (4.15)$$

$$\hat{s}^k = \frac{i}{2} \epsilon_{kjl} (b_j^\dagger b_l - b_l^\dagger b_j), \quad \hat{W}^2 = -m^2 \hat{n} (3 - \hat{n}). \quad (4.16)$$

The operator \hat{n} commutes with \hat{H} , \hat{p}_μ and $\hat{M}_{\mu\nu}$, that means that states with a fixed n form invariant subspaces. In this realization the equation (4.3) imposes only restrictions on the vectors v from \mathcal{R}_{spin} ,

$$\hat{n}v = \left(\kappa + \frac{3}{2}\right)v,$$

they have to be eigenstates of the operator \hat{n} . That implies that κ takes on the values $-3/2$, $-1/2$, $1/2$, $3/2$. Due to (4.16) theories with $\kappa = \pm 1/2$ describe particles with spin one, whereas theories with $\kappa = \pm 3/2$ describe spinless particles. The canonical quantization of the latter case was described in [14], thus, we consider here only the former case. First, let us take $\kappa = -1/2$. In this case $n = 1$ and a general form of the time dependent state vector $f \in \mathcal{R}$ is

$$f = v_k^{(1)} f^k(x). \quad (4.17)$$

Due to (4.6) each component $f^k(x)$ obeys the Klein-Gordon equation,

$$(\square + m^2) f^k(x) = 0, \quad \square = \partial_\mu \partial^\mu. \quad (4.18)$$

We interpret $f_{(+)}^k(x) = f_1^k(x)$ as the wave function of a particle and $f_{(-)}^k(x) = f_2^{k*}(x)$ as the wave function of antiparticle with spin one. According to (4.11) the scalar product of two state vectors has the following form

$$(f, g) = \int [f_1^{k*} g_1^k + g_2^{k*} f_2^k] dx = \int f_{(\zeta)}^{k*} g_{(\zeta)}^k dx, \quad \zeta = \pm. \quad (4.19)$$

Now one can find a correspondence between the quantum mechanics constructed and the classical Proca field, which describe particles of spin one in the field theory. To this end we construct a vector field $\mathcal{A}_\mu(x)$ from the functions $f^k(x)$ in the following way

$$\mathcal{A}_\mu(x) = \frac{1}{\sqrt{2\omega}} \xi_\mu^{(k)}(\hat{\mathbf{p}}) (f_1^k(x) + f_2^k(x)), \quad (4.20)$$

with polarization vectors $\xi_\mu^{(k)}(\mathbf{p})$, having the form

$$\xi_0^{(k)}(\mathbf{p}) = \frac{p_0 p_k}{m\omega}, \quad \xi_i^{(k)}(\mathbf{p}) = \delta_i^k + \frac{p_i p_k}{m(m + \omega)}, \quad (4.21)$$

$$\xi_\mu^{(k)}(\mathbf{p}) p^\mu = 0, \quad \xi_\mu^{(k)}(\mathbf{p}) \xi_\nu^{(k)}(\mathbf{p}) = -\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}, \quad p_0 = -\zeta\omega. \quad (4.22)$$

One can check, using (4.18) and (4.20), that the field $\mathcal{A}_\mu(x)$ obeys the equations

$$(\square + m^2) \mathcal{A}_\mu(x) = 0, \quad \partial_\mu \mathcal{A}^\mu(x) = 0, \quad (4.23)$$

which are just equations for the Proca field [17]. Moreover, one can find the action of the generators (4.7) on the field $\mathcal{A}_\mu(x)$, calculating their action on the vector (4.17) and using the representation (4.20),

$$\hat{M}_{\alpha\beta} \mathcal{A}_\mu(x) = (x_\alpha p_\beta - x_\beta p_\alpha) \mathcal{A}_\mu(x) - i(\eta_{\alpha\mu} \mathcal{A}_\beta(x) - \eta_{\beta\mu} \mathcal{A}_\alpha(x)), \quad p_\alpha = -i\partial_\alpha. \quad (4.24)$$

That result reproduces the transformation properties of a vector field under the Lorentz rotations with $\delta x^\mu = \omega^{\mu\nu} x_\nu$,

$$\delta \mathcal{A}_\mu(x) = \frac{i}{2} \hat{M}_{\alpha\beta} \mathcal{A}_\mu(x) \omega^{\alpha\beta}. \quad (4.25)$$

It is also instructive to point out a correspondence between the quantum mechanics constructed and one particle sector of the quantum theory of the Proca field. In this quantum theory the Proca field appears to be the operator

$$\hat{A}_\mu(x) = \int \frac{d\mathbf{p}}{\sqrt{2\omega} (2\pi)^3} [e^{-ipx} a_k(\mathbf{p}) \xi_\mu^{(k)}(\mathbf{p}) + e^{ipx} d_k^\dagger(\mathbf{p}) \xi_\mu^{(k)*}(\mathbf{p})],$$

where $a_k(\mathbf{p})$, $a_k^\dagger(\mathbf{p})$, $d_k(\mathbf{p})$, $d_k^\dagger(\mathbf{p})$, $k = 1, 2, 3$ are two kinds of Bose, annihilation and creation operators, $p_0 = \omega$, and the polarization vectors $\xi_\mu^{(k)}(\mathbf{p})$ obey just the conditions (4.22). If we choose for them real expressions (4.21), then the relations hold

$$\begin{aligned} \mathcal{A}_\mu(x) &= \langle 0 | \hat{A}_\mu(x) | f_1 \rangle + \langle f_2 | \hat{A}_\mu(x) | 0 \rangle, \\ |f_1 \rangle &= \int d\mathbf{p} \tilde{f}_1^k(\mathbf{p}) a_k^\dagger(\mathbf{p}) | 0 \rangle, \quad \tilde{f}_1^k(\mathbf{p}) = \int \frac{dx}{(2\pi)^{(3/2)}} e^{-i\mathbf{p}\mathbf{x}} f_{(+)}^k(x) \Big|_{x^0=0}, \\ |f_2 \rangle &= \int d\mathbf{p} \tilde{f}_2^k(\mathbf{p}) d_k^\dagger(\mathbf{p}) | 0 \rangle, \quad \tilde{f}_2^k(\mathbf{p}) = \int \frac{dx}{(2\pi)^{(3/2)}} e^{-i\mathbf{p}\mathbf{x}} f_{(-)}^k(x) \Big|_{x^0=0}, \end{aligned}$$

so that $\mathcal{A}_\mu(x)$ is the classical Proca field (4.20). In fact, by such a choice of the polarization vectors, we have a direct correspondence between the wave functions of particles and antiparticles $f_{(\pm)}^k$ in the quantum mechanics and the states $|f_{1,2} \rangle$ in the quantum field theory.

Finally, let us consider the case $\kappa = 1/2$, which also describes a particle spin one. In this case $n = 2$ and a general form of the time dependent state vector $\mathbf{f} \in \mathcal{R}$ is

$$\mathbf{f} = v_k^{(2)} f^k(x). \quad (4.26)$$

One can check by straightforward calculations that $f^k(x)$ from the eq. (4.26) obeys the same equations and appears in the same form in all the constructions as $f^k(x)$ from the eq. (4.17). Moreover, the action of the generators $\hat{M}_{\mu\nu}$ on the basis vectors $v_i^{(1)}$ and $v_i^{(2)}$ is equal. That provides equal transformation properties for the field (4.18) constructed by $f^k(x)$ in both cases. All that testifies that both theories with $\kappa = \pm 1/2$ describe spin one particles.

V. MASSLESS CASE. QUANTUM MECHANICS OF PHOTON.

Here we are going to discuss the problem of quantization of massless particles spin one half and spin one. In this connection, one can consider the limit $m = 0$ of the above constructed quantum mechanics and compare it with an independent quantization of classical action, describing massless particles at the beginning. As to the limit, one can remark that all formulas are nonsingular in the mass and admit such a limit. On the classical level, after

the gauge fixing, it is possible to use, on the surface of the second-class constraints, the variables x^i , p_i , ζ , $\psi_a^{i\perp}$, ψ_a^5 or the variables X^i , p_i , ζ , θ_a^i , the Dirac brackets of the latter do not contain mass at all and expressions of the former via the latter are nonsingular in the mass. The first set of the variables at $m = 0$ splits into two (anti) commuting one with another groups x^i , p_i , $\psi_a^{i\perp}$, and ψ_a^5 . The Poincare generators are only expressed via the first group of variables and commute with ψ_a^5 . Instead of the Casimir operator W^2 , which vanishes at $m = 0$, appears a new one, helicity Λ ,

$$\Lambda = \hat{p}^{-1} \hat{p}_k \hat{s}^k, \quad (5.1)$$

It turns out that at $m = 0$ the variable ψ_a^5 can be omitted from the action (2.1). The quantization of such modified action reproduces the physical sector (in particular, quantum mechanics of the transversal photons) of the limit of the massive quantum mechanics. Below we adduce details of the limit $m = 0$ for two cases: of spin one half and spin one, taking into account general properties mentioned above, and emphasizing mainly differences from the massive case.

A. Massless particle spin one half

As we have mentioned above, the Dirac brackets for the variables X^i , p_i , ζ , θ_a^k do not depend on the mass, that means that realization (4.1), (4.2) remains in the limit $m = 0$. It is clear that the realization does not depend on the presence of the operator $\hat{\psi}^5$. In the limit we have $\hat{\psi}^5 = i\hat{p}^{-1} \hat{p}_k \epsilon_{kjl} \hat{\psi}^{l\perp} \hat{\psi}^{j\perp} = \Lambda$, where Λ is the helicity operator. The Schrödinger equation (4.6) with $m = 0$ gives the Dirac equation with $m = 0$ after the corresponding FW transformation. The total Hilbert space forms now a reducible representation of the Poincare group (right and left neutrinos). It follows from the described structure of the quantum mechanics that in the limit $m = 0$ one does not need the variable ψ^5 at the theory. Indeed, one can take the action (2.1) at $m = 0$ and omit ψ^5 in the beginning. In such a theory, after the same gauge fixing (in particular, $\psi_0 = 0$), we have only the variables

$x^i, p_i, \zeta, \psi^{i\perp}$ on the constraint surface. Their Dirac brackets and the expressions of the Poincare generators coincide with the corresponding expressions of the massive theory at $m = 0$. The same realization is available. If one introduces the operator $i\hat{p}^{-1}\hat{p}_k\epsilon_{kji}\hat{\psi}^{i\perp}\hat{\psi}^{j\perp}$, which is in fact the operator $\hat{\psi}^5$ of the massive case, then the theory literally coincides with the limit of the massive case. In this connection one can remark that the dimensionality of the Hilbert space in the discussed realization does not depend on the presence of the variable ψ^5 at $m = 0$ and coincide with dimensionality of the massive case.

B. Quantum mechanics of photon

Now let us turn to the massless case $N = 2$, which, according to our expectations has to describe a photon. First, we consider the limit $m = 0$ of the massive spin one case with $\kappa = -1/2$. According to our interpretation, states with $\zeta = +1$ correspond to particles and with $\kappa = -1$ to antiparticles. Because of our aim is a photon, which is neutral, we may restrict ourselves to consider the limit of massive quantum mechanics of neutral spin one particle. To get such a quantum mechanics one needs to replace the gauge condition $x_0 = \zeta\tau$ by the one $x_0 = \tau$, the latter fixes, besides the reparametrization gauge freedom, the discrete variable ζ ($\zeta = 1$) as well [14]. Thus, the operator $\hat{\zeta}$ disappears from the consideration and elements of the \mathcal{R}_{scal} are merely functions $f(\mathbf{x})$ from L_2 with the scalar product $(f, g) = \int f^*gdx$. The realization for $\hat{X}^i = \hat{x}^i, \hat{p}_i, \hat{\theta}_a^k$ remains the same as at $m \neq 0$. The operator of helicity and its square have the form

$$\Lambda = i\hat{p}^{-1}\hat{p}_i\epsilon_{ijl}\hat{b}_j^{+\perp}\hat{b}_l^{\perp}, \quad \Lambda^2 = \hat{n}^{\perp}(2 - \hat{n}^{\perp}), \quad \hat{n}^{\perp} = \hat{b}_j^{+\perp}\hat{b}_j^{\perp}.$$

The total Hilbert space splits into the two invariant subspaces, with $\Lambda^2 = 1, \Lambda = \pm 1$ and with $\Lambda = 0$. The first subspace can be created by the operators $\hat{x}^i, \hat{p}_i, \hat{\psi}_a^{i\perp} = \hat{\theta}_a^{i\perp}$, whereas the second one by the operators $\hat{x}^i, \hat{p}_i, \hat{\psi}_a^5 = -\hat{p}^{-1}\hat{\theta}_a^{\parallel}$. We treat the subspace with $\Lambda^2 = 1$ as the Hilbert space of transversal photons with helicity $\Lambda = \pm 1$. The subspace with $\Lambda = 0$ we treat as the Hilbert space of longitudinal photons with helicity 0. To exclude the longitudinal

photons from the consideration one needs to impose a supplementary condition $\Lambda^2 = 1$. On the other hand, to get a theory, containing only the transversal photons, one can start from the action (2.1) $N = 2, m = 0$, without the variables ψ_a^5 ,

$$\int_0^1 \left[-\frac{1}{2e} (\dot{x}^\mu - i\psi_a^\mu \chi_a)^2 + \frac{i}{2} f_{ab} [\psi_{a\mu}, \psi_b^\mu]_- - i\psi_{a\mu} \dot{\psi}_a^\mu \right] d\tau.$$

In this case one can have the same realization for the operators $\hat{x}^i, \hat{p}_i, \hat{\psi}_a^{i\perp}$ as in quantum mechanics with ψ_a^5 at $m = 0$. Instead of the operator \hat{n} in the condition (4.3) appears the operator \hat{n}^{\perp} ,

$$\hat{n}^{\perp} \mathbf{f} = (\kappa + 1) \mathbf{f}. \quad (5.2)$$

Its eigenvalues n^{\perp} can be only 0, 1, 2, so that κ takes now on the values 0, ± 1 . The cases $n^{\perp} = 0, 2; \kappa = \pm 1$ correspond to the spinless particles; the case $n^{\perp} = 1; \kappa = 0$ corresponds to the limit $m = 0$ of the quantum theory with the action (2.1) with $\kappa = -1/2$, sector $\Lambda^2 = 1$, and reproduces the quantum mechanics of the transversal photons.

Finally, we can demonstrate that the quantum mechanics of the transversal photons reproduces in a sense the classical Maxwell theory and is equivalent to one-particle sector of quantum theory of Maxwell field. To this end let us rewrite the representation (4.17) in the form

$$\mathbf{f} = v_k^{(1)\perp} f^{k\perp}(x) + v_k^{(1)\parallel} f^{k\parallel}(x),$$

where the transversal and longitudinal components are defined by means of the corresponding projectors, $\Pi_k^{\perp}(\hat{\mathbf{p}}), L_k^{\parallel}(\hat{\mathbf{p}})$. After the limit $m = 0$ one can interpret $f^{k\perp}(x)$ as the wave function of transversal photons. To construct the classical electromagnetic field we have to use the wave functions $f^{k\perp}(x)$ in the same way we had used the wave functions $f^{k\parallel}(x)$ in the previous section to construct the Proca field. Namely, we define a vector field $\mathcal{A}_\mu(x)$ in the following way

$$\mathcal{A}_\mu(x) = \frac{1}{\sqrt{2\hat{\omega}}} \xi_\mu^{(k)\perp} [f^{k\perp}(x) + f^{k\perp*}(x)], \quad (5.3)$$

where $\xi_\mu^{(k)\perp}$ are transversal components of the polarization vectors (4.21),

$$\xi_\mu^{(k)\perp} = \delta_\mu^i \Pi_i^k(\hat{\mathbf{p}}). \quad (5.4)$$

Due to the equation (4.18) and the structure of the polarization vectors (5.4), the field (5.3) obeys the Maxwell equations in the Coulomb gauge,

$$\square^2 \mathcal{A}_\mu(x) = 0, \quad \partial_j \mathcal{A}^j(x) = 0, \quad \mathcal{A}_0(x) = 0.$$

Let us turn to the quantum theory of the Maxwell field. In the Coulomb gauge the operator of the vector potential has the form

$$\hat{A}_k(x) = \int \frac{d\mathbf{p}}{\sqrt{2p_0} (2\pi)^3} \left[e^{-ipx} c_\lambda(\mathbf{p}) + e^{ipx} c_\lambda^\dagger(\mathbf{p}) \right] e_k^{(\lambda)}(\mathbf{p}), \quad \lambda = 1, 2, \quad p_0 = |\mathbf{p}|,$$

where $c_\lambda^\dagger(\mathbf{p})$, $c_\lambda(\mathbf{p})$ are creation and annihilation operators of transversal photons and $e_k^{(\lambda)}(\mathbf{p})$ are two polarization vectors, which are selected here to be real,

$$e_k^{(\lambda)}(\mathbf{p}) e_k^{(\lambda')}(\mathbf{p}) = \delta_{\lambda\lambda'}, \quad e_k^{(\lambda)} p_k = 0.$$

Classical vector potential $\mathcal{A}_k(x)$ can be constructed as

$$\mathcal{A}_k(x) = \langle 0 | \hat{A}_k(x) | f \rangle + \langle f | \hat{A}_k(x) | 0 \rangle, \quad (5.5)$$

$$c_\lambda^\dagger(\mathbf{p}) | 0 \rangle = 0, \quad | f \rangle = \int \tilde{f}^{(\lambda)}(\mathbf{p}) c_\lambda^\dagger(\mathbf{p}) | 0 \rangle, \quad \tilde{f}^{(\lambda)}(\mathbf{p}) = \int \frac{d\mathbf{p}}{(2\pi)^{3/2}} e^{-ipx} e_k^{(\lambda)}(\mathbf{p}) f^{k\perp}(x) \Big|_{x^0=0},$$

so that $\mathcal{A}_k(x)$ are three-dimensional components of the classical Maxwell field (5.3). The last formulas establish a correspondence between the wave functions $f^{k\perp}(x)$ of the transversal photons in the quantum mechanics and states $|f\rangle$ of the photons in quantum electrodynamics. One can verify, similar to the massive case, that the actions of the Poincare generators on the fields (5.4) and (5.5) coincide in the both theories.

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