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POINCARÉ GROUP AND RELATIVISTIC WAVE EQUATIONS IN 2+1 DIMENSIONS

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Using the generalized regular representation, an explicit construction of the unitary irreducible representations of the (2+1)-Poincaré group is presented. A detailed description of the angular momentum and spin in 2+1 dimensions is given. On this base the relativistic wave equations for all spins (including fractional) are constructed.

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I. INTRODUCTION

At the present time a great attention is devoted to field theoretical models in $2+1$-dimensional space-time [1]. There is a possibility to exist particles with fractional spin and exotic statistics in this space. These particles, which are called anyons, may have a relation to the physics of planar phenomena, for example, to the fractional quantum Hall effect [2].

The corresponding Poincaré group, which will be further denoted as $M(2,1)$, was studied in [3] and from the the field theoretical point of view in [4]. Importance of the $M(2,1)$ group investigation of is also stressed by the fact that, being a subgroup of the Poincaré group in $3+1$ dimensions $M(3,1)$, it retains many properties of the latter. In this connection, a part of results, which can be derived for the $M(2,1)$ group, may also be valid for $M(3,1)$ group. One has to remark that in contrast with $M(1,1)$, discussed in details in [5], $M(2,1)$ has a non-Abelian and non-compact subgroup of rotations, similar to $M(3,1)$, that leads to a nontrivial structure of the spinning space.

The aim of the present work is to construct a detailed theory of the $M(2,1)$ group representations in the form which may be convenient for the physical applications. Namely, we try to emphasize the problem of spin description and relativistic wave equations construction. Whereas numerous papers and books are devoted to the representation theory of the $M(3,1)$, see e.g. [5,9,10], there is, in fact, only one work where the representation theory of $M(2,1)$ is studied directly. Thus, we hope that the present paper can add some important details to the latter theory.

Usually, doing classification of representations of semi-direct products, one uses the method of the little group (see for example [9,10]). That method was also applied to $M(2,1)$ in [3]. However, for our purposes of detailed and explicit construction of representations it is more convenient to use the method of the harmonic analysis and, in particular, the method of the generalized regular representation (GRR). It is known that any irreducible representation (IR) of a Lie group is equivalent to a sub-representation of the left (right) GRR [11–13].

The harmonic analysis allows one to give the most complete description of representatio
of a group Lie, using explicit realizations in spaces of functions on the group. The ideas
the method are presented for example in [9], there one can also meet its application to t
motion group of the plane $M(2)$. One can find the harmonic analysis for the $M(3,1)$ gro
in the papers [14,15].

In the present work we are using the quasi-regular and generalized regular represe
tations to construct explicitly all unitary IR of $M(2,1)$ and to analyze on this basis t
relativistic wave equations for higher spins (including fractional) and the corresponding c
herent states. Studying the quasi-regular representation of $M(2,1)$, we introduce the scal
fields and construct the relativistic theory of 2+1 angular orbital momentum. Prese
ing (2+1)-dimensional vectors by means of 2×2 matrices, we introduce a parameteri
tion of the $M(2,1)$ group, where the rotations are given by two complex numbers z_1 a
z_2, $|z_1|^2-|z_2|^2=1$, which are analogs of Cayley-Klein parameters of the compact ca
The representation space of the left GRR consists of scalar functions $f(x,z)$, whereas t
spinning operators can be presented as first order differential operators in the variables
It is convenient to classify representations not only with respect to the Casimir operat
$\hat{p}^2=p_\mu p^\mu$ and $\hat{W}=\hat{p}_\mu\hat{J}^\mu$, but also with respect to the operator of square of the spin, wh
commutes with all generators of the left GRR. The latter operator marks representations
2+1 Lorentz group.

In the framework of such an approach one can naturally construct relativistic w
equations for particles with arbitrary spin. The fixation of the value of the square of
spin $S(S+1)$ defines the structure of z-dependence of the functions $f(x,z)$, namely, t
appear to be (quasi-)polynomials of the power $2S$ on z. Coefficients of these polynomi
are interpreted as components of finite(infinite)-dimensional wave functions of relativi
particles with higher spins. Fixation of the values of the Casimir operators provides equati
for these components.

In such a way, for example, both $2+1$ Dirac equation, equation for spin 1, and equati
for particles with fractional spins, which are related to the discret series of the Lore
group, (see [4,16,17]) appear. Thus, using GRR one gets an unique approach to descr
particles with different spins and gets also a possibility to establish a relation betw
different descriptions of these spins, for example, in terms of scalar functions $f(x,z)$ o
terms of multicomponent columns $\psi(x)$.

A detailed description of angular momentum and spin in $2+1$ dimensions is given
the base of the representation theory of $SU(1,1)$, which is summarized in the Appen
In particular, multivalued unitary IR of $SO(2,1)\sim SU(1,1)$ and corresponding cohe
states (CS) are considered. It is interesting to discover that $2+1$ Dirac equation app
also in the latter case as an equation for CS evolution.

II. PARAMETERIZATION

$M(2,1)$ is a six-parametric group of motions of $2+1$-dimensional pseudo-Euclid
space, it preserves the interval $\eta_{\mu\nu}\Delta x^\mu\Delta x^\nu$, where $x=(x^\mu)$, $\mu=0,1,2$, are coordina
and $\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1)$ is Minkowski tensor. The transformation of the vector x u
the action of the group (vector representation) is given by the formula

$$x' = gx, \quad g \in M(2,1), \quad x'^{\nu} = \Lambda^{\nu}_{\mu} x^{\mu} + a^{\nu}, \tag{2.1}$$

where Λ is a 3×3 rotation matrix of 2+1 Lorentz group $O(2,1)$. The transformations can also be presented in the four-dimensional form,

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ 1 \end{pmatrix} = \begin{pmatrix} & & & a^0 \\ & \Lambda(\alpha) & & a^1 \\ & & & a^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ 1 \end{pmatrix}, \tag{2.2}$$

with the composition low $(a_2, \Lambda_2)(a_1, \Lambda_1) = (a_2 + \Lambda_2 a_1, \Lambda_2 \Lambda_1)$. The latter means that $M(2,1)$ is the semi-direct product of the 2+1 translation group $T(3)$ and the Lorentz group $O(2,1)$,

$$M(2,1) = T(3) \times) O(2,1).$$

As it is known the group $O(2,1)$ contains four disjoint sets $O^{\uparrow}_{+}$ ($\det \Lambda = +1$, $\Lambda^0_0 > 0$), $O^{\downarrow}_{+}$ ($\det \Lambda = +1$, $\Lambda^0_0 < 0$), $O^{\uparrow}_{-}$ ($\det \Lambda = -1$, $\Lambda^0_0 > 0$), $O^{\downarrow}_{-}$ ($\det \Lambda = -1$, $\Lambda^0_0 < 0$), where only $O^{\uparrow}_{+} = SO_0(2,1)$ is connected to the identity continuously. Two sets $O^{\uparrow\downarrow}_{+}$ are equivalent to the group $SO(2,1)$. The corresponding continuously connected part of $M(2,1)$ is $T(3) \times) SO_0(2,1)$.

Consider first the group $SO_0(2,1)$. One-parametrical subgroups of $SO_0(2,1)$, which correspond to the rotations around axes x^0, x^1, x^2, are given by the matrices

$$\Lambda_{x^0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_0 & -\sin\alpha_0 \\ 0 & \sin\alpha_0 & \cos\alpha_0 \end{pmatrix}, \quad \Lambda_{x^1} = \begin{pmatrix} \cosh\alpha_1 & 0 & \sinh\alpha_1 \\ 0 & 1 & 0 \\ \sinh\alpha_1 & 0 & \cosh\alpha_1 \end{pmatrix},$$
$$\Lambda_{x^2} = \begin{pmatrix} \cosh\alpha_2 & -\sinh\alpha_2 & 0 \\ -\sinh\alpha_2 & \cosh\alpha_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.3}$$

The general transformation can be written in the form $\Lambda_{x^\mu} = \exp(-i\alpha_\mu J^\mu)$, where generators $J^\mu = i\frac{d}{d\alpha_\mu}(\Lambda_{x^\mu})\,|_{\alpha=0}$ are

$$J^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J^1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

They obey the commutation relations

$$[J^\mu, J^\nu] = -i\epsilon^{\mu\nu\eta} J_\eta,$$

where $\epsilon^{\mu\nu\eta}$ is totally antisymmetric Levi-Civita symbol, $\epsilon^{012} = 1$.

It is also possible to write the finite transformations by means of $SL(2,R)$ matrices [3] or $SU(1,1)$ matrices. Consider below the latter possibility in details, taking into account that $SO_0(2,1)$ is equivalent to $SU(1,1)/Z_2$, $Z_2 = \{I, -I\}$, where Z_2 is a multiplicative group consisting of two elements, I is the unit matrix. Thus, we are going, in fact, to study the group $\tilde{M}(2,1) = T(3) \times) SU(1,1)$. The classification and construction of representations of $\tilde{M}(2,1)$ allow one to describe representations of the group $M(2,1)$.

There is one-to-one correspondence between the 2+1 Lorentz vectors x^μ and 2×2 matrices X. Let σ^0 is the unit 2×2 matrix and σ^1, σ^2 are two first Pauli matrices. Then

$$X = x^\mu \sigma^\mu = \begin{pmatrix} x^0 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 \end{pmatrix}, \quad \det X = X^2 = x_\mu x^\mu, \qquad ($$
$$x^\mu = \frac{1}{2}\mathrm{Tr}(X\sigma^\mu).$$

In term of the matrix representation the transformation (2.1) can be written in the forn

$$X' = UXU^\dagger + A, \qquad ($$

where the matrices X', X, A correspond to the vectors x'^μ, x^μ, a^μ, and $SU(1,1)$ matrix

$$U = \begin{pmatrix} u_1 & u_2 \\ \bar{u}_2 & \bar{u}_1 \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} \bar{u}_1 & u_2 \\ \bar{u}_2 & u_1 \end{pmatrix},$$
$$|u_1|^2 - |u_2|^2 = 1, \; u_1 = \cosh(\theta/2)e^{i(-\phi-\omega)/2}, \; u_2 = -\sinh(\theta/2)e^{i(-\phi+\omega)/2},$$
$$0 \le \theta < \infty, \quad -2\pi \le \phi < 2\pi, \quad 0 \le \omega < 2\pi, \qquad ($$

provides the Lorentz rotations. Its relation with the matrix Λ from $SO_0(2,1)$ is given the formula

$$\Lambda = \begin{pmatrix} u_1\bar{u}_1 + u_2\bar{u}_2 & 2\mathrm{Re}(u_1\bar{u}_2) & 2\mathrm{Im}(u_1\bar{u}_2) \\ 2\mathrm{Re}(u_1u_2) & \mathrm{Re}(u_1^2 + u_2^2) & \mathrm{Im}(u_1^2 - u_2^2) \\ -2\mathrm{Im}(u_1u_2) & -\mathrm{Im}(u_1^2 + u_2^2) & \mathrm{Re}(u_1^2 - u_2^2) \end{pmatrix}.$$

One can remark that U and $-U$ correspond to one and the same Λ, so that to paramete the rotations it is enough to use $\phi \in [0, 2\pi]$.

In the representation (2.6) u_1 and u_2 are analogs of Cayley-Klein parameters, and ϕ, θ are ones of Euler angles, $U = U(\phi, \theta, \omega)$. It is possible to see that matrices $U(\phi,$ and $U(0,0,\omega)$ correspond to the rotations around the axis x^0, $U(0,\theta,0)$ correspond to rotations around the axis x^2 and $U(\phi,\theta,\omega) = U(\phi,0,0)U(0,\theta,0)U(0,0,\omega)$, i.e. the gen transformation can be presented as the ω-rotation around the axis x^0, then the θ-rotal around the axis x^2, and again the ϕ-rotation around the axis x^0.

The following sets of the parameters (ϕ,θ,ω): $(\alpha_0, 0, 0)$, $(-\pi/2, \alpha_2, \pi/2)$, $(0, \alpha_1, 0)$, co spond to the one parametrical subgroups $\Lambda_{x^0}(\alpha_0)$, $\Lambda_{x^1}(\alpha_1)$, $\Lambda_{x^2}(\alpha_2)$ respectively. The ma Λ in the Euler angles parameterization can be presented as $\Lambda(\phi,\theta,\omega) = \Lambda_{x^0}(\phi)\Lambda_{x^2}(\theta)\Lambda_{x^0}$

Further we are going to use the latter parameterization of elements g of $M(2,1)$ by m of matrices A and $SU(1,1)$ matrices U, $g = (A, U)$. In this representation the composi low and inverse elements have the form

$$g = (A, U) = (A_2, U_2)(A_1, U_1) = (U_2 A_1 U_2^\dagger + A_2, \; U_2 U_1),$$
$$g^{-1} = (-U^{-1}A(U^{-1})^\dagger, \; U^{-1}). \qquad ($$

III. QUASI-REGULAR REPRESENTATION AND THEORY OF ORBITAI MOMENTUM

A. Quasi-regular representation and scalar field

Let us consider a quasi-regular representation $T(g)$, which is acting on the coset s $M(2,1)/O(2,1) = \tilde{M}(2,1)/SU(1,1)$, i.e. in the space of functions $f(x)$,

$$f'(x) = T(g)f(x) = f(g^{-1}x). \tag{3.1}$$

The representation (3.1) corresponds to a scalar field transformation low, $f'(gx) = f'(x') = f(x)$. The explicit form of $g^{-1}x$ is given by the formulas

$$(g^{-1}x)^\nu = (\Lambda^{-1})^\nu_\mu(x^\mu - a^\mu), \quad g^{-1}x = U^{-1}(X - A)(U^{-1})^\dagger, \tag{3.2}$$

in the parameterizations (2.1) and (2.6) respectively. The Lie algebra of $M(2,1)$ contains six generators $\hat{p}_\mu$ and $\hat{L}^\mu$, which correspond to the parameters a^μ and $-\alpha_\mu$. They have a form

$$\hat{p}_\mu = i\partial/\partial x^\mu, \quad \hat{L}^\eta = \epsilon^{\eta\mu\nu}\hat{x}_\mu\hat{p}_\nu = i\epsilon^{\eta\mu\nu}x_\mu\partial/\partial x^\nu, \tag{3.3}$$

in the representation in question, and obey the commutation relations

$$[\hat{p}_\mu, \hat{p}_\nu] = 0\,, \quad [\hat{p}^\mu, \hat{L}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{p}_\eta\,, \quad [\hat{L}^\mu, \hat{L}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{L}_\eta\,. \tag{3.4}$$

Finite transformations in the parameterizations (2.4) and (2.7) can be written as

$$T(g)f(x) = e^{-i\phi\hat{L}^0}e^{-i\theta\hat{L}^2}e^{-i\omega\hat{L}^0}e^{ia\hat{p}}f(x). \tag{3.5}$$

The eigenvalue m^2 of the Casimir operator[1] $\hat{\mathrm{p}}^2$ can, in particular, characterize the IR, $\hat{\mathrm{p}}^2 f_m(x) = m^2 f_m(x)$. For unitary representations, where the generators $\hat{p}_\mu$ and $\hat{L}^\mu$ are hermitian, m^2 is real. It follows from the commutation relations (3.4) that $\hat{p}\hat{L}$ is also a Casimir operator, which is, however, zero in the representation under consideration.

To find all IR, which are contained in the representation (3.1), we go over to the space of functions dependent on momenta, doing the Fourier transformation,

$$\varphi(p) = (2\pi)^{-3/2}\int f(x)e^{ipx}dx. \tag{3.6}$$

In this space the expressions for the generators have the form

$$\hat{p}_\mu = p_\mu, \quad \hat{L}^\eta = \epsilon^{\eta\mu\nu}\hat{x}_\mu p_\nu = i\epsilon^{\eta\mu\nu}p_\mu\partial/\partial p^\nu. \tag{3.7}$$

The form of $\hat{L}^\mu$ in the space of functions $\varphi(p)$ coincides with one in the space of functions $f(x)$ if one replaces $p^\mu \to x^\mu$, and, therefore, the rotations result in: $\varphi(p) \to \varphi(p')$, where $p'_\mu = (\Lambda^{-1})^\nu_\mu p_\nu$. In the parameterization (2.6),

$$P' = U^{-1}P(U^{-1})^\dagger, \quad P = p^0 I + p^1\sigma^2 + p^2\sigma^2. \tag{3.8}$$

Translations affect only the phase of the functions, so we get an analog of eq.(3.1),

$$T(g)\varphi(p) = e^{iap'}\varphi(p'). \tag{3.9}$$

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[1]Here and in what follows $\hat{\mathrm{p}}^2 = \hat{p}_\mu\hat{p}^\mu$ and so on.

IR are related to orbits in the space of functions $\varphi(p)$ and are marked by the values p^2 $(p')^2 = m^2$. Representations with a given m we denote as $T_m(g)$. Below we consider th possible cases.

1. $m \neq 0$ and is real. In this case the representations $T_m(g)$ are acting in the space functions on two-sheeted hyperboloid,

$$p_0 = \pm m\cosh\theta, \quad p_1 = \mp m\sinh\theta\cos\phi, \quad p_2 = \mp m\sinh\theta\sin\phi. \qquad (3.$$

At $m > 0$ it is decomposed in two IR, one $T^+_m(g)$, which corresponds to particles (up sheet, $p_0 > 0$), and another one $T^-_m(g)$, which corresponds to antiparticles (lower she $p_0 < 0$). One can consider only IR with $m > 0$ because of $T^+_m(g)$ and $T^-_{-m}(g)$ are equivale The scalar product at a fixed m is given by the equation

$$\langle f_1 \mid f_2\rangle = \int_0^{2\pi}d\phi\int_0^{+\infty}\overline{\varphi_1(\theta,\phi)}\varphi_2(\theta,\phi)\sinh\theta d\theta, \qquad (3.$$

and the generators L^μ have the form

$$\hat{L}^0 = -i\partial_\phi, \quad \hat{L}^1 = -i(\coth\theta\cos\phi\partial_\phi + \sin\phi\partial_\theta),$$
$$\hat{L}^2 = i(-\coth\theta\sin\phi\partial_\phi + \cos\phi\partial_\theta). \qquad (3.$$

2. $m = 0$. In this case the representations $T_m(g)$ are acting in the space of functions the cone,

$$p_0 = p, \quad p_1 = -p\cos\phi, \quad p_2 = -p\sin\phi. \qquad (3.$$

The representation $T_0(g)$ is split into three IR: one-dimensional $T^0_0(g)$, which correspo to the invariant $p = 0$ (vertex of the cone), and $T^+_0(g)$ and $T^-_0(g)$, which are acting on upper and lower sheets of the cone. The scalar product is given by the formula

$$\langle f_1 \mid f_2\rangle = \int_0^{2\pi}d\phi\int_0^{+\infty}\overline{\varphi_1(p,\phi)}\varphi_2(p,\phi)dp, \qquad (3.$$

and the generators L^μ have the form

$$\hat{L}^0 = -i\partial_\phi, \quad \hat{L}^1 = i(\cos\phi\partial_\phi + p\sin\phi\partial_p), \quad \hat{L}^2 = i(-\sin\phi\partial_\phi + p\cos\phi\partial_p). \qquad (3.$$

3. m is imaginary, that corresponds to tachyons. The representations $T_m(g)$ are act in the space of functions on one-sheeted hyperboloid,

$$p_0 = im\,\sinh\theta, \quad p_1 = -im\,\cosh\theta\cos\phi, \quad p_2 = -im\,\cosh\theta\sin\phi. \qquad (3.$$

The scalar product is given by the formula

$$\langle f_1 \mid f_2\rangle = \int_0^{2\pi}d\phi\int_0^{+\infty}\overline{\varphi_1(\theta,\phi)}\varphi_2(\theta,\phi)\cosh\theta d\theta, \qquad (3.$$

and the generators L^μ have the form

$$\hat{L}^0 = -i\partial_\phi, \quad \hat{L}^1 = -i(\tanh\theta\cos\phi\partial_\phi + \sin\phi\partial_\theta),$$
$$\hat{L}^2 = i(-\tanh\theta\sin\phi\partial_\phi + \cos\phi\partial_\theta). \qquad (3.$$

B. Angular momentum

We have considered three types of scalar representations of $\tilde{M}(2,1)$, which correspond to a real mass, zero mass, and imaginary mass. In each case the functional representation spaces are different, these are functions on one- or two-sheeted hyperboloid and on the cone. Respectively, expressions for the angular momentum operators $\hat{L}^\mu$ are different. Here we are going to analyze the eigenvalue problem for the square of this operator and its projection in all the cases, using p-representation (3.6) and the consideration of the Appendix. In particular, we will use bases of unitary IR $SO(2,1)$ to decompose functions on one- and two-sheeted hyperboloid and cone.

1. $m \neq 0$ and is real. The operators $\hat{L}^\mu$ are acting in the space of functions on two-sheeted hyperboloid with the scalar product (3.11). The arising and lowering operators $\hat{L}_\pm$ and the operator of square of the angular momentum $\hat{\mathbf{L}}^2$ have the form

$$\hat{L}_\pm = e^{\pm i l \phi}(\pm i \coth\theta\partial/\partial\phi + \partial/\partial\theta)\,,$$
$$\hat{\mathbf{L}}^2 = \frac{\partial^2}{\partial\theta^2} + \coth\theta\frac{\partial}{\partial\theta} + \frac{1}{\sinh^2\theta}\frac{\partial^2}{\partial\phi^2}\,. \tag{3.19}$$

Let us suppose that a representation of the $SO(2,1)$ subgroup has the highest weight $f(\theta,j)e^{ij\phi}$. Then

$$\hat{L}_+ f_j(\theta)e^{ij\phi} = e^{i(j+1)\phi}(-j\coth\theta f_j(\theta) + \partial f_j(\theta)/\partial\theta) = 0. \tag{3.20}$$

and, therefore, the highest weight has the form $(\sinh\theta)^j e^{ij\phi}$. It is easy to remark that at $j < -1/2$ (that would correspond to a discrete series) the norm of the state has a power divergence due to a singularity at $\theta = 0$, and at $j > -1/2$ the integrand of the norm is growing exponentially with the growth of θ (the case of double-valued IR with $j = -1/2$ is considered below). That means that single-valued unitary IR with a highest (lowest) weight are absent in the decomposition of $T^\pm_m$.

In general case the wave function (3.5) in the p-representation, which are eigenvectors of the operators $\hat{\mathbf{L}}^2$, $\hat{L}^0$,

$$\hat{\mathbf{L}}^2 \mid jl\rangle = j(j+1) \mid jl\rangle, \quad \hat{L}^0 \mid jl\rangle = l \mid jl\rangle, \tag{3.21}$$

can be written in the form $NP^l_j(\cosh\theta)e^{il\phi}$, where $P^l_j(\cosh\theta)$ is adjoint Legendre function and N does not depend on θ and ϕ. Below we are going to use the functions $\tilde{P}^l_j(\cosh\theta) = (\Gamma(j+1)/\Gamma(j+l+1))P^l_j(\cosh\theta))$. The representation is unitary and single-valued at $j = -1/2 + i\lambda/2$ and integer l. (see [13]). Thus, IR $T^\pm_m$ of $\tilde{M}(2,1)$ are decomposed in course of the reduction into the representations of the principal series,

$$\mid \lambda l\rangle = \tilde{P}^l_j(\cosh\theta)e^{il\phi}/\sqrt{2\pi}, \quad j = -1/2 + i\lambda/2, \tag{3.22}$$
$$\langle \lambda l \mid \lambda' l'\rangle = (1/2\pi^2)\lambda\tanh(\pi\lambda/2)\delta(\lambda-\lambda')\delta_{ll'}, \tag{3.23}$$
$$\sum_{l=-\infty}^{+\infty} \mid \lambda l\rangle\langle \lambda' l \mid = \delta_{\lambda\lambda'}/2\pi.$$

The representations of the principal series $T_{\lambda,\varepsilon}$ with arbitrary nonzero ε can be constructed in terms of multivalued functions on a sheet of the hyperboloid ($\varepsilon = 0$ corresponds

to the single-valued representations). The eigenfunctions of $\hat{\mathbf{L}}^2$ and $\hat{L}^0$ are the same adj Legendre functions (3.22) with $l = n + \varepsilon$, n - integer, and with scalar product (3.23), w the factor $\tanh(\pi\lambda/2)$ has to be replaced by one $\tanh(\pi(\lambda/2 + i\varepsilon))$ [13]. At $\varepsilon = 1/2$ (dou valued representations) and $j = -1/2$, the representation is reducible and is split into representations with the highest weight $l = -1/2$ and with the lowest weight $l = 1/2$, corresponding functions have the form $(\sinh\theta)^{-1/2}e^{\mp i\phi/2}$, according to (3.20).

2. $m = 0$. The operators $\hat{L}^\mu$ (3.15), and ones

$$\hat{L}_\pm = e^{\pm i l\phi}(p\partial/\partial p \pm \partial/\partial\phi), \quad \hat{\mathbf{L}}^2 = p\partial/\partial p(p\partial/\partial p + 1), \quad (3$$

are acting in the space of functions on the cone $\mathbf{p}^2 = 0$. One can remark that the expres (3.24) for $\hat{L}_\pm$ passes into the expression (6.20) on the complex cone (6.9) after the repl ment p by ρ^2. The scalar products on these manifolds differ only by limits of integra over the angle ϕ ([-2π, 2π] or $[0, 2\pi]$). Thus, the representations of the principal series can be constructed in the space of functions on the cone, however, only the representa with $\varepsilon = 0$ are single-valued and the representation with $\varepsilon = 1/2$ are double-valued.

According to (6.21), the wave function of a massless particle with the fixed $j = -1$, $i\lambda/2$ and with the projection l has the form in the momentum representation

$$\mid \lambda l\rangle = p^{-1/2+i\lambda/2}e^{il\phi}/2\pi,$$
$$\langle \lambda l \mid \lambda' l'\rangle = \delta((\lambda-\lambda')/2)\delta_{ll'}. \quad (\text{:}$$

3. $m \neq 0$ and is imaginary. The operators $\hat{L}^\mu$ (3.18) and ones

$$\hat{L}_\pm = e^{\pm i\phi}(\pm i\tanh\theta\partial/\partial\phi + \partial/\partial\theta),$$
$$\hat{\mathbf{L}}^2 = \frac{\partial^2}{\partial\theta^2} + \tanh\theta\frac{\partial}{\partial\theta} - \frac{1}{\cosh^2\theta}\frac{\partial^2}{\partial\phi^2}, \quad (\text{:}$$

are acting in the space of functions on one-sheeted hyperboloid. Unitary IR of the dis series can be realized in such a space. The result of the action of the arising operator $\hat{L}$ the highest weights $f_j(\theta)e^{ij\phi}$ of the discrete negative series IR must be zero,

$$\hat{L}_+ f_j(\theta)e^{ij\phi} = e^{i(j+1)\phi}(-j\tanh\theta f_j(\theta) + \partial f_j(\theta)/\partial\theta) = 0,$$

thus, $f_j(\theta) = (\cosh\theta)^j$. By analogy, we get the expression $(\cosh\theta)^j e^{-ij\phi}$ for the lowest w of the discrete positive series. Normalizing these functions by means of the scalar pro (3.17) and denoting them as $Y_{j\,j}(\theta,\phi)$ and $Y_{j\,-j}(\theta,\phi)$, we can write

$$Y_{j\,\pm j}(\theta,\phi) = \left(\frac{(-2j-2)!!}{\pi^2(-2j-3)!!}\right)^{1/2}(\cosh\theta)^j e^{\pm ij\phi}. \quad ($$

The functions $Y_{j\,l}(\theta,\phi)$, $l < j$ (IR T^-_j) can be derived by the action of the lowering ope $\hat{L}_-$ on the highest weight $Y_{j\,-j}(\theta,\phi)$, and the functions $Y_{j\,l}(\theta,\phi)$, $l > -j$ (IR T^+_j) ca derived by the action of the arising operator $\hat{L}_+$ on the lowest weight $Y_{j\,j}(\theta,\phi)$. By an with the spherical functions we will call (3.27) the functions of one-sheeted hyperb The wave functions of tachyons in $2+1$ dimensions have the form,

$$| jl\rangle = Y_{jl}(\theta, \phi), \quad \langle \lambda l \mid \lambda' l' \rangle = \delta_{\lambda\lambda'}\delta_{ll'}, \tag{3.28}$$

where $j \leq -1$ and is integer (for the multivalued IR $j < -1/2$, and non-integer), whereas the momentum projection $l \geq |j|$. The functions (3.28), similar to the ordinary spherical functions, differ from the adjoint Legendre functions P_l^j by a factor only.

In general case one has to consider eigenfunctions of the operators $\hat{\mathbf{L}}^2$ and $\hat{L}^0$ with the eigenvalues $j(j+1)$ and l. These functions have the form $f(\theta)e^{il\phi}$, where $f(\theta)$ obeys the equation

$$\left(\frac{\partial^2}{\partial\theta^2} + \tanh\theta\frac{\partial}{\partial\theta} + \frac{1}{\cosh^2\theta}l^2\right) f(\theta) = j(j+1)f(\theta), \tag{3.29}$$

which coincides with one for the adjoint Legendre functions,

$$\left((1-z^2)\frac{\partial^2}{\partial z^2} - 2z\frac{\partial}{\partial z} - \frac{l^2}{(1-z^2)}\right) P_j^l(z) = -j(j+1)P_j^l(z),$$

at $z = i\sinh\theta$. At $j \leq -1$ we get the above considered IR of the discrete series. The functions $P_j^l(i\sinh\theta)$ at $j = -1/2 + i\lambda/2$ could correspond to the principal series of the unitary IR, but the corresponding norm is divergent in this case.

Thus, our consideration shows: in course of the reduction on the subgroup $SO(2,1)$ the representations $T_m^{\pm}(g)$ and $T_0^{\pm}(g)$ of $\tilde{M}(2,1)$ with real (in particular zero) mass are split into IR of the principal series, $j = -1/2 + i\lambda$, $\hat{\mathbf{L}}^2 \leq -1/4$, whereas l are arbitrary integer. For tachyons, the representations $T_m(g)$ are split into IR of the discrete series, $j \leq -1$ and integer, $\hat{\mathbf{L}}^2 = j(j+1) \geq 0$ (i.e. the space component of the angular momentum L^0 is greater than the bust ones). For the tachyons the absolute value of the projection l can not be less than $|j|$, in particular, l can not be zero.

Below we present three sets of wave functions of scalar particles, which are eigenfunctions for the commuting operators, $\{\hat{p}_\mu\}$, $\{\hat{\mathbf{p}}^2,\ \hat{p}_0,\ \hat{L}^0\}$ and $\{\hat{\mathbf{p}}^2,\ \hat{\mathbf{L}}^2,\ \hat{L}^0\}$ respectively:

1. States with a given momentum, $f(x) = e^{-ipx}$.
2. States with a given energy p_0 and angular momentum projection l (in x-representation),

$$f(x) = e^{ip_0x^0 + il\phi} J_l\left(\rho\sqrt{p_0^2 - m^2}\right), \tag{3.30}$$

where ρ, ϕ are the polar coordinates in the x^1, x^2 plane, and J_l are Bessel functions.

3. States (3.21) in the p-representation with a given orbital momentum j and its projection l. According to the eq. (3.22), (3.25), and (3.28), we have three cases:

$$m > 0, \qquad | \lambda l\rangle = \tilde{P}_j^l(\cosh\theta)e^{il\phi}, \quad j = -1/2 + i\lambda/2, \tag{3.31}$$

where θ and ϕ are coordinates on two sheet hyperboloid $p^2 = m^2 > 0$, and $\tilde{P}_j^l$ are adjoint Legendre functions;

$$m = 0, \qquad | \lambda l\rangle = p^{-1/2 + i\lambda/2}e^{il\phi}, \tag{3.32}$$

where θ and ϕ are coordinates on the light cone $p^2 = 0$;

$$m \text{ -- imaginary}, \qquad | jl\rangle = Y_{jl}(\theta, \phi), \tag{3.33}$$

where θ and ϕ are coordinates on one sheet hyperboloid $p^2 = m^2 < 0$, and $Y_{jl}(\theta,\phi)$ are one sheet hyperboloid functions (3.28).

IV. GENERALIZED REGULAR REPRESENTATION AND 2+1 SPIN

In the previous section we have considered the quasi-regular representation, which duces description of scalar fields or spinless particles. To get a complete picture of all pos representations one has to turn to the so called generalized regular representation (G [11–13]. The GRR is acting in the space of functions $f(g)$ on the group. The left C $T_L(g)$ and the right GRR $T_R(g)$ are defined as

$$T_L(g)f(g_0) = f(g^{-1}g_0), \qquad ($$
$$T_R(g)f(g_0) = f(g_0g). \qquad ($$

It is known that any IR of a group is equivalent to one of sub-representation of the (right) GRR [11]. Taking this into account, we are going to construct GRR of $\tilde{M}(2,1$ the parameterization (2.5)-(2.7), where $g_0 \leftrightarrow (x,z) \leftrightarrow (X,Z)$, $g \leftrightarrow (x,z) \leftrightarrow (A,U)$,

$$X = \begin{pmatrix} x^0 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 & z_2 \\ \bar{z}_2 & \bar{z}_1 \end{pmatrix},$$
$$A = \begin{pmatrix} a^0 & a^1 - ia^2 \\ a^1 + ia^2 & a^0 \end{pmatrix}, \quad U = \begin{pmatrix} u_1 & u_2 \\ \bar{u}_2 & \bar{u}_1 \end{pmatrix}. \qquad ($$

Using the composition law (2.8), one can get

$$T_L(g)f(x,z) = f(g^{-1}x,\ g^{-1}z),$$
$$g^{-1}x \leftrightarrow U^{-1}(X - A)(U^{-1})^\dagger,\ \ g^{-1}z \leftrightarrow U^{-1}Z, \qquad ($$
$$T_R(g)f(x,z) = f(xg,\ zg), \quad xg \leftrightarrow X + ZAZ^\dagger,\ \ zg \leftrightarrow ZU. \qquad ($$

According to (4.4), X is transformed with respect to the adjoint (vector) representation Z with respect to the spinor representation of $SU(1,1)$. One can also see that Z is inva under translations. If one restricts itself by Z-independent functions (i.e. by the func on the coset space $\tilde{M}(2,1)/SU(1,1)$), then (4.4) reduces to the quasi-regular represent (3.1), which corresponds to the scalar field case. If one restricts itself by X-indepen functions, then (4.4) and (4.5) reduce to the left and the right GRR of $SU(1,1)$.

Calculating generators, which correspond to the parameters a^μ and $-\alpha_\mu$, in the left ((4.4), we get

$$\hat{p}_\mu = i\partial/\partial x^\mu, \quad \hat{J}^\mu = \hat{L}^\mu + \hat{S}^\mu,$$

where $\hat{L}^\mu$ are the angular momentum operators (3.3), and $\hat{S}^\mu$ are spin operators,

$$\hat{S}^0 = -\frac{1}{2}V\sigma^3\partial_V + \frac{1}{2}\bar{V}\sigma^3\partial_{\bar{V}},$$
$$\hat{S}^1 = \frac{i}{2}V\sigma^2\partial_V - \frac{i}{2}\bar{V}\sigma^2\partial_{\bar{V}},$$
$$\hat{S}^2 = \frac{i}{2}V\sigma^1\partial_V + \frac{i}{2}\bar{V}\sigma^1\partial_{\bar{V}},$$
$$[\hat{S}^\mu, \hat{S}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{S}_\eta, \quad [\hat{S}^\mu, \hat{p}_\nu] = 0,$$

and $V = (z_1\ \bar{z}_2)$, $\bar{V} = (\bar{z}_1\ z_2)$. The algebra of the generators (4.6) has the form

$$[\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{p}^\mu, \hat{J}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{p}_\eta, \quad [\hat{J}^\mu, \hat{J}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{J}_\eta\,. \tag{4.8}$$

Generators of the right GRR we denote by the same letters but underlined below. The generators $\underline{\hat{J}}^\mu$ do not depend on x and are only expressed in terms of z,

$$\underline{\hat{p}}_\mu = -(\Lambda^{-1})^\nu_\mu \hat{p}_\nu \quad (\text{or } \underline{\hat{P}} = -Z^{-1}\hat{P}(Z^{-1})^\dagger), \qquad \underline{\hat{J}}^\mu = \underline{\hat{S}}^\mu, \tag{4.9}$$

$$\begin{aligned}
\underline{\hat{S}}^0 &= \frac{1}{2}\chi\sigma^3\partial_\chi - \frac{1}{2}\bar{\chi}\sigma^3\partial_{\bar{\chi}}, \\
\underline{\hat{S}}^1 &= \frac{i}{2}\chi\sigma^2\partial_\chi - \frac{i}{2}\bar{\chi}\sigma^2\partial_{\bar{\chi}}, \\
\underline{\hat{S}}^2 &= -\frac{i}{2}\chi\sigma^1\partial_\chi - \frac{i}{2}\bar{\chi}\sigma^1\partial_{\bar{\chi}},
\end{aligned} \tag{4.10}$$

where $\chi = (z_1\ z_2)$, $\bar{\chi} = (\bar{z}_1\ \bar{z}_2)$. All the right generators commute with all the left generators and obey the same commutation relations (4.8). The operator $\hat{\mathbf{p}}^2 = \underline{\hat{\mathbf{p}}}^2$ and Pauli-Lubanski scalar $\hat{W} = \hat{p}\hat{J} = \underline{\hat{p}}\underline{\hat{J}}$ are the Casimir operators. Thus, IR of $\tilde{M}(2,1)$ can be marked by their eigenvalues.

It follows from (3.3) that $\hat{p}\hat{L} = 0$, so that always $\hat{W} = \hat{p}\hat{S}$. The operator $\hat{W}$ commutes with the total angular momentum operators $\hat{J}^\mu = \hat{L}^\mu + \hat{S}^\mu$, but not with the orbital momentum operators $\hat{L}^\mu$ and spin operators $\hat{S}^\mu$ separately. The operator of spin square $\hat{\mathbf{S}}^2 = \underline{\hat{\mathbf{J}}}^2$ commutes with all the generators of the left GRR. That means that objects, which are transformed under the left GRR or under its sub-representations, can also be marked by eigenvalues of this operator. However, that operator does not commute with the generators $\underline{\hat{p}}_\mu$ of the right GRR, $[\underline{\hat{p}}^\mu, \underline{\hat{\mathbf{J}}}^2] = i\epsilon^{\mu\nu\eta}(\underline{\hat{p}}_\nu\underline{\hat{J}}_\eta + \underline{\hat{J}}_\eta\underline{\hat{p}}_\nu)$, similar to the left GRR case, $[\hat{p}^\mu, \hat{\mathbf{J}}^2] = i\epsilon^{\mu\nu\eta}(\hat{p}_\nu\hat{J}_\eta + \hat{J}_\eta\hat{p}_\nu)$. Thus, the square of spin is not a conserved quantity in all the right representations, but $\hat{\mathbf{J}}^2$ is.

Making Fourier transformation (3.6) in the variables x, i.e. considering representations in the space of functions $\varphi(p,z)$, one can get an analog of the formulas (4.4,4.5) in this representation,

$$T_L(g)\varphi(p,z) = e^{iap'}\varphi(p', g^{-1}z), \quad p' = g^{-1}p \leftrightarrow P' = U^{-1}P(U^{-1})^\dagger, \tag{4.11}$$

$$T_R(g)\varphi(p,z) = e^{-ia'p}\varphi(p, zg), \quad a' \leftrightarrow A' = ZAZ^\dagger, \tag{4.12}$$

where P defined by (3.8). It is seen that the combination $|z_1|^2 - |z_2|^2$ and p^2 are conserved under the transformations (4.11) and (4.12). The former is always equal to 1 and the latter to m^2, and depends on the representation. Z and P are defined by six real parameters. Three of them (namely, $\underline{P} = -Z^{-1}P(Z^{-1})^\dagger$ for the left GRR or P for the right GRR) are fixed and only three of them vary under the group transformations (for the left GRR two of them set the direction of the momentum).

The classification of the orbits with respect to the eigenvalues of the operator $\hat{p}^2$ is completely similar to one was done in Sect.3 for the spinless case. These are orbits $O_m^\pm$ for real $m \neq 0$, $O_0^\pm$ and O_0^0 for $m = 0$, and finally O_m for imaginary m. However, to describe IR it is not enough only one parameter m, one needs to know characteristics connected with the spin.

Remark that the left and the right GRR are equivalent, $\hat{C}T_R(g) = T_L(g)\hat{C}$, wh $\hat{C}f(g_0) = f(g_0^{-1})$. Because of that, and also since the left representations are more a quate to describe physical fields, we are going to consider further in detail only the left G of $\tilde{M}(2,1)$.

Consider the left GRR, which acts in the space of functions $f(x,z)$, $f'(x,z)$ $T_L(g)f(x,z) = f(g^{-1}x,\ g^{-1}z)$. It is easy to remark that

$$f'(x',z') = f(x,z), \tag{4.}$$

where

$$x' = gx = \Lambda x + a \leftrightarrow U(X+A)U^\dagger, \quad z' = gz \leftrightarrow UZ. \tag{4.}$$

Thus, one can reduce the problem of the classification of left representations to one of scalar functions (4.13)-(4.14), using the general scheme of the harmonic analysis [9,11].

To classify the functions $f(x,z)$ we are going to use besides the Casimir operators $\hat{W}$ the operator of spin square $\hat{\mathbf{S}}^2$, which commutes with all the generators of the left G. By means of this operator it is convenient to select IR from the set of equivalent ones, a moreover, to classify IR in the special case of zero eigenvalues of the Casimir operat where the functions (4.13) do not depend on x. In the latter case IR of the Poincaré gr coincide, in fact, with ones of the Lorentz group.

Let us consider in this connection the discrete basis $\overline{R}_{S\zeta}(z)$ of the Lorentz group re sentation $T_S(g)$,

$$\begin{aligned}
&\hat{\mathbf{S}}^2\overline{R}_{S\zeta}(z) = S(S+1)\overline{R}_{S\zeta}(z), \quad \hat{S}^0\overline{R}_{S\zeta}(z) = \zeta\overline{R}_{S\zeta}(z) \\
&\overline{R}'_S(z) = T_S(g)\overline{R}_S(z) = \overline{R}_S(g^{-1}z),
\end{aligned} \tag{4}$$

where $\overline{R}_S(z)$ is a column with the components $\overline{R}_{S\zeta}(z)$. The number S marks IR of Lorentz group and further we will call S the Lorentz spin. The possible values of S and corresponding spectrum of ζ depends on the type of the Lorentz group representation Appendix and the Table 2. The eigenvectors $f(x,z)$ of the operator $\hat{\mathbf{S}}^2$ can be presente the form

$$f(x,z) = \sum_\zeta \overline{\psi}_\zeta(x)\overline{R}_{S\zeta}(z) = \overline{\psi}(x)\overline{R}_S(z), \tag{4}$$

where $\overline{\psi}(x)$ is a line with components $\overline{\psi}_\zeta(x)$. On the other hand one can introduce a l $R_{S\zeta}(z)$ of the contragradient [9] to $T_S(g)$ representation . In terms of this basis a fun $f(x,z)$ can be presented by the decomposition

$$f(x,z) = \sum_\zeta \psi_\zeta(x)R_{S\zeta}(z) = \psi(x)R_S(z), \quad R'_S(z) = R'_S(z)T_S(g^{-1}), \tag{4}$$

where $R_S(z)$ is a line with the components $R_{S\zeta}(z)$ and $\psi_\zeta(x)$ is a column with the compo $\psi_\zeta(x)$. In case if the representation $T_S(g)$ and its contragradient are equivalent, whi valued for example for finite-dimensional IR of Lorentz group, one and the same fun has both representations (4.16) and (4.17). Using (4.16) and (4.17), one can find

$$\overline{\psi}'(x') = \overline{\psi}(x)T_S(g), \quad \psi'(x') = T_S(g^{-1})\psi(x).$$

The product $\overline{\psi}(x)\psi(x)$ is Poincaré invariant.

Thus, the eigenvectors of $\hat{\mathbf{S}}^2$ can be described by the columns $\psi(x)$ (lines $\overline{\psi}(x)$) with the components $\psi_\zeta(x)$ ($\overline{\psi}_\zeta(x)$). Their dimensionality depends on the representation of the Lorentz group. Further we will call $\psi(x)$ the wave function in S-representation or simple the wave function. In such a form all the spinning operators can be realized as discrete matrices. Their explicit form can be easily found.

As is demonstrated in the Appendix any IR of the Lorentz group can be constructed on the elements of the first column of the matrix Z (4.4). Thus, one can restrict himself by the functions $f(x,z)$, with $z = \{z_1, \bar{z}_2\}$ only. In this case eigenvectors of the operator $\hat{\mathbf{S}}^2$ are homogeneous functions in the variables z_1 and $\bar{z}_2$ of the power $2S$, and the discret basis can be chosen in the form

$$R_{S\zeta}(z) = N_{S\zeta} z_1^{S-\zeta} \bar{z}_2^{S+\zeta}. \tag{4.18}$$

The Lorentz IR with $2S$ integer and positive are non-unitary and finite-dimensional, whereas unitary infinite-dimensional IR correspond to $S < 0$ (discret and supplementary series) and $S = -1/2 + i\lambda/2$ (principal series).

Let $2S$ is integer and positive. (The case $S = 0$ corresponds to the scalar functions (3.1), which do not depend on z.) First consider $S = 1/2$. In this case the decomposition (4.17) can be written in the form

$$f(x,z) = \overline{\psi}_{-1/2}(x) z_1 + \overline{\psi}_{1/2}(x)\bar{z}_2, \quad \hat{\mathbf{S}}^2 f = \frac{3}{4} f. \tag{4.19}$$

Applying the transformation (4.4) to this function

$$f'(x,z) = \left(\overline{\psi}'_{-1/2}(x)\ \ \overline{\psi}'_{1/2}(x)\right)\begin{pmatrix} z_1 \\ \bar{z}_2 \end{pmatrix} = \left(\overline{\psi}_{-1/2}(g^{-1}x)\ \ \overline{\psi}_{1/2}(g^{-1}x)\right) U^{-1} \begin{pmatrix} z_1 \\ \bar{z}_2 \end{pmatrix},$$

we conclude that the line $\overline{\psi}(x) = \left(\overline{\psi}_{-1/2}(x)\ \ \overline{\psi}_{1/2}(x)\right)$ is transformed under the spinor representation of the Lorentz group,

$$\overline{\psi}'(x') = \overline{\psi}(x)U^{-1}.$$

Taking into account the relation $U^{-1} = \sigma^3 U^\dagger \sigma^3$, which is valued for the $SU(1,1)$ matrices, we get the transformation low for the columns $\psi(x) = (\psi_{1/2}(x)\ \psi_{-1/2}(x))^T = \sigma^3\overline{\psi}^\dagger$,

$$\psi'(x') = U\psi(x).$$

One can find that the same spinor ψ appears from the decomposition

$$f(x,z) = \psi_{1/2}(x)\bar{z}_2 - \psi_{-1/2}(x)z_1 = (\bar{z}_2\ \ -z_1)\begin{pmatrix} \psi_{1/2}(x) \\ \psi_{-1/2}(x) \end{pmatrix}, \quad \hat{\mathbf{S}}^2 f = \frac{3}{4} f. \tag{4.20}$$

Thus, in the case under consideration, we have two equivalent descriptions. One in terms of functions (4.13) and another one in terms of lines $\overline{\psi}(x)$ or columns $\psi(x)$. One can find the action of the operators $\hat{S}^\mu$ in the latter representation,

$$\hat{S}^\mu\psi(x) = \frac{1}{2}\gamma^\mu\psi(x),$$

where

$$\gamma^\mu = (\sigma^3, i\sigma^2, -i\sigma^1), \quad [\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \quad [\gamma^\mu, \gamma^\nu] = -2i\epsilon^{\mu\nu\lambda}\gamma_\lambda, \qquad (4$$

are 2×2 γ-matrices in $2+1$ dimensions. The functions $\psi = (\psi_{1/2}\ 0)^T$ and $\psi = (0\ \psi_{-1}$
are eigenvectors for the operator $\hat{S}^0$ with the eigenvalues $(\pm 1/2)$.

The product $\overline{\psi}(x)\psi(x) = \overline{\psi}'(x')\psi'(x')$ is the scalar density, which is not positive defi
The polynomials of the power $2S$ can be written in the form

$$f(x,z) = \sum_{n=0}^{2S} \overline{\psi}_{n-S}(x)\,(C_{2S}^n)^{1/2}\, {z_1}^{2S-n}\bar{z}_2^n = \overline{\psi}(x)\overline{R}_S(z), \qquad (4$$

where $\overline{\psi}(x)$ is $(2S+1)$-component line, $\overline{R}_S(z)$ is a column with elements $(C_{2S}^n)^{1/2}\, {z_1}^{2S-}$
$n = 0, 1, ..., 2S$, which is transformed with respect finite-dimensional IR $T_S(g^{-1})$ of
Lorentz group, $\overline{R}'_S(z) = T_S(g^{-1})\overline{R}_S(z)$, or in the form

$$f(x,z) = \sum_{n=0}^{2S} \psi_{S-n}(x)\,(C_{2S}^n)^{1/2}\,(-z_1)^n \bar{z}_2^{2S-n} = R_S(z)\psi(x), \quad \hat{\mathbf{S}}^2 f = S(S+1)f, \qquad (4$$

where $\psi(x)$ is $(2S+1)$-component column, $\psi(x) = \Gamma\overline{\psi}^\dagger(x)$, and $(\Gamma)_{nn'} = (-1)^n\delta_{nn'}$.

In analogy with the case $S = 1/2$ one cane get

$$\overline{\psi}'(x') = \overline{\psi}(x)T_S(g), \quad \psi'(x') = T_S(g^{-1})\psi(x). \qquad (4$$

Here the scalar density has the form $\overline{\psi}(x)\psi(x) = \psi^\dagger(x)\Gamma\psi(x)$. The operators $\hat{S}^\mu$ are (2
$1) \times (2S+1)$ spin matrices S^μ in the space of columns $\psi(x)$, and are generators of $SU($
in the representation T_S,

$$\begin{aligned}
(S^0)_{nn'} &= \delta_{nn'}(S-n), \qquad n = 0, 1, \ldots, 2S, \\
(S^1)_{nn'} &= -\frac{1}{2}\left(\delta_{n\ n'+1}\sqrt{(2S-n+1)n} - \delta_{n+1\ n'}\sqrt{(2S-n)(n+1)}\right), \\
(S^2)_{nn'} &= -\frac{i}{2}\left(\delta_{n\ n'+1}\sqrt{(2S-n+1)n} + \delta_{n+1\ n'}\sqrt{(2S-n)(n+1)}\right). \qquad (4
\end{aligned}$$

For the infinite-dimensional unitary IR of $SU(1,1)$ the values of S can be non-int
$S < -1/2$ (discrete series), $-1/2 < S < 0$ (supplementary series), or complex, $S = -1$
$i\lambda/2$ (principal series), see Appendix. Consider first representations with highest or lo
weights. These are all representations of the discrete series $T_S^\pm$ and two representatio
the principal series $T_{S,\varepsilon}$, which correspond to $S = -1/2$ and $\varepsilon = 1/2$, i.e. to half-integer
projections. The eigenfunctions of the operator $\hat{\mathbf{S}}^2$ in the representations $T_S^\pm$ are neg
power S quasi-polynomials (see (6.15)),

$$\begin{aligned}
f^+(x,z) &= \sum_{n=0}^{\infty} \psi_n^+(x)\,(C_{2S}^n)^{1/2}\,(-z_1)^{2S-n}\bar{z}_2^n, \\
f^-(x,z) &= \sum_{n=0}^{\infty} \psi_n^-(x)\,(C_{2S}^n)^{1/2}\,(-z_1)^n\,\bar{z}_2^{2S-n}, \qquad (4
\end{aligned}$$

$$\psi^{\pm\prime}(x') = T_S^\pm(g^{-1})\psi^\pm(x), \quad C_{2S}^n = \left(\frac{(-1)^n\Gamma(n-2S)}{n!\Gamma(-2S)}\right)^{1/2}.$$

The representations of the positive and negative series are conjugated,

$$(T_S^+(g))^\dagger = T_S^-(g), \quad \left(\psi^{\pm\prime}(x')\right)^\dagger = \left(\psi^\pm(x)\right)^\dagger T_S^\mp(g).$$

In contrast with the case of the finite-dimensional representations, here the scalar density is positively defined,

$$(\psi^+(x))^\dagger\psi^+(x) = \sum_{n=0}^{\infty} |\psi^+_{-S+n}(x)|^2, \quad (\psi^-(x))^\dagger\psi^-(x) = \sum_{n=0}^{\infty} |\psi^-_{S-n}(x)|^2.$$

The possible eigenvalues ζ of the operator $\hat{S}^0$ obey the inequality $|\zeta| \geq |S| > 1/2$ for the IR of the discret series. The spin projection ζ can take on only positive values for the representations T_S^+, $\zeta = -S + n$, and negative values for ones T_S^-, $\zeta = S - n$.

For the representations T_S^+ the spin matrices S^μ are

$$\begin{aligned}
(S^0)_{nn'} &= \delta_{nn'}(-S+n), \qquad n = 0,1,2,\dots, \\
(S^1)_{nn'} &= -\frac{i}{2}\left(\delta_{n\ n'+1}\sqrt{(n-1-2S)n} - \delta_{n+1\ n'}\sqrt{(n-2S)(n+1)}\right), \\
(S^2)_{nn'} &= \frac{1}{2}\left(\delta_{n\ n'+1}\sqrt{(n-1-2S)n} + \delta_{n+1\ n'}\sqrt{(n-2S)(n+1)}\right).
\end{aligned} \tag{4.27}$$

For T_S^- representations S^1 is the same and S^0, S^2 change the sign only.

In the case of unitary representations of the principal series, $S = -1/2 + i\lambda/2$, the functions $f(x,z)$ are presented by the infinite sum,

$$f(x,z) = \sum_{n=-\infty}^{+\infty} \psi_{\varepsilon+n}(x) i^n (-z_1)^{-1/2-i\lambda/2-(\varepsilon+n)}\bar{z}_2^{-1/2-i\lambda/2+(\varepsilon+n)}, \tag{4.28}$$

$$\hat{\mathbf{S}}^2 f = -\frac{1}{4}(1+\lambda^2)f.$$

The spin projection ζ can take on the values $\varepsilon+n$, where $\varepsilon \in [-1/2,\ 1/2]$, $n = 0,\ \pm1,\ \dots$. In the space of infinite-dimensional columns ψ with the elements $\psi_{\varepsilon+n}(x)$ the operators $\hat{S}^\mu$ have the form of corresponding infinite-dimensional matrices S^μ,

$$\begin{aligned}
(S^0)_{nn'} &= \delta_{nn'}(\varepsilon+n), \qquad n = 0,\pm1,\pm2,\dots, \\
(S^1)_{nn'} &= -\frac{i}{2}\big(\delta_{n\ n'+1}(-1/2+\varepsilon+n-i\lambda/2) - \delta_{n+1\ n'}(1/2+\varepsilon+n+i\lambda/2)\big), \\
(S^2)_{nn'} &= \frac{1}{2}\big(\delta_{n\ n'+1}(-1/2+\varepsilon+n-i\lambda/2) + \delta_{n+1\ n'}(1/2+\varepsilon+n+i\lambda/2)\big).
\end{aligned} \tag{4.29}$$

Due to unitarity of the representations under consideration, the corresponding scalar density

$$\psi^\dagger(x)\psi(x) = \sum_{n=-\infty}^{\infty} |\psi_{\varepsilon+n}(x)|^2$$

is positively defined.

In case of the unitary infinite-dimensional representations of the principal and discret series the matrices S^1 and S^2 are Hermitian, whereas in case of the finite-dimensional non-unitary representations considered above they are anti-Hermitian. In the space of columns with elements ψ_ζ the matrices S^1 and S^2 have nonzero elements only on the secondary diagonals.

The spin projection ζ can take on non-integer values for some IR of the principal and discret series. These IR can be used to describe the anions [4].

V. RELATIVISTIC WAVE EQUATIONS AND IR OF $\tilde{M}(2,1)$

A. Relativistic wave equations

As is known, wave functions of relativistic particles are identified with vectors o spaces of the corresponding Poincaré group. Thus, the problem of the construction of relativistic wave equations for particles with different spins can be solved by means decomposition of the left GRR of the $\tilde{M}(2,1)$ group.

Consider functions $f(x,z)$, which are transformed under the left GRR of $\tilde{M}(2,1)$, which are eigenvectors for the Casimir operators $\hat{\mathbf{p}}^2$, $\hat{W} = \hat{p}\hat{S}$, and for the operator $\hat{\mathbf{S}}^2$, w commutes with all the generators of the left GRR,

$$\begin{aligned}
&(\hat{\mathbf{p}}^2 - m^2)f(x,z) = 0, \\
&(\hat{p}_\mu\hat{S}^\mu - K)f(x,z) = 0, \\
&(\hat{\mathbf{S}}^2 - S(S+1))f(x,z) = 0.
\end{aligned}$$

The equations (5.1)–(5.3) define some sub-representation of the left GRR of $M(2,1)$, w is characterized by mass m, Lorentz spin S, and by the eigenvalue K of Lubanski-I operator. Possible values of K can be easily described in the massive case. Here we can a rest frame, where $\hat{p}_\mu\hat{S}^\mu = \hat{S}^0 m\,\mathrm{sign}\,p_0$. Thus, for particles $K = sm$ and for antipart $K = -sm$, where the spectrum s coincides with one of the operator $\hat{S}^0$. The latter spect depends on the representation of the Lorentz group, see Appendix and the table 2. At m we suppose $K = 0$, that is true for IR with finite number of spinning degrees of free The general cases $m = 0$ and m imaginary will be discussed below.

At S fixed and in the S-representation the equations (5.1)-(5.2) have the form

$$\begin{aligned}
&(\hat{p}^2 - m^2)\psi(x) = 0, \\
&(\hat{p}_\mu S^\mu - sm)\psi(x) = 0,
\end{aligned}$$

where $\psi(x)$ are columns and S^μ are matrices, described in the previous section. They the commutation relations of the $SU(1,1)$ group,

$$[S^\mu, S^\nu] = -i\epsilon^{\mu\nu\eta}S_\eta.$$

Let us describe possible cases, which correspond to finite-dimensional non-unitary IR, to infinite-dimensional unitary IR of the latter group.

1. Consider finite-dimensional and non-unitary IR of $SU(1,1)$. In this case S has t positive, integer or half-integer. According to (5.5),

$$\psi^\dagger(x)(iS^{\dagger\mu}\overleftarrow{\partial_\mu} + sm) = 0.$$

It follows from the explicit expressions for S^μ (4.22) that $S^{\dagger\mu} = \Gamma S^\mu \Gamma$, where $(\Gamma)_n$ $(-1)^n\delta_{nn'}$. The function $\overline{\psi} = \psi^\dagger\Gamma$ obeys the equation

$$\overline{\psi}(x)(iS^\mu\overleftarrow{\partial_\mu} + sm) = 0.$$

As a consequence of (5.5) and (5.6), the continuity equation holds

$$\partial_\mu j^\mu = 0, \quad j^\mu = \overline{\psi} S^\mu \psi. \tag{5.7}$$

At $S = 1/2$ the density $j^0 = \overline{\psi} S^0 \psi$ is positively defined (the scalar density $\overline{\psi}\psi$ is not positively defined, as was mentioned before).

At $S = 1/2$ the equation (5.5) can be rewritten in the form of $2+1$ Dirac equation,

$$(\hat{p}_\mu \gamma^\mu - m)\psi(x) = 0, \tag{5.8}$$

where $\gamma^\mu = 2S^\mu$ are γ-matrices in $2+1$ dimensions (4.21).

Let us consider the states $f(x,z) = e^{-ipx}(Az_1 + Bz_2)$ with a definite momentum. The combination $|A|^2 - |B|^2 = C$ remains constant under the $\tilde{M}(2,1)$ transformations. One can set A or B to be zero in a certain reference frame, depending on the sign of C. In the rest frame we get two wave functions, which can not be connected by any $\tilde{M}(2,1)$ transformation, $e^{-ip_0x^0}z_1 \quad (C>0), \qquad e^{-ip_0x^0}z_2 \quad (C<0)$. They correspond to two different directions of the spin projection on the axis x^0. Representations of $\tilde{M}(2,1)$ at $m>0$ and $S=1/2$ are split into two IR, which correspond to particles with spin projections $s = 1/2$ and $s = -1/2$.

The case $C = 0$, $f(x,z) = Ae^{-ip^0x^0}(e^{i\phi_1}z_1 + e^{i\phi_2}z_2)$, $\quad A \neq 0$, corresponds to the massless particle. Indeed, a straightforward calculation shows that the action of the operator $\hat{p}\hat{S}$ on the function $(e^{i\phi_1}z_1 + e^{i\phi_2}z_2)$ gives zero at $\hat{p}^0 = p$, $\hat{p}^1 = p\cos\varphi$, $\hat{p}^2 = p\sin\varphi$, $\varphi = \varphi_1 - \varphi_2$ (see also (5.35)). Thus, at $S = 1/2$ we have three cases in accordance with possible values of the Casimir operator $\hat{p}\hat{S}$ $(\pm m/2,\ 0)$.

At $S = 1$ the decomposition (4.17) has the following form

$$f(x,z) = \psi_1(x)\bar{z}_2^2 - \psi_0(x)\sqrt{2}z_1\bar{z}_2 + \psi_{-1}(x)z_1^2, \tag{5.9}$$

where $\psi(x) = (\psi_1(x)\ \psi_0(x)\ \psi_{-1}(x))^T$ is subjected to the equation (5.5)

$$(\hat{p}_\mu S^\mu - sm)\psi(x) = 0\ , \tag{5.10}$$

$$S^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S^1 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^2 = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where the spin projection s takes on the values $\pm 1,\ 0$. If one introduces the new (Cartesian) components $\mathcal{F}_\mu$, $\mathcal{F}_1 = -(\psi_{-1} + \psi_1)/\sqrt{2}$, $\mathcal{F}_2 = -i(\psi_1 - \psi_{-1})/\sqrt{2}$, $\mathcal{F}_0 = \psi_0$, instead of the components $\psi_1(x)$, $\psi_0(x)$, $\psi_{-1}(x)$ (cyclic components), then the eq. (5.5) takes the form

$$\partial_\mu \varepsilon^{\mu\nu\eta}\mathcal{F}_\eta + sm\mathcal{F}^\nu = 0. \tag{5.11}$$

A transversality condition follows from (5.11),

$$\partial_\mu \mathcal{F}^\mu = 0. \tag{5.12}$$

One can see now that the equations (5.11) are in fact field equations of the so called "self-dual" free massive field theory [18], with the Lagrangian

$$\mathcal{L}_{SD} = \frac{1}{2}\mathcal{F}_\mu\mathcal{F}^\mu - \frac{s}{2m}\varepsilon^{\mu\nu\lambda}\mathcal{F}_\mu\partial_\nu\mathcal{F}_\lambda = 0. \tag{5.13}$$

As remarked in [19] this theory is equivalent to the topologically massive gauge theory with the Chern-Simons term. Indeed, the transversality condition (5.12) can be viewed a Bianchi identity, which allows introducing gauge potentials A_μ, namely a transverse vec can be written (in topologically trivial space-time) as a curl:

$$\mathcal{F}^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu A_\nu = \frac{1}{2}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda},$$

where $F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu$ is the field strength. Thus, $\mathcal{F}^\mu$ appears to be dual field streng whish is a tree-component vector in 2+1 dimensions. Then (5.11) implies the follow equations for $F_{\mu\nu}$

$$\partial_\mu F^{\mu\nu} + \frac{sm}{2}\varepsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0, \qquad (5.$$

which are the field equations of the topologically massive gauge theory with the Lagrang

$$\mathcal{L}_{CS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{sm}{4}\varepsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda. \qquad (5.$$

One can find that finite transformations of $\tilde{M}(2,1)$ act on the Cartesian component $\mathcal{F}'^\nu(x') = \Lambda^\nu_\mu\mathcal{F}^\mu(x)$. Here the combination $\bar{\mathcal{F}}_\mu\mathcal{F}^\mu = C(x)$ is preserved. C does not dep on x for states with a definite momentum. The case $C > 0$ corresponds to particles v real mass $m \neq 0$, the case $C = 0$ corresponds to massless particles. The correspondent w functions will be presented below.

If a particle has integer or half-integer spin projection s, then the correspondent re sentation of $SU(1,1)$ of a minimal dimension is the finite-dimensional $T_S(g)$, where $S =$ and $\dim T_S = 2S+1$. To describe states with fractional spin projections one has to cons infinite-dimensional representations $SU(1,1)$.

2. Consider now unitary infinite-dimensional IR of $SU(1,1)$. In this case S can non-integer, $S < -1/2$ (discrete series), $-1/2 < S < 0$ (supplementary series), or comp $S = -1/2 + i\lambda/2$ (principal series), see Appendix. Matrices S^μ are hermitian and accor to (5.5) the conjugated equation has the form

$$\psi^\dagger(x)(iS^\mu\overleftarrow{\partial_\mu} + sm) = 0. \qquad (5$$

As a consequence of (5.5) and (5.16) the continuity equation holds

$$\partial_\mu j^\mu = 0, \quad j^\mu = \psi^\dagger S^\mu\psi. \qquad (5$$

In IR of discrete positive (negative) series $j^0 = \psi^\dagger S^0\psi$ is positively (negatively) defi Besides, for unitary IR the scalar density $\psi^\dagger\psi$ is also positively defined in contrast the finite-dimension case. For discrete positive series s can take on only positive val $s = -S + n$, and for negative one only negative $s = S - n$, $n = 0,1,2,\ldots$. The case $s =$ was considered earlier in [4,16,17].

There are cases when the equations (5.4) and (5.5) are dependent. Indeed, multipl the equation (5.5) by $\hat{p}_\mu S^\mu + ms$ one gets

$$(\hat{p}_\mu S^\mu + ms)(\hat{p}_\mu S^\mu - ms)\psi(x) = \left(\hat{p}_\mu\hat{p}_\nu\{S^\mu,\ S^\nu\} - m^2s^2\right)\psi(x) = 0. \qquad (5$$

In the particular case $S = 1/2$ we have $s = \pm 1/2$, $S^{\mu} = \gamma^{\mu}/2$ and (5.18) is merely the Klein-Gordon equation (5.4). In general case the matrices S^{μ} are not γ-matrices in higher dimensions and the squared equation (5.18) do not coincide with the Klein-Gordon equation.

As one can see from the consideration presented, the construction of the relativistic wave equations in $2+1$ dimensions is, in a sense, simpler then one in $3+1$ dimensions. That is connected with the vectorial nature of the operators of the angular momentum and of the spin. In $3+1$-dimensional case the above mentioned operators are tensors, and namely this circumstance complicates the problem.

Different IR of $\tilde{M}(2,1)$ with $m \neq 0$ are marked by the spin projection s. However, how one can see from the previous consideration, the classification by the Lorentz spin S, is also useful. S define the dimension of matrix representation of the spin operators in the equations (5.4) and (5.5).

One can easily see that massive particles have only one polarization state. Indeed, in the rest frame the equation (5.5) has the form

$$(S^0 - s)\psi = 0. \tag{5.19}$$

The spectrum s coincides with the spectrum of the operator S^0, which is not degenerated as was demonstrated above. Thus, a fixation of s leads to only one solution of the equation (5.5). For $S = 1/2$ and $S = 1$ that property was demonstrated explicitly in [4]. One can make the same conclusion, remarking that the non-relativistic group of movements is $M(2) = T(2)\times)SO(2)$, where the group $SO(2)$, which describes the spin, is Abelian one and has only one-dimensional IR.

In case of the infinite-dimensional unitary representations of $2+1$ Lorentz group, it is easier to deal with the functions $f(x,z)$, but not with infinite number of their components $\psi_{\zeta}(x)$ in S-representation.

As an example let us consider the plane wave solutions at $m > 0$. For $S = 1/2$ and $S = 1$ such solutions were analyzed in [4]. There was remarked that, in fact, all the components are connected, that means that the number of spinning degrees of freedom is one. Here we are going to present similar consideration for all the representations of $2+1$ Lorentz group, which have lowest weights, namely, for finite-dimensional T_S ($S > 0$, integer or half-integer), and for infinite-dimensional unitary representations T_S^+ ($S \leq -1/2$).

The wave function in the rest frame, which corresponds to the spin projection $s = -S$, has the form $z_1^{2S}\Psi(p_0)$, $p_0 = E = \pm m$. Acting on it by finite transformations, we get at $E > 0$ a solution in the form of the plane wave, which is characterized by the momentum p,

$$\begin{aligned} f(p,z) &= (z_1\bar{u}_1 - \bar{z}_2 u_2)^{2S}\,\Psi(p), \\ P &= U^{-1}P_0(U^{-1})^{\dagger}, \qquad P_0 = mI. \end{aligned} \tag{5.20}$$

The momentum p does not depend on the parameter ϕ, $p^0 = E = m\cosh\theta$, $-p_1 + ip_2 = m\sinh\theta e^{i\omega}$. Let us put $\phi = -\omega$ (in this case u_1 is real). Using the relations (2.7), one can express the parameters $\bar{u}_1$ and u_2 via the momentum p,

$$\begin{pmatrix} u_2 \\ \bar{u}_1 \end{pmatrix} = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} -p_1 + ip_2 \\ E + m \end{pmatrix}. \tag{5.21}$$

In case of finite-dimensional representations one can get $2S+1$ components $\psi_{\zeta}(p)$ as coefficients in the decomposition of the function (5.20),

$$\psi(p) = \begin{pmatrix} \psi_S \\ \dots \\ \psi_{-S} \end{pmatrix} = \begin{pmatrix} u_2^{2S} \\ \dots \\ \bar{u}_1^{2S} \end{pmatrix} \Psi(p), \qquad (5.$$

$$\begin{aligned} \psi_{\zeta}(p) &= \left(C_{2S}^{S+\zeta}\right)^{1/2} \bar{u}_1^{S-\zeta} u_2^{S+\zeta} \\ &= \left(C_{2S}^{S+\zeta}\right)^{1/2} \frac{(E+m)^{S-\zeta}(-p_1 + ip_2)^{S+\zeta}}{(2m(E+m))^{S}} \Psi(p). \qquad (5. \end{aligned}$$

In the particular case $S = 1/2$ we get [4],

$$\psi(p) = \frac{1}{\sqrt{2m(E-m)}} \begin{pmatrix} -p_2 + ip_1 \\ E + m \end{pmatrix} \Psi(p). \qquad (5.$$

For representations of discrete and principal series similar results holds. For example, in former case one can get the formula (5.23), where C_{2S}^{n} are the coefficients from (4.26) a $\zeta = -S, -S+1, \dots$.

Among the above considered relativistic wave equations are ones which describe pa cles with fractional real spin. These equations are connected with unitary multivalued of the Lorentz group and can be used to describe anyons. In spite of the fact that the nu ber of independent polarization states for massive 2+1 particles is one, the vectors of corresponding representation space have infinite number of component in S-representati Thus, z-representation is more convenient in this case.

B. Dirac equation and CS evolution

It turns out that $2+1$ Dirac equation appears also in the case of infinite-dimensi unitary IR of 2+1 Lorentz group (discrete and principal series with highest or lowest weig as an equation for CS evolution. To see that, let us take, for example, spinning CS, rela to the highest (lowest) weight of IR T_S^- (T_S^+) (see Appendix),

$$\psi_u^-(x,z) = (z_1\bar{u}^2(x) + \bar{z}_2\bar{u}^1(x))^{2S}, \qquad (5$$

$$\psi_u^+(x,z) = (z_1 u^1(x) + \bar{z}_2 u^2(x))^{2S}, \quad |u^1|^2 - |u^2|^2 = 1. \qquad (5$$

Here S can take on the value $-1/2$, that corresponds to the principal series of $SU(1,1)$, or values $S < -1/2$, that corresponds to the discrete series of the group. At S integer or h integer the representations are single-valued. We demand $\psi_u^+(x,z)$ to be an eigenfunc for the Lubanski-Pauli operator $\hat{W} = \hat{p}\hat{S}$,

$$\hat{W}\psi_u^+(x,z) = ms\psi_u^+(x,z). \qquad (5$$

The left side of the equation (5.27) takes the form after the action of the operator $\hat{W}$,

$$\begin{aligned} &S\left(\hat{p}_0(\bar{z}_2 u^2 - z_1 u^1) - \hat{p}_1(z_1 u^2 - \bar{z}_2 u^1) - i\hat{p}_2(z_1 u^2 + \bar{z}_2 u^1)\right)(z_1 u^1 + \bar{z}_2 u^2)^{2S-1} \\ &= S\,(\bar{z}_2 \;\; z_1) p_{\mu}\gamma^{\mu} \begin{pmatrix} u^2(x) \\ u^1(x) \end{pmatrix} (z_1 u^1 + \bar{z}_2 u^2)^{2S-1}. \end{aligned}$$

Thus, we obtain an equation for the parameters of CS (5.26),

$$(\hat{p}_\mu \gamma^\mu - \frac{s}{S} m) \begin{pmatrix} u^2(x) \\ u^1(x) \end{pmatrix} = 0, \tag{5.28}$$

which is, in fact, $2+1$ Dirac equation with mass $m' = \frac{s}{S} m$. The same equation controls the evolution of the parameters of CS (5.25), and appears also both in case $S = -1/2$, and for arbitrary $S < -1/2$.

C. IR of $\tilde{M}(2,1)$: classification and bases

Here we are going to derive explicit forms of eigenfunctions for sets of commuting operators of $\tilde{M}(2,1)$, decomposing GRR in IR. A classification and a description of unitary IR of the group will also be given.

It is possible to construct bases for particles with spin, which consist of eigenvectors for different sets of commuting operators. For example, for sets of operators: $(\hat{p}_\mu, \hat{W}, \hat{\mathbf{S}}^2)$, $(\hat{\mathbf{p}}^2, \hat{W}, \hat{\mathbf{S}}^2, \hat{\mathbf{J}}^2, J^0)$, $(\hat{\underline{p}}_\mu, \hat{W}, \hat{\mathbf{J}}^2)$, $(\hat{\mathbf{p}}^2, \hat{\mathbf{S}}^2, \hat{p}_0, \hat{L}^0, \hat{S}^0$ (we did not include the Casimir operator $\hat{W}$ in this set since it does not commute with the operators $\hat{L}^\mu$ and $\hat{S}^\mu$ separately)), $(\hat{p}_\mu, \hat{\underline{p}}_\mu, \hat{W})$, and so on.

Let us consider states, which are eigenvectors for the operators $\hat{p}_\mu$, $\hat{W}$, $\hat{\mathbf{S}}^2$ (plane waves). They can be written in the following form

$$f_{p,S}(x,z) = e^{-ipx} f_S(p,z), \tag{5.29}$$

where $f_S(z)$ is a homogeneous function on the variables z_1, $\bar{z}_2$ of the power $2S$. These states are important to classify IR of $\tilde{M}(2,1)$ by means of the little group method.

It is known that IR of the motion groups of the pseudo-Euclidean spaces (Poincaré groups) are marked completely by means of parameters of orbits in the space of momenta and by numbers, which characterize IR of a stationary subgroup of a state, belonging to the orbit (little group) [9]. Thus, let us consider three cases: $m > 0$ (orbits O_m^+, O_m^-), $m = 0$ (orbits O_0^+, O_0^-, O_0^0), and $m^2 < 0$ (orbits O_m).

1. At $m > 0$, in the rest frame, $\hat{p}\hat{S} = \pm m \hat{S}^0$, so that the eigenvectors of this operator with the eigenvalues $\pm ms$ are

$$f_{p,S}(x,z) = e^{-ip_0 x^0} \bar{z}_2^{S+s} (-z_1)^{S-s}. \tag{5.30}$$

One can find the stationary subgroup of the state (5.30) from the condition $U^{-1} P_0 (U^{-1})^\dagger = P_0$, where $P_0 = \mathrm{diag}(m, m)$. The matrices $U = \mathrm{diag}(e^{-i\varphi/2}, e^{i\varphi/2})$ obey the condition and form a one-parametric subgroup, which is isomorphic to the group $U(1)$ with the generator $\hat{J}^0 = \hat{L}^0 + \hat{S}^0$. The eigenvalues s of this operator together with the characteristic of the orbit mark IR of $\tilde{M}(2,1)$. Let us denote such representations as $T^+_{m,s}$ and $T^-_{m,s}$. They are single-valued at s integer and half-integer, whereas ms and $-ms$ are the eigenvalues of the operator $\hat{p}\hat{S}$ in these representations respectively. Subjecting the state (5.30) to a finite transformations of $\tilde{M}(2,1)$, we get the function

$$f_{p',S,s}(x,z) = e^{-ip'x} N_{S,s} (\bar{z}_2 u_1 - z_1 \bar{u}_2)^{S+s} (\bar{z}_2 u_2 - z_1 \bar{u}_1)^{S-s}, \quad P' = U^{-1} P_0 (U^{-1})^\dagger . \tag{5.31}$$

The spinning part of the function is CS of $SU(1,1)$. The parameters u_1, $\bar{u}_2$ are expre via the momentum p' (see (5.21)). This function describes a particle with real mass m momentum p', Lorentz spin S, and the spin projection s. The normalization coefficient depends on IR series, see Appendix.

The wave function of a massive particle with Lorentz spin S, energy p_0, angular mentum projection l, and spin projection ζ on the axis x^0, have the form, according the (3.30),

$$f_{p_0,S,\zeta,l}(x,z) = e^{-ip_0 x^0 + il\phi} J_l \left(\rho \sqrt{p_0^2 - m^2} \right) N_{S,\zeta} \bar{z}_2^{S+\zeta} (-z_1)^{S-\zeta}. \quad (5$$

2. The wave function of a massless particle with $p^\mu = p(1,1,0)$ is

$$f_{p,S}(x,z) = e^{-ip(x^0 - x^1)} f_S(z), \qquad \hat{W} f_{p,S}(x,z) = p e^{-ip(x^0 - x^1)} (\hat{S}^0 - \hat{S}^1) f_S(z) .$$

The operator $\hat{S}^0 - \hat{S}^1$ is the generator of the stationary subgroup of the state. Th matrices, which correspond to the subgroup, obey the condition

$$U^{-1} P_{01} (U^{-1})^\dagger = P_{01} , \quad P_{01} = \begin{pmatrix} p & p \\ p & p \end{pmatrix} ,$$

and have the form

$$U = \pm \begin{pmatrix} 1 + ia & ia \\ -ia & 1 - ia \end{pmatrix} .$$

They form $R \otimes Z$ group, where R is the additive group of the real numbers, and Z is multiplicative group, which consist of two elements $\{1, -1\}$. These two elements corresp to the identical transformation and to $\varphi = 2\pi$ rotation around the axis x^0, respecti $U = I$ and $U = -I$, where I is the unite matrix. One can see from (4.4) that the la rotation does not change x but changes the sign of z, $T(2\pi) f(x,z) = f(x,-z)$.

The eigenvectors of the operator $\hat{S}^0 - \hat{S}^1$, which correspond to the eigenvalues λ, l the form

$$f_\lambda(z) = F(z_1 - \bar{z}_2) \exp \left(\lambda \frac{z_1 + \bar{z}_2}{\bar{z}_2 - z_1} \right). \quad (5$$

The wave functions of a massless particle with the momentum $(p, p, 0)$, Lorentz spin S, the spin projection λ on the direction of the momentum can be written as

$$f_{p,S,\lambda}(x,z) = e^{-ip(x^0 - x^1)} (z_1 - \bar{z}_2)^{2S} \exp \left(\lambda \frac{z_1 + \bar{z}_2}{\bar{z}_2 - z_1} \right). \quad (5$$

They are eigenvectors of the operators $\hat{W}$ and $\hat{\mathbf{S}}^2$ with the eigenvalues $K = p\lambda$ and $S(S$ These functions change the sign under the Z-transformations (rotations on 2π) at half-int S and remain unchanged at S integer. We denote IR, which correspond to $m = 0$, as T and $T^-_{0,\epsilon,K}$. Here $\epsilon = 0$ (S integer) or $\epsilon = 1$ (S half-integer) mark IR of Z group. One ca that $(\hat{S}^0 - \hat{S}^1)^n = [(\bar{z}_2 - z_1)(\partial/\partial z_1 + \partial/\partial \bar{z}_2)/2]^n = [(\bar{z}_2 - z_1)/2]^n (\partial/\partial z_1 + \partial/\partial \bar{z}_2)^n$, therefore, the operator $\hat{S}^0 - \hat{S}^1$ can have only zero eigenvalues in the space of polyno

Thus, as was remarked before in [3], eigenvalues of the Casimir operator $\hat{W}$ are zero for the finite-dimensional in spin wave functions of the massless particles. That can be seen directly using the explicit form of the states (5.33),(5.34). At $\lambda \neq 0$ there is an exponential factor dependent on z, its z-decomposition leads to infinite number of wave function components, similar states appear in the tachyon case.

At $\lambda = 0$, $f_S(z) = (z_1 - \bar{z}_2)^{2S}$ and if $S \geq 0$ integer or half-integer, then the number of components is finite (is equal to $2S+1$). We denote IR at $\lambda = 0$ via $T^+_{0,\varepsilon}$ and $T^-_{0,\varepsilon}$, where $\varepsilon = 0$ corresponds to the integer and $\varepsilon = 1$ to half-integer S. The case of an arbitrary direction of movement, $p'^\mu = p(1, \cos\varphi, \sin\varphi)$, can be derived by a rotation around the axis x^0, $U = \mathrm{diag}(e^{i\varphi/2}, e^{-i\varphi/2})$, then $z_1' = z_1 e^{-i\varphi/2}$, $\bar{z}_2' = \bar{z}_2^{i\varphi/2}$. In particular, at $\lambda = 0$,

$$f_{p',S}(x,z) = e^{-ip'x}(z_1 e^{-i\varphi/2} - \bar{z}_2 e^{i\varphi/2})^{2S}. \tag{5.35}$$

This function describes a massless particle with the momentum p' and Lorentz spin S.

3. In case of tachyons, the state with $p_0 = p_2 = 0$, $p_1 = im$,

$$f_{p,S}(x,z) = e^{-ip_1 x^1} f_S(z),$$

has the stationary subgroup, which can be found from the condition $U^{-1}P_1(U^{-1})^\dagger = P_1$, where

$$U = \pm\begin{pmatrix} \cosh\theta/2 & i\sinh\theta/2 \\ -i\sinh\theta/2 & \cosh\theta/2 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & -im \\ -im & 0 \end{pmatrix}.$$

This subgroup is isomorphic to $R \otimes Z$ and has the generator $\hat{J}^1$. The eigenvectors $f_{p,S}(x,z)$ for the operators $\hat{S}^1$ and $\hat{\mathbf{S}}^2$, with the eigenvalues σ and $S(S+1)$ respectively, have the form

$$f_{p,S,\sigma}(x,z) = e^{-ip_1 x^1}(\bar{z}_2 + iz_1)^{S+i\sigma}(\bar{z}_2 - iz_1)^{S-i\sigma} = e^{-ip_1 x^1}(\bar{z}_2^2 + z_1^2)^S \left(\frac{\bar{z}_2 - iz_1}{\bar{z}_2 + iz_1}\right)^{-i\sigma}. \tag{5.36}$$

Functions $f_{p,S,\sigma}(x,z)$ are the eigenvectors for the Casimir operators $\hat{W}$ and $\hat{\mathbf{p}}^2$ with the eigenvalues $p_1\sigma$ and $-p_1^2$ respectively. σ has to be real for unitary IR, therefore, for $\sigma \neq 0$, representations, which correspond to the imaginary mass case, are infinite-dimensional in the spin. The case of arbitrary direction of the momentum can be derived by means of a rotation, as was done above for the real and zero mass.

4. Unitary IR of $\tilde{M}(2,1)$, which are connected with the orbit O^0_0, are IR of $SU(1,1)$.

The classification of the single-valued unitary IR of the $\tilde{M}(2,1) = T(3)\times)SU(1,1)$ group can be summarized in a table, which we present below.

TABLE I. Unitary single-valued IR of $\tilde{M}(2,1)$.

mass, orbits	IR	eigenv. $\hat{W} = \hat{p}\hat{S}$	states	remarks
$m > 0$,	$T^+_{m,s}$	ms	(5.31)	$s \geq 0$, integer or half-integer
O^+_m, O^-_m	$T^-_{m,s}$	$-ms$		
$m = 0$,	$T^+_{0,\varepsilon}$	0	(5.35)	$\varepsilon = 0, 1$
O^+_0, O^-_0	$T^-_{0,\varepsilon}$	0		
	$T^+_{0,K,\varepsilon}$	$K = p\lambda$	(5.34)	$K \neq 0$, real, infinite-dimensional IR
	$T^-_{0,K,\varepsilon}$	$K = p\lambda$		
$m^2 < 0$,	$T_{m,0,\varepsilon}$	0	(5.36)	
O_m	$T_{m,\sigma,\varepsilon}$	$im\sigma$	(5.36)	$\sigma \neq 0$, real, infinite-dimensional IR
$m = 0$,	T^+_S, T^-_S	0	see	discrete series of $SU(1,1)$
O^0_0	$T_{S,\varepsilon}$	0	Appendix	principal series of $SU(1,1)$
	T_S	0		supplementary series of $SU(1,1)$
	T^0_0	0		invariant

The IR states of $SU(1,1)$, correspondent to the orbit O_0^0, do not depend on x and are invariant under translations. The sign (+ or −) at T is related to the sign of p_0. The characteristics "infinite-dimensional" mean infinite-dimensionality in the spin space.

The finite-dimensional in spin wave functions of massless particles and tachyons are zero modes of the operator $\hat{W}$.

To complete the picture one has to add to this table multivalued representations $T^+_{m,s}$ and $T^-_{m,s}$ at non-integer $2s$, and multivalued IR of $SU(1,1)$, described in the Appendix. The explicit form of states, which are transformed under the representations $T^+_{m,s}$ and $T^-_{m,s}$ at non-integer $2s$, can also be given by the formula (5.31), however, in this case, z-decomposition generates infinite number of components. Just those IR are used to describe anyons.

VI. APPENDIX: UNITARY IR AND COHERENT STATES OF THE $SU(1,1)$ GROUP

The 2+1 Lorentz group $SO(2,1)$, and close related groups $SU(1,1)$ and $SL(2,R)$ with the same algebra, where studied in numerous papers [11–13,20–37]. Their finite-dimensional IR and unitary IR (discrete series) are used to describe spin in $2+1$-dimensions [4]. As is known, $SO(3,1)$ has only principal and supplementary series of unitary representations, and the principal series is used to describe spin in $3+1$ dimensions [38,39]. In this connection, besides of all, it is important to consider the same series of $SO(2,1)$ or $SU(1,1)$.

We are going to describe unitary IR of $SU(1,1)$, their discrete bases and corresponding CS. The consideration, to be complete, is going to repeat some known results, but also to present some new ones. For example, we are constructing CS in unitary IR of the principal series at arbitrary fractional projections of the angular momentum in addition to [35], where only integer ones were considered. We construct unitary IR, including multivalued, in spaces of functions on various manifolds connected with $SO(2,1)$ or $SU(1,1)$, whereas usually they restrict themselves to the unit disk or to a circle. In particular, we consider decompositions of functions on a cone and one-sheeted hyperboloid with respect to unitary IR of $SO(2,1)$.

Consider the left representation $T(U)$, $U \in SU(1,1)$, acting in the space of functions $f(v)$,

$$T(U)f(v) = f(U^{-1}v), \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{6.1}$$

The matrices U^{-1} can be parameterized by two complex numbers u^1, u^2,

$$U^{-1} = \begin{pmatrix} \bar{u}_1 & -u_2 \\ -\bar{u}_2 & u_1 \end{pmatrix}, \quad |u^1|^2 - |u^2|^2 = 1. \tag{6.2}$$

The combination

$$|v_1|^2 - |v_2|^2 = C \tag{6.3}$$

remains invariant under the SU(1,1) transformations. Generators $\hat{J}^\mu$, which correspond to one-parametrical subgroups with parameters $-\alpha^\mu$ (see (2.3)), and arising $\hat{J}_+$ and lowering $\hat{J}_-$ operators have the following form in this representation

$$\begin{aligned} \hat{J}^0 &= -(1/2)(v_1\partial/v_1 - v_2\partial/v_2), \quad \hat{J}_- = v_1\partial/v_2, \quad \hat{J}_+ = v_2\partial/v_1, \\ \hat{J}^1 &= (1/2)(\hat{J}_+ - \hat{J}_-) = (1/2)(v_2\partial/v_1 - v_1\partial/v_2), \\ \hat{J}^2 &= (i/2)(\hat{J}_+ + \hat{J}_-) = (i/2)(v_1\partial/v_2 + v_2\partial/v_1). \end{aligned} \tag{6.4}$$

They obey the commutation relations

$$[\hat{J}^\mu, \hat{J}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{J}_\eta, \quad [\hat{J}_+, \hat{J}_-] = 2\hat{J}^0, \quad [\hat{J}^0, \hat{J}_\pm] = \pm\hat{J}_\pm,$$

so that $\hat{\mathbf{J}}^2$ is a Casimir operator,

$$\hat{\mathbf{J}}^2 = \hat{J}_\mu\hat{J}^\mu = (\hat{J}_0)^2 + \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) = \frac{1}{4}(v_1\partial/\partial v_1 + v_2\partial/\partial v_2)(v_1\partial/\partial v_1 + v_2\partial/\partial v_2 + 2).$$

Let us take functions of the form $f_{n_1n_2}(v) = v_1^{n_1}v_2^{n_2}$. The action of the generators on these functions can be found[2],

$$\begin{aligned} &\hat{J}^0 f_{n_1n_2} = m f_{n_1n_2}, \quad \hat{\mathbf{J}}^2 f_{n_1n_2} = j(j+1) f_{n_1n_2}, \quad m = \frac{n_2 - n_1}{2}, \ j = \frac{n_1 + n_2}{2}, \\ &\hat{J}_- f_{n_1n_2} = n_2 f_{n_1+1, n_2-1}, \quad \hat{J}_+ f_{n_1n_2} = n_1 f_{n_1-1, n_2+1}. \end{aligned} \tag{6.5}$$

Thus, quasi-polynomials of the power $2j$ form a IR space (j characterizes the IR). $\hat{J}_+$ and $\hat{J}_-$ are arising and lowering operators for the projection of the angular momentum $m = (n_2 - n_1)/2$. If $n_2 \geq 0$ and is integer then $f_{n_1n_2}$ belongs to IR, which has the lowest weight v_1^{2j}; if $n_1 \geq 0$, and is integer then IR has the highest weight v_2^{2j}; if both $n_i \geq 0$, $i = 1,2$ and are integer then IR in finite-dimensional (has both the highest and lowest weights). For unitary IR of $SU(1,1)$: $(\hat{J}^0)^+ = \hat{J}^0$, $\hat{J}^+_\pm = -\hat{J}_\mp$, that means $n_2 - n_1$ is real, and $n_1(n_2+1) \leq 0$, $n_2(n_1+1) \leq 0$, whereas for IR of $SU(2)$: $\hat{J}^+_\pm = \hat{J}_\mp$ and $n_1(n_2+1) \geq 0$, $n_2(n_1+1) \geq 0$ [37]. At a given j one can select

$$N_{n_1n_2} v_1^{n_1} v_2^{n_2}, \tag{6.6}$$

as elements of a discrete basis in the space of functions $f_j(v)$, where $N_{n_1n_2}$ is the normalization constant, and $n_1 = j - m$, $n_2 = j + m$.

A classification and weight structure of unitary infinite-dimensional and non-unitary finite-dimensional IR of $SU(1,1)$ is presented on the Fig.1.

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[2]We are going to use here the notation m for the angular momentum projection (the same was used for the mass), hoping that this will not lead to a misunderstanding.

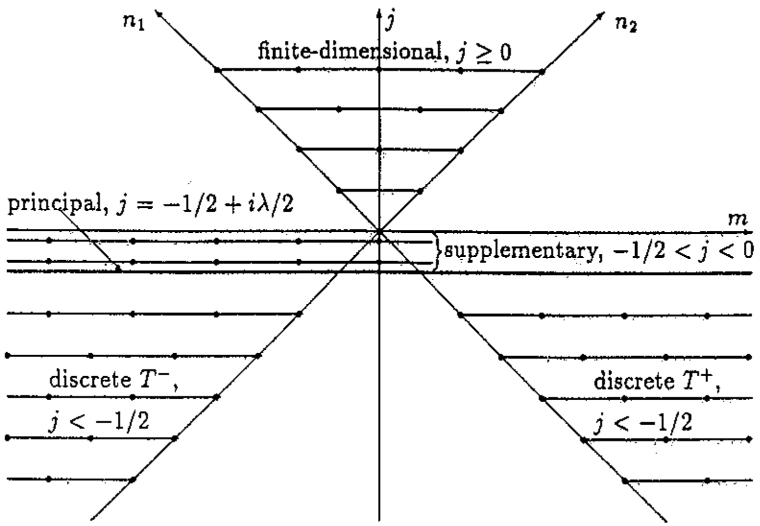

FIG. 1. Weight diagrams for unitary and finite-dimensional IR of $SU(1,1)$

To describe IR of different series one has to define in more detail the space of functions $f(v)$. At different C in eq.(6.3) one can use the following parameterization of v_1 and v_2 :

$$\begin{aligned} C = 0: \quad & v_1 = \rho e^{i(\varphi+\omega)/2}, \quad v_2 = \rho e^{i(\omega-\varphi)/2}, \\ & 0 < \rho < +\infty, \quad 0 \leq \varphi < 4\pi, \quad 0 \leq \omega < 2\pi; \end{aligned} \tag{6.7}$$

$$\begin{aligned} C = 1: \quad & v_1 = \cosh(\theta/2) e^{i(\varphi+\omega)/2}, \quad v_2 = \sinh(\theta/2) e^{i(\omega-\varphi)/2}, \\ & 0 \leq \theta < +\infty, \quad 0 \leq \varphi < 4\pi, \quad 0 \leq \omega < 2\pi\ . \end{aligned} \tag{6.8}$$

The case of negative C ($C = -1$) is reduced to (6.8) by the replacement $v_1 \leftrightarrow v_2$. The parameter ω is not changed under the group transformations in the case (6.7), thus, there are two complex manifolds, on which the group is acting transitive: the complex hyperboloid (6.8) and the cone,

$$C = 0: \ v_1 = \rho e^{i\varphi/2}, \ v_2 = \rho e^{-i\varphi/2}, \ 0 < \rho < +\infty, \ 0 \leq \varphi < 4\pi. \tag{6.9}$$

Using the components (v_1, v_2) of the spinor and the complex conjugate components $(\overline{v}_1, \overline{v}_2)$, one can construct objects (x^0, x^1, x^2), which are transformed under three-dimensional vector IR with $j = 1$,

$$x^0 = (|v_1|^2 + |v_2|^2)/2, \quad x^1 = (\overline{v}_1 v_2 + v_1 \overline{v}_2)/2, \quad x^2 = (v_1 \overline{v}_2 - \overline{v}_1 v_2)/2i, \tag{6.10}$$

$$x^0 = v_1 v_2, \quad x^1 = (v_1^2 + v_2^2)/2, \quad x^2 = (v_1^2 - v_2^2)/2i. \tag{6.11}$$

The vectors (6.10) and (6.11) have the same transformation properties, since the spinors (v_1, v_2) and $(\overline{v}_2, \overline{v}_1)$ are transformed equally. The latter can be easily checked, using the explicit form of the matrix (6.2). Substituting (6.9) into (6.10) or (6.11), we get the cone

$$x^0 = \rho^2, \quad x^1 = -\rho^2 \cos\varphi, \quad x^2 = -\rho^2 \sin\varphi, \qquad x_0^2 - x_1^2 - x_2^2 = 0. \tag{6.12}$$

Substituting (6.8) into (6.10), we get two-sheeted hyperboloid

$$x^0 = \cosh\theta, \quad x^1 = -\sinh\theta\cos\varphi, \quad x^2 = -\sinh\theta\sin\varphi, \qquad x_0^2 - x_1^2 - x_2^2 = 1.$$

If v_k are periodic in φ with the period 4π, then x_μ are also periodic with the period

Let us turn first to IR of the discrete series T_j^+ ($m = -j, -j+1, -j+2, \ldots$) and T_j^- $j, j-1, j-2, \ldots$), $j < -1/2$, the theory of which is quite similar to the one of the dimensional IR. The IR T_j^+ and T_j^- can be realized in the space of functions $f(v)$, w and v_2 belong to the case (6.8). The scalar product of functions on the complex hyper

$$\begin{aligned} \langle f_1 | f_2 \rangle &= \frac{1}{8\pi^2} \int \overline{f}_1 f_2 \delta(|v_1|^2 - |v_2|^2 - 1) d^2v_1 d^2v_2 \\ &= \frac{1}{8\pi^2} \int_0^{2\pi} d\omega \int_{-2\pi}^{2\pi} d\varphi \int_0^\infty \overline{f}_1 f_2 \sinh\theta d\theta, \ \ d^2v = d\Re v \ d\Im v\ , \end{aligned}$$

allows one to normalize the elements of the discrete basis T_j^+ at $j < -1/2$,

$$\begin{aligned} \psi_{j,m}(v) = \langle v \mid jm \rangle &= \left(\frac{(-1)^{n_2}\Gamma(-n_1)}{n_2!\Gamma(-2j)} \right)^{1/2} v_1^{n_1} v_2^{n_2} \\ &= \left(\frac{(-1)^{n_2}\Gamma(-n_1)}{n_2!\Gamma(-2j)} \right)^{1/2} (\cosh(\theta/2))^{n_1} (\sinh(\theta/2))^{n_2} e^{im(\varphi+4\pi k)} e^{ij(\omega+2\pi k)}. \end{aligned}$$

The projection m, and therefore j ($j = m_{\max}$ in T_j^-, $j = -m_{\min}$ in T_j^+), have to run integer and half integer, $j = -1, -3/2, -2, \ldots$, for representations in spaces of single functions.

The lowest weight $\langle v | j\ {-j} \rangle = v_2^{2j}$ has a stationary subgroup $U(1)$ and CS are eterized by dots of the upper sheet of two-sheeted hyperboloid $SU(1,1)/U(1)$. An form of CS can be obtained by the action of finite transformations on the lowest we

$$\psi_{j,u}(v) = \langle v \mid ju \rangle = (\bar{u}_1 v_1 + u_2 v_2)^{2j},$$

where $u = (\bar{u}_1, -u_2)$, $\bar{u}_1 = \cosh(\theta_1/2) e^{im\varphi_1/2}$, $-u_2 = \sinh(\theta_1/2) e^{-im\varphi_1/2}$ are element matrix (6.2). The CS overlapping has the form

$$\langle j'u' \mid ju \rangle = \delta_{jj'} (u_1' \bar{u}_1 - \bar{u}_2' u_2)^{2j}.$$

A detailed description of CS of the discrete series of $SU(n,1)$ one can find in [36] $SU(1,1)$ in [35–37]. The representations T_j^+ and T_j^- are conjugate; the discrete basis be derived by means of the complex conjugation from (6.15) or by the replacement

For the functions, which are transformed with respect to one and the same repres T_j^+, the integral over ω in (6.14) gives 2π. The completeness relation at a given j written both in terms of the discrete basis and in terms of CS,

$$\hat{1}_j = \sum_{m=-\infty}^{j} \mid jm \rangle\langle jm \mid = \frac{-2j-1}{4\pi} \int_{-2\pi}^{2\pi} d\varphi_1 \int_0^\infty \mid j\theta_1\varphi_1 \rangle \langle j\theta_1\varphi_1 \mid \sinh\theta_1 d\theta_1.$$

The parameter j takes discrete values and the basis functions are orthonormalize Kronecker symbol $\delta_{jj'}$ for the single-valued IR of the discrete series, whereas for the series the condition of the orthonormality contains the δ-function $\delta(j - j')$. Principal series can be constructed both in the space of functions on the complex hyperboloid (6.8), and on the cone (6.9).

One can construct the principal series on the cone (6.9) with the scalar product

$$\langle f_1 \mid f_2 \rangle = (1/8\pi^2) \int_{-2\pi}^{2\pi} d\varphi \int_0^{\infty} \overline{f_1(\rho,\varphi)} f_2(\rho,\varphi) \rho \, d\rho. \tag{6.19}$$

We get $C_{n_1 n_2} = 1$, $n_1 + n_2 = 2j = -1 + i\lambda$, $2m = n_2 - n_1$, for the elements of the discrete basis (6.6) in case of the principal series,

$$\hat{J}_{\pm} = e^{\pm i\varphi}((1/2)\rho\partial/\partial\rho \pm i\partial/\partial\varphi), \quad \hat{J}_0 = -i\partial/\partial\varphi, \tag{6.20}$$

$$\langle \rho\varphi \mid \lambda m \rangle = v_1^{n_1} v_2^{n_2} = \rho^{-1+i\lambda} e^{im(\varphi+4\pi k)}, \quad \langle \lambda m \mid \lambda' m' \rangle = \delta(\lambda - \lambda')\delta_{mm'}, \tag{6.21}$$

$$\langle \rho\varphi \mid \rho'\varphi' \rangle = (1/\rho\rho')\delta(\ln\rho - \ln\rho')\delta(\varphi - \varphi') = (1/\rho)\delta(\rho - \rho')\delta(\varphi - \varphi').$$

Two IR in the space of single-valued functions (with integer and half-integer m, the first and the second principal series accordingly to the terminology of the work [13]) correspond to each given λ,

$$\hat{1} = \frac{1}{8\pi^2} \int_{-2\pi}^{2\pi} d\varphi \int_0^{\infty} \mid \rho\varphi\rangle\langle\rho\varphi \mid \rho d\rho = \frac{1}{8\pi^2} \int_{-\infty}^{+\infty} d\lambda \sum_m \mid \lambda m\rangle\langle\lambda m \mid .$$

The summation in the last equation is running over all integer and half-integer m. Multi-valued IR are characterized not only by λ but also by a number ε, $|\varepsilon| \leq 1/2$, which gives the nearest to zero value of m (for single-valued IR, $\varepsilon = 0$ or $\varepsilon = \pm 1/2$). Elements of the infinite-valued IR space are not periodic in φ. Thus, an arbitrary representation of the principal series is defined by two numbers (λ, ε), where $j = (-1 + i\lambda)/2$ characterizes the angular momentum square, $\mathbf{J}^2 = j(j+1) = (-1 - \lambda^2)/4$, and ε characterizes possible values of the momentum projection $m = \varepsilon + [m]$. There is a certain analogy with IR of the principal series of $SO(3,1)$, which are defined by two numbers (λ, S), where S corresponds to the spin [38,39], and λ defines the square of the four-dimensional angular momentum.

The representation of the principal series $T_{-1/2}$ is reducible at $\lambda = 0$ and $|\varepsilon| = 1/2$, and is split into two IR: $T^{+}_{-1/2}$ ($\varepsilon = -1/2$) and $T^{-}_{-1/2}$ ($\varepsilon = 1/2$); $\varepsilon = \pm 1/2$ corresponds to one and the same IR at $\lambda \neq 0$.

One can remark that, according to (6.21), ρ-dependence of functions on the cone is the same at a fixed j, and it is possible to consider the space of functions $f(\varphi)$ on the circle, what they usually are doing, considering the principal series of IR. However, such a reduction of the representation space is not always reasonable because of the space of functions on the cone appears sometimes naturally in different physical problems.

To construct CS one has to consider orbits in the representation space, factorized with respect to stationary subgroups [35]. The stationary subgroup of the state $|\lambda \; m = 0\rangle = \rho^{-1+i\lambda}$ is $U(1)$, and CS, which correspond to integer m ($\varepsilon = 0$), are parameterized by the dots (θ, ψ) on the upper sheet of the hyperboloid $SU(1,1)/U(1)$. (Such CS were constructed in [35,40] in the space of functions on a circle.) Substituting $\bar{u}_1 = \cosh(\theta/2)e^{i\psi/2}$, $-u_2 = \sinh(\theta/2)e^{-i\psi/2}$, $\rho' = \rho(\cosh\theta + \sinh\theta\cos(\psi + \varphi))^{1/2}$ in (6.1),(6.2), we get CS in the form

$$\langle \rho\varphi \mid \lambda\theta\psi \rangle = (\rho')^{-1+i\lambda} = \rho^{-1+i\lambda}(\cosh\theta + \sinh\theta\cos(\psi + \varphi))^{-1/2+i\lambda/2},$$

$$\begin{aligned}\langle \lambda m \mid \lambda_1\theta\psi \rangle &= \frac{1}{8\pi^2} \int\int \langle \lambda m \mid \rho\varphi\rangle\langle\rho\varphi \mid \lambda_1\theta\psi\rangle \rho d\rho d\varphi \\ &= (1/2\pi)\delta(\lambda - \lambda_1) \int_0^{2\pi} e^{im\varphi}(\cosh\theta + \sinh\theta\cos(\psi + \varphi))^{-1/2+i\lambda/2} d\varphi \\ &= \delta(\lambda - \lambda_1) \frac{\Gamma(m+1)}{\Gamma(m + 1/2 + i\lambda/2)} P^m_{-1/2+i\lambda/2}(\cosh\theta) e^{-im\psi},\end{aligned} \tag{6.22}$$

where $P^m_{-1/2+i\lambda/2}(\cosh\theta)$ is adjoint Legendre function. At $m = 0$ the latter goes over to zona harmonic $P_{-1/2+i\lambda/2}(\cosh\theta)$ (it is also called cone function [13,41]). To get CS at arbitrar ε one has to act by means of finite transformations on the state $|\lambda \; m = \varepsilon\rangle = \rho^{-1+i\lambda} e^{i\varepsilon\varphi}$,

$$\begin{aligned}\langle \rho\varphi \mid \lambda\varepsilon\theta\psi \rangle &= \left((v_1\bar{u}_1 - v_2 u_2)(-v_1\bar{u}_2 + v_2 u^1)\right)^{-1/2+i\lambda/2} \left(\frac{-v_1\bar{u}_2 + v_2 u_1}{v_1\bar{u}_1 - v_2 u_2}\right)^{\varepsilon} \\ &= \rho^{-1+i\lambda}(\cosh\theta + \sinh\theta\cos(\varphi + \psi))^{-1/2+i\lambda/2} \\ &\times \left(\frac{\cosh(\theta/2)\exp[-i(\varphi - \psi)/2] + \sinh(\theta/2)\exp[i(\varphi - \psi)/2]}{\cosh(\theta/2)\exp[i(\varphi - \psi)/2] + \sinh(\theta/2)\exp[-i(\varphi - \psi)/2]}\right)^{\varepsilon}.\end{aligned} \tag{6.23}$$

The case $\varepsilon = 0$, which we have considered above, and $\varepsilon = \pm 1/2$, correspond to representa tions in spaces of single-valued functions. In the latter case at $m = \pm 1/2$ we get

$$\begin{aligned}\langle \rho\varphi \mid \lambda \; 1/2, \theta\psi \rangle &= (v_1\bar{u}_1 - v_2 u_2)^{-1} \mid v_1\bar{u}_1 - v_2 u_2 \mid^{i\lambda}, \\ \langle \rho\varphi \mid \lambda \; -1/2, \theta\psi \rangle &= (-v_1\bar{u}_2 + v_2 u_1)^{-1} \mid v_1\bar{u}_1 - v_2 u_2 \mid^{i\lambda}.\end{aligned} \tag{6.24}$$

At $\lambda = 0$ the CS take a simple form

$$\begin{aligned}\langle \rho\varphi \mid 0 \; 1/2, \theta\psi \rangle &= (v_1\bar{u}_1 - v_2 u_2)^{-1}, \\ \langle \rho\varphi \mid 0 \; -1/2, \theta\psi \rangle &= (-v_1\bar{u}_2 + v_2 u_1)^{-1},\end{aligned} \tag{6.25}$$

which coincides with the explicit form of CS of the discrete series (6.16) (in this case, all th difference between CS of different series consists in different domains of v_1 and v_2, see (6.8 and (6.9)).

Let us turn to IR of supplementary series. The integral in (6.19) is divergent at real. However, one can use a convergent "non-local" scalar product

$$\langle f_1 \mid f_2 \rangle = \int\int \overline{f_1(x_1)} f_2(x_2) I(x_1, x_2) dx_1 dx_2, \tag{6.26}$$

where the kernel function $I(x_1, x_2)$ has to be invariant with respect to the group transforma tions. For the cone one can select an invariant expression $(v_1\bar{v}_1' - \bar{v}_2 v_2') = 2i\sin(\varphi/2 - \varphi'/2)\rho\rho$ At a fixed j representation functions have the form $\rho^{2j} f(\varphi)$. Let us select $I(x_1, x_2)$ $|(v_1 v_1' - v_2 v_2')/2|^{-2j}$, then the integrand in (6.26) is $\overline{f_1(\varphi)} f_2(\varphi') |\sin(\varphi/2 - \varphi'/2)|^{-2j}$. It do not depend on ρ, so that at a fixed j (6.26) takes the form

$$\langle f_1 | f_2 \rangle = \int_{-2\pi}^{2\pi} \int_{-2\pi}^{2\pi} \overline{f_1(\varphi)} f_2(\varphi') |\sin(\varphi/2 - \varphi'/2)|^{-2j} d\varphi d\varphi', \tag{6.27}$$

where $-1/2 < j < 0$, the latter is necessary for the scalar product to be convergent ar positive defined.

For the single-valued representations of the supplementary series m is integer, for the multi-valued representations one has to introduce ε, $|\varepsilon| \le |j|$ (restrictions on ε follow from the unitarity of the representation, see Fig.1). Matrix elements of the supplementary series IR are expressed via so called torus function [41].

An invariant dispersion with respect to $SO(2,1)$ transformations can be written as

$$\Delta \mathbf{J}^2 = \langle \hat{J}_\mu \hat{J}^\mu \rangle \langle \hat{J}_\mu \rangle \langle \hat{J}^\mu \rangle = (\Delta J^0)^2 - (\Delta J^1)^2 - (\Delta J^2)^2. \tag{6.28}$$

It has the value $j(j+1) - m^2$ on the states $|jm\rangle$. At a given j CS minimize the absolute value of the dispersion (6.28). For CS of the discrete series $\Delta J^2 = j$, and for the principal series $\Delta J^2 = -1/4 - \lambda^2/4 - \varepsilon^2$.

Below we present a short summary of IR studied.

For single-valued unitary IR of $SO(2,1)$ the angular momentum projection m is integer, for single-valued IR of $SU(1,1)$ it is integer or half-integer. For multivalued unitary IR the projection m can take any real values. Here we meet an essential difference with the Lorentz group in four dimensions, for unitary representations of the group this projection is always integer or half-integer. That is connected with the existence of non-Abelian compact subgroup $SU(2) \sim SO(3)$. Representations of the discrete series $T_j^{\pm}(g)$ of $SU(1,1)$ at real, integer and half-integer $j < -1/2$ are single-valued and have the highest and lowest weights $m = \pm j$. Representations of the principal series $T_{j,\varepsilon}(g)$, $j = -1/2 + i\lambda$, $-1/2 < \varepsilon \le 1/2$, are single-valued at $\varepsilon = 0$ and at $\varepsilon = 1/2$. At $\varepsilon \ne 1/2$ representations are irreducible and have neither highest nor lowest weights; at $\varepsilon = 1/2$ the representation is split in two ones: $T^-_{j,1/2}(g)$ with the highest weight $m = -1/2$ and $T^+_{j,1/2}(g)$ with the lowest weight $m = 1/2$.

Now we have to make some technical remark. As it follows from our consideration, representatives of all non-equivalent finite-dimensional and unitary IR of $SU(1,1)$ can be constructed in the space of functions on two complex variables v_1 and v_2 only. At the same time, studying the left GRR (4.4) of the $M(2,1)$ group, it is convenient to use functions on the elements z_1, $\bar{z}_2$ of the first column of the matrix Z. In such a space the spin generators (4.7) are reduced to the form

$$\begin{aligned} \hat{S}^0 &= (1/2)(z_1\partial/z_1 - \bar{z}_2\partial/\bar{z}_2), \quad \hat{S}^1 = (i/2)(z_1\partial/\bar{z}_2 + \bar{z}_2\partial/z_1), \\ \hat{S}^2 &= -(1/2)(z_1\partial/\bar{z}_2 - \bar{z}_2\partial/z_1), \end{aligned} \tag{6.29}$$

In fact, after the re-notation $z_1 \to v_1$, $\bar{z}_2 \to v_2$ they go over to the generators (6.4).

All said about IR of $SU(1,1)$ is summarized for the case of spin operators in Table 2. We denote the eigenvalue of $\hat{S}^0$ as ζ and eigenvalue of $\hat{\mathbf{S}}^2$ as $S(S+1)$, that, in fact, correspond to re-notation $j \to S$, $m \to \zeta$. The parameter n in Table 2 is integer and $n \ge 0$; s-v or m-v signify single-valued or multivalued IR respectively.

TABLE II. Unitary and finite-dimensional IR of $SU(1,1)$.

series	S	ζ	s-v or m-v
finite-dimensional : T_S	$S \ge 0$, integer or half-integer	$S - n$, $n \le 2S$	s-v
discrete:	$S < -1/2$		s-v at $S = -1 - n/2$
T_S^+		$-S + n$	
T_S^-		$S - n$	
principal:			
$T_{S,\varepsilon}$, $-1/2 < \varepsilon \le 1/2$	$S = -1/2 + i\lambda/2$	$\varepsilon \pm n$	s-v at $\varepsilon = 0, 1/2$
$T_{-1/2,1/2} = T^+_{-1/2} \oplus T^-_{-1/2}$,			
$T^+_{-1/2}$	$S = -1/2$	$1/2 + n$	s-v
$T^-_{-1/2}$	$S = -1/2$	$-1/2 - n$	s-v
supplementary:	$-1/2 < S < 0$		
$T_{S,\varepsilon}$, $\lvert\varepsilon\rvert < \lvert S\rvert$		$\varepsilon \pm n$	s-v at $\varepsilon = 0$
T_S^+ $(\varepsilon = S)$		$\varepsilon + n$	m-v
T_S^- $(\varepsilon = -S)$		$\varepsilon - n$	m-v

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