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2+1 POINCARE GROUP, RELATIVISTIC WAVE EQUATIONS, COHERENT STATES, AND SO ON

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Abstract

Poincare group in 2 + 1-dimensional space-time is considered. An explicit realization of all unitary irreducible representations of the group is constructed, using the generalized regular representation. On this base one presents relativistic wave equations for higher spins (including fractional). A detailed description of angular momentum and spin in 2 + 1 dimensions is given and corresponding coherent states are constructed.

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I. INTRODUCTION

At the present time a great attention is devoted to field theoretical models in 2 + 1-dimensional space-time [1]. In the space there is a possibility to exist particles with fractional spin and exotic statistics. Such particles, which are called anyons, have an interest in connection with different applications in physics of planar phenomenon. One can mention in this connection quantum Hall effect and high temperature conductivity [2].

The corresponding Poincare group, which will be further denoted as $M(2,1)$, is the semi-direct product of the translation group $T(3)$ and the rotation group $SO(2,1)$, $M(2,1) = T(3) \times) SO(2,1)$. It was studied in [3] and from the the field theoretical point of view in [4].

Significance of the investigation of $M(2,1)$ is also stressed by the fact that, being a subgroup of the Poincare group in 3 + 1 dimensions $M(3,1)$, it retains many properties of the latter. In this connection, a part of results, which can be derived for $M(2,1)$, may also be valid for $M(3,1)$. One has to remark that in contrast with $M(1,1)$, discussed in details in [5], $M(2,1)$ has a non-Abelian and non-compact subgroup of rotations, similar to $M(3,1)$, that leads to a nontrivial structure of spinning space.

Usually, doing classification of representations of semi-direct products, they are using the method of little group (see for example [6,7]). That method was also applied to $M(2,1)$ [3]. Nevertheless, for our purposes of detailed and explicit construction of representations it is more convenient to use the method of harmonic analysis and method of generalized regular representation (GRR). It is known that any irreducible representation (IR) of a Lie group is equivalent to a subrepresentation of left (right) GRR [8–10].

In this work we present an explicit description of all unitary IR of $M(2,1)$. On this base we construct relativistic wave equations for higher spins (including fractional), and corresponding coherent states. To this end we are studying the left GRR in the parameterization where the rotations are given by two complex numbers z_1 and z_2, $|z_1|^2 - |z_2|^2 = 1$, which are analogs of Cayley-Klein parameters of the compact case. The representation space consists of scalar functions $f(x, z)$, whereas the spinning operators can be presented as first order differential operators in the variables z. It is convenient to classify representations not only with respect to the Casimir operators $\hat{p}^2$ and $\hat{p}\hat{J}$, but also with respect to the operator of square of the spin $\hat{S}^2$, which commutes with all generators of the left GRR.

In the frame of such an approach one can naturally construct relativistic wave equations for particles with arbitrary spin. The fixation of the value of the square of the spin $S(S+1)$ defines the structure of z-dependence of the functions $f(x,z)$, namely, they appear to be (quasi-)polynomials of the power $2S$ on z. Coefficients of these polynomials are interpreted as components of finite(infinite)-dimensional wave functions of relativistic particles with higher spins. Fixation of the values of Casimir operators produces equations for these components. In such a way, for example, $2+1$ Dirac equation for a particle with spin projection $1/2$ appears. In the same way one can write explicitly equations for higher spins and equations for functions, which are transformed under infinite-dimensional unitary representations of 2+1 Lorentz group $SO(2,1)$, and which correspond, in particular, to particles with fractional spins, i.e. to anyons.

A detailed description of angular momentum and spin in $2+1$ dimensions is given on the base of the representation theory of $SU(1,1)$. In particular, multivalued unitary IR of $SO(2,1) \sim SU(1,1)$ and corresponding coherent states (CS) are considered. It is interesting to discover that $2+1$ Dirac equation appears also in the latter case as an equation for CS evolution.

II. PARAMETERIZATION

$M(2,1)$ is a six-parametric group of motions of $2+1$-dimensional pseudo-Euclidean space, it preserves the interval $\eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$, where $x=(x^\mu)$, $\mu=0,1,2$, are coordinates, and $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1)$ is Minkovski tensor. The transformation of the vector x under the action of the group (vector representation) is given by the formula

$$x' = gx, \quad g \in M(2,1), \quad x'^\nu = \Lambda^\nu_\mu x^\mu + a^\nu, \tag{2.1}$$

where Λ is a 3×3 rotation matrix of $SO(2,1)$. The transformations can also be presented in the four-dimensional form,

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ 1 \end{pmatrix} = \begin{pmatrix} & \Lambda(\alpha) & & a^0 \\ & & & a^1 \\ & & & a^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ 1 \end{pmatrix}, \tag{2.2}$$

with the composition low $(a_2,\Lambda_2)(a_1,\Lambda_1) = (a_2+\Lambda_2 a_1, \Lambda_2\Lambda_1)$.

Different parameterizations of the rotations are possible. For example, by means of three angles α_μ,

$$\Lambda(\alpha) = \exp(i\alpha J), \quad [J^\mu, J^\nu] = i\epsilon^{\mu\nu\eta} J_\eta,$$
$$J^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad J^1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.3}$$

where $\epsilon^{\mu\nu\eta}$ is totally antisymmetric Levi-Civita tensor. The one-parametrical subgroups can be written as

$$\exp(i\alpha_0 J^0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_0 & -\sin\alpha_0 \\ 0 & \sin\alpha_0 & \cos\alpha_0 \end{pmatrix}, \quad \exp(i\alpha_1 J^1) = \begin{pmatrix} \cosh\alpha_1 & 0 & -\sinh\alpha_1 \\ 0 & 1 & 0 \\ -\sinh\alpha_1 & 0 & \cosh\alpha_1 \end{pmatrix},$$
$$\exp(i\alpha_2 J^2) = \begin{pmatrix} \cosh\alpha_2 & -\sinh\alpha_2 & 0 \\ -\sinh\alpha_2 & \cosh\alpha_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.4}$$

It is also possible to write finite transformations by means of matrices from $SL(2,R)$ [3].

There exists another possibility to write finite transformations of $M(2,1)$, which is similar to the cases of $M(3)$ and $SO(3)$. Remember [10-14] that the transformations from the group $M(3)$ can be presented in the form $X' = UXU^{-1} + A$, where $X = \mathbf{i}x^1 + \mathbf{j}x^2 + \mathbf{k}x^3$, $A = \mathbf{i}a^1 + \mathbf{j}a^2 + \mathbf{k}a^3$ are vectorial quaternions, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $X^2 = -\sum (x^i)^2$, $i = 1,2,3$, and U is an unimodular quaternion, $|U| = 1$. If one presents the quaternions by complex 2×2 matrices, then $\mathbf{i} = i\sigma^1$, $\mathbf{j} = i\sigma^2$, $\mathbf{k} = i\sigma^3$, where σ^i are Pauli matrices, and the transformations can be written in the form

$$
\begin{pmatrix} ix'^3 & x'^2 + ix'^1 \\ -x'^2 + ix'^1 & -ix'^3 \end{pmatrix} = \begin{pmatrix} ia^3 & a^2 + ia^1 \\ -a^2 + ia^1 & -ia^3 \end{pmatrix}
$$
$$
+ \begin{pmatrix} u_1 & u_2 \\ -\overline{u}_2 & \overline{u}_1 \end{pmatrix} \begin{pmatrix} ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & -ix^3 \end{pmatrix} \begin{pmatrix} \overline{u}_1 & -u_2 \\ \overline{u}_2 & u_1 \end{pmatrix} ,
$$
$$
|u_1|^2 + |u_2|^2 = 1, \; u_1 = \cos(\theta/2)e^{-i(\varphi+\omega)/2}, \; u_2 = \sin(\theta/2)e^{i(\varphi-\omega)/2},
$$
$$
0 \le \theta \le \pi, \quad -2\pi \le \varphi < 2\pi, \quad 0 \le \omega < 2\pi \, . \tag{2.5}
$$

where u_1 and u_2 are Cayley-Klein parameters, and φ, θ, ω are Euler angles.

For $M(2,1)$ the transformations (2.1) can be written in a similar form,

$$
X' = UXU^{-1} + A \, , \tag{2.6}
$$

$X = \mathbf{i}x^1 + \mathbf{j}x^2 + \mathbf{k}x^0$, $A = \mathbf{i}a^1 + \mathbf{j}a^2 + \mathbf{k}a^0$, $\mathbf{k}^2 = 1$, $\mathbf{i}^2 = \mathbf{j}^2 = -1$, $X^2 = x_\mu x^\mu$. In this case U are matrices from $SU(1,1)$. Clear that A is responsible for translations and U for Lorentz rotations. If we present $\mathbf{i}, \mathbf{j}, \mathbf{k}$ by 2×2 matrices, $\mathbf{i} = i\sigma^1$, $\mathbf{j} = i\sigma^2$, $\mathbf{k} = \sigma^3$, then

$$
A = \begin{pmatrix} a^0 & a^2 + ia^1 \\ -a^2 + ia^1 & -a^0 \end{pmatrix}, \qquad U = \begin{pmatrix} u_1 & u_2 \\ \overline{u}_2 & \overline{u}_1 \end{pmatrix} ,
$$

and the transformations (2.6) can be written in the form

$$
\begin{pmatrix} x'^0 & x'^2 + ix'^1 \\ -x'^2 + ix'^1 & -x'^0 \end{pmatrix} = \begin{pmatrix} a^0 & a^2 + ia^1 \\ -a^2 + ia^1 & -a^0 \end{pmatrix}
$$
$$
+ \begin{pmatrix} u_1 & u_2 \\ \overline{u}_2 & \overline{u}_1 \end{pmatrix} \begin{pmatrix} x^0 & x^2 + ix^1 \\ -x^2 + ix^1 & -x^0 \end{pmatrix} \begin{pmatrix} \overline{u}_1 & -u_2 \\ -\overline{u}_2 & u_1 \end{pmatrix} ,
$$
$$
|u_1|^2 - |u_2|^2 = 1, \; u_1 = \cosh(\theta/2)e^{-i(\varphi+\omega)/2}, \; u_2 = \sinh(\theta/2)e^{i(\varphi-\omega)/2},
$$
$$
0 \le \theta < \infty, \quad -2\pi \le \varphi < 2\pi, \quad 0 \le \omega < 2\pi \, . \tag{2.7}
$$

Here (for $SO(2,1)$) u_1 and u_2 are analogs of Cayley-Klein parameters, and φ, θ, ω are ones of Euler angles. It is possible to see that $U(\varphi,\theta,\omega) = U(\varphi,0,0)U(0,\theta,0)U(0,0,\omega)$, i.e. the general transformation can be presented as ω-rotation around the axis x^0, then the θ-rotation around the axis x^1, and again φ-rotation around the axis x^0. The following sets of the parameters (φ,θ,ω): $(\alpha_0,0,0)$, $(0,\alpha_1,0)$, $(\pi/2,\alpha_2,-\pi/2)$ correspond to the one parametrical subgroups (2.4) with the angles $(\alpha_0,0,0)$, $(0,\alpha_1,0)$, $(0,0,\alpha_2)$ respectively. The

A matrix from (2.1) in Euler angles parameterization can be presented as $\Lambda(\varphi,\theta,\omega) = \Lambda(\varphi,0,0)\Lambda(0,\theta,0)\Lambda(0,0,\omega)$.

One can remark that U and $-U$ correspond to one and the same rotation in $2+1$-dimensional space, so that to parameterize the rotations it is enough to use $\varphi \in [0,2\pi]$, the matrices U in (2.7) belong to the fundamental spinor IR of $SU(1,1)$, which is the double covering of $SO(2,1)$.

Further we are going to use the latter parameterization of elements of $M(2,1)$ by means of matrices A (vector a) and $SU(1,1)$ matrices U, $g = (A,U)$, where the composition low and inverse elements have the form

$$
\begin{aligned}
g &= (A,U) = (A_2,U_2)(A_1,U_1) = (U_2A_1U_2^{-1} + A_2, \; U_2U_1) \, , \\
g^{-1} &= (-U^{-1}AU, \; U^{-1}) \, .
\end{aligned} \tag{2.8}
$$

III. QUASI-REGULAR REPRESENTATION

Let us consider a quasi-regular representation $T(g)$, which is acting on the homogeneous space $M(2,1)/SO(2,1)$, i.e. in the space of functions $f(x)$,

$$
f'(x) = T(g)f(x) = f(g^{-1}x). \tag{3.1}
$$

It is easy to remark that the representation (3.1) corresponds to a scalar field, $f'(gx) = f'(x') = f(x)$. The explicit form of $g^{-1}x$ is given by the formulas

$$
(g^{-1}x)^\nu = (\Lambda^{-1})^\nu_\mu(x^\mu - a^\mu), \quad g^{-1}x = U^{-1}(X-A)U, \tag{3.2}
$$

in the parameterizations (2.1) and (2.6) respectively. Lie algebra of $M(2,1)$ contains six generators $\hat{p}_\mu$ and $\hat{L}^\mu$, which correspond to the parameters a^μ and α_μ. They have a form

$$
\hat{p}_\mu = i\partial/\partial x^\mu, \quad \hat{L}^\eta = \epsilon^{\eta\mu\nu}\hat{x}_\mu\hat{p}_\nu = i\epsilon^{\eta\mu\nu}x_\mu\partial/\partial x^\nu, \tag{3.3}
$$

in the representation in question, and obey the commutation relations

$$[\hat{p}_\mu,\hat{p}_\nu]=0\,,\quad [\hat{L}^\mu,\hat{p}^\nu]=i\epsilon^{\mu\nu\eta}\hat{p}_\eta\,,\quad [\hat{L}^\mu,\hat{L}^\nu]=i\epsilon^{\mu\nu\eta}\hat{L}_\eta\,. \tag{3.4}$$

Finite transformations in the parameterizations (2.3) and (2.7) can be written as

$$T(g)f(x)=e^{-i\alpha_2\hat{L}^2-i\alpha_1\hat{L}^1-i\alpha_0\hat{L}^0}e^{-ia\hat{p}}f(x), \tag{3.5}$$

$$T(g)f(x)=e^{-i\varphi\hat{L}^0}e^{-i\theta\hat{L}^1}e^{-i\omega\hat{L}^0}e^{-ia\hat{p}}f(x). \tag{3.6}$$

The eigenvalue m^2 of the Casimir operator $\hat{p}^2$ can, in particular, characterize the IR, $\hat{p}^2 f_m(x)=m^2 f_m(x)$. For unitary representations, where the generators $\hat{p}_\mu$ and $\hat{L}^\mu$ are hermitian, m^2 is real. It follows from the commutation relations (3.4) that $\hat{p}\hat{L}$ is also a Casimir operator, which is, however, zero in the representation under consideration.

To find all IR, which are contained in the representation (3.1), we go over to the space of functions dependent on momenta, doing Fourier transformation,

$$\tilde{f}(p)=(2\pi)^{-3/2}\int f(x)e^{ipx}dx. \tag{3.7}$$

In this space expressions for the generators have the form

$$\hat{p}_\mu=p_\mu,\quad \hat{L}^\eta=\epsilon^{\eta\mu\nu}\hat{x}_\mu p_\nu=i\epsilon^{\eta\mu\nu}p_\mu\partial/\partial p^\nu. \tag{3.8}$$

The form of $\hat{L}^\mu$ in the space of functions $\tilde{f}(p)$ coincides with one in the space of functions $f(x)$ if one replaces $p^\mu\to x^\mu$, and, therefore, the rotations result in: $\tilde{f}(p)\to\tilde{f}(p')$, where $p'_\mu=(\Lambda^{-1})_\mu^{\ \nu}p_\nu$. In the parameterization (2.6),

$$P'=U^{-1}PU,\quad P=\mathbf{i}p^1+\mathbf{j}p^2+\mathbf{k}p^0. \tag{3.9}$$

Translations affect only the phase of the functions, so we get an analog of eq.(3.1),

$$T(g)\tilde{f}(p)=e^{-iap'}\tilde{f}(p'),\quad p^2=(p')^2=m^2. \tag{3.10}$$

Representations with a given m we denote as $T_m(g)$. Below we consider three possible cases.

1. $m\neq 0$ and is real. In this case the representations $T_m(g)$ are acting in the space of functions on two sheet hyperboloid,

$$p_0=\pm m\cosh\theta,\quad p_1=\mp m\sinh\theta\cos\varphi,\quad p_2=\mp m\sinh\theta\sin\varphi. \tag{3.11}$$

At $m>0$ it is decomposed in two IR, one $T_m^+(g)$, which corresponds to particles (upper sheet, $p_0>0$), and another one $T_m^-(g)$, which corresponds to antiparticles (lower sheet, $p_0<0$). One can consider only IR with $m>0$ because of $T_m^+(g)$ and $T_{-m}^-(g)$ are equivalent. The scalar product at a fixed m is given by the equation

$$\langle f_1\mid f_2\rangle=\int_0^{2\pi}d\varphi\int_0^{+\infty}\overline{f_1(\theta,\varphi)}f_2(\theta,\varphi)\sinh\theta d\theta, \tag{3.12}$$

and the generators $\hat{L}^\mu$ have the form

$$\hat{L}^0=-i\partial_\varphi,\quad \hat{L}^1=-i(\coth\theta\cos\varphi\partial_\varphi+\sin\varphi\partial_\theta),$$
$$\hat{L}^2=i(-\coth\theta\sin\varphi\partial_\varphi+\cos\varphi\partial_\theta). \tag{3.13}$$

2. $m=0$. In this case the representations $T_m(g)$ are acting in the space of functions on the cone,

$$p_0=p,\quad p_1=-p\cos\varphi,\quad p_2=-p\sin\varphi. \tag{3.14}$$

The representation $T_0(g)$ is split into three IR: one-dimensional $T_0^0(g)$, which corresponds to the invariant $p=0$ (vertex of the cone), and $T_0^+(g)$ and $T_0^-(g)$, which are acting on the upper and lower sheets of the cone. The scalar product is given by the formula

$$\langle f_1\mid f_2\rangle=\int_0^{2\pi}d\varphi\int_0^{+\infty}\overline{f_1(p,\varphi)}f_2(p,\varphi)dp, \tag{3.15}$$

and the generators $\hat{L}^\mu$ have the form

$$\hat{L}^0=-i\partial_\varphi,\quad \hat{L}^1=i(\cos\varphi\partial_\varphi+p\sin\varphi\partial_p),\quad \hat{L}^2=i(-\sin\varphi\partial_\varphi+p\cos\varphi\partial_p). \tag{3.16}$$

3. m is imaginary, that corresponds to tachyons. The representations $T_m(g)$ are acting in the space of functions on one sheet hyperboloid,

$$p_0=im\ \sinh\theta,\quad p_1=-im\ \cosh\theta\cos\varphi,\quad p_2=-im\ \cosh\theta\sin\varphi. \tag{3.17}$$

The scalar product is given by the formula

$$\langle f_1 \mid f_2\rangle = \int_0^{2\pi} d\varphi \int_0^{+\infty} \overline{f_1(\theta,\varphi)} f_2(\theta,\varphi)\cosh\theta d\theta, \tag{3.18}$$

and the generators $\hat{L}^\mu$ have the form

$$\hat{L}^0 = -i\partial_\varphi, \quad \hat{L}^1 = -i(\tanh\theta\cos\varphi\partial_\varphi + \sin\varphi\partial_\theta),$$
$$\hat{L}^2 = i(-\tanh\theta\sin\varphi\partial_\varphi + \cos\varphi\partial_\theta). \tag{3.19}$$

IV. GENERALIZED REGULAR REPRESENTATION

In the previous section we have considered the quasi-regular representation, which produces description of scalar fields or spinless particles. To get a complete picture of all possible representations one has to turn to so called generalized regular representation (GRR) [8–10]. The GRR is acting in the space of functions $f(g)$ on the group, $g \in G$. The left GRR $T_L(g)$ and the right GRR $T_R(g)$ are defined as

$$T_L(g)f(g_0) = f(g^{-1}g_0), \tag{4.1}$$
$$T_R(g)f(g_0) = f(g_0 g). \tag{4.2}$$

It is known that any IR of a group is equivalent to one of subrepresentations of the left (right) GRR [8]. Taking this into account, we are going to the construct GRR of $M(2,1)$ in the parameterization (2.6), where we put $g_0 = (X,Z)$, and $g = (A,U)$,

$$X = \begin{pmatrix} x^0 & x^2 + ix^1 \\ -x^2 + ix^1 & -x^0 \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 & z_2 \\ \bar{z}_2 & \bar{z}_1 \end{pmatrix},$$
$$A = \begin{pmatrix} a^0 & a^2 + ia^1 \\ -a^2 + ia^1 & -a^0 \end{pmatrix}, \quad U = \begin{pmatrix} u_1 & u_2 \\ \bar{u}_2 & \bar{u}_1 \end{pmatrix}. \tag{4.3}$$

Using the composition law (2.8), one can get

$$T_L(g)f(X,Z) = f((A,U)^{-1}(X,Z)) = f(U^{-1}(X-A)U, U^{-1}Z), \tag{4.4}$$
$$T_R(g)f(X,Z) = f((X,Z)(A,U)) = f(X + ZAZ^{-1}, ZU). \tag{4.5}$$

According to (4.4), X is transformed with respect to the adjoined (vectorial) representation and Z with respect to the spinor representation of $SO(2,1)$. One can also see that Z is

invariant under translations. If one restricts itself by Z-independent functions (i.e. by the functions on the homogeneous space $M(2,1)/SO(2,1)$), then (4.4) reduces to the quasi-regular representation (3.1), which corresponds to a scalar field. If one restricts itself by X-independent functions, then (4.4) and (4.5) reduce to the left and the right GRR of $SU(1,1)$.

Calculating generators, which correspond to the parameters a^μ and α_μ, (see (2.4)), in the left GRR (4.4), we get

$$\hat{p}_\mu = i\partial/\partial x^\mu, \quad \hat{J}^\mu = \hat{L}^\mu + \hat{S}^\mu, \tag{4.6}$$

where $\hat{L}^\mu$ are angular momentum operators (3.3), and $\hat{S}^\mu$ are spin operators,

$$\hat{S}^0 = -(1/2)(z_1\partial/z_1 - \bar{z}_2\partial/\bar{z}_2) + (1/2)(\bar{z}_1\partial/\bar{z}_1 - z_2\partial/z_2),$$
$$\hat{S}^1 = -(i/2)(z_1\partial/\bar{z}_2 + \bar{z}_2\partial/z_1) - (i/2)(z_2\partial/\bar{z}_1 + \bar{z}_1\partial/z_2),$$
$$\hat{S}^2 = -(1/2)(z_1\partial/\bar{z}_2 - \bar{z}_2\partial/z_1) + (1/2)(\bar{z}_1\partial/z_2 - z_2\partial/\bar{z}_1),$$
$$[\hat{S}^\mu, \hat{S}^\nu] = i\epsilon^{\mu\nu\eta}\hat{S}_\eta, \quad [\hat{S}^\mu, \hat{p}_\nu] = 0. \tag{4.7}$$

The algebra of the generators has the form

$$[\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{J}^\mu, \hat{p}^\nu] = i\epsilon^{\mu\nu\eta}\hat{p}_\eta, \quad [\hat{J}^\mu, \hat{J}^\nu] = i\epsilon^{\mu\nu\eta}\hat{J}_\eta\,. \tag{4.8}$$

Generators of the right GRR we denote by the same letters but underlined below. The generators $\underline{\hat{J}}^\mu$ do not depend on x and are only expressed in terms of z,

$$\underline{\hat{p}}_\mu = -(\Lambda^{-1})^\nu_\mu \hat{p}_\nu \quad (\text{or } \underline{\hat{P}} = -Z^{-1}\hat{P}Z), \qquad \underline{\hat{J}}^\mu = \underline{\hat{S}}^\mu, \tag{4.9}$$

where

$$\underline{\hat{S}}^0 = (1/2)(z_1\partial/z_1 - z_2\partial/z_2) - (1/2)(\bar{z}_1\partial/\bar{z}_1 - \bar{z}_2\partial/\bar{z}_2),$$
$$\underline{\hat{S}}^1 = (i/2)(z_1\partial/z_2 + z_2\partial/z_1) + (i/2)(\bar{z}_1\partial/\bar{z}_2 + \bar{z}_2\partial/\bar{z}_1),$$
$$\underline{\hat{S}}^2 = -(1/2)(z_1\partial/z_2 - z_2\partial/z_1) + 1/2(\bar{z}_1\partial/\bar{z}_2 - \bar{z}_2\partial/\bar{z}_1). \tag{4.10}$$

All the right generators commute with all the left generators and obey the same commutation relations (4.8). The operator $\hat{p}^2 = \hat{\underline{p}}^2$ and Pauli-Lubanski operator $\hat{J}\hat{p} = \hat{\underline{J}}\hat{\underline{p}}$ are Casimir operators. Thus, IR of $M(2,1)$ can be marked by their eigenvalues.

It follows from (3.3) that $\hat{L}\hat{p} = 0$, so that always $\hat{J}\hat{p} = \hat{S}\hat{p}$. The operator $\hat{J}\hat{p}$ commutes with the total angular momentum operators $\hat{J}^\mu = \hat{L}^\mu + \hat{S}^\mu$, but not with the orbital momentum operators $\hat{L}^\mu$ and spin operators $\hat{S}^\mu$ separately. The operator of spin's square[1] $\hat{\mathbf{S}}^2 = \hat{\underline{\mathbf{J}}}^2$ commutes with all the generators of the left GRR. That means that objects, which are transformed under the left GRR or under its subrepresentations, can also be marked by eigenvalues of this operator. However, that operator does not commute with the generators $\hat{\underline{p}}_\mu$ of the right GRR, $[\hat{\underline{\mathbf{J}}}^2, \hat{\underline{p}}^\mu] = i\epsilon^{\mu\nu\eta}(\hat{\underline{p}}_\eta \hat{\underline{J}}_\nu + \hat{\underline{J}}_\nu \hat{\underline{p}}_\eta)$, similar to the left GRR case, $[\hat{\mathbf{J}}^2, \hat{p}^\mu] = i\epsilon^{\mu\nu\eta}(\hat{p}_\eta \hat{J}_\nu + \hat{J}_\nu \hat{p}_\eta)$. Thus, the spin is not a conserved quantity in all the right GRR, but $\hat{\mathbf{J}}^2$ is.

Remark that the left and the right GRR are equivalent, $\hat{C}T_R(g) = T_L(g)\hat{C}$, where $\hat{C}f(g_0) = f(g_0^{-1})$. Because of that, and also because of the left representations are more adequate to describe physical fields, we are going to consider in detail only the left GRR of $M(2,1)$.

Making Fourier transformation in the variables x, i.e. considering representations in the space of functions $f(P,Z)$,

$$P = \begin{pmatrix} p_0 & -p_2 - ip_1 \\ p_2 - ip_1 & -p_0 \end{pmatrix},$$

one can get an analog of formulas (4.1,4.2) in this representation,

$$T_L(g)f(P,Z) = e^{-iap'}f(U^{-1}PU, U^{-1}Z), \qquad P' = U^{-1}PU, \tag{4.11}$$

$$T_R(g)f(P,Z) = e^{-ia'p}f(P, ZU), \qquad A' = ZAZ^{-1}. \tag{4.12}$$

It is seen that the combination $|z_1|^2 - |z_2|^2$ and p^2 are preserved under the transformations (4.11) and (4.12). The former is always equal to 1 and the latter to m^2, and depends on the

[1]Here and in what follows $\mathbf{S}^2 = S_\mu S^\mu$ and so on.

representation.

The classification of orbits with respect to the eigenvalues of the operator $\hat{p}^2$ is completely similar to one was done in Sect.3 for spinless case. These are orbits $O_m^\pm$ for real $m \neq 0$, $O_0^\pm$ and O_0^0 for $m = 0$, and finally O_m for imaginary m. However, to describe IR it is not enough only one parameter m, one needs to know characteristics connected with the spin. Z and P are defined by six real parameters. Three of them (the mass m and the momentum direction, namely, $\underline{P} = -Z^{-1}PZ$ for the left GRR or P for the right GRR) are fixed and three are changed under group transformations (for the left GRR two of them set the direction of the momentum).

The variables z_1 and z_2, are transformed under the spinor representation of $SU(1,1)$. Moreover, IR of this group are at the same time ones of $M(2,1)$ connected with the orbit O_0^0. That is why we are going to describe below representations of the former group in detail.

V. UNITARY IR AND COHERENT STATES OF THE $SU(1,1)$ GROUP

Lorentz group $SO(2,1)$, and close related groups $SU(1,1)$ and $SL(2,R)$ with the same algebra, where investigated in numerous papers [8–10,15–32]. Their finite-dimensional IR and unitary IR (discrete series) are used to describe spin in 2 + 1-dimensions [4]. As is known, $SO(3,1)$ has only principal and supplementary series of unitary representations, and the principal series is used to describe spin in 3 + 1 dimensions [33,34]. In this connection, besides of all, it is important to consider the same series of $SO(2,1)$ or $SU(1,1)$.

We are going to describe unitary IR of $SU(1,1)$, their discrete bases and corresponding CS. The consideration, to be complete, is going to repeat some known results, but also to present some new ones. For example, we are constructing CS in unitary IR of the principal series at arbitrary fractional projections of the angular momentum, in addition to [30] where only integer ones were considered. We construct unitary IR, including multivalued, in spaces of functions on various manifolds connected with $SO(2,1)$ or $SU(1,1)$, whereas usually they restrict themselves to the unit disk or to a circle. In particular, we consider decompositions of functions on a cone and one sheet hyperboloid with respect to unitary IR of $SO(2,1)$.

Consider a representation $T(g)$, $g \in SU(1,1)$, acting in the space of functions $f(v)$ defined on the columns $v = \{v_1, v_2\}$,

$$T(g)f(v) = f(v'), \quad v' = g^{-1}v\,. \tag{5.1}$$

The matrices g^{-1} can be parameterized by two complex numbers u^1, u^2,

$$g^{-1} = \begin{pmatrix} u^1 & u^2 \\ \bar{u}^2 & \bar{u}^1 \end{pmatrix}, \quad |u^1|^2 - |u^2|^2 = 1, \tag{5.2}$$

Taking into account that the combination

$$|v_1'|^2 - |v_2'|^2 = |v_1|^2 - |v_2|^2 = C \tag{5.3}$$

remains invariant, one can use C to specify different subrepresentations. Generators, which correspond to one-parametrical subgroups (see (2.4), have the following form in this representation

$$\hat{J}^0 = -(1/2)(v_1\partial/v_1 - v_2\partial/v_2), \quad \hat{J}_- = v_1\partial/v_2, \quad \hat{J}_+ = v_2\partial/v_1, \tag{5.4}$$
$$\hat{J}^1 = -(i/2)(\hat{J}_+ + \hat{J}_-) = -(i/2)(v_1\partial/v_2 + v_2\partial/v_1),$$
$$\hat{J}^2 = (1/2)(\hat{J}_+ - \hat{J}_-) = (1/2)(v_2\partial/v_1 - v_1\partial/v_2).$$

They obey the commutation relations (4.8), so that $\hat{\mathbf{J}}^2 = \hat{J}_\mu \hat{J}^\mu$ is a Casimir operator.

Let us take functions of the form $f_{n_1n_2}(v) = v_1^{n_1}v_2^{n_2}$. The action of the generators on these functions can be found[2],

$$\hat{J}^0 f_{n_1n_2} = m f_{n_1n_2}, \quad \hat{\mathbf{J}}^2 f_{n_1n_2} = j(j+1)f_{n_1n_2}, \quad m = \frac{n_2 - n_1}{2}, \; j = \frac{n_1 + n_2}{2},$$
$$\hat{J}_- f_{n_1n_2} = n_2 f_{n_1+1,n_2-1}, \quad \hat{J}_+ f_{n_1n_2} = n_1 f_{n_1-1,n_2+1}. \tag{5.5}$$

Thus, all quasi-polynomials of the power $2j$ form a IR space (j characterizes the IR). $\hat{J}_+$ and $\hat{J}_-$ are arizing and lowering operators for the projection of the angular momentum

[2]We are going to use here the notation m for the anglular momentum projection (the same was used for the mass), hoping that this will not lead to a misunderstanding.

$m = (n_2 - n_1)/2$. If $n_2 \geq 0$ and is integer then $f_{n_1n_2}$ belongs to IR, which has the lowest weight v_1^{2j}; if $n_1 \geq 0$, and is integer then IR has the highest weight v_2^{2j}; if both $n_i \geq 0$, $i = 1,2$, and are integer then IR in finite-dimensional (has both the highest and lowest weights). For unitary IR of $SU(1,1)$: $(\hat{J}^0)^+ = \hat{J}^0$, $\hat{J}_\pm^+ = -\hat{J}_\mp$, that means $n_2 - n_1$ is real, and $n_1(n_2+1) \leq 0$, $n_2(n_1+1) \leq 0$, whereas for IR of $SU(2)$: $\hat{J}_\pm^+ = \hat{J}_\mp$ and $n_1(n_2+1) \geq 0$, $n_2(n_1+1) \geq 0$, [32]. At a given j one can select

$$N_{n_1n_2} v_1^{n_1} v_2^{n_2}, \tag{5.6}$$

as elements of a discrete basis in the space of functions $f_j(v)$, where $N_{n_1n_2}$ is the normalization constant, and $n_1 = j - m$, $n_2 = j + m$.

A classification and weight structure of unitary infinite-dimensional and non-unitary finite-dimensional IR of $SU(1,1)$ is presented on the Fig.1.

FIGURES

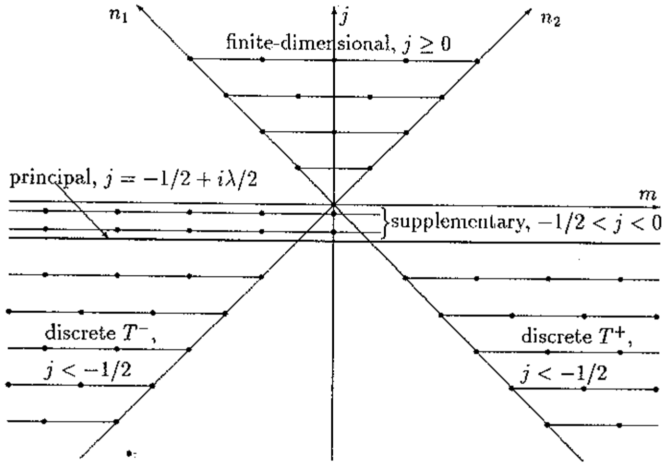

FIG. 1. Weight diagrams for unitary and finite-dimensional IR of $SU(1,1)$

To describe IR of different series one has to define in more details the space of functions $f(v)$. At different C in eq.(5.3) one can use the following parameterization of v_1 and v_2 :

$$C = 0: \quad v_1 = \rho e^{i(\varphi+\omega)/2}, \quad v_2 = \rho e^{i(\omega-\varphi)/2},$$
$$0 < \rho < +\infty, \quad 0 \leq \varphi < 4\pi, \quad 0 \leq \omega < 2\pi; \tag{5.7}$$
$$C = 1: \quad v_1 = \cosh(\theta/2) e^{i(\varphi+\omega)/2}, \quad v_2 = \sinh(\theta/2) e^{i(\omega-\varphi)/2},$$
$$0 \leq \theta < +\infty, \quad 0 \leq \varphi < 4\pi, \quad 0 \leq \omega < 2\pi\,. \tag{5.8}$$

The case of negative C ($C = -1$) is reduced to (5.8) by the replacement $v_1 \to v_2$. The parameter ω is not changed under the group transformations in the case (5.7), thus, there are two complex manifolds, on which the group is acting transitive: the complex hyperboloid (5.8) and the cone,

$$C = 0: \; v_1 = \rho e^{i\varphi/2}, \;\; v_2 = \rho e^{-i\varphi/2}, \;\; 0 < \rho < +\infty, \;\; 0 \leq \varphi < 4\pi. \tag{5.9}$$

Using the components (v_1, v_2) of the spinor and the conjugate components $(\overline{v}_1, \overline{v}_2)$, one can

construct objects (x^0, x^1, x^2), which are transformed under three-dimensional vector IR with $j = 1$,

$$x^0 = (|v_1|^2 + |v_2|^2)/2, \quad x^1 = (\overline{v}_1 v_2 + v_1 \overline{v}_2)/2, \quad x^2 = (v_1 \overline{v}_2 - \overline{v}_1 v_2)/2i, \tag{5.10}$$
$$x^0 = v_1 v_2, \quad x^1 = (v_1^2 + v_2^2)/2, \quad x^2 = (v_1^2 - v_2^2)/2i. \tag{5.11}$$

The vectors (5.10) and (5.11) have the same transformation properties, because of the spinors (v_1, v_2) and $(\overline{v}_2, \overline{v}_1)$ are transformed equally. The latter can be easily checked, using the explicit form of the matrix (5.2). Substituting (5.9) into (5.10) or (5.11), we get the cone

$$x^0 = \rho^2, \quad x^1 = -\rho^2 \cos\varphi, \quad x^2 = -\rho^2 \sin\varphi, \qquad x_0^2 - x_1^2 - x_2^2 = 0. \tag{5.12}$$

Substituting (5.8) into (5.10), we get two sheets hyperboloid

$$x^0 = \cosh\theta, \quad x^1 = -\sinh\theta\cos\varphi, \quad x^2 = -\sinh\theta\sin\varphi, \quad x_0^2 - x_1^2 - x_2^2 = 1. \tag{5.13}$$

If v_k are periodic in φ with the period 4π, then x_μ are also periodic with the period 2π.

Let us turn first to IR of the discrete series T_j^+ ($m = -j, -j+1, -j+2, ...$) and T_j^- ($m = j, j-1, j-2, ...$), $j < -1/2$, the theory of which is quite similar to the one of the finite-dimensional IR. The IR T_j^+ and T_j^- can be realized in the space of functions $f(v)$, where v_1 and v_2 belong to the case (5.8). The scalar product of functions on the complex hyperboloid,

$$\langle f_1 | f_2 \rangle = \frac{1}{8\pi^2} \int \overline{f}_1 f_2 \delta(|v_1|^2 - |v_2|^2 - 1) d^2 v_1 d^2 v_2$$
$$= \frac{1}{8\pi^2} \int_0^{2\pi} d\omega \int_{-2\pi}^{2\pi} d\varphi \int_0^{\infty} \overline{f}_1 f_2 \sinh\theta d\theta, \;\; d^2 v = d\Re v \; d\Im v\,, \tag{5.14}$$

allows one to normalize the elements of the discrete basis T_j^+ at $j < -1/2$,

$$\psi_{j,m}(v) = \langle v \mid jm \rangle = \left(\frac{(-1)^{n_2} \Gamma(-n_1)}{n_2! \Gamma(-2j)} \right)^{1/2} v_1^{n_1} v_2^{n_2}$$
$$= \left(\frac{(-1)^{n_2} \Gamma(-n_1)}{n_2! \Gamma(-2j)} \right)^{1/2} (\cosh(\theta/2))^{n_1} (\sinh(\theta/2))^{n_2} e^{im(\varphi+4\pi k)} e^{ij(\omega+2\pi k)}. \tag{5.15}$$

The projection m, and therefore j ($j = m_{\max}$ in T_j^-, $j = -m_{\min}$ in T_j^+), have to run over the integer and half integer, $j = -1, -3/2, -2, \ldots$, for representations in spaces of single-valued functions.

The lowest weight $\langle v|j\ -j\rangle = v_2^{2j}$ has a stationary subgroup $U(1)$ and CS are parameterized by dots of the upper sheet of two sheet hyperboloid $SU(1,1)/U(1)$. An explicit form of CS can be obtained by the action of finite transformations on the lowest weight,

$$\psi_{j,u}(v) = \langle v \mid ju\rangle = (u^1 v_1 - u^2 v_2)^{2j}, \tag{5.16}$$

where $u = (u^1, u^2)$, $u^1 = \cosh(\theta_1/2)e^{im\varphi_1/2}$, $u^2 = \sinh(\theta_1/2)e^{-im\varphi_1/2}$ are elements of the matrix (5.2). The CS overlapping has the form

$$\langle j'u' \mid ju\rangle = \delta_{jj'}(\bar{u}_1' u^1 - \bar{u}_2' u^2)^{2j}. \tag{5.17}$$

A detailed description of CS of the discrete series of $SU(n,1)$ one can find in [31], and of $SU(1,1)$ in [30–32]. The representations T_j^+ and T_j^- are conjugate; the discrete basis T_j^- can be derived by means of the complex conjugation from (5.15) or by the replacement $v_1 \leftrightarrow v_2$.

For the functions, which are transformed with respect to one and the same representation T_j^+, the integral over ω in (5.14) gives 2π. The completeness relation at a given j can be written in terms of the discrete basis as well in terms of CS,

$$\hat{1}_j = \sum_{m=-\infty}^{j} \mid jm\rangle\langle jm \mid = \frac{-2j-1}{4\pi}\int_{-2\pi}^{2\pi} d\varphi_1 \int_0^{\infty} \mid j\theta_1\varphi_1\rangle\langle j\theta_1\varphi_1 \mid \sinh\theta_1 d\theta_1. \tag{5.18}$$

The parameter j takes on discrete values and the basis functions are orthonormalized on the Kronecker symbol $\delta_{jj'}$ for the single-valued IR of the discrete series, whereas for the principal series the condition of orthonormality contains the δ-function $\delta(j-j')$. Principal series can be constructed both in the space of functions on the complex hyperboloid (5.8), and on the cone (5.9).

One can construct the principal series on the cone (5.9) with the scalar product

$$\langle f_1 \mid f_2\rangle = (1/8\pi^2)\int_{-2\pi}^{2\pi} d\varphi \int_0^{\infty} \overline{f_1(\rho,\varphi)} f_2(\rho,\varphi)\rho\, d\rho. \tag{5.19}$$

We get $C_{n_1 n_2} = 1$, $n_1 + n_2 = 2j = -1 + i\lambda$, $2m = n_2 - n_1$, for the elements of the discrete basis (5.6) in case of the principal series,

$$\hat{J}_\pm = e^{\pm im\varphi}((1/2)\rho\partial/\partial\rho \pm i\partial/\partial\varphi), \quad \hat{J}_0 = -i\partial/\partial\varphi, \tag{5.20}$$

$$\langle\rho\varphi \mid \lambda m\rangle = v_1^{n_1} v_2^{n_2} = \rho^{-1+i\lambda} e^{im(\varphi+4\pi k)}, \quad \langle\lambda m \mid \lambda' m'\rangle = \delta(\lambda-\lambda')\delta_{mm'}, \tag{5.21}$$

$$\langle\rho\varphi \mid \rho'\varphi'\rangle = (1/\rho\rho')\delta(\ln\rho - \ln\rho')\delta(\varphi-\varphi') = (1/\rho)\delta(\rho-\rho')\delta(\varphi-\varphi').$$

Two IR in the space of single-valued functions with integer and half-integer m (the first and the second principal series accordingly to the terminology of the work [10]) correspond to each given λ,

$$\hat{1} = \frac{1}{8\pi^2}\int_{-2\pi}^{2\pi} d\varphi \int_0^{\infty} \mid \rho\varphi\rangle\langle\rho\varphi \mid \rho d\rho = \frac{1}{8\pi^2}\int_{-\infty}^{+\infty} d\lambda \sum_m \mid \lambda m\rangle\langle\lambda m \mid .$$

The summation in the last equation is running over all integer and half-integer m. Multi-valued IR are characterized not only by λ but also by a number ε, $|\varepsilon| \le 1/2$, which gives the nearest to zero value of m (for single-valued IR, $\varepsilon = 0$ or $\varepsilon = \pm 1/2$). Elements of the infinite-valued IR space are not periodic in φ. Thus, an arbitrary representation of the principal series is defined by two numbers (λ, ε), where $j = (-1 + i\lambda)/2$ characterizes the angular momentum square, $\mathbf{J}^2 = j(j+1) = (-1-\lambda^2)/4$, and ε characterizes possible values of the momentum projection $m = \varepsilon + [m]$. There is a certain analogy with IR of the principal series of $SO(3,1)$, which are defined by two numbers (λ, S), where S corresponds to the spin [33,34], and λ defines the square of the four-dimensional anglular momentum.

The representation of the principal series $T_{-1/2}$ is reducible at $\lambda = 0$ and $|\varepsilon| = 1/2$, and is split into two IR: $T_{-1/2}^+$ ($\varepsilon = -1/2$) and $T_{-1/2}^-$ ($\varepsilon = 1/2$); $\varepsilon = \pm 1/2$ corresponds to one and the same IR at $\lambda \neq 0$.

One can remark that, according to (5.21), ρ-dependence of functions on the cone is the same at a fixed j, and it is possible to consider the space of functions $f(\varphi)$ on the circle, what they usually are doing, considering the principal series of IR. However, such a reduction of the representation space is not always reasonable because of the space of functions on the cone appears sometimes naturally in different physical problems.

To construct CS one has to consider orbits in the representation space, factorized with respect to stationary subgroups [30]. The stationary subgroup of the state $|\lambda\ m = 0\rangle =$

$\rho^{-1+i\lambda}$ is $U(1)$, and CS, which correspond to integer m ($\varepsilon = 0$), are parameterized by the dots (θ, ψ) on the upper sheet of the hyperboloid $SU(1,1)/U(1)$. (Such CS were constructed in [30,35] in the space of functions on a circle.) Substituting $u^1 = \cosh(\theta/2)e^{i\psi/2}$, $u^2 = \sinh(\theta/2)e^{-i\psi/2}$, $\rho' = \rho(\cosh\theta + \sinh\theta\cos(\psi+\varphi))^{1/2}$ in (5.1),(5.2), we get CS in the form

$$\begin{aligned}
\langle\rho\varphi \mid \lambda\theta\psi\rangle &= (\rho')^{-1+i\lambda} = \rho^{-1+i\lambda}(\cosh\theta + \sinh\theta\cos(\psi+\varphi))^{-1/2+i\lambda/2}, \\
\langle\lambda m \mid \lambda_1\theta\psi\rangle &= \frac{1}{8\pi^2}\int\int\langle\lambda m \mid \rho\varphi\rangle\langle\rho\varphi \mid \lambda_1\theta\psi\rangle\rho d\rho d\varphi \\
&= (1/2\pi)\delta(\lambda-\lambda_1)\int_0^{2\pi} e^{im\varphi}(\cosh\theta + \sinh\theta\cos(\psi+\varphi))^{-1/2+i\lambda/2}d\varphi \\
&= \delta(\lambda-\lambda_1)\frac{\Gamma(m+1)}{\Gamma(m+1/2+i\lambda/2)}P^m_{-1/2+i\lambda/2}(\cosh\theta)e^{-im\psi},
\end{aligned} \tag{5.22}$$

where $P^m_{-1/2+i\lambda/2}(\cosh\theta)$ is adjoint Legendre function. At $m = 0$ the latter goes over to zonal harmonic $P_{-1/2+i\lambda/2}(\cosh\theta)$ (it is also called cone function [10,36]). To get CS at arbitrary ε one has to act by means of finite transformations on the state $|\lambda\ m = \varepsilon\rangle = \rho^{-1+i\lambda}e^{i\varepsilon\varphi}$,

$$\begin{aligned}
\langle\rho\varphi \mid \lambda\varepsilon\theta\psi\rangle &= \left((v_1u^1 + v_2u^2)(v_1\bar{u}^2 + v_2\bar{u}^1)\right)^{-1/2+i\lambda/2}\left(\frac{v_1\bar{u}^2 + v_2\bar{u}^1}{v_1u^1 + v_2u^2}\right)^{\varepsilon} \\
&= \rho^{-1+i\lambda}(\cosh\theta + \sinh\theta\cos(\varphi+\psi))^{-1/2+i\lambda/2} \\
&\times\left(\frac{\cosh(\theta/2)\exp[-i(\varphi-\psi)/2] + \sinh(\theta/2)\exp[i(\varphi-\psi)/2]}{\cosh(\theta/2)\exp[i(\varphi-\psi)/2] + \sinh(\theta/2)\exp[-i(\varphi-\psi)/2]}\right)^{\varepsilon}.
\end{aligned} \tag{5.23}$$

The case $\varepsilon = 0$, which we have considered above, and $\varepsilon = \pm 1/2$, correspond to representations in spaces of single-valued functions. In the latter case at $m = \pm 1/2$ we get

$$\begin{aligned}
\langle\rho\varphi \mid \lambda\ 1/2, \theta\psi\rangle &= (v_1u^1 + v_2u^2)^{-1} \mid v_1u^1 + v_2u^2 \mid^{i\lambda}, \\
\langle\rho\varphi \mid \lambda\ -1/2, \theta\psi\rangle &= (v_1\bar{u}^2 + v_2\bar{u}^1)^{-1} \mid v_1u^1 + v_2u^2 \mid^{i\lambda}.
\end{aligned} \tag{5.24}$$

At $\lambda = 0$ the CS take a simple form

$$\begin{aligned}
\langle\rho\varphi \mid 0\ 1/2, \theta\psi\rangle &= (v_1u^1 + v_2u^2)^{-1}, \\
\langle\rho\varphi \mid 0\ -1/2, \theta\psi\rangle &= (v_1\bar{u}^2 + v_2\bar{u}^1)^{-1},
\end{aligned} \tag{5.25}$$

which coincides with the explicit form of CS of the discrete series (5.16) (in this case, all the difference between CS of different series consists in different domains of v_1 and v_2, see (5.8) and (5.9)).

Let us turn to IR of supplementary series. The integral in (5.19) is divergent at real j. However, one can use a convergent "non-local" scalar product

$$\langle f_1 \mid f_2\rangle = \int\int\overline{f_1(x_1)}f_2(x_2)I(x_1,x_2)dx_1dx_2, \tag{5.26}$$

where the kernel function $I(x_1, x_2)$ has to be invariant with respect to finite transformations of the group. For the cone one can select an invariant expression $(v_1\bar{v}'_1 - \bar{v}_2v'_2) = 2i\sin(\varphi/2 - \varphi'/2)\rho\rho'$. At a fixed j representation functions have the form $\rho^{2j}f(\varphi)$. Let us select $I(x_1, x_2) = |(v_1v'_1 - v_2v'_2)/2|^{-2j}$, then the integrand in (5.26) is $\overline{f_1(\varphi)}f_2(\varphi')|\sin(\varphi/2 - \varphi'/2)|^{-2j}$. It does not depend on ρ, so that at a fixed j (5.26) takes the form

$$\langle f_1|f_2\rangle = \int_{-2\pi}^{2\pi}\int_{-2\pi}^{2\pi}\overline{f_1(\varphi)}f_2(\varphi')|\sin(\varphi/2 - \varphi'/2)|^{-2j}d\varphi d\varphi', \tag{5.27}$$

where $-1/2 < j < 0$, the latter is necessary for the scalar product to be convergent and positive defined.

For the single-valued representations of the supplementary series m is integer, for the multi-valued representations one has to introduce ε, $|\varepsilon| \le |j|$ (restrictions on ε follow from the unitarity of the representation, see Fig.1). Matrix elements of the supplementary series IR are expressed via so called torus function [36].

An invariant dispersion with respect to $SO(2,1)$ transformations can be written as

$$\Delta\mathbf{J}^2 = \langle\hat{J}_\mu\hat{J}^\mu\rangle\langle\hat{J}_\mu\rangle\langle\hat{J}^\mu\rangle = (\Delta J^0)^2 - (\Delta J^1)^2 - (\Delta J^2)^2. \tag{5.28}$$

It has the value $j(j+1) - m^2$ on the states $|jm\rangle$. At a given j CS minimize the absolute value of the dispersion (5.28). For CS of the discrete series $\Delta J^2 = j$, and for the principal series $\Delta J^2 = -1/4 - \lambda^2/4 - \varepsilon^2$.

Below we present a short summary of IR studied.

For single-valued unitary IR of $SO(2,1)$ the amgular momentum projection m is integer, for single-valued IR of $SU(1,1)$ it is integer or half-integer. For multivalued unitary IR the projection m can take any real values. Here we meet an essential difference with the Lorentz group in four dimensions, for unitary representations of the group this projection is always integer or half-integer. That is connected with the existence of non-Abelian compact subgroup $SU(2) \sim SO(3)$. Representations of the discrete series $T_j^{\pm}(g)$ of $SU(1,1)$ at real, integer and half-integer $j < -1/2$ are single-valued and have the highest and lowest weights $m = \pm j$. Representations of the principal series $T_{j,\varepsilon}(g)$, $j = -1/2 + i\lambda$, $-1/2 < \varepsilon \le 1/2$, are single-valued at $\varepsilon = 0$ and at $\varepsilon = 1/2$. At $\varepsilon \ne 1/2$ representations are irreducible and have neither highest nor lowest weights; at $\varepsilon = 1/2$ the representation is split in two ones: $T^-_{j,1/2}(g)$ with the highest weight $m = -1/2$ and $T^+_{j,1/2}(g)$ with the lowest weight $m = 1/2$. All said about IR of $SU(1,1)$ is summarized in the table below. The parameter n there is integer and $n \ge 0$; S or M signify single-valued or multivalued IR respectively.

series	j	m	S,M
finite-dimensional : T_j	$j \ge 0$, integer	$j-n,\ n \le 2j$	S
discrete:	$j < -1/2$		S at $j = -1 - n/2$
T_j^+		$-j+n$	
T_j^-		$j-n$	
principal:			
$T_{j,\varepsilon},\ -1/2 < \varepsilon \le 1/2$	$j = -1/2 + i\lambda/2$	$\varepsilon \pm n$	S at $\varepsilon = 0,\ 1/2$
$T_{-1/2,1/2} = T^+_{-1/2} \oplus T^-_{-1/2}$,			
$T^+_{-1/2}$	$j = -1/2$	$1/2 + n$	S
$T^-_{-1/2}$	$j = -1/2$	$-1/2 - n$	S
supplementary:	$-1/2 < j < 0$		
$T_{j,\varepsilon},\ \lvert\varepsilon\rvert < \lvert j\rvert$		$\varepsilon \pm n$	S at $\varepsilon = 0$
$T_j^+\ (\varepsilon = j)$		$\varepsilon + n$	M
$T_j^-\ (\varepsilon = -j)$		$\varepsilon - n$	M

Now we have to make some technical remark. As it follows from our consideration, representatives of all non-equivalent finite-dimensional and unitary IR of $SU(1,1)$ can be construced in the space of functions on two complex variables v_1 and v_2 only. At the same time, studing the left GRR (4.4) of the $M(2,1)$ group, it is convenient to use functions on

the elements z_1, $\bar{z}_2$ of the first column of the matrix Z. In such a space the spin generators (4.7) are reduced to the form

$$\hat{S}^0 = (1/2)(z_1\partial/z_1 - \bar{z}_2\partial/\bar{z}_2),\quad \hat{S}^1 = (i/2)(z_1\partial/\bar{z}_2 + \bar{z}_2\partial/z_1),$$
$$\hat{S}^2 = -(1/2)(z_1\partial/\bar{z}_2 - \bar{z}_2\partial/z_1), \tag{5.29}$$

In fact, after the renotation $z_1 \to v_1$, $\bar{z}_2 \to v_2$ they go over to the generators (5.4).

VI. RELATIVISTIC WAVE EQUATIONS FOR HIGHER SPINS

As is known, wave functions of relativistic particles are identified with vectors of representation spaces of the corresponding Poincare groups. Thus, the problem of the construction of the relativistic wave equations for particles with different spins is connected with a decomposition of the (left) GRR of the $M(2,1)$ group.

Consider functions $f(x,z)$, which are transformed under the left GRR of $M(2,1)$, and which are eigenfunctions for the Casimir operators $\hat{p}^2$, $\hat{p}\hat{J} = \hat{p}\hat{S}$, and for the operator $\hat{\mathbf{S}}^2$, which commutes with all the generators of the left GRR,

$$(\hat{p}^2 - m^2)f(x,z) = 0, \tag{6.1}$$
$$(\hat{p}\hat{S} - K)f(x,z) = 0, \tag{6.2}$$
$$(\hat{\mathbf{S}}^2 - S(S+1))f(x,z) = 0. \tag{6.3}$$

The equations (6.1)–(6.3) define some subrepresentation of the left GRR of $M(2,1)$, which is characterized by mass m, spin S, and by the eigenvalue K of Lubanski-Pauli operator, the latter is connected with the spin projection s on the direction of the momentum. The equation (6.1) is $2+1$-dimensional Klein-Gordon equation. For particles with $m > 0$ one can see that $K = \pm ms$. Indeed, in the rest frame $\hat{p}\hat{S} = p_0 s^0$, $p_0 s^0 - K = \pm ms - K = 0$. At $m = 0$ we suppose $K = 0$, that is true for IR with finite number of spinning degrees of freedom. The cases $m = 0$ and m imaginary will be discussed in Sect.9 in detail. The equation (6.3) defines z dependence of the wave functions. The eigenfunctions of the operator

$\hat{S}^2$ are (quasi)polynomials in the variables z_1, $\bar{z}_2$ of the power $2S$, on which representations of $SU(1,1)$ can be realized (see the previous section). Consider possible cases, which correspond to finite-dimensional non-unitary IR and to infinite-dimensional unitary IR of the latter group.

1. The value of S has to be positive, integer or half-integer for finite-dimensional and non-unitary IR of $SU(1,1)$. In this case $f(x,z) = f_S(x,z)$ are polynomials in z_1, $\bar{z}_2$ of the power $2S$. Solutions of the equation (6.3) have the form at $S = 1/2$,

$$f_{1/2}(x,z) = -\psi_{1/2}(x)\bar{z}_2 + \psi_{-1/2}(x)z_1 = (-\bar{z}_2 \;\; z_1)\begin{pmatrix} \psi_{1/2}(x) \\ \psi_{-1/2}(x) \end{pmatrix}. \qquad (6.4)$$

Using eq. (5.29), we find the action of Lubanski-Pauli operator $\hat{p}\hat{S}$ on the functions $f_{1/2}(x,z)$,

$$\hat{p}\hat{S} f_{1/2}(x,z) = \frac{1}{2}(-\bar{z}_2 \;\; z_1)\hat{p}_\mu\gamma^\mu \begin{pmatrix} \psi_{1/2}(x) \\ \psi_{-1/2}(x) \end{pmatrix},$$

where $\gamma^\mu = (\sigma^3, i\sigma^1, -i\sigma^2)$ are 2×2 γ-matrices in $2+1$ dimensions, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $[\gamma^\mu, \gamma^\nu] = 2i\epsilon^{\mu\nu\eta}\gamma_\eta$. Then, the equation (6.2) can be rewritten in the form of $2+1$ Dirac equation

$$(\hat{p}_\mu\gamma^\mu \pm m)\begin{pmatrix} \psi_{1/2}(x) \\ \psi_{-1/2}(x) \end{pmatrix} = 0. \qquad (6.5)$$

The column $(\psi_{1/2}(x),\ \psi_{-1/2}(x))^T$ is transformed under the spinor representation of $SU(1,1)$. Indeed, let us subject the function $f_{1/2}(x,z)$ to a finite transformation of $M(2,1)$,

$$T(g)f_{1/2}(x,z) = -\psi_{1/2}(g^{-1}x)(g^{-1}\bar{z}_2) + \psi_{-1/2}(g^{-1}x)(g^{-1}z_1),$$

$$x' = gx, \quad (-g^{-1}\bar{z}_2\; g^{-1}z_1) = (-\bar{z}_2 \;\; z_1)\begin{pmatrix} u_1 & u_2 \\ \bar{u}_2 & \bar{u}_1 \end{pmatrix}.$$

On the other hand, $f'_{1/2}(x,z) = -\psi'_{1/2}(x)\bar{z}_2 + \psi'_{-1/2}(x)z_1$, that means that $\psi'(x') = U\psi(x)$. Thus, $\psi_{1/2}(x)$, $\psi_{-1/2}(x)$ are transformed as components of a spinor field. The combination $|\psi'_{1/2}(x')|^2 - |\psi'_{-1/2}(x')|^2 = |\psi_{1/2}(x)|^2 - |\psi_{-1/2}(x)|^2 = C(x)$ is preserved under the transformations.

Let us consider the states $f_{1/2}(x,z) = e^{-ipx}(Az_1 + Bz_2)$ with $S = 1/2$ and a definite momentum. The combination $|A|^2 - |B|^2 = C$ remains constant under the $M(2,1)$ transformations. One can set A or B to be zero in a certain reference frame, depending on the sign of C. In the rest frame we get two wave functions, which can not be connected by any continuous $M(2,1)$ transformation, $e^{-ip_0x^0}z_1 \;\; (C > 0)$, $\quad e^{-ip_0x^0}z_2 \;\; (C < 0)$. They correspond to two different directions of the spin projection on the axis x^0. The case $C = 0$, $Ae^{-ip^0x^0}(e^{i\phi_1}z_1 + e^{i\phi_2}z_2)$, $A \neq 0$, corresponds to the massless particle. Indeed, a straightforward calculation shows that the action of the operator $\hat{p}\hat{S}$ on the function $(e^{i\phi_1}z_1 + e^{i\phi_2}z_2)$ gives zero at $\hat{p}^0 = p$, $\hat{p}^1 = p\cos\varphi$, $\hat{p}^2 = p\sin\varphi$, $\varphi = \varphi_1 - \varphi_2$ (see also (9.11)). Thus, at $S = 1/2$ we have three cases in accordance with possible values of the Casimir operator $\hat{p}\hat{S}$ $(\pm m/2,\ 0)$. Representations of $M(2,1)$ at $m > 0$ and $S = 1/2$ are split into two IR, which correspond to particles with spin projections $s = 1/2$ and $s = -1/2$, whereas the representation, which corresponds to the massless particles, is irreducible.

For the $S = 1$ the equation (6.3) has the following solutions

$$f_1(x,z) = F_1(x)\bar{z}_2^2 - F_0(x)\sqrt{2}z_1\bar{z}_2 + F_{-1}(x)z_1^2, \qquad (6.6)$$

with $F(x) = (F_1(x)\; F_0(x)\; F_{-1}(x))^T$ subjected the equation

$$(\hat{p}_\mu S^\mu - sm)F(x) = 0\,, \qquad (6.7)$$

$$S^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S^1 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

where the projection s takes on the values ± 1, 0. The matrices S^1 and S^2 are connected with the corresponding spin matrices $\tilde{S}^1$ and $\tilde{S}^2$ of $SU(2)$ by the relations $S^1 = i\tilde{S}^1$ and $S^2 = -i\tilde{S}^2$. If one introduces the new (Cartesian) components $\tilde{F}^\mu$, $\tilde{F}^1 = i(F_{-1} - F_1)/\sqrt{2}$, $\tilde{F}^2 = -(F_1 + F_{-1})/\sqrt{2}$, $\tilde{F}^0 = F_0$, instead of the components $F_1(x)$, $F_0(x)$, $F_{-1}(x)$ (cyclic components), then in such terms the eq. (6.7) takes the form

$$\partial_\mu\varepsilon^{\mu\nu\eta}\tilde{F}_\eta - sm\tilde{F}^\nu = 0. \qquad (6.8)$$

One can find, by analogy with the $S = 1/2$ case, that finite transformations of $M(2,1)$ act on the Cartesian components as $\tilde{F}'^{\nu}(x') = \Lambda^{\nu}_{\mu}\tilde{F}^{\mu}(x)$. Here the combination $|\tilde{F}^0(x)|^2 - |\tilde{F}^1(x)|^2 - |\tilde{F}^2(x)|^2 = C(x)$ is preserved. C does not depend on x for states with a definite momentum. The case $C > 0$ corresponds to particles with real mass $m \neq 0$, the case $C = 0$ corresponds to massless particles. Correspondent wave functions are presented in Sect.9.

Consideration of higher spins can be done by analogy. The functions $f_S(x,z)$ have the form

$$f_S(x,z) = \sum_{s^0=-S}^{S} F_{s^0}(x)\,(C^n_{2S})^{1/2}\,(-\bar{z}_2)^{S+s^0} z_1^{S-s^0},$$

where $F(x)$ is $2S+1$ component column, obeying the equation

$$(\hat{p}_\mu S^\mu - sm)F(x) = 0, \tag{6.9}$$

C^n_{2S} are binomial coefficients, $n = S + s^0$, and $(2S+1)\times(2S+1)$ spin matrices S^μ are generators of $SU(1,1)$ in the representation T_S,

$$S^0 = \mathrm{diag}(-S,\ -S+1, \dots S-1,\ S), \tag{6.10}$$

$$S^1 = \frac{i}{2}\begin{pmatrix} 0 & \sqrt{2S} & 0 & \dots & 0 & 0 \\ \sqrt{2S} & 0 & \sqrt{(2S-1)2} & \dots & 0 & 0 \\ 0 & \sqrt{(2S-1)2} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \sqrt{2S} \\ 0 & 0 & 0 & \dots & \sqrt{2S} & 0 \end{pmatrix},$$

$$S^2 = \frac{1}{2}\begin{pmatrix} 0 & -\sqrt{2S} & 0 & \dots & 0 & 0 \\ \sqrt{2S} & 0 & -\sqrt{(2S-1)2} & \dots & 0 & 0 \\ 0 & \sqrt{(2S-1)2} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -\sqrt{2S} \\ 0 & 0 & 0 & \dots & \sqrt{2S} & 0 \end{pmatrix}.$$

The functions $F(x)$ are transformed under IR T_S, of $M(2,1)$, $F'(x') = T_S(g)F(x)$.

In $2+1$ dimensions, in the rest frame, a particle has only one polarization state,

$$f(x,z) = e^{-imx^0}\bar{z}_2^{S+s}z_1^{S-s} = e^{-imx^0}(\cosh\theta/2)^{S+s}(\sinh\theta/2)^{S-s}e^{is\alpha}.$$

Indeed, the non-relativistic group of movements is $M(2) = T(2)\times)SO(2)$, where the group $SO(2)$, which describes the spin, is Abelian and has only one-dimensional IR, to which the functions $e^{is\alpha}$ correspond.

If in the rest frame a particle has integer or half-integer spin projection s, then the correspondent representation of $SU(1,1)$ of a minimal dimension is finite-dimensional $T_S(g)$, where $S = |s|$, and $\dim T_S = 2S+1$. To describe states with fractional spin projections one has to consider infinite-dimensional representations of $2+1$ Lorentz group.

2. For infinite-dimensional unitary IR of $SU(1,1)$, the values of S have to be non-integer, $S < -1/2$ (discrete series), $-1/2 < S < 0$ (supplementary series), or complex, $S = -1/2 + i\lambda/2$ (principal series), see Sect.5 . The correspondent equations describe, in particular, anyones. Consider first representations with highest and lowest weights. These are all representations of the discrete series $T^{\pm}_S$ and two representations of the principal series $T_{S,\varepsilon}$, which correspond to $S = -1/2$ and $\varepsilon = 1/2$, i.e. to half-integer spin projections. The eigenfunctions of the operator $\hat{S}^2$ in the representations $T^{\pm}_S$ are negative power S quasi-polynomials (see (5.15)),

$$f^+_S(x,z) = \sum_{n=0}^{\infty} F_{-S+n}(x)\,(C^n_{2S})^{1/2}\,(-\bar{z}_2)^n z_1^{2S-n},\quad C^n_{2S} = \left(\frac{(-1)^n\Gamma(n-2S)}{n!\Gamma(-2S)}\right)^{1/2},$$

$$f^-_S(x,z) = \sum_{n=0}^{\infty} F_{S-n}(x)\,(C^n_{2S})^{1/2}\,(-\bar{z}_2)^{2S-n} z_1^n, \tag{6.11}$$

Taking this into account, in (6.2), we get an equation for the infinite-component column $F(x)$,

$$(\hat{p}_\mu S^\mu - sm)F(x) = 0. \tag{6.12}$$

The matrices S^0 are diagonal, whereas S^1 and S^2 have only non-zeroth elements on the secondary diagonals. For the representations T^+_S

$$S^0 = \mathrm{diag}(-S,\ -S+1,\ -S+2,\ \ldots),$$

$$S^1 = -\frac{1}{2}\begin{pmatrix} 0 & \sqrt{-2S} & 0 & \ldots \\ \sqrt{-2S} & 0 & \sqrt{2(1-2S)} & \ldots \\ 0 & \sqrt{2(1-2S)} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

$$S^2 = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{-2S} & 0 & \ldots \\ \sqrt{-2S} & 0 & -\sqrt{2(1-2S)} & \ldots \\ 0 & \sqrt{2(1-2S)} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \tag{6.13}$$

In case of unitary infinite-dimensional representations the matrices S^1 and S^2 are Hermitian, whereas in case of finite-dimensional non-unitary representations, considered above, they are anti-Hermitian. As it follows from (6.11), the functions $F_{-S+n}(x)$ are transformed under representations of positive series T_S^+ ($F_{S-n}(x)$ under the negative series T_S^-) and therefore, the sum $\sum_{n=0}^{+\infty} |F_{S-n}(x)|^2$ is invariant under the transformations of $M(2,1)$.

The spin projection s can take on only positive values for the representations T_S^+ and negative values for ones T_S^-.

Representations of the principal series $T_{S,\varepsilon}$, $S = -1/2 + i\lambda/2$, $-1/2 < \varepsilon \leq 1/2$, have neither highest nor lowest weights, with the exception of the above mentioned case.

In case of representations of the principal series, the functions $f_S(x,z)$ are presented by the infinite sum

$$f_S(x,z) = \sum_{n=-\infty}^{+\infty} F_{\varepsilon+n}(x)(-\bar{z}_2)^{S+(\varepsilon+n)} z_1^{S-(\varepsilon+n)}. \tag{6.14}$$

Equations for the components $F_{\varepsilon+n}(x)$ have the form analogous to (6.12), where the matrices S^0 are diagonal, and S^1, S^2 have non-zeroth elements only on the secondary diagonals,

$$S^0 = \delta_{nn'}(\varepsilon + n), \qquad n = 0, \pm 1, \pm 2, \ldots,$$

$$S^1 = \frac{i}{2}\left(\delta_{n\ n'+1}(S - \varepsilon - n) + \delta_{n+1\ n'}(S + \varepsilon + n + 1)\right),$$

$$S^2 = \frac{1}{2}\left[-\delta_{n\ n'+1}(S - \varepsilon - n) + \delta_{n+1\ n'}(S + \varepsilon + n + 1)\right]. \tag{6.15}$$

The functions $F_{\varepsilon+n}(x)$ are transformed under representations of the principal series of $SU(1,1)$, and the sum $\sum_{n=-\infty}^{+\infty} |F_{\varepsilon+n}(x)|^2$ is invariant under the transformations of $M(2,1)$.

In all the cases considered above the matrices S^μ obey the commutation relations $[S^\mu, S^\nu] = i\epsilon^{\mu\nu\eta} S_\eta$, and anticommute at different indices, $\{S^\mu, S^\nu\} = 0$, $\mu \neq \nu$.

As one can see from the consideration presented, the construction of the relativistic wave equations in $2+1$ dimensions is, in a sense, simpler then one in $3+1$ dimensions. That is connected with the vectorial nature of the operators of the angular momentum and of the spin. In $3+1$-dimensional case the above mentioned operators are tensors, and namely this circumstance complicates the problem.

Different IR of $M(2,1)$ are marked by the spin projection s. However, how one can see from the previous consideration, the classification by the value S, connected with the square of the spin operator, is also useful.

In case of the infinite-dimensional unitary representations of $2+1$ Lorentz group, it is easier to deal with the functions $f(x,z)$, but not with infinite number of their components $F_{s^0}(x)$ in z-decomposition.

As an example let us consider the plane wave solutions at $m > 0$. For $S = 1/2$ and $S = 1$ such solutions were analyzed in [4]. There was remarked that, in fact, all the components are connected, that means that the number of spinning degrees of freedom is one. Here we are going to present similar consideration for all the representations of $2+1$ Lorentz group, which have lowest weights, namely, for finite-dimensional T_S ($S \geq 0$, integer or half-integer), and for infinite-dimensional unitary representations T_S^+ ($S \leq -1/2$).

The wave function in the rest frame, which corresponds to the spin projection $s = -S$, has the form $z_1^{2S}\Psi(p_0)$, $p_0 = E = \pm m$. Acting on it by finite transformations, we get at $E > 0$ a solution in the form of the plane wave, which is characterized by the momentum p,

$$f(p,z) = (z_1\bar{u}_1 - \bar{z}_2 u_2)^{2S}\,\Psi(p), \tag{6.16}$$

$$P = U^{-1}MU, \qquad M = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}.$$

Without loss of generality, we select $u_1 = \cosh\theta/2e^{-i\omega/2}$, $u_2 = \sinh\theta/2e^{-i\omega/2}$, because of φ-rotations correspond to the stationary subgroup of the state (6.16), and lead only to variations of the phase. Using the relations $p^0 = E = m\cosh\theta$, $-p_2 + ip_1 = m\sinh\theta e^{-i\omega}$, one can express the parameters $\bar{u}_1$ and u_2 via the momentum p,

$$\begin{pmatrix} u_2 \\ \bar{u}_1 \end{pmatrix} = \frac{e^{i\omega/2}}{\sqrt{2m(E+m)}} \begin{pmatrix} -p_2 + ip_1 \\ E+m \end{pmatrix}. \tag{6.17}$$

In case of finite-dimensional representations one can get the $2S+1$ components $F_s(p)$ as coefficients in z-decomposition of the function (6.16),

$$F(p) = \begin{pmatrix} F_S \\ \dots \\ F_{-S} \end{pmatrix} = \begin{pmatrix} u_2^{2S} \\ \dots \\ \bar{u}_1^{2S} \end{pmatrix} \Psi(p), \tag{6.18}$$

$$\begin{aligned} F_s(p) &= \left(C_{2S}^{S+s}\right)^{1/2} \bar{u}_1^{S-s} u_2^{S+s} \\ &= \left(C_{2S}^{S+s}\right)^{1/2} \frac{(E+m)^{S-s}(-p_2+ip_1)^{S+s}}{(2m(E+m))^S} \Psi(p). \end{aligned} \tag{6.19}$$

(Here we have omitted the phase factor $e^{-i\omega/2}$.) In the particular case $S = 1/2$ we get [4],

$$\psi(p) = \frac{1}{\sqrt{2m(E-m)}} \begin{pmatrix} -p_2 + ip_1 \\ E+m \end{pmatrix} \Psi(p). \tag{6.20}$$

For representations of discrete and principal series similar results holds. For example, in the former case one can get

$$\begin{aligned} F_{-S+n}(p) &= C_{2S}^{n} \bar{u}_1^{S-s} u_2^{S+s} \\ &= (C_{2S}^{n})^{1/2} \frac{(E+m)^{2S-n}(-p_2+ip_1)^n}{(2m(E+m))^S} \Psi(p). \end{aligned} \tag{6.21}$$

VII. DIRAC EQUATION AND CS EVOLUTION

It turns out that $2+1$ Dirac equation appears also in case of infinite-dimensional unitary IR (discrete and principal series with highest and lowest weights) as an equation for CS

evolution. To see this, let us take spinning CS of a massive particle, related to the highest (lowest) weight of IR T_S^- (T_S^+),

$$\psi_u^-(x,z) = (z_1\bar{u}^2(x) + \bar{z}_2\bar{u}^1(x))^{2S}, \tag{7.1}$$

$$\psi_u^+(x,z) = (z_1 u^1(x) + \bar{z}_2 u^2(x))^{2S}, \quad |u^1|^2 - |u^2|^2 = 1. \tag{7.2}$$

Here S can take on the value $-1/2$, that corresponds to the principal series of $SU(1,1)$, or the values $S < -1/2$, that corresponds to the discrete series of the group. At S integer or half-integer the representations are single-valued. We demand $\psi_u^+(x,z)$ to be an eigenfunction for Lubanski-Pauli operator $\hat{p}\hat{S}$,

$$\hat{p}\hat{S}\psi_u^+(x,z) = \mp mS\psi_u^+(x,z). \tag{7.3}$$

The left side of the equation (7.3) takes the form after the action of the operator,

$$\begin{aligned} &S\left(-\hat{p}_0(z_1u^1 - \bar{z}_2u^2) - i\hat{p}_1(z_1u^2 + \bar{z}_2u^1) - \hat{p}_2(z_1u^2 - \bar{z}_2u^1)\right) \\ &\times(z_1u^1 + \bar{z}_2u^2)^{2S-1} = S\,(z_1\ \bar{z}_2)\left[-\hat{p}_0\sigma^0 - i\hat{p}_1\sigma^1 - i\hat{p}_2\sigma^2\right]\begin{pmatrix} u^1(x) \\ u^2(x) \end{pmatrix} \\ &\times(z_1u^1 + \bar{z}_2u^2)^{2S-1} = \mp mS(z_1u^1 + \bar{z}_2u^2)^{2S}. \end{aligned}$$

Taking into account the explicit form of the wave function, we obtain an equation for the parameters of CS,

$$(\hat{p}_\mu\gamma^\mu \pm m)\begin{pmatrix} u^1(x) \\ u^2(x) \end{pmatrix} = 0, \tag{7.4}$$

which is, in fact, $2+1$ Dirac equation. The same equation controls the evolution of the parameters of CS (7.1), and appears also both in case of $S = -1/2$, and for arbitrary $S < -1/2$.

Consider an example. Let a massive particle in the rest frame is in the spinning CS, related with the lowest weight of the unitary IR T_S^+ or T_S^+. The corresponding wave functions have the form

$$\psi_0^+(x,z) = e^{-ip_0x^0}(z_1)^{2S}, \tag{7.5}$$

$$\psi_0^-(x,z) = e^{-ip_0x^0}(\bar{z}_2)^{2S}. \tag{7.6}$$

The wave function of the particle with arbitrary momentum can be derived from (7.5) or (7.6) by means of an appropriate rotation,

$$\psi_u^+(x,z) = e^{-ip'x}(z_1u^1 + \bar{z}_2u^2)^{2S}, \tag{7.7}$$

$$\psi_u^-(x,z) = e^{-ip'x}(z_1\bar{u}^2 + \bar{z}_2\bar{u}^1)^{2S}, \quad P' = U^{-1}P_0U. \tag{7.8}$$

It is easy to verify that the functions $u^k(x) = e^{-ip'x}u^k$ obey the Dirac equation (7.4).

VIII. ANGULAR MOMENTUM

In Sect.3 we have considered three types of representations of $M(2,1)$, which correspond to a real mass, zero mass, and imaginary mass. In each case the functional representation spaces are different, these are functions on one or two sheet hyperboloid and on the cone. Respectively, are different expressions for the angular momentum operator. Here we are going to analize the eigenvalue problem for the square of this operator and its projection in all the cases, using essentially the consideration of Sect.5. In particular, we will use bases of unitary IR $SO(2,1)$ to decompose functions on one and two sheet hyperboloid and cone.

1. $m \neq 0$ and is real. The operators $\hat{L}^\mu$ are acting in the space of functions on two sheet hyperboloid with the scalar product (3.12). The arising and lowering operators $\hat{L}_\pm$ and the operator of square of the angular momentum $\hat{\mathbf{L}}^2$ have the form

$$\hat{L}_\pm = e^{\pm i l\phi}(\pm i \coth\theta\partial/\partial\phi + \partial/\partial\theta)\,,$$
$$\hat{\mathbf{L}}^2 = \frac{\partial^2}{\partial\theta^2} + \coth\theta\frac{\partial}{\partial\theta} + \frac{1}{\sinh^2\theta}\frac{\partial^2}{\partial\phi^2}\ . \tag{8.1}$$

Let us suppose that a representation of the $SO(2,1)$ subgroup has the highest weight $f(\theta,j)e^{ij\phi}$, then

$$\hat{L}_+ f_j(\theta)e^{ij\phi} = e^{i(j+1)\phi}(-j\coth\theta f_j(\theta) + \partial f_j(\theta)/\partial\theta) = 0. \tag{8.2}$$

and, therefore, the highest weight has the form $(\sinh\theta)^j e^{ij\phi}$. It is easy to remark that at $j < -1/2$ (that would correspond to a discrete series) the norm of the state has a power divergence because of a singularity at $\theta = 0$, and at $j > -1/2$ the integrand of the norm is growing exponentially with the growth of θ (the case of double-valued IR with $j = -1/2$ is considered below). That means that single-valued unitary IR with a highest (lowest) weight are absent in the decomposition of $T_m^\pm$.

In general case the wave function, which corresponds to the state $|\,jl\rangle$ can be written in the form $NP_j^l(\cosh\theta)e^{il\phi}$, where N does not depend on θ and ϕ. The equation

$$\hat{\mathbf{L}}^2 P_j^l(\cosh\theta)e^{il\phi} = j(j+1)P_j^l(\cosh\theta)e^{il\phi}$$

defines adjoint Legendre functions $P_j^l(\cosh\theta)$ (below we are going to use the functions $\tilde{P}_j^l(\cosh\theta) = (\Gamma(j+1)/\Gamma(j+l+1))P_j^l(\cosh\theta))$. The representation is unitary at $j = -1/2 + i\lambda/2$ (see [10]). Thus, IR $T_m^\pm$ of $M(2,1)$ are decomposed in course of the reduction into representations of the principal series,

$$\langle\theta\phi\mid\lambda l\rangle = \tilde{P}_j^l(\cosh\theta)e^{il\phi}/\sqrt{2\pi}, \quad j = -1/2 + i\lambda/2, \tag{8.3}$$

$$\langle\lambda l\mid\lambda' l'\rangle = (1/2\pi^2)\lambda\tanh(\pi\lambda/2)\delta(\lambda-\lambda')\delta_{ll'}, \tag{8.4}$$

$$\sum_{l=-\infty}^{+\infty}\langle\theta\phi\mid\lambda l\rangle\langle\lambda' l\mid\theta\phi\rangle = \delta_{\lambda\lambda'}/2\pi.$$

The representations of the principal series (λ,ε) with arbitrary non-zeroth ε can be constructed in terms of multivalued functions on a sheet of the hyperboloid ($\varepsilon = 0$ corresponds to the single-valued representations). The functions have the form $\langle\theta\phi\mid\lambda\varepsilon l\rangle = \tilde{P}_j^{l+\varepsilon}(\cosh\theta)e^{i(l+\varepsilon)\phi}/\sqrt{2\pi}$, with the scalar product (8.4), where the factor $\tanh(\pi\lambda/2)$ has to be replaced by one $\tanh(\pi(\lambda/2+i\varepsilon))$ [10]. At $\varepsilon = 1/2$ (double-valued representations) and $j = -1/2$ the representation is reducible and is split into two representations with the highest weight $l = -1/2$ and with the lowest weight $l = 1/2$, the corresponding functions have the form $(\sinh\theta)^{-1/2}e^{\mp i\phi/2}$, according to (8.2).

2. $m = 0$. The operators $\hat{L}^\mu$ (3.16), and ones

$$\hat{L}_\pm = e^{\pm il\phi}(p\partial/\partial p \pm \partial/\partial\phi), \quad \hat{\mathbf{L}}^2 = p\partial/\partial p(p\partial/\partial p + 1), \tag{8.5}$$

are acting in the space of functions on the cone $p_0^2 - p_1^2 - p_2^2 = 0$. One can remark that after the replacement p by ρ^2 the expression (8.5) for $\hat{L}_\pm$ passes into the expression (5.20) on the complex cone (5.9). The scalar products on these manifolds differ only by the limits of integration over the angle ϕ ([-2π, 2π] or $[0, 2\pi]$). Thus, the representations of the principal series (λ, ε) can be constructed in the space of functions on the cone, however, only the representation with $\varepsilon = 0$ will be single-valued and the representation with $\varepsilon = 1/2$ will be double-valued.

According to (5.21), the wave function of a massless particle with the fixed momentum $j = -1/2 + i\lambda/2$ and with the projection l has the form in the momentum representation

$$\begin{aligned} \langle p\phi \mid \lambda l\rangle &= p^{-1/2+i\lambda/2} e^{il\phi}/2\pi, \\ \langle \lambda l \mid \lambda' l'\rangle &= \delta((\lambda - \lambda')/2)\delta_{ll'}. \end{aligned} \tag{8.6}$$

3. $m \neq 0$ and is imaginary. The operators $\hat{L}^\mu$ (3.19) and ones

$$\begin{aligned} \hat{L}_\pm &= e^{\pm i\phi}(\pm i \tanh\theta \partial/\partial\phi + \partial/\partial\theta), \\ \hat{\mathbf{L}}^2 &= \frac{\partial^2}{\partial\theta^2} + \tanh\theta\frac{\partial}{\partial\theta} - \frac{1}{\cosh^2\theta}\frac{\partial^2}{\partial\phi^2}, \end{aligned} \tag{8.7}$$

are acting in the space of functions on one sheet hyperboloid. Unitary IR of the discrete series can be realized in such a space. The result of the action of the arising operator $\hat{L}_+$ on the highest weights $f_j(\theta)e^{ij\phi}$ of the discrete negative series IR must be zero,

$$\hat{L}_+ f_j(\theta)e^{ij\phi} = e^{i(j+1)\phi}(-j\tanh\theta f_j(\theta) + \partial f_j(\theta)/\partial\theta) = 0,$$

thus, (with accuracy to a normalization factor) $f_j(\theta) = (\cosh\theta)^j$. By analogy, we get the expression $(\cosh\theta)^j e^{-ij\phi}$ for the lowest weight ($l = -j$) of the discrete positive series. Normalizing these functions by means of the scalar product (3.18) and denoting them as $Y_{jj}(\theta,\phi)$ and $Y_{j\,-j}(\theta,\phi)$, we can write

$$Y_{j\,\pm j}(\theta,\phi) = \left(\frac{(-2j-2)!!}{\pi^2(-2j-3)!!}\right)^{1/2} (\cosh\theta)^j e^{\pm ij\phi}. \tag{8.8}$$

The functions $Y_{jl}(\theta,\phi)$, $l < j$ (IR T_j^-) can be derived by the action of the lowering operator $\hat{L}_-$ on the highest weight $Y_{j\,-j}(\theta,\phi)$, and ones $Y_{jl}(\theta,\phi)$, $l > -j$ (IR T_j^+) by the action of the

arising operator $\hat{L}_+$ on the lowest weight $Y_{jj}(\theta,\phi)$. By analogy with the spherical functions we will call (8.8) the functions of one sheet hyperboloid. The wave functions of tachyons in $2+1$ dimensions have the form,

$$\langle\theta\phi \mid jl\rangle = Y_{jl}(\theta,\phi), \quad \langle\lambda l \mid \lambda' l'\rangle = \delta_{\lambda\lambda'}\delta_{ll'}, \tag{8.9}$$

where $j \leq -1$ and is integer (for the multivalued IR $j < -1/2$, and non-integer), whereas the momentum projection $l \geq |j|$. The functions (8.9), similar to the ordinary spherical functions, differ from adjoint Legendre functions P_l^j by a factor only.

In general case one has to consider eigenfunctions of the operators $\hat{\mathbf{L}}^2$ and $\hat{L}^0$ with the eigenvalues $j(j+1)$ and l. These functions have the form $f(\theta)e^{il\phi}$, where $f(\theta)$ obeys the equation

$$\left(\frac{\partial^2}{\partial\theta^2} + \tanh\theta\frac{\partial}{\partial\theta} + \frac{1}{\cosh^2\theta}l^2\right) f(\theta) = j(j+1)f(\theta), \tag{8.10}$$

which coincides with one for the adjoint Legendre functions,

$$\left((1-z^2)\frac{\partial^2}{\partial z^2} - 2z\frac{\partial}{\partial z} - l^2/(1-z^2)\right) P_j^l(z) = -j(j+1)P_j^l(z),$$

at $z = i\sinh\theta$. At $j \leq -1$ we get the above considered IR of the discrete series. The functions $P_j^l(i\sinh\theta)$ at $j = -1/2 + i\lambda/2$ could correspond to the principal series of unitary IR, but the corresponding norm is divergent in this case.

Thus, our consideration shows: in course of the reduction on the subgroup $SO(2,1)$ the representations $T_m^\pm(g)$ and $T_0^\pm(g)$ of $M(2,1)$ with real (in particular zeroth) mass are split into IR of the principal series, $j = -1/2 + i\lambda$, $\hat{\mathbf{L}}^2 \leq -1/4$, whereas l are arbitrary integer. For tachyons, the representations $T_m(g)$ are split into IR of the discrete series, $j \leq -1$ and integer, $\hat{\mathbf{L}}^2 = j(j+1) \geq 0$ (i.e. the space component of the angular momentum L^0 is greater than the bust ones). For the tachyons the absolute value of the projection l can not be less than $|j|$, in particular, l can not be zero.

IX. HARMONIC ANALYSIS ON $M(2,1)$

The harmonic analysis allows on to give the most complete description of representations of a group Lie, using explicit realizations in spaces of functions on the group. The bases of the method are presented for example in [6], there one can also meet a detailed application to the group of movements of the plane $M(2)$. The harmonic analysis for $M(3,1)$ one can find in [37,38].

The consideration we have presented in Sect.4,5, and 8 is, in fact, a component part of the harmonic analysis on $M(2,1)$. Here we are going to derive explicit forms of eigenfunctions for different sets of commuting operators of $M(2,1)$, decomposing GRR in IR. A classification and a description of unitary IR of the group will also be given.

Let us turn first to wave functions of spinless particles.

1. States with a given momentum, $\langle x \mid p\rangle = e^{-ipx}$.

2. States with a given energy p_0 and angular momentum projection l,

$$\langle x \mid m\ p_0\ l\rangle = e^{ip_0x^0+il\phi} J_l\left(\rho\sqrt{p_0^2-m^2}\right), \tag{9.1}$$

where ρ, ϕ are the polar coordinates in the x^1, x^2 plane, and J_l are Bessel functions.

3. States with a given orbital momentum L and with its given projection l. According to the eq. (8.3), (8.6), and (8.9), we have three case in the momentum representation:

$$m>0, \qquad \langle\theta\phi \mid \lambda l\rangle = \tilde{P}_L^l(\cosh\theta)e^{il\phi}, \quad L=-1/2+i\lambda/2, \tag{9.2}$$

where θ and ϕ are coordinates on two sheet hyperboloid $p^2=m^2>0$, and $\tilde{P}_j^l$ are adjoint Legendre functions;

$$m=0, \qquad \langle p\phi \mid \lambda l\rangle = p^{-1/2+i\lambda/2}e^{il\phi}, \tag{9.3}$$

where θ and ϕ are coordinates on the light cone $p^2=0$;

$$m - \text{imaginary}, \qquad \langle\theta\phi \mid jl\rangle = Y_{jl}(\theta,\phi), \tag{9.4}$$

where θ and ϕ are coordinates on one sheet hyperboloid $p^2=m^2<0$, and $Y_{jl}(\theta,\phi)$ are one sheet hyperboloid functions (8.9).

It is possible to construct bases for particles with spin, which consist of eigenfunctions for different sets of commuting operators. For example, for sets of operators: $(\hat{p}^2, \hat{p}\hat{S}, \hat{\mathbf{S}}^2, \hat{\mathbf{J}}^2, J^0)$, $(\hat{\underline{p}}_\mu, \hat{p}\hat{S}, \hat{\mathbf{J}}^2)$, $(\hat{p}^2, \hat{\mathbf{S}}^2, \hat{p}_0, \hat{L}^0, \hat{S}^0$ (we did not include the Casimir operator $\hat{p}\hat{S}$ in this set because of it does not commute with the operators $\hat{L}^\mu$ and $\hat{S}^\mu$ separately)), $(\hat{p}_\mu, \hat{\underline{p}}_\mu, \hat{p}\hat{S})$, and so on.

Let us consider states, which are eigenfunctions for the operators $\hat{p}_\mu$, $\hat{p}\hat{S}$, $\hat{\mathbf{S}}^2$ (plane waves). They can be written in the following form

$$f_{p,S}(x,z) = e^{-ipx} f_S(p,z), \tag{9.5}$$

where $f_S(z)$ is a homogeneous function on the variables $z_1, \bar{z}_2$ of the power $2S$. These states are important to classify IR of $M(2,1)$ by means of the little group method.

It is known that IR of the motion groups of the pseudo-Euclidean spaces (Poincare groups) are marked completely by means of parameters of orbits in the space of momenta and by numbers, which characterize IR of a stationary subgroup of a state, belonging to the orbit (little group) [6]. Thus, let us consider three cases: $m>0$ (orbits O_m^+, O_m^-), $m=0$ (orbits O_0^+, O_0^-, O_0^0), and $m^2<0$ (orbits O_m).

1. At $m>0$, in the rest frame, $\hat{p}\hat{S} = \pm m\hat{S}^0$, so that the eigenfunctions of this operator with the eigenvalues $\pm ms$ are

$$e^{-ip_0x^0}(-\bar{z}_2)^{S+s}z_1^{S-s}. \tag{9.6}$$

One can find the stationary subgroup of the state (9.6) from the condition $UP_0U^{-1}=P_0$, where $P_0 = \mathrm{diag}(m,-m)$. The matrices $U = \mathrm{diag}(e^{-i\varphi/2}, e^{i\varphi/2})$ obey the condition and form a one-parametric subgroup, which is isomorphic to the group $U(1)$ with the generator $\hat{J}^0 = \hat{L}^0 + \hat{S}^0$. The eigenvalues s of this operator together with the characteristic of the orbit mark IR of $M(2,1)$. Let us denote such representations as $T_{m,s}^+$ and $T_{m,s}^-$. They are single-valued at s integer and half-integer, whereas ms and $-ms$ are the eigenvalues of the operator $\hat{p}\hat{S}$ in these representations respectively. Subjecting the state (9.6) to a finite transformations of $M(2,1)$, we get the function

$$\langle x\ z \mid m\ S\ s\ p'\rangle = e^{-ip'x} N_{S,s}(z_1\bar{u}^2 - \bar{z}_2 u^1)^{S+s}(z_1\bar{u}^1 - \bar{z}_2 u^2)^{S-s},\ P' = U^{-1}P_0 U\ . \tag{9.7}$$

The spinning part of the function is CS of $SU(1,1)$. The parameters $u_1,\ u_2$ are expressed via the momentum p' (see (6.17)). This function describes a particle with real mass $m \neq 0$, momentum p', spin S, and spin projection s. The normalization coefficient $N_{S,s}$ depends on IR series, see Sect.5.

The wave function of a massive particle with spin S, energy p_0, angular momentum projection s, and spin projection l on the axis x^0, have the form, according the eq. (9.1),

$$\langle x\ z \mid m\ S\ s\ p_0\ l\rangle = e^{ip_0x^0+il\phi} J_l\left(\rho\sqrt{p_0^2 - m^2}\right) N_{S,s}(\bar{\ } - z_2)^{S+s} z_1^{S-s}. \tag{9.8}$$

2. The wave function of a massless particle, moving along the axis x^1, is

$$f_{p,S}(x,z) = e^{-ip(x^0-x^1)} f_S(z), \qquad \hat{p}\hat{S} f_S(x,z) = p e^{-ip(x^0-x^1)}(\hat{S}^0 - \hat{S}^1) f_S(z)\ .$$

The operator $\hat{S}^0 - \hat{S}^1$ is the generator of the stationary subgroup of the state with $p_2 = 0$, $p_0 = p_1 = p$. The U matrices, which correspond to the subgroup, obey the condition

$$U P_{01} U^{-1} = P_{01}\ ,\quad P_{01} = \begin{pmatrix} p & ip \\ -ip & -p \end{pmatrix},$$

and have the form

$$U = \pm \begin{pmatrix} 1 - ia & ia \\ -ia & 1 + ia \end{pmatrix}.$$

They form the $R \otimes Z$ group, where R is the additive group of the real numbers, and Z is the multiplicative group, consistent of two elements $\{1, -1\}$. These two elements correspond to the identical transformation and to $\varphi = 2\pi$ rotation around the axis x^0, respectively $U = I$ and $U = -I$, where I is the unite matrix. One can see from (4.4) that the latter rotation does not change x but changes the sign of z, $T(2\pi)f(x,z) = f(x,-z)$.

The eigenfunctions of the operator $\hat{S}^0 - \hat{S}^1$, which correspond to the eigenvalues λ, have the form

$$f_\lambda(z) = F(z_1 - \bar{z}_2)\exp\left(\lambda\frac{z_1 + \bar{z}_2}{-z_1 + \bar{z}_2}\right). \tag{9.9}$$

The wave functions of the massless particle with the momentum $(p,p,0)$, spin S, and the spin projection λ on the direction of the movement can be written as

$$f_{p,S,\lambda}(z) = e^{-ip(x^0-x^1)}(z_1 - \bar{z}_2)^{2S}\exp\left(\lambda\frac{z_1 + \bar{z}_2}{-z_1 + \bar{z}_2}\right). \tag{9.10}$$

They are eigenfunctions of the operators $\hat{p}\hat{S}$ and $\hat{S}^2$ with the eigenvalues $\Lambda = p\lambda$ and $S(S+1)$. These functions change the sign under Z-transformations (rotations on 2π) at half-integer S and remain unchanged at S integer. We denote IR, which correspond to $m = 0$ as $T^+_{0,\varepsilon,\Lambda}$ and $T^-_{0,\varepsilon,\Lambda}$. Here $\varepsilon = 0$ (S integer) or $\varepsilon = 1$ (S half-integer) mark IR of Z group. One can see that

$$(\hat{S}^0 - \hat{S}^1)^n = \left((z_1 - \bar{z}_2)\left(-\frac{\partial}{\partial z_1} - \frac{\partial}{\partial \bar{z}_2}\right)\right)^n = (z_1 - \bar{z}_2)^n\left(-\frac{\partial}{\partial z_1} - \frac{\partial}{\partial \bar{z}_2}\right)^n,$$

and, therefore, the operator $\hat{S}^0 - \hat{S}^1$ can have only zeroth eigenvalues in the space of polynomials. Thus, as was remarked before in [3], eigenvalues of the Casimir operator $\hat{p}\hat{S}$ are zero for the finite-dimensional in spin wave functions of the massless particles. That can be seen directly, using the explicit form of the states (9.9),(9.10). At $\lambda \neq 0$ there is an exponential factor dependent on z, its z-decomposition leads to infinite number of wave function components; similar states appear in the tachyon case. At $\lambda = 0$, $f_S(z) = (z_1 - \bar{z}_2)^{2S}$ and if $S \geq 0$ integer or half-integer, then the number of components is finite (is equal to $2S$). We denote IR at $\lambda = 0$ via $T^+_{0,\varepsilon}$ and $T^-_{0,\varepsilon}$, where $\varepsilon = 0$ corresponds to the integer and $\varepsilon = 1$ to half-integer S. The case of an arbitrary direction of movement, $p'^1 = p\cos\varphi$, $p'^2 = p\sin\varphi$, $p'^0 = p$, can be derived by a rotation around the axis x^0, then $z_1' = z_1 e^{i\varphi/2}$, $\bar{z}_2' = \bar{z}_2^{-i\varphi/2}$. In particular, at $\lambda = 0$,

$$\langle x\ z \mid m = 0\ \ S\ \ p'\rangle = e^{-ip'x}((z_1 e^{-i\varphi/2} - \bar{z}_2^{i\varphi/2})^{2S}. \tag{9.11}$$

This function describes a massless particle with the momentum p' and spin S. It is easy to remark that the states (9.11) are eigenfunctions for the helicity operator in $2+1$ dimensions,

$$(p_1\hat{S}^1 + p_2\hat{S}^2)/i|p|, \tag{9.12}$$

with the eigenvalues $-S$ or S, depending of the sign p_0. Thus, massless particles, which are described by the finite component wave functions, have the spin directed antiparallel to the spatial components of the momentum, whereas antiparticles have one parallel. For IR $T^+_{0,\varepsilon}$ and $T^-_{0,\varepsilon}$ (wave functions (9.11)) the helicity as well as the square of the spin are invariants, the corresponding operators commute with the generators of translations and rotations of the left GRR.

In particular case, $S = 1/2$, taking into account the explicit form of the γ-matrices (see Sect.6), we get for the operator of helicity

$$(p_1\gamma^1 + p_2\gamma^2)/i|p| = (-p_1\sigma^1 + p_2\sigma^2)/|p|. \tag{9.13}$$

(Another possible choice of the γ matrices, $\gamma^\mu = \{-\sigma^3, -i\sigma^1, -i\sigma^2\}$, leads to the helicity operator $(p_1\sigma^1 + p_2\sigma^2)/|p|$, which is a dimensional reduction of $3+1$-dimensional case.)

There exist unitary IR of $M(2,1)$, which are connected with the orbit O^0_0 and are IR of $SU(1,1)$.

3. In case of tachyons, the state with $p_0 = p_2 = 0$, $p_1 = m/i$,

$$f_{p,S}(x,z) = e^{ip_1x^1} f_S(z),$$

has the stationary subgroup, which can be found from the condition $UP_1U^{-1} = P_1$, where

$$U = \pm \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & ip \\ ip & 0 \end{pmatrix}.$$

This subgroup is isomorphic to $R \otimes Z$ and has the generator $\hat{J}^1$. The eigenfunctions for the operators $\hat{S}^1$ and $\hat{\mathbf{S}}^2$, with the eigenvalues is and $S(S+1)$ respectively, have the form

$$f_S(z) = (z_1 - \bar{z}_2)^{S+s}(-z_1 - \bar{z}_2)^{S-s} = (\bar{z}_2^2 - z_1^2)^S \left(\frac{z_1 - \bar{z}_2}{-z_1 - \bar{z}_2}\right)^s,$$
$$f_{p,S}(x,z) = e^{ip_1x^1}(z_1 - \bar{z}_2)^{S+s}(-z_1 - \bar{z}_2)^{S-s}. \tag{9.14}$$

The value of s has to be zero or imaginary for unitary IR, therefore, for $s \neq 0$, representations, which correspond to the imaginary mass case, are infinite-dimensional in the spin. The case of arbitrary direction of the movement can be derived by means of a rotation, as was done above for the real mass.

The classification of the single-valued unitary IR of the $T(3)\times)SU(1,1)$ group can be summarized in a table, which we present below.

mass, orbits	IR	eigenv. $\hat{p}\hat{S}$	states	remarks
$m > 0$,	$T^+_{m,s}$	ms	(9.7)	$s \geq 0$,
O^+_m, O^-_m	$T^-_{m,s}$	$-ms$		integer or half-integer
$m = 0$,	$T^+_{0,\varepsilon}$	0	(9.11)	$\varepsilon = 0, 1$
O^+_0, O^-_0	$T^-_{0,\varepsilon}$	0		
	$T^+_{0,\Lambda,\varepsilon}$	$\Lambda = p\lambda$	(9.10)	$\Lambda \neq 0$, real,
	$T^-_{0,\Lambda,\varepsilon}$	$\Lambda = p\lambda$		infinite-dimensional IR
$m^2 < 0$,	$T_{m,0,\varepsilon}$	0	(9.14)	
O_m	$T_{m,s,\varepsilon}$	ms		$s \neq 0$, imaginary, infinite-dimensional IR
$m = 0$,	T^+_S, T^-_S	0	see	discrete series of $SU(1,1)$
O^0_0	$T_{S,\varepsilon}$	0	sect.5	principal series of $SU(1,1)$
	T_S	0		supplementary series of $SU(1,1)$
	T^0_0	0		invariant

The IR states of $SU(1,1)$, correspondent to the orbit O^0_0, do not depend on x and are invariant under translations. The sign (+ or −) at T is related to the sign of p_0. The characteristics "infinite-dimensional" mean infinite-dimensionality in the spin space.

As was already remarked in [3], the finite-dimensional in spin wave functions of massless particles are zeroth modes of the operator $\hat{p}\hat{S}$.

To complete the picture one has to add to this table multivalued representations $T^+_{m,s}$ and $T^-_{m,s}$ at non-integer $2s$, and multivalued IR of $SU(1,1)$, described in the Sect.5. The explicit form of states, which are transformed under the representations $T^+_{m,s}$ and $T^-_{m,s}$ at non-integer $2s$, can also be given by the formula (9.7), however, in this case, z-decomposition generates infinite number of components.

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REFERENCES

[1] R. Jackiw and S. Templeton, Phys. Rev. D **23** (1981) 2291; J. Schonfeld, Nucl. Phys. **B185** (1981) 157; S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48** (1982) 975; Ann. Phys. (NY) **140** (1982) 372; **185** (1988) 406(E); S. Deser and R. Jackiw, Phys. Lett. **139B** (1984) 371; C.Hagen, Ann. Phys. (N.Y.) **157**, 342 (1984); Phys. Rev. D **31**, 848 (1985); **31**, 2135 (1985); D. Arovas, J. Schrieffer, F. Wilczek and A. Zee, Nucl. Phys. **B251**, 117 (1985); S. Deser and A. N. Redlich, Phys. Rev. Lett. **61**, 1541 (1988); P. Gerbert, Int. J. Mod. Phys. A **6**, 173 (1991); S. Forte, Rev. Mod. Phys. **64**, 193 (1992); H.O. Girrotti, M. Gomes, et al, Phys. Rev. Lett. **69**, 2623 (1992); G. Scharf, W.F. Wreszinski, B.M. Pimentel, and J.L. Tomazelli, Ann. of Phys. **231**, 185 (1994)

[2] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific Publishing, Singapore 1990)

[3] B. Binegar, J. Math. Phys. **23** (1982) 1511

[4] R. Jackiw and V.P. Nair, Phys. Rew. D **43** (1991) 1933

[5] D.V. Lvov, A.L. Shelepin and L.A. Shelepin, Yad. Fiz. **57** (1994) 1147

[6] A.O. Barut, R. Raczka, *Theory of Group Representations and Applications* (PWN, Warszawa, 1977)

[7] M.B. Menski, *Method of Induced Representations* (Nauka, Moscow, 1976)

[8] D.P. Zhelobenko and A.I. Schtern, *Representations of Groups Lie*, (Nauka, Moscow, 1983)

[9] C.Fronsdal, *High Energy Physics and Elementary Particles*, (IAEA, Viena, 1965) 585

[10] N.Ja. Vilenkin, *Special Functions and Theory of Group Representations*, (Nauka, Moscow, 1965)

[11] A.V. Berezin, Yu.A. Kurotchkin and E.A. Tolkachev, *Quaternions in Relativistic*

Physics, (Nauka e Technika, Minsk, 1989)

[12] E. Cartan, *The Theory of Spinors* (MIT press, Cambrige, 1966)

[13] D.A. Varshalovich, D.A. Moskalev and V.K. Khersonskiy, *Quantum Theory of Angular Momentum*, (Nauka, Leningrad, 1975)

[14] L.C. Biedenharn and J.D. Louck, *Angular Momentum in Quantum Physics*, (Addison-Wesley, Massachusetts, 1981)

[15] V. Bargmann, Ann. Math. **48** (1947) 568

[16] L. Pukanszky, Trans. Am. Math. Soc. **100** (1961) 116; Math. Annal. **156** (1964) 96

[17] I. Gel'fand, V. Graev and N. Vilenkin, *Generalized Functions*, (Academic, N.Y., 1966)

[18] W.J. Holman and L.C. Biedenharn, Ann. Phys. (NY) **39** (1966) 1; Ann. Phys. (NY) **47** (1968) 205

[19] A.O.Barut, C.Fronsdal, Proc. Roy. Soc. A **287** (1965) 532

[20] P.J. Sally, Bull. Am. Math. Soc. **72** (1966) 269

[21] W. Miller, *Lie Theory and Special Functions* (Academic, New York 1972)

[22] N.J. Mukunda, J. Math. Phys. **8** (1967) 2210; **9** (1968) 417; **10** (1973) 2068; 2092

[23] J.G. Kuriyan, N.J. Mukunda, and E.C.G. Sudarshan, J. Math. Phys. **9** (1968) 2100; Commun. Math. Phys. **8** (1968) 204

[24] J.D. Talman, *Special Functions, a Group Theoretical Approach* (Benjamin, New York 1968)

[25] E.G. Kalnins and W. Miller, J. Math. Phys. **15** (1974) 1263

[26] K.B. Wolf, J. Math. Phys. **15** (1974) 1295; 2102

[27] B.G. Wybourne, *Classical Groups for Physicists*, (Wiley, New York 1974)

[28] D. Basu and K.B. Wolf, J. Math. Phys. **23** (1982) 189

[29] S. Lang, SL_2R, (Springer, Berlin, 1985)

[30] A.M. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin 1986)

[31] D.M. Gitman and A.L. Shelepin, J. Phys. A **26** (1993) 7003

[32] Ya.A.Smorodinski, A.L.Shelepin and L.A.Shelepin, Uspekhi Fiz. Nauk **162** (1992) 1

[33] I.S. Schapiro, Docl. Acad. Nauk USSR **106** No 4 (1956) 647

[34] V.S. Popov, Zh. Eksp. Teor. Fiz. **37** (1959) 1116

[35] A.M.Perelomov, Comm. Math. Phys. **26** (1972) 222

[36] H. Bateman and A. Erdelyi, *Higher Transcendental Functions*, Vol.1, (McGraw-Hill, New York 1953)

[37] G.Rideau, Comm. Math. Phys. **3** (1966) 218

[38] Nghiem Xuan Hai, Comm. Math. Phys. **12** (1969) 331