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NON-LINEAR ELECTROMAGNETIC INTERACTIONS IN THERMAL QED

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Non-linear electromagnetic interactions in thermal QED

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We examine the behavior of the non-linear interactions between electromagnetic fields at high temperature. It is shown that, in general, the $\log(T)$ dependence on the temperature of the Green functions is simply related to their UV behavior at zero-temperature. We argue that the effective action describing the nonlinear thermal electromagnetic interactions has a finite limit as $T \to \infty$. This thermal action approaches, in the long wavelength limit, the negative of the corresponding zero-temperature action.

The effective thermal action due to the electron-positron box, which is fourth order in the electromagnetic field, has been studied previously in the literature [1-3]. In reference [4] it was shown that this effective action has a finite limit at high temperatures, when $T \to \infty$. The main purpose of this paper is to extend the analysis in [4] on the non-linear interactions, to all orders in the electromagnetic field.

At high temperatures, individual contributions contain power dependence on T, but the T^3 and T terms cancel by symmetry. Gauge invariance imposes strong constraints, which lead to the cancellation of the T^2 contributions in QED [5,6], except in the case of the photon self-energy. Note that, in general, the dependence upon T for high T is not necessarily connected to the UV divergence or convergence of the zero-temperature amplitude. In QCD, for example, all the N-gluon functions behave like T^2, although they are UV finite for $N > 4$.

This work addresses the problem of possible $\log(T)$ contributions. We will present a

simple argument showing that in thermal field theories, these are related to the UV behavior of the Green functions at $T = 0$. Consequently, in the $N \geq 4$ photon Green functions, which are UV finite, the $\log(T)$ contributions must be absent. We have verified this behavior by explicit computation of the electron-positron 6-point function, in the long wavelength limit of the external photons. Therefore, the effective action describing the nonlinear interactions between electromagnetic fields at high temperature must have a finite limit as $T \to \infty$. Like the zero-temperature action, it must be gauge invariant. But this thermal action may be more complicated, because it is not necessarily Lorentz-invariant.

Typical graphs contributing to the nonlinear electromagnetic interactions are shown in Fig. 1. We must consider only diagrams with an even number of external photon lines, since for N-odd, their contribution vanishes by charge conjugation.

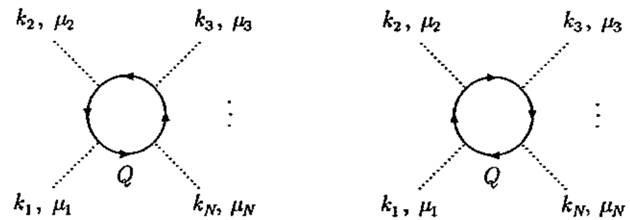

FIG. 1. Diagrams which contribute to the nonlinear interactions between electromagnetic fields. Dotted lines represent photons, and solid lines stand for electrons or positrons.

In order to discuss the logarithmic temperature dependence, we use the analytically continued imaginary-time thermal perturbation theory [7]. Then, we can express the complete thermal amplitude, which includes the zero-temperature part, in the form

$$A^{\mu_1\mu_2\cdots\mu_N}(\mathbf{k}_i, k_i^0, T) = M^\epsilon\, T \sum_{Q_0=\pi i T(2n+1)} \int d^{3-\epsilon}\mathbf{Q}\, F^{\mu_1\mu_2\cdots\mu_N}(Q^0, \mathbf{Q}, \mathbf{k}_i, k_i^0). \tag{1}$$

Here M is the UV renormalization scale, $k_i^0/2\pi i T$ are integers and n runs over all integers. For fixed n, the Q-integral is UV finite, having no poles at $\epsilon = 0$.

Our argument requires the identification of those terms which can yield a pole at $\epsilon = 0$, when performing in (1) the summation over the frequencies $Q^0 = i\pi T(2n+1)$. To this end, we consider the high temperature limit

$$T >> |\mathbf{k}_i|, m \tag{2}$$

(m is the electron mass) and examine a relevant contribution involving a sum like [8]

$$S = T \sum_{n=-\infty}^{\infty} \frac{1}{|Q^0 + k^0|^{1+\epsilon}}. \tag{3}$$

Here k^0 is some linear combination of the external energies, with integral coefficients. We can thus set: $k^0 = 2\pi i l T$, where l is some integer. Considering the contributions from the regions $|n| < |l|$ and $|n| \geq |l|$, we obtain in the limit $\epsilon \to 0$ the expression

$$\begin{aligned} S = \frac{T^{-\epsilon}}{(2\pi)^{1+\epsilon}} \Big[& 2\Psi(|l+1/2|) - 2\Psi(|l+1/2| - |l|) + \frac{1}{|l+1/2|} + \\ & \zeta(1+\epsilon, l+1/2) + \zeta(1+\epsilon, -l-1/2) \Big] \end{aligned} \tag{4}$$

where Ψ is the Euler psi-function and $\zeta(\alpha, z)$ denotes the Riemann zeta-function [9]

$$\zeta(\alpha, z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^\alpha}. \tag{5}$$

The only singular point of this function occurs as $\alpha \to 1$, where it obeys the relation

$$\zeta(1+\epsilon, z) = \frac{1}{\epsilon} - \Psi(z). \tag{6}$$

Hence, using Eq. (4), we see that the divergent part of S is given simply by

$$S_\epsilon = \frac{1}{\pi\epsilon}. \tag{7}$$

This contribution arises from the summation over the region $|n| >> |l|$, i.e. where $|Q_0| >> |k_0|$. It is associated with the leading term obtained by expanding (3) in powers of k^0/Q^0.

Consequently, for the purpose of evaluating the pole part of the complete amplitude, we *can* expand the integral in (1) in powers of Q_0^{-1}. Proceeding systematically in this way, and identifying all terms proportional to $(Q^0)^{-1-\epsilon}$, we obtain contributions of the form

$$A^{\mu_1\mu_2\cdots\mu_N}(\mathbf{k}_i, k_i^0, T) = \sum_{n=-\infty}^{\infty} \left[\left(\frac{M}{T}\right)^\epsilon f^{\mu_1\mu_2\cdots\mu_N}(\mathbf{k}_i, k_i^0, \epsilon) \frac{1}{|2n+1|^{1+\epsilon}} + \mathcal{O}(T^{-1-\epsilon} n^{-2-\epsilon}) \right]. \tag{8}$$

If $f^{\mu_1\mu_2\cdots\mu_N}(\mathbf{k}_i, k_i^0, 0)$ is nonzero, this sum diverges and the corresponding zero-temperature Euclidean field theory would be UV divergent, having a pole like $1/\epsilon$. In this case we obtain

$$A_\epsilon^{\mu_1\mu_2\cdots\mu_N}(k_i, T) = f^{\mu_1\mu_2\cdots\mu_N}(k_i, 0) \left[\frac{1}{\epsilon} + \log\left(\frac{M}{T}\right)\right]. \tag{9}$$

We note that the $\log(T)$ term always combines with the $\log(M)$ term to yield a $\log(M/T)$ contribution. Something similar happens for the photon self-energy in QED, or the gluon two-point function in QCD [8,10]. Since $f^{\mu_1\mu_2\cdots\mu_N}(k_i, 0)$ is gauge and Lorentz invariant, we expect the $\log(T)$ contributions to be Lorentz invariant quantities, despite the presence of the heat bath. This was argued explicitly in Ref. [4].

Because the Green functions with $N \geq 4$ external photon lines are UV convergent, the functions $f^{\mu_1\mu_2\cdots\mu_N}(k_i, 0)$ must vanish. Consequently, the $\log(T)$ terms should be absent in the non-linear electromagnetic interactions at high temperature.

We now verify by explicit computation, the cancellation of the $\log(T)$ contributions to the 6-point photon function, which is UV finite. For simplicity we consider here the long wavelength limit of the external photons, when $\mathbf{k_i} \to 0$.

The analytically continued imaginary-time formalism can be formulated [11,12] so as to express the thermal amplitude (having subtracted the zero-temperature part) in terms of amplitudes for forward scattering of a thermal electron or positron in an external electromagnetic field, as depicted in Fig. 2. There are 6! diagrams like this one, which are obtained by permutations of the external momenta and polarizations. The corresponding analytic expression has the form

$$\frac{e^6}{(2\pi)^3} \int_0^\infty \frac{q^2}{2q^0} \frac{\mathrm{d}q}{\exp(q^0/T)+1} \int \mathrm{d}\Omega \sum_{(\mathrm{ijklmn})} B^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}_{(\mathrm{ijklmn})}(k_1, k_2, k_3, k_4, k_5, k_6, Q). \tag{10}$$

Here $q = |\mathbf{Q}|$, $q^0 = (q^2 + m^2)^{1/2}$, $\int \mathrm{d}\Omega$ is an integral over the directions of $\mathbf{Q}$, and the sum is over the permutations (ijklmn) of (123456). Each B has a numerator which is a Dirac trace containing projection operators $P(k) = [\gamma \cdot (Q+k) + m]$. For example

$$B^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}_{(123456)} = \frac{\mathrm{tr}\,[P(0)\ \gamma^{\mu_1}\ P(k_1)\ \gamma^{\mu_2}\ P(k_{12})\ \gamma^{\mu_3}\ \cdots\ P(k_{12345})\ \gamma^{\mu_6}]}{(2Q\cdot k_1 + k_1^2)(2Q\cdot k_{12} + k_{12}^2)\cdots(2Q\cdot k_{12345} + k_{12345}^2)}, \tag{11}$$

where $k_{12} = k_1 + k_2$, etc.

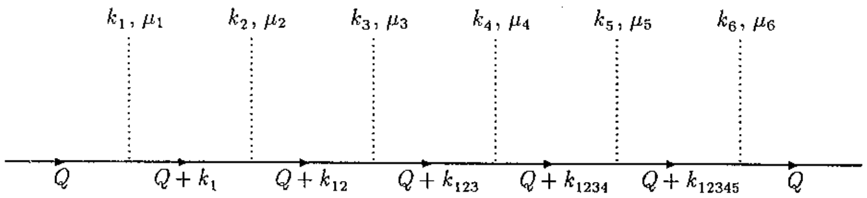

FIG. 2. A typical contribution to the forward scattering amplitude. Dotted lines represent photons, and solid lines stand for electrons or positrons.

In order to obtain the high-temperature limit of Eq. (10) we use the expansion

$$\left(2Q \cdot k + k^2\right)^{-1} = (2Q \cdot k)^{-1}\left[1 - k^2(2Q \cdot k)^{-1} + k^4(2Q \cdot k)^{-2} + \cdots\right] \tag{12}$$

in (11). For our purposes, we can neglect the electron mass in (10). One thus gets in Eq. (11) terms which are homogeneous in Q of degrees 1, 0, -1, -2. The expansion cannot be taken further without introducing spurious infra-red divergences. Since for each term in the sum in Eq. (10) there is a corresponding one with $Q \leftrightarrow -Q$, terms which are odd in Q will cancel. The terms of degree 0 would produce $\mathcal{O}(T^2)$. However, as we have already mentioned, these contributions cancel as a consequence of gauge invariance. We have checked explicitly this cancellation at the integrand level in Eq. (10).

Terms of degree -2 would produce possible $\log(T)$ contributions. The explicit computation of this logarithmic contribution is lengthy and can only be performed using a computer program for symbolic manipulations. Even with the help of a computer, one would find very difficult angular integrals for general values of the external photon momenta. In the long wavelength limit, one has to consider only the space components of Eq. (10), because any other component must vanish by gauge invariance. The angular integrations can then be easily computed. There are in fact no angular dependences left in Eq. (11) apart from simple numerators like $Q^{i_1} Q^{i_2} \cdots Q^{i_n}$, where $n = 0, 2, 4, 6$ and $i_1, \cdots, i_n$ represent the space directions. Even in this relatively simple case the number of terms resulting from the expansion of a single contribution such as (11) is 975. After adding the 6! permutations of all

the above contributions, and using the conservation of the photon momenta, we obtain zero.

In conclusion, we discuss the finite contributions to the effective thermal action at high temperature. To this end, we convert the frequency sum in the complete amplitude (1) to contour integrals, expressed at zero chemical potential as [7]

$$A(\mathbf{k}_i, k_i^0, T) = \frac{1}{2\pi i}\int_{-i\infty+\delta}^{i\infty+\delta} dQ^0 \left(I(Q^0, \mathbf{k}_i, k_i^0) + I(-Q^0, \mathbf{k}_i, k_i^0)\right)\left[\frac{1}{2} - \frac{1}{\exp(Q^0/T) + 1}\right], \tag{13}$$

where I is given by the $\mathbf{Q}$-integral in Eq. (1), and we have omitted for simplicity of notation all Lorentz indices. The first term in the square bracket of (13) leads to the zero temperature Euclidean amplitude, while the second one contains the temperature-dependent Fermi distribution $N(Q^0/T)$.

We now evaluate the Q^0-integral by residues in terms of the poles inside the contour. In general, these poles will be situated at $\left[(\mathbf{Q} + \mathbf{k})^2 + m^2\right]^{1/2} + k^0$, where $\mathbf{k}$ and k^0 denote some linear combinations with integral coefficients, of the external momenta and energies. An important simplification occurs in the long wavelength limit $\mathbf{k}_i \to 0$, when the contributions from all poles to the Fermi distribution factors become equal to

$$N\left((q^0 + k^0)/T\right) = N(q^0/T) = \frac{1}{\exp(q^0/T) + 1}, \tag{14}$$

where $q^0 = (\mathbf{Q}^2 + m^2)^{1/2}$. Denoting by R_+ the residues at the poles of the function $F(Q^0, \mathbf{Q}, k_i^0) + F(-Q^0, \mathbf{Q}, k_i^0)$ (see Eq. (1)), we get from (13) the result

$$A(k_i^0, T) = \int d^3\mathbf{Q} R_+\left(\mathbf{Q}, q^0, k_i^0\right)\left[-\frac{1}{2} + \frac{1}{\exp(q^0/T) + 1}\right]. \tag{15}$$

Here we have set $\epsilon = 0$, since the nonlinear Green functions are UV finite. At this point, we can analytically continue the complete amplitude to general values of the energies k_i^0.

We use next the important fact that the thermal contributions have a finite limit as $T \to \infty$. Thus, we can take in (15) the limit $N(q^0/T) \to 1/2$, without affecting the convergence of the $\mathbf{Q}$-integral. It follows that in this case, the nonlinear thermal action evaluated in the long wavelength limit, approaches the negative of the zero-temperature action. In this limit, we expect important effects due the collective behavior of the thermal medium.

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