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# PUBLICAÇÕES

**IFUSP/P-1094**



## A PSEUDOCLASSICAL MODEL FOR WEYL PARTICLE

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Dezembro/1993

# A pseudoclassical model for Weyl particle.

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(December 16, 1993)

## Abstract

A pseudoclassical model to describe a Weyl particle is proposed. Its canonical quantization leads to the massless Dirac equation and to the Weyl condition for the wave function. In spite of the classical theory is not Lorentz covariant the corresponding quantum theory is. That can be treated as an anomaly, which restore a symmetry broken on the classical level.

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In recent years numerous classical models of relativistic particles and superparticles have been discussed intensively in different contexts. First, the interest in such models was initiated by the close relationship with problems in string theory and gravity, but now it is clear that it is an important question itself whether there exist classical models for any relativistic particle whose quantization reproduces, in a sense, the corresponding field theory or one particle sector of the corresponding quantum field theory. In this paper we propose a pseudoclassical model, whose canonical quantization reproduces the quantum theory of Weyl particles. The story of the question is the following. As known, first, a pseudoclassical action of spin one half relativistic particle was proposed by Berezin and Marinov [1] and just after that was discussed and investigated in papers [2–6]. The action has the form

$$S_1 = \int_0^1 \left[ -\frac{1}{2e}\left(\dot{x}^\mu - i\psi^\mu\chi\right)^2 - \frac{e}{2}m^2 - im\psi^5\chi - i\left(\psi_\mu\dot{\psi}^\mu - \psi_5\dot{\psi}_5\right)\right] d\tau \,, \tag{1}$$

where $x^\mu$, $e$ are even and $\psi^\mu$, $\chi$ are odd variables, dependent on a parameter $\tau \in [0,1]$, $\mu = \overline{0,3}$, $\eta_{\mu\nu} = \mathrm{diag}(1 -1 -1 -1)$. Because of the reparametrization invariance of the action, the Hamiltonian of the model is equal to zero on the constraints surface. In the papers [7–10] devoted to the quantization of the model, they tried to avoid this difficulty, using the so called Dirac method of quantization of theories with first-class constraints [14], treating the first-class constraints in the sense of restrictions on state vectors. Unfortunately, in general case, this scheme of quantization creates many questions, e.g. with Hilbert space construction, what is Schrödinger equation and so on. A consistent, but more complicated technically way is to work in the physical sector, namely, first, on the classical level, to impose gauge conditions to all first class-constraints to reduce the theory to one with second-class constraints only, and then quantize by means of the Dirac brackets (we will call such a method as canonical quantization). First canonical quantization for a relativistic spin one half particle was done in [12]. The quantum mechanics constructed there allowes one the limit $m = 0$ and as a result one gets the quantum theory of massless particle [13], which is described by the Dirac equation with $m = 0$,

$$\hat{p}_\mu\gamma^\mu\psi = 0 \,, \quad \hat{p}_\mu = i\partial_\mu \,, \quad [\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu} \,. \tag{2}$$

It turns out that the variable $\psi^5$ can be omitted from the action (1) at $m = 0$. The quantization of such a modified action reproduces the physical sector of the limit $m = 0$ of the massive quantum mechanics. Unfortunately, such a way of quantization gives a quantum theory, discribing massless spin one half particle with the all possible values of helicity (right and left neutrinos). As it is known, the right (left) neutrino is described by four-spinor, which obeys, besides the Dirac equation (2), the Weyl condition as well,

$$\left(\gamma^5 - \alpha\right)\psi(x) = 0\,, \quad \alpha = 1\,(-1)\,, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{3}$$

There were several attempts to modify the action (1) so that in course of quantization one can get a quantum mechanics with wave functions obeing both equations (2) and (3) at the same time. So, in the work [10] they modified the action (1) at $m = 0$ to the following form

$$S_2 = \int_0^1 \left[-\frac{1}{2e}\left(\dot{x}^\mu + g_1 Q^\mu - i\psi^\mu\chi\right)^2 - i\psi_\mu\dot{\psi}^\mu + g_2\left(\Lambda - \alpha\right)\right] d\tau\,, \tag{4}$$

where $g_a$ are Lagrange multipliers and $\Lambda = i\epsilon^{\mu\nu\rho\varsigma}\psi_\mu\psi_\nu\psi_\rho\psi_\varsigma/3$, $Q^\mu = \epsilon^{\mu\nu\rho\varsigma}\psi_\nu\psi_\rho\psi_\varsigma$ . The theory has additional first-class constraints. Quantization by means of the Dirac method in the realization $\psi^\mu = i/2\gamma^\mu$ gives both equations (2) and (3) as restrictions on states vectors. As we mentioned before, this way of quantization is not well grunded jet. Moreover, attempts to quantize this model canonically fail, since as soon as one chooses any gauge condition linear in $\psi$, the function $\Lambda$ vanishes and only Dirac equations remains after quantization. Another possibility was discussed in [16]. They considered the theory with the action

$$S_3 = \int_0^1 \left\{-\frac{1}{2e}\left[\dot{x}^\mu - i\left(\psi^\mu - \frac{2i\alpha}{3}\epsilon^{\mu\nu\rho\varsigma}\psi_\nu\psi_\rho\psi_\varsigma\right)\chi\right]^2 - i\psi_\mu\dot{\psi}^\mu\right\} d\tau\,. \tag{5}$$

A formal quantization of the theory following the Dirac method leads to the equation

$$\hat{p}_\mu\gamma^\mu\left(\gamma^5 - \alpha\right)\psi(x) = 0$$

for state vectors, which is not equivalent to the both equations (2), (3). The canonical quantization gives Dirac equation (2) but without any additional restrictions for helicity. That is in the agreement with the fact that classically actions (5) and (1) at $m = 0$ are equivalent [16].



In this paper we propose the following pseudoclassical acction for the Weyl particle

$$S = \int_0^1 \left\{\left[-\frac{1}{2e}\dot{x}^\mu - i\psi^\mu\chi - g\left(\epsilon^{\mu\nu\rho\varsigma}n_\nu\psi_\rho\psi_\varsigma + \frac{i\alpha}{2}n^\mu\right)\right]^2 - i\psi_\mu\dot{\psi}^\mu\right\} d\tau\,, \tag{6}$$

where $g$ is an even variable, $\alpha$ is an even constant and $n^\mu$ an external given four vector, wich we select in the form $n^\mu = (1,\ 0,\ 0,\ 0)$. Due to the existence of this vector the action (6) is not Lorentz covariant (the theory is formulated in an given reference frame), but, as we will see further, the corresponding quantum mechanics is Lorentz covariant. The are three types of gauge transformations under which the action (6) is invariant: reparametrizations

$$\delta x^\mu = \dot{x}^\mu\xi\,, \quad \delta e = \frac{d}{d\tau}(e\xi)\,, \quad \delta g = \frac{d}{d\tau}(g\xi)\,, \quad \delta\psi^\mu = \dot{\psi}^\mu\xi\,, \quad \delta\chi = \frac{d}{d\tau}(\chi\xi)\,,$$

with an even parameter $\xi(\tau)$, supertransformation

$$\delta x^\mu = i\psi^\mu\epsilon\,, \quad \delta e = i\chi\epsilon\,, \quad \delta g = 0\,, \quad \delta\chi = \dot{\epsilon}\,, \quad \delta\psi^\mu = \frac{1}{2e}z^\mu\epsilon\,,$$
$$z^\mu = \dot{x}^\mu - i\psi^\mu\chi - g\epsilon^{\mu\nu\rho\varsigma}n_\nu\psi_\rho\psi_\varsigma - \frac{i\alpha g n^\mu}{2},$$

with an odd parameter $\epsilon(\tau)$, and additional (in comparasion with the action (1)) gauge transformations

$$\delta x^\mu = \left(\epsilon^{\mu\nu\rho\varsigma}n_\nu\psi_\rho\psi_\varsigma + \frac{i\alpha}{2}n^\mu\right)\kappa\,, \quad \delta e = 0\,, \quad \delta g = \dot{\kappa}\,, \quad \delta\chi = 0\,, \quad \delta\psi^\mu = \frac{i}{e}\epsilon^{\mu\nu\rho\varsigma}n_\nu z_\rho\psi_\varsigma\kappa\,,$$

with an even parameter $\kappa(\tau)$. The equations of motion have the form

$$\begin{aligned}
&\frac{\delta S}{\delta x_\mu} = \frac{d}{d\tau}\left[\frac{1}{e}z^\mu\right] = 0\,, \quad \frac{\delta S}{\delta e} = \frac{1}{2e^2}z^2 = 0\,, \quad \frac{\delta S}{\delta g} = \frac{1}{e}z_\mu\left(\epsilon^{\mu\nu\rho\varsigma}n_\nu\psi_\rho\psi_\varsigma + \frac{i\alpha}{2}n^\mu\right) = 0\,, \\
&\frac{\delta_r S}{\delta\chi} = \frac{i}{e}z_\mu\psi^\mu = 0\,, \quad \frac{\delta_r S}{\delta\psi_\mu} = 2i\dot{\psi}^\mu - \frac{1}{e}z_\rho\left(ig^{\rho\mu}\chi + 2g\epsilon^{\mu\nu\rho\varsigma}n_\nu\psi_\varsigma\right) = 0\,.
\end{aligned} \tag{7}$$

Going over to the hamiltonian formulation, we introduce the canonical momenta:

$$\begin{aligned}
&p_\mu = \frac{\partial L}{\partial\dot{x}_\mu} = -\frac{1}{e}z_\mu\,, \quad P_e = \frac{\partial L}{\partial\dot{e}} = 0\,, \\
&P_\chi = \frac{\partial_r L}{\partial\dot{\chi}} = 0 \quad \pi_\mu = \frac{\partial_r L}{\partial\dot{\psi}^\mu} = -i\psi_\mu\,, \quad P_g = \frac{\partial L}{\partial\dot{g}} = 0\,.
\end{aligned} \tag{8}$$

It follows from (8) that there exist primary constraints $\Phi^{(1)} = 0$, $\Phi^{(1)} = (P_e,\ P_\chi,\ P_g,\ \pi_\mu + i\psi_\mu)$. We construct the Hamiltonian $H^{(1)}$ according to the standard procedure (we are using the notations of the book [12]), $H^{(1)} = H + \lambda_A\phi_A^{(1)}$, where

$$H = -\frac{e}{2}p^2 + ip_\mu\psi^\mu\chi + g\left(\epsilon^{\mu\nu\rho\varsigma}p_\mu n_\nu\psi_\rho\psi_\varsigma + \frac{i\alpha}{2}p_\mu n^\mu\right) . \tag{9}$$

From the conditions of the conservation of the primary constraints in time, $\dot{\Phi}^{(1)} = 0$, we find secondary constraints $\Phi^{(2)} = 0$, $\Phi^{(2)} = \left(p^2 , p_\mu\tilde{\psi}^\mu , \epsilon^{\mu\nu\rho\varsigma}p_\mu n_\nu\tilde{\psi}_\rho\tilde{\psi}_\varsigma + \frac{i\alpha}{2}p_\mu n^\nu\right)$, where $\tilde{\psi}^\mu = i/2(\pi^\mu - i\psi^\mu) = \psi^\mu + i/2(\pi^\mu + i\psi^\mu)$, and determine $\lambda$, which correspond to the primary constraints $\pi_\mu + i\psi_\mu = 0$. Thus, the Hamiltonian $H$ appears to be proportional to the constraints and vanishes on the constraints surface. No more secondary constraints arise from the Dirac procedure. Primary constraints $P_e$ , $P_\chi$ , $P_g$ and all the secondary ones are firt-class. We impose the following gauge conditions $\Phi^G = 0$, $\Phi^G = \left(\chi , g , x^0 - \zeta\tau , e + \zeta p_0^{-1} , \psi_0\right)$, where $\zeta = -\text{sign}\, p_0$. (The gauge $x_0 - \zeta\tau = 0$ was first proposed in papers [12] as a conjugated gauge condition to the constraint $p^2 = m^2$ in the case of scalar and spining particles. In constrast with the gauge $x_0 = \tau$, which together with the continuous reparametrization symmetry breaks the time reflection symmetry and therefore fixes the variables $\zeta$, the former gauge breaks only the continuous symmetry, so that the variable $\zeta$ remains in the theory to describe states of particles $\zeta = +1$ and states of antiparticles $\zeta = -1$. Namely this circumstance allowed one to get Klein-Gordon and Dirac equations as Schrödinger ones in course of the canonical quantization. To break the supergauge symmetry the gauge condiction $\psi^5 = 0$ was used in [12]. In [14] the general class of gauge conditions of the form $\alpha\psi^0 + \beta\psi^5 = 0$ was investigated in case of $D-$dimensional spining particle). To go over to a time-independent set of constraints we introduce the variable $x_0'$, $x_0' = x_0 - \zeta\tau$, instead of $x_0$ without changing the rest of the variables. That is a canonical transformation in the space of all variables with the generating function $W = x_0p_0' + \tau\,|p_0'| + W_0$, where $W_0$ is the generating function of the identity transformation with respect to all variables except $x_0$, $p_0$. The transformed Hamiltonian $H^{(1)'}$ is of the form $H^{(1)'} = H^{(1)} + \partial W/\partial\tau = H + \{\Phi\}$ , where $\{\Phi\}$ are terms proportional to the constraints and $H$ is the physical Hamiltonian,

$$H = \omega = |\mathbf{p}| \ , \ \mathbf{p} = (p_k) \ . \tag{10}$$

One can present all the constraints of the theory (including the gauge conditions), after the canonical transformations, in the following equivalent form: $K = 0$, $\phi = 0$, $T = 0$,



$$K = \left(\chi , g , e - \omega^{-1} , x'_0 , \psi_0 , P_\chi , P_g , P_e , \pi_0\right) ,$$

$$\phi = \left(\psi^{\parallel} , \pi_j + i\psi_j\right) , \ T = \epsilon^{jkl}p_j\psi^{k\perp}\psi^{l\perp} - \frac{i\alpha}{2}\zeta\omega .$$

(We are using the following notations $a^{i\perp} = \Pi^i_j(\mathbf{p})a^j$ , $a^{\parallel} = p_ja^j$ , $\Pi^i_j(\mathbf{p}) = \delta^i_j - \omega^{-2}p_ip_j$ .) The both sets of constraints $K$ and $\phi$ are of second-class, only $T$ are now first-class constraints. The set $K$ has the so called special form [12]; in this case if we eliminate the variables $\chi$, $g$, $e$, $x'_0$, $\psi_0$, $P_\chi$, $P_g$, $P_e$, $\pi_0$ from the consideration, using these constraints, the Dirac brackets for the rest of variables with respect to all second-class constraints $(K, \phi)$ reduce to ones with respect to the constraints $\phi$ only. Thus, one can consider the variables $x^i$, $p_i$, $\zeta$, $\psi^{\parallel}$, $\psi^{i\perp}$, $\pi_k$ and two sets of constraints, second-class one $\phi$ and first-class one $T$ only. Nonzero Dirac brackets between all the variables have the form

$$\begin{aligned} &\left\{x^k, p_j\right\}_{D(\phi)} = \delta^k_j , \quad \left\{x^k, x^j\right\}_{D(\phi)} = \frac{i}{\omega^2}\left[\psi^{k\perp}, \psi^{j\perp}\right]_- , \\ &\left\{x^i, \psi^{j\perp}\right\}_{D(\phi)} = -\frac{\psi^{i\perp}p_j}{\omega^2} , \quad \left\{\psi^{i\perp}, \psi^{j\perp}\right\}_{D(\phi)} = -\frac{i}{2}\Pi^i_j(\mathbf{p}) . \end{aligned} \tag{11}$$

Thus, on this stage we have a theory with only one first-class constraint $T$. This constraint is quadratic in the fermionic variables. On the one hand, that circumstance makes it difficult to impose a conjugated gauge condition, on the other hand, imposing these constraints on states vectors does not creates problems with Hilbert space construction since the corresponding operator of constraint has a discrete spectrum. Thus, we suppose to treat only the constraint $T$ in sense of the Dirac method. Namely, commutation relations between the operators $\hat{x}^i$, $\hat{p}_i$, $\hat{\psi}^{j\perp}$, which are related to the corresponding classical variables, we calculate by means of Dirac brackets (11), so that the nonzero commutators are

$$\begin{aligned} &\left[\hat{x}^k, \hat{p}_j\right]_- = i\delta^k_j , \quad \left[\hat{x}^k, \hat{x}^j\right]_- = -\hat{\omega}^{-2}\left[\hat{\psi}^{k\perp}, \hat{\psi}^{j\perp}\right]_- , \\ &\left[\hat{x}^i, \hat{\psi}^{j\perp}\right]_- = -i\frac{\hat{\psi}^{i\perp}\hat{p}_j}{\hat{\omega}^2} , \quad \left[\hat{\psi}^{i\perp}, \hat{\psi}^{j\perp}\right]_+ = \frac{1}{2}\Pi^i_j(\hat{\mathbf{p}}) . \end{aligned} \tag{12}$$

We assume also the operator $\hat{\zeta}$ to have the eigenvalues $\zeta = \pm 1$ by analogy with the classical theory, so that $\hat{\zeta}^2 = 1$. One can construct the realization of the algebra (12) and operator equation for $\hat{\zeta}$ in the Hilbert space $\mathcal{R}$, whose elements $\mathbf{f} \in \mathcal{R}$ are four-component columns,

$$\mathbf{f} = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{pmatrix},$$

so that $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are two components columns. We seek all the operators in the block-diagonal form, namely

$$\hat{\zeta} = \gamma^0\,, \quad \hat{p}_k = -i\partial_k \mathbf{I}\,, \quad \hat{x}^i = x^i\mathbf{I} + \frac{1}{2\hat{\omega}^2}\epsilon^{ijk}\hat{p}_j\Sigma^k\,, \quad \hat{\psi}^{j\perp} = \frac{1}{2}\Pi^j_k(\hat{\mathbf{p}})\,\Sigma^k\,, \tag{13}$$

where $\mathbf{I}$ is $4\times 4$ unit matrix, $\boldsymbol{\Sigma} = \mathrm{diag}\,(\boldsymbol{\sigma},\ \boldsymbol{\sigma})$, and $\sigma^k$ are Pauli matrices. The operator $\hat{T}$, which corresponds to the first-class constraint $T$, has the form in the realization $\hat{T} = \hat{p}_0^{-1}\hat{p}_k\Sigma^k + \alpha$. Physical state vectors have to obey the condition

$$\hat{T}\mathbf{f} = 0 \tag{14}$$

and Schrödinger equation $\left(i\partial/\partial\tau - \hat{H}\right)\mathbf{f} = 0$, which being written in term of the physical time $x^0 = \zeta\tau$ (see [12]), has the form

$$\left(i\frac{\partial}{\partial x^0} - \hat{\zeta}\hat{\omega}\right)\mathbf{f} = 0\,. \tag{15}$$

The quantum mechanics constructed appears to be equivalent to the theory of Weyl particle, namely it is connected with the latter by the unitary Foldy-Wouthuysen transformation [15]. Doing this transformation

$$\mathbf{f} = \mathcal{U}\Psi\,, \quad \mathcal{U} = \frac{\hat{\omega} + \gamma\hat{\mathbf{p}}}{\hat{\omega}\sqrt{2}}\,, \quad \mathcal{U}^{+}\mathcal{U} = 1\,,$$

we obtain from the equation (14) the Weyl condition (3) and from the equation (15) we obtain the Dirac equation (2). Thus, the action (6) with $\alpha = 1$ describes right and with $\alpha = -1$ left neutrino. In spite of the classical action (6) is not Lorentz covariant, the corresponding quantum theory is explicitly Lorentz covariant since it coincides with the Dirac-Weyl theory of a massless spin one half particle in the Foldy-Wouthuysen representation. Moreover, if we take Poincare generators, which correspond to the theories with the covariant actions (4) or (5), then their action on the state vector of the quantum mechanics in question reproduces



the transformation properties of the Dirac field [14]. Besides, they commute with the first-class constraint $\hat{T}$, that means that the Weyl condition is Lorentz-invariant. In classical theory the constraint surface is not Lorentz-covariant. We interprete this situation as the presence of an anomaly in the quantum mechanics. The presence of anomalies in quantum mechanics with gauge symmetries (one-dimensional gauge field theory) was remarked by some authors [17]. However, usualy the anomalies either break a classical symmetry or deform a classical algebra of symmetry [10]. The model under consideration is an example when anomalies play "positive" role, they restore the symmetry broken in classical theory. In fact, one can say that the classical theory has anomaly.

## ACKNOWLEDGMENTS

One of the authors (I. V. Tyutin) thanks FAPESP (Brasil) for support and Departamento de Física Matemática of the Universidade de São Paulo for hospitality and (A. E. Gonçalves) thanks CAPES for support.

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