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EFFECT OF DELAY ON THE BOUNDARY OF THE BASIN OF ATTRACTION IN A SYSTEM OF TWO NEURONS

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Abstract

The behavior of neural networks may be influenced by transmission delays and many studies have derived constraints on network parameters such as connection weights and neuron transfer functions to ensure that the asymptotic dynamics of a network with delay remains similar to that of the corresponding system without delay. However, even when the delay does not affect the asymptotic behavior of the system, it may still influence other important features in the system's dynamics such as the boundary of the basin of attraction of the stable equilibria. As a first step towards a better understanding of the influence of delay, we study the dynamics of a system constituted by two

neurons interconnected through delayed excitatory connections. In this case, the system with delay has exactly the same stable equilibrium points as the associated system without delay. Moreover, in both the network with delay and the corresponding one without delay, most trajectories converge to these stable equilibria. So that, the network with delay displays an asymptotic behavior similar to the corresponding system without delay. However, we show that even in this simple system, the boundary of the basin of attraction of a stable equilibrium point depends on the value of the delays.

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I. INTRODUCTION

In many neural network applications, such as associative memories, the network is designed so that stable equilibrium points represent the stored information [1,2]. In this framework, relevant information is retrieved by initializing the network at a point within the basin of attraction of a stable equilibrium point, and allowing it to evolve to its stationary state. Therefore, many studies have been concerned with the asymptotic behavior of neural networks and have derived conditions on network parameters to ensure that all or almost all trajectories eventually converge to equilibria, thus avoiding spurious undamped oscillations [1,2]. Such networks are referred to as convergent or quasi-convergent. However, loss of stability may arise in hardware implementations of (quasi-) convergent neural networks due to the presence of finite time transmission times, referred to here as delay, between units. This phenomenon, as well as possible applications of networks implementing delay, have motivated a number of studies dealing with the effect of interneural transmission times on the asymptotic behavior of neural networks [3–16]. Many of these studies derive conditions on the network parameters, such as the connection weights, the neuron transfer functions as well as the delays to ensure that the network with delay behaves in a way similar to the associated network when the delays are set to zero. These studies are mainly concerned with two aspects of the dynamics. *i)* To ensure that the delay does not induce the loss of information stored in the stable equilibrium points. This is satisfied when the system with delay has exactly the same stable equilibria as the one without delay, that is, the delay does not alter the local stability of the stable equilibrium points [3–6]. *ii)* To ensure that the delay does not induce spurious stable undamped oscillations. This is satisfied when both the system without delay and the one implementing delay are (quasi-) convergent, that is, the delay does not alter the global stability of the system [4–14].

The above constraints on network parameters avoid delay induced changes in the asymptotic behavior of the system. However even under such constraints, the delay may influence important features in the system's dynamics. For instance, changing the delay can alter the

boundary of the basins of attraction of the stable equilibrium points. This can be of prime importance in associative memory networks in which the position of the basin boundaries determines which information is retrieved for a given initial condition. Thus changing the shape of the basin boundaries alters the classification.

In this paper, we study the dynamics of a network of two neurons connected through delayed excitatory connections. This system has been chosen because it is simple enough so that thorough theoretical and numerical analysis of its dynamics are possible. Indeed, we show that it satisfies both of the conditions stated above, that is, the delay does not affect the local stability of its stable equilibria, and, no matter what value the delays take, the two-neuron network remains quasi-convergent. Yet, we show that the boundary separating the basins of attraction of two stable equilibria depends on the delays.

In section II we present the two-neuron network model. The boundary between the basins of attraction of two stable equilibria of the system is studied in section IV. In section V the boundary is numerically estimated for several values of the delays. A discussion is presented in section VI.

II. THE MODEL

In the nonlinear graded response model (NGRM), a neuron is described by its activation at time t and a sigmoidal output function σ that depends on the activation. A constant decay rate of the activation is also taken into account. For details and references on the NGRM see [1,2]. In this paper, we consider a system of two neurons connected to each other. Denoting the activation of the neurons by x and y, their decay rates by γ and γ', their connection weights by W and W', the delay related to each connection by A and A' and the constant input received by each neuron by K and K' (schematically represented in Fig. 1), we write the following system of delayed differential equations (DDEs) for the evolution of x and y:

$$\begin{cases} \frac{dx}{dt}(t) = -\gamma x(t) + K + W\sigma(y(t-A)) \\ \frac{dy}{dt}(t) = -\gamma' y(t) + K' + W'\sigma(x(t-A')), \end{cases} \tag{1}$$

where σ is given by:

$$\sigma(x) = \frac{1}{1+e^{-x}} \quad . \tag{2}$$

In many cases, the dynamics of DDEs [17–19] are more complicated than that of the related ordinary differential equations (ODEs) obtained by setting the delays equal to zero. This difference comes from the fact that, no matter how small the delays are, DDEs always generate infinite dimensional dynamical systems. Indeed, in order to determine the evolution of the variables x and y in Eq. (1), it is necessary to give the initial condition for $x(t)$ and $y(t)$ in the intervals $[-A',0]$ and $[-A,0]$, respectively. So, it is natural to consider as the phase space of Eq. (1) the set $S = \mathcal{C}([-A',0],\mathbb{R}) \times \mathcal{C}([-A,0],\mathbb{R})$, where $\mathcal{C}(I,\mathbb{R})$ designates the space of continuous functions from the interval I to the real line $\mathbb{R}$. The set S, with the usual supremum norm, is an infinite dimensional Banach space.

For an initial condition $\phi = (\phi_1, \phi_2)$ in S, there exists a unique solution of Eq. (1), noted $z(t,\phi) = (x(t,\phi), y(t,\phi))$, such that $x(\theta',\phi) = \phi_1(\theta')$ for $-A' \leq \theta' \leq 0$, $y(\theta,\phi) = \phi_2(\theta)$ for $-A \leq \theta \leq 0$ and $z(t,\phi)$ satisfies Eq. (1) for $t \geq 0$. For such a solution of the DDE, we note $z_t(\phi) = (x_t(\phi), y_t(\phi))$ the element of S defined by $x_t(\phi)(\theta') = x(t+\theta',\phi)$, for $\theta' \in [-A',0]$, and $y_t(\phi)(\theta) = y(t+\theta,\phi)$, for $\theta \in [-A,0]$. z_t is the differentiable semi-flow generated by Eq. (1) in the space S [20]. From this point on, the dependence on the initial condition ϕ will not be indicated explicitly in the notations when no confusion results from this omission.

In spite of the differences between DDEs and their related ODEs mentioned above, there are cases in which their dynamics are, in some sense, very similar. For Eq. (1), this happens, for instance, when both connection weights are strictly positive, $W > 0$ and $W' > 0$ (positive feedback). In the rest of the paper we will only consider this case.

At this point, in order to be more precise, we need to introduce some definitions. For $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2)$ in S, we say that ϕ is larger (resp. strictly larger) than ψ, noted $\phi \geq \psi$ (resp. $\phi > \psi$), if $\phi_1(\theta') \geq \psi_1(\theta')$ (resp. $\phi_1(\theta') > \psi_1(\theta')$) for all θ' in $[-A',0]$

and $\phi_2(\theta) \geq \psi_2(\theta)$ (resp. $\phi_2(\theta) > \psi_2(\theta)$) for all θ in $[-A,0]$. When $\psi < \phi$, the set defined by $\{U \in S, \psi < U < \phi\}$ is an open subset of S.

The positive feedback ($W > 0$, $W' > 0$) condition and theorem 2.5 in [21] (see also [9]) imply that Eq. (1) generates a strongly monotone semi-flow, that is:

$$\text{If} \quad \phi \leq \phi' \quad \text{and} \quad \phi \neq \phi' \quad \text{then} \quad z_t(\phi) < z_t(\phi') \quad \text{for} \quad t \geq 2\max\{A', A\}. \tag{3}$$

Throughout this paper, constant functions in S are identified with the value they take in $\mathbb{R}^2$. A constant solution of system (1) is referred to as an equilibrium point. $z(t) = (x(t), y(t))$ taking the value (a,b), that is $x(t') = a$ for $t' \geq -A'$ and $y(t) = b$ for $t \geq -A$, is a solution of Eq. (1) if and only if (a,b) satisfies the system:

$$\begin{cases} -\gamma a + K + W\sigma(b) = 0 \\ -\gamma' b + K' + W'\sigma(a) = 0 \quad . \end{cases} \tag{4}$$

The number and the value of the solutions of system (4) depend on the values of the parameters $(\gamma, \gamma', W, W', K, K')$. The parameter set can be separated into two regions: one in which the system has a unique solution, and another such that it has three solutions. These solutions do not depend on the delays A' and A. Geometrically, they are the intersection points between the curves $-\gamma a + K + W\sigma(b) = 0$ and $-\gamma' b + K' + W'\sigma(a) = 0$ in the (a,b)-plane. An example is shown in figure 2. In the rest of the paper we will restrict our attention to the case where system (4) has three solutions. The constant functions associated with these solutions constitute the equilibria of Eq. (1) and will be denoted by r_1, r_2 and r_3, so that, when considered as constant functions in S, they are ordered as $r_1 < r_2 < r_3$.

The monotonicity property (3) and the results in [21,9] imply that the asymptotic dynamics of Eq. (1) and its related ODE (obtained by setting $A' = A = 0$) is essentially the same in the following sense:

(P1) The equilibrium r_k of (1) is locally asymptotically stable if, and only if, the same is true of the related ODE;

(P2) The union of the basins of attraction of the stable equilibria of DDE (1) is an open and dense set in the phase space S, the same being true of the ODE related to it.

We remind that the basin of attraction of a stable equilibrium point is the set of initial conditions in S whose trajectories eventually converge to the equilibrium point.

From the above results it can be deduced that r_1 and r_3 are locally asymptotically stable equilibrium points and that r_2 is unstable, and the union of the basins of attraction of r_1 and r_3 is an open dense set in S. So that neither DDE (1) nor its related ODE have stable non-constant solutions.

Some authors have studied the effect of delays on the dynamics of neural networks composed of "shunting" units [22–27]. The difference between the shunting model and the NGRM resides in a term representing the reversal potential that multiplies the inputs a neuron receives from other units. So that Eq. (1) is changed into:

$$\begin{cases} \frac{dx}{dt}(t) = -\gamma x(t) + K + (E - x(t))W\sigma(y(t-A)) \\ \frac{dy}{dt}(t) = -\gamma' y(t) + K' + (E' - y(t))W'\sigma(x(t-A')), \end{cases} \tag{5}$$

where E and E' are the reversal potentials of the first and second neuron respectively. Under the assumption $\gamma E > K$ and $\gamma' E' > K'$, and positive connection weights $W > 0$, $W' > 0$, for any initial condition ϕ in S, there is a time T, such that the solution $z_t(\phi)$ of (5) satisfies: $z_t(\phi) < (E, E')$, for $t > T$. As the semiflow generated by (5) is strictly monotonous when the phase space is restricted to $S' = \{\phi \in S, \phi < (E, E')\}$, the results established for the two-NGRM network can be extended to the two-shunting-neuron network.

III. NUMERICAL RESOLUTION

Numerical solutions of DDE (1) are obtained by discretization of time. The equation is then integrated by using the GEAR corrector formula which can be easily adapted to integrating DDEs when nonlinearities are restricted to the terms which contain the delay [28]. The time step used was 10^{-4} (smaller time steps had no effect on the results). The

calculations were carried with double precision on a 64-bit DEC AXP 3000/500 running DECOSF/1 v3.2.

IV. THE BOUNDARY OF THE BASIN OF ATTRACTION

Our goal in this section is to study the boundary B separating the basin of attraction of r_1 from that of r_3. Any neighborhood of a point in B intersects the basins of attraction of both r_1 and r_3.

Theorem. *a) Let u be in S, such that $u > 0$. There exists a continuous, strictly decreasing (with respect to the order defined above) map, b_u, from S to $\mathbb{R}$ such that:*

1. *For all ϕ in S, $\phi + b_u(\phi).u$ is the unique intersection between the line going through ϕ and directed by u (i.e. the set $\{\phi + \lambda u,\ \lambda \in \mathbb{R}\}$) with the boundary separating the two basins of attraction.*
2. *the set $\{\phi \in S, b_u(\phi) > 0\}$ is exactly the basin of attraction of r_1;*
3. *the set $\{\phi \in S, b_u(\phi) < 0\}$ is exactly the basin of attraction of r_3;*
4. *the set $\{\phi \in S, b_u(\phi) = 0\}$ is exactly the boundary separating the two basins of attraction;*

b) Let ϕ belong to the boundary B, then $z_t(\psi)$ converges to r_1 (resp. r_3) as t tends to ∞ for all $\psi \leq \phi$ (resp. $\psi \geq \phi$) such that $\psi \neq \phi$.

In order to prove the theorem, we need the following lemma:

Lemma. *The solution going through an initial condition smaller (resp. larger) than r_2 and different from r_2 converges to r_1 (resp. r_3), that is, for ϕ in S, if $\phi \leq r_2$ (resp. $\phi \geq r_2$) and $\phi \neq r_2$ then $z_t(\phi)$ converges to r_1 (resp. r_3) as t tends to ∞.*

Proof of the lemma. *Let ϕ be in S, such that $\phi \leq r_2$ and $\phi \neq r_2$. Then, from the monotonicity of the semi-flow, we deduce that $z_T(\phi) < r_2$ for T sufficiently large. The set*

$Q = \{U \in S, z_T(\phi) < U < r_2\}$ is an open set. From property (P2) we deduce that Q intersects the union of the basins of attraction of the two stable equilibrium points: there is ψ in Q such that $z_t(\psi)$ converges to either r_1 or r_3. However, solutions going through the points in Q are upper bounded by r_2, so that $z_t(\psi)$ necessarily converges to r_1. In the same way, it can be shown that there is ψ' smaller than ϕ such that $z_t(\psi')$ tends to r_1. Therefore, for t sufficiently large, $z_t(\phi)$ is bounded by two solutions converging to r_1, so that $z_t(\phi)$ tends to r_1. In a similar way, it can be shown that for $\phi \geq r_2$, and $\phi \neq r_2$, $z_t(\phi)$ converges to r_3.

Proof of the theorem. *a) Let u in S be strictly larger than zero ($u > 0$). For a given ϕ in S, and a real number c, we define the translated function ϕ_c given by $\phi_c = \phi + c.u$. Given $\phi \in S$, there exist real numbers c' and c'' with $c' \leq c''$ such that $\phi_c < r_2$ if $c < c'$ and $\phi_c > r_2$ if $c > c''$. From the lemma we deduce that, as $t \to \infty$, $z_t(\phi_c) \to r_1$ if $c < c'$ and $z_t(\phi_c) \to r_3$ if $c > c''$. Now, let us denote by $b'(\phi)$ the supremum of the set of values of c such that $z_t(\phi_c) \to r_1$ as $t \to \infty$ and by $b''(\phi)$ the infimum of the set of values of c such that $z_t(\phi_c) \to r_3$ as $t \to \infty$. We claim that $b'(\phi) = b''(\phi)$. Indeed, suppose this is false. Then, there exists an open set Q of initial conditions $w \in S$ such that $\phi + b'(\phi)u < w < \phi + b''(\phi)u$. According to the monotonicity property, solutions going through initial conditions in Q converge neither to r_1 nor to r_3. This implies that the open set Q does not intersect the union of the basins of attraction of r_1 and r_3, which contradicts property (P2). So, $b'(\phi) = b''(\phi)$ and we denote this value by $b_u(\phi)$. As the basin of attraction of either of the two stable equilibria r_1 and r_3, is an open set, $\phi + b_u(\phi).u$ does not belong to either of the basins and is necessarily in the boundary. The characterization of the basins and the boundary reported in the theorem are derived from the construction of b_u. The fact that the map b_u from S to $\mathbb{R}$ is continuous and strictly decreasing stems from the continuous dependence of the solutions of DDE (1) on initial conditions, and from the monotonicity of the semi-flow.*

b) Let ϕ be on the boundary, for a ψ in S, such that $\psi \leq \phi$ and $\psi \neq \phi$, we have $z_T(\psi) < z_T(\phi)$, for T sufficiently large. Let u be defined by $u = z_T(\phi) - z_T(\psi) > 0$. Then we have $z_T(\phi) = z_T(\psi) + u$. As $z_T(\phi)$ belongs to the boundary we have $b_u(z_T(\psi)) = 1 > 0$, so that $z_T(\psi)$ and therefore ψ belong to the basin of attraction of r_1. In a similar way it can be shown that $z_t(\psi)$ converges to r_3 for $\psi \geq \phi$ and $\psi \neq \phi$.

From the above theorem, we can deduce that the boundary has a "regular" structure. Indeed, let $u > 0$ be in S, and H be a hyperplane supplementary to u. So that for all ϕ in S, we can write in a unique way: $\phi = h + \lambda u$, where $h \in H$ and $\lambda \in \mathbb{R}$. λu is the projection of ϕ onto the line $\mathbb{R}u$ along the direction H. We note $p_u(\phi) = -\lambda$. Then, $p_u(\phi)$ is the unique real number such that $\phi + p_u(\phi)u$ belongs to H.

Let ϕ belong to the boundary B, we can write $\phi = h - p_u(\phi)u$. From the definition of b_u, we know that $b_u(h)$ is the unique real number such that $h + b_u(h)u$ belongs to B, so that we have necessarily: $p_u(\phi) = -b_u(h)$. From this we can deduce that the boundary is homeomorphic to the linear hyperplane H.

Corollary 1. *The map $\phi \to \phi + p_u(\phi)u$ from B to H is a homeomorphism with inverse: $h \to h + b_u(h)u$ from H to B.*

Moreover, the theorem indicates that the boundary cannot be folded in a way that it be crossed more than once by any line with a strictly positive (for the order defined in S) direction. This property can be expressed as follows.

Corollary 2. *The boundary B is the graph of a map from the hyperplane H to the line $\mathbb{R}u$.*

Proof of corollary 2. *As the map $(h, \lambda u) \to h + \lambda u$ from $H \times \mathbb{R}u$ to S is an isomorphism, we identify these two spaces. From corollary 1, we obtain that the boundary B is the subset $H \times \mathbb{R}u$ defined by $\{(h, b_u(h)u),\ h \in H\}$, which, by definition, is the graph of the map L from H to $\mathbb{R}u$ defined by: $L(h) = b_u(h)u$.*

The basin boundary can also be characterized in terms of the dynamics of the solutions. We introduce the notion of oscillating functions in S. This is an extension of the definition of real valued functions oscillating on an interval.

Definition. *Let $c = (c_1, c_2)$ be in $\mathbb{R}^2$ and $\phi = (\phi_1, \phi_2)$ be in S. We say that ϕ oscillates around c if it satisfies at least one of the properties below:*

1. *$\phi_1 - c_1$ has at least one zero on the interval $[-A', 0]$,*
2. *$\phi_2 - c_2$ has at least one zero on the interval $[-A, 0]$,*
3. *$(\phi_1(0) - c_1)(\phi_2(0) - c_2) < 0$.*

Therefore, a function $\phi = (\phi_1, \phi_2)$ in S is oscillating when either at least one of its two coordinates ϕ_1 or ϕ_2 is oscillating or one is larger and the other is smaller than the coordinates of c. This is a weak definition for oscillation [29] that also includes the constant function taking the value c. Furthermore, if ϕ in S does not oscillate around c, then we have necessarily either $\phi > c$ or $\phi < c$.

Theorem. *The boundary B between the basins of attraction of the two locally stable equilibria r_1 and r_3 of system (1) is constituted by the solutions z_t that oscillate around r_2 for all $t \geq 0$.*

Proof. *This characterization stems from the definition of oscillating solutions and the previous lemma.*

The above theorems show that the space S is partitioned into three disjoint subsets that are positively invariant by the semi-flow generated by Eq. (1): the two basins of attraction of the stable equilibrium points and the boundary separating them. In terms of the temporal evolution of solutions of system (1), this means that there are only three types of asymptotic behaviors: solutions tend to either r_1 or r_3 or oscillate around r_2. Although transient oscillations can be easily obtained in numerical investigations, solutions oscillating indefinitely around r_2 are not likely to be observed as they are unstable.

The first theorem provides a method to estimate numerically the boundary for any class of initial conditions. A given initial condition can be translated "up" or "down", along a "positive" direction, until it crosses the boundary (Fig. 3). However, carrying such a

task requires extensive computations as solutions close to the boundary tend to have long oscillatory transients before they converge to either of the stable equilibria; moreover, the duration of this transient regime increases with the delay [30]. To overcome this problem, we present an explicit linear approximation of b_u, for an appropriately chosen u.

We first remark that W^s, the stable manifold of the unstable point r_2, is included in the boundary B. Moreover, when the sum of the delays, $A + A'$, is small enough, this manifold is of codimension-one in the neighborhood of r_2, so that the two sets, B and W^s, coincide at least in a neighborhood of r_2. Therefore, in the neighborhood of r_2, and $A + A'$ small enough, the boundary B can be approximated by the linear hyperplane, noted E, that is tangent to the stable manifold W^s at r_2. Based on the above remark, for ϕ in S, we approximate $b_u(\phi)$, where u is the tangent to the unstable manifold at r_2, by $p_u(\phi)$, where $-p_u(\phi)u$ is the projection on the line $\mathbb{R}u$ along the direction E. That is, $p_u(\phi)$ is the unique real number such that $\phi + p_u(\phi)u$ belongs to the hyperplane E (Fig. 3). The expression of p_u is given in appendix A.

Numerical investigations indicate that E remains a satisfactory approximation of B close to r_2, even when $A + A'$ is large enough so that the stable manifold of r_2, W^s, is not codimension-one and does not locally coincide with B.

V. COMPARISON OF BASIN BOUNDARIES

The phase space of DDE (1) depends on the values of A and A'. One method to compare the boundaries separating the basins of attraction of the equilibria r_1 and r_3 of DDE (1), for different values of A and A', is to consider a restricted set of initial conditions of the DDE such that there exists one-to-one correspondence between initial conditions for different values of A and A'. A natural way of doing this is to consider constant functions as initial conditions.

In the following, we will restrict our attention to this class. For a given real number c_1, let $\beta(c_1)$ be the unique real number such that $(c_1, \beta(c_1))$ is on the basin boundary. From the characterization of the basin boundary described in the previous section, we know that the function β is a decreasing continuous function defined on the real line. The graph of β divides the (c_1, c_2)-plane into two regions. Points "below" this graph correspond exactly to the constant initial conditions lying in the basin of attraction of r_1, and those "above" it to the ones lying in the basin of attraction of r_3. Thus B_c, the graph of β in the (c_1, c_2)-plane, can be considered as the boundary separating the basins of attraction of the two stable equilibrium points.

In order to see the effect of the delays on B_c, we considered two cases: identical neurons connected with either symmetrical weights or symmetrical delays. In both situations, the inputs are set to: $K = -W/2$ and $K' = -W'/2$.

For these parameters, the coordinates of the equilibria r_1, r_2 and r_3 are:

$$r_1 = (-a, -b), \quad r_2 = (x_u, y_u) = (0, 0), \quad r_3 = (a, b),$$

where (a, b) is the strictly positive solution of system (4). Note that in this case, the system is invariant under the change $(x, y) \to (-x, -y)$. This implies that B_c is also symmetric under the same transformation.

A. Symmetrical weights

For identical neurons, receiving the same inputs, and connected through symmetrical weights and delays, the neurons are indistinguishable one from the other. Therefore, the basin boundary B_c is the straight line defined by: $c_1 - x_u + c_2 - y_u = 0$ (dashed-dotted line in Fig. 4). The theoretical approximation, given by Eq. (A6) in appendix A, yields the same result. Hence, in this situation, varying both delays together does not bring any change to the basin boundary. However, when the delays are no longer equal, B_c undergoes changes depending on the values of the delays. In Fig. 4, the basin boundary (thick lines) and its theoretical approximation (thin lines) are represented for three different values of the delay A', when the delay A is kept constant. The dashed-dotted line corresponds to equal delays

$(A = A')$. The dotted lines correspond to $A = 5$ and $A' = 2.2$ and the solid lines to $A = 5$ and $A' = 0.2$. It can be seen that when A' decreases from A down to zero, the slope of the boundary at each point increases from -1 to a strictly negative limit value which depends on A. A similar modification is visible on the slope of the straight line representing the theoretical approximation (thin lines) which is tangent to the boundary at r_2. For $c_1 < x_u$ (resp. $c_1 > x_u$) fixed, $c_2 = \beta(c_1)$ decreases (increases) from $y_u + x_u - c_1$ to a limit value strictly larger (smaller) than y_u as the delay A' decreases from A to zero.

The symmetry of the boundaries results from the special choice of the parameters, as explained above.

B. Symmetrical delays

For neurons connected through symmetrical delays but non-symmetrical weights, it is not possible to obtain an analytical description of B_c, even when the delays are set to zero.

Figure 5 shows how the boundary B_c changes for non-symmetric weights, when the delay is changed. In the figure only the case of symmetrical delays is considered. The dashed-dotted line corresponds to $A = A' = 2$, the dotted line corresponds to $A = A' = 1$ and the thick solid line to $A = A' = 0.1$. The thin solid line is the theoretical approximation. The slope of the boundary decreases as the delays $A = A'$ are increased from 0 to 2, except at the unstable point r_2 (thick lines). This is reflected in the theoretical approximation. The zeros of the functional (A6) in appendix A form the straight line $c_2 - y_u + (c_1 - x_u)\sqrt{\frac{W'}{W}} = 0$, and do not depend on the value of the delay when the delays are identical $(A = A')$. Therefore, the three different boundaries shown in Fig. 5 have the same linear approximation at r_2 (thin line). For $c_1 < x_u$ (resp. $c_1 > x_u$) fixed, $c_2 = \beta(c_1)$ increases (decreases) from the value it takes for the ODE associated to Eq. (1) as both delays $A = A'$ are increased from zero.

Here again, the symmetry of the boundaries results from the special choice of the parameters, as explained above.

VI. DISCUSSION

In this paper, we studied the dynamics of a pair of neurons with delayed excitatory inter-connection. The asymptotic behavior of the network we have considered is similar to that of the associated system without delay, in the sense the network with delay has exactly the same locally stable equilibria as the one without delay and most trajectories converge to these equilibria for all values of the delays. By relying on the fact that DDE (1) generates a strongly monotonous semi-flow (3), we were able to carry out a precise theoretical and numerical characterization of the boundary separating the basins of attraction of two locally stable equilibrium points. We showed that the shape of the basin boundary was modified as the values of the delays were changed. The examples we considered showed this modification depended on several factors. The first example, a network with symmetrical weights (section V A), showed that the difference between the two delay values was highly influential in the alterations of the basin boundary. This suggests that, in general, the distribution of delay values in a network plays an important role in shaping the basin boundaries. In this example, the local linear approximation used to estimate the boundary gave satisfactory indications on the direction of modifications as the delays were changed. Therefore, linear approximations may be one method to investigate the influence of delays on basin boundaries near the unstable equilibrium. However, the fact that the linear approximation does not depend on the delays does not rule out such an influence on the basin boundary. This was illustrated in the example studied in section V B, where the linear approximation does not exhibit any dependence on the delay since near the unstable equilibrium the basin boundary does not depend on the delay. But, as we move away from the unstable equilibrium, the basin boundary changes with the delay and the linear approximation is not valid anymore. Therefore, near the unstable equilibrium, the linear approximation can be used to investigate the influence of the delay on the basin boundary however, away from the unstable equilibrium, the linear approximation ceases to be valid and we have to resort to numerical investigations to shed light on this phenomenon.

Our results suggest that, in general, the boundary of basins of attraction of equilibria, or of more complicated objects (like periodic orbits), of systems with delay, depend on the value of the delay. However, the methodology used in this paper relied on the properties of the monotonous semi-flow and may not be adequate for the study of networks commonly used in applications that may have both positive and negative connection weights. In fact, it is well-known that even scalar equations with delayed feedback can display complex dynamics (see, for example, [17–19]) and that the basin boundaries in such systems can have an intricate structure [31]. Therefore, a network made of a large number of units, with both delayed excitatory and inhibitory connections, may display more complex behaviors than those of the two-neuron network studied in this paper [15,16].

General considerations

In our work, we restricted the set of initial conditions to constant functions. This is appropriate for the system we are considering, as in many applications the network is started from such an initial condition. However, it should be remarked that the choice of the class of initial conditions is not unique. Different biological or physical problems may require other restrictions. For instance another choice of initial conditions for this system is the following class of functions:

$$\phi_1(\theta') = x_s, \text{ for } \theta' \in [-A', 0), \quad \phi_2(\theta) = y_s, \text{ for } \theta \in [-A, 0),$$
$$\phi_1(0) = p, \quad \phi_2(0) = p',$$

where (x_s, y_s) are the coordinates of one of the two stable equilibrium points r_1 or r_3 and $p \neq x_s$ and $p' \neq y_s$ are arbitrary real numbers. Such an initial condition corresponds to the situation in which the system is initially in a stable equilibrium state and is then suddenly (at $t = 0$) perturbed.

VII. CONCLUSION

For convergent neural networks used in applications, such as associative memories, the shape of the basin of attraction of the equilibrium points determines to which information a given initial condition is associated. In this paper we have shown that even when precautions are taken, so that the local stability of the equilibria and the quasi convergence of the system are preserved in presence of delays, these delays may still modify the boundary of the basins of attraction of the stable equilibria. This can deteriorate the performance of the network. However, our work was based on the behavior of a two-neuron network and represents only the first step towards the study of the influence of delay on the basin boundaries in neural networks.

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APPENDIX A: APPROXIMATION OF THE BOUNDARY

We present here some general results on linear DDEs (see [20] for details). Let us consider the following linear DDE:

$$\frac{dx}{dt}(t) = q_{11}x(t) + q_{12}y(t-A)$$

$$\frac{dy}{dt}(t) = q_{22}y(t) + q_{21}x(t-A'), \tag{A1}$$

where $q_{11}, q_{22} < 0$ and $q_{12}, q_{21} > 0$. This equation is defined in the same phase space S as Eq. (1).

The characteristic equation of this linear system is given by:

$$(q_{11} - \lambda)(q_{22} - \lambda) - q_{12}q_{21}e^{-\lambda(A'+A)} = 0. \tag{A2}$$

Let us order the complex roots λ_k, $k = 1, 2, \ldots$, of this equation in such a way that $\Re(\lambda_k)$ (the real part of λ_k) is larger than or equal to $\Re(\lambda_j)$ whenever $k \leq j$. Let $f_k = (f_k^1, f_k^2)$ be an eigenfunction associated with λ_k, f_k is defined by $f_k^1(\theta') = e^{\lambda_k\theta'}c_k^1$ for $-A' \leq \theta' \leq 0$ and $f_k^2(\theta) = e^{\lambda_k\theta}c_k^2$ for $-A \leq \theta \leq 0$ such that $c_k = (c_k^1, c_k^2)$ is a non trivial solution of the linear equation $G(\lambda_k)c = 0$, where

$$G(\lambda) = \begin{pmatrix} q_{11} - \lambda & q_{12}e^{-\lambda A} \\ q_{21}e^{-\lambda A'} & q_{22} - \lambda \end{pmatrix} \quad . \tag{A3}$$

Our choice of signs for the coefficients q_{ij} implies that λ_1 is a simple real root of Eq. (A2) and that c_1 can be chosen with both components strictly positive. In this case [20] it is possible to write any function $\phi \in S$ as $\phi + P(\phi)f_1 = \hat{\phi}$ where $P : S \to \mathbb{R}$ is a linear functional such that $P(f_k) = 0$ for all $k > 1$ and $\hat{\phi}$ belongs to E, the generalized eigenspace of equation (A1) associated with the eigenvalues $\{\lambda_j\}_{j>1}$. E is a linear hyperplane of S and all $f_k \in E$, $k > 1$. Using the results in [20] (section 7.3) and setting $\lambda_1 = \nu$ we write $P(\phi)$ as:

$$P(\phi) = \phi_1(0)v_1 + \phi_2(0)v_2 + q_{12}v_1\int_{-A}^{0} e^{-A(s+\nu)}\phi_2(s)\,ds + q_{21}v_2\int_{-A'}^{0} e^{-A'(s+\nu)}\phi_1(s)\,ds, \tag{A4}$$

where the vector $v = (v_1, v_2)$ is a solution of $G^\dagger(\nu)v = 0$, with $G^\dagger(\nu)$ being the transpose of the matrix defined in (A3), normalized so that $P(f_1) = -1$.

S is partitioned into three different regions that are stable by the semi-flow generated by Eq. (A1): these are the subsets $\{\phi,\ P(\phi) > 0\}$, $E = \{\phi,\ P(\phi) = 0\}$ and $\{\phi,\ P(\phi) < 0\}$.

Now, if we linearize Eq. (1) at the unstable equilibrium point $r_2 = (x_u, y_u)$ we get Eq. (A1) with $q_{11} = -\gamma < 0$, $q_{22} = -\gamma' < 0$ and $q_{12} = W\sigma'(y_u) > 0$ and $q_{21} = W'\sigma'(x_u) > 0$. This implies that $\lambda_1 = \nu > 0$ and it can be verified that if $A + A'$ is sufficiently small then λ_1 is the only root of Eq. (A2) with real part larger than or equal to zero. In this case, r_2 has a one-dimensional unstable manifold and a codimension-one stable manifold W^s. The stable manifold is tangent to E at r_2 and can be locally described as a graph of a function from E to S [20]. In this case the boundary B coincides locally with W^s. Therefore, B can be approximated by E as well as W^s is. So, near r_2, the set B is approximately given by $\{\phi \in S | P(\phi) = 0\}$, where $P(\phi)$ is defined in Eq. (A4). Any initial condition ϕ in the neighborhood of r_2 with $P(\phi) < 0$ is likely to generate a solution that converges to r_3 and any initial condition $\phi \in S$ with $P(\phi) > 0$ is likely to generate a solution that converges to r_1.

Let $u = (u_1, u_2)$ in S be defined by $u_1(\theta') = W\sigma'(y_u)e^{\nu(\theta'-A)}$ for $-A' \leq \theta' \leq 0$ and $u_2(\theta) = (\nu + \gamma)e^{\nu\theta}$ for $-A \leq \theta \leq 0$. u is strictly larger than zero. We note v_1 and v_2 the real numbers:

$$v_1 = \frac{-(\nu+\gamma')e^{\nu A}}{W\sigma'(y_u)(2\nu+\gamma+\gamma'+(A+A')(\nu+\gamma)(\nu+\gamma'))} \quad ; \qquad \text{(A5)}$$

$$v_2 = \frac{-1}{2\nu+\gamma+\gamma'+(A+A')(\nu+\gamma)(\nu+\gamma')} \quad .$$

From the above considerations we derive an explicit linear approximation of the map b_u.

Approximation of b_u. *When $A + A'$ is small enough, for $\phi = (\phi_1, \phi_2)$ in the neighborhood of r_2, $b_u(\phi)$ can be approximated by the following expression:*

$$\begin{aligned} p_u(\phi) = \; & (\phi_1(0) - x_u)v_1 + (\phi_2(0) - y_u)v_2 + W\sigma'(y_u)v_1 \textstyle\int_{-A}^{0} e^{-A(s+\nu)}(\phi_2(s) - y_u)\, ds + \\ & W'\sigma'(x_u)v_2 \textstyle\int_{-A'}^{0} e^{-A'(s+\nu)}(\phi_1(s) - x_u)\, ds. \end{aligned} \qquad \text{(A6)}$$

FIGURE LEGENDS

and

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FIGURES

FIG. 1. *Two interconnected neurons*

The system formed by two neurons, labeled after their activation x and y. The neurons interact through connections with weights W and W', and delays A and A'. K and K' are the constant inputs the neurons receive.

FIG. 2. *Equilibrium points of the two-neuron system*

The equilibrium points of (1) are the intersection points between the curves $-\gamma a + K + W\sigma(b) = 0$ (dotted line) and $-\gamma' b + K' + W'\sigma(a) = 0$ (solid line). In the example shown in the figure, there are three equilibrium points $r_1 < r_2 < r_3$. Abscissae: a and ordinates: b. Parameters used: $\gamma = \gamma' = 1$, $W = 10$, $W' = 5$, and $K = K' = -3$.

FIG. 3. *Schematic representation of the map b_u*

The three equilibrium points r_1, r_2 and r_3 are represented in a plane. The strictly positive function u in S is indicated with a 2-dimensional vector with strictly positive coordinates. The regions with vertical and horizontal grey stripes correspond to initial conditions smaller and larger than r_2 respectively. The decreasing curved line, $\mathbf{B}$, is the basin boundary. All points below the boundary are in the basin of r_1, whereas all points above the boundary are in the basin of r_3. A given initial condition, represented by a point in the plane, has to be translated in the same direction as u (indicated by the arrow) to reach the boundary if it is in the basin of r_1 (the case of $\mathbf{f}$ in the figure), and in the opposite direction if it is in the basin of r_3 (the case of $\mathbf{g}$ in the figure). The amount by which the initial condition has to be translated to reach the basin boundary represents the value of b_u. The straight line E represents the tangent to the boundary B at r_2. An approximation of b_u close to r_2 is given by p_u, the amount by which an initial condition has to be translated to reach E. Note that the arrows do not represent trajectories in the phase space.

FIG. 4. *Boundary of the basin of attraction for symmetrical weights*

The graph of the boundary B_c between the basin of attractions of the equilibrium points r_1 and r_3 for constant initial conditions (c_1, c_2) is shown for $\gamma = \gamma' = 1$, $W = W' = 6$, $K = K' = -3$, $A = 5$ and $A' = 0.2$ (solid lines), $A' = 2.2$ (dotted lines) and $A' = 5$ (dashed-dotted line). In each case, the thick line corresponds to the numerical estimation and the thin line corresponds to the theoretical approximation (Eq. (A6)). For $A' = 5$ both lines coincide with the straight line: $c_1 + c_2 = 0$. Abscissae: c_1, ordinates c_2.

FIG. 5. *Boundary of the basin of attraction for non symmetrical weights*

The graph of the boundary B_c between the basin of attractions of the equilibrium points r_1 and r_3 for constant initial conditions (c_1, c_2) is shown for $\gamma = \gamma' = 1$, $W = 1$, $W' = 36$, $K = -0.5$, $K' = -18$, $A = A' = 0.1$ (thick solid line), $A = A' = 1$ (dotted line) $A = A' = 2$ (dashed-dotted lines. The thin solid line corresponds to the theoretical approximation (Eq. (A6)), which in this situation is the same for all delays. Abscissae: c_1, ordinates c_2.

Figure 1

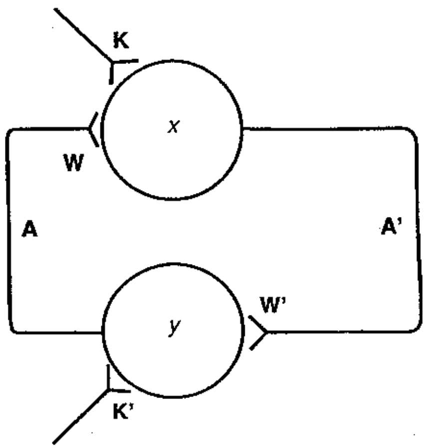

Figure 2

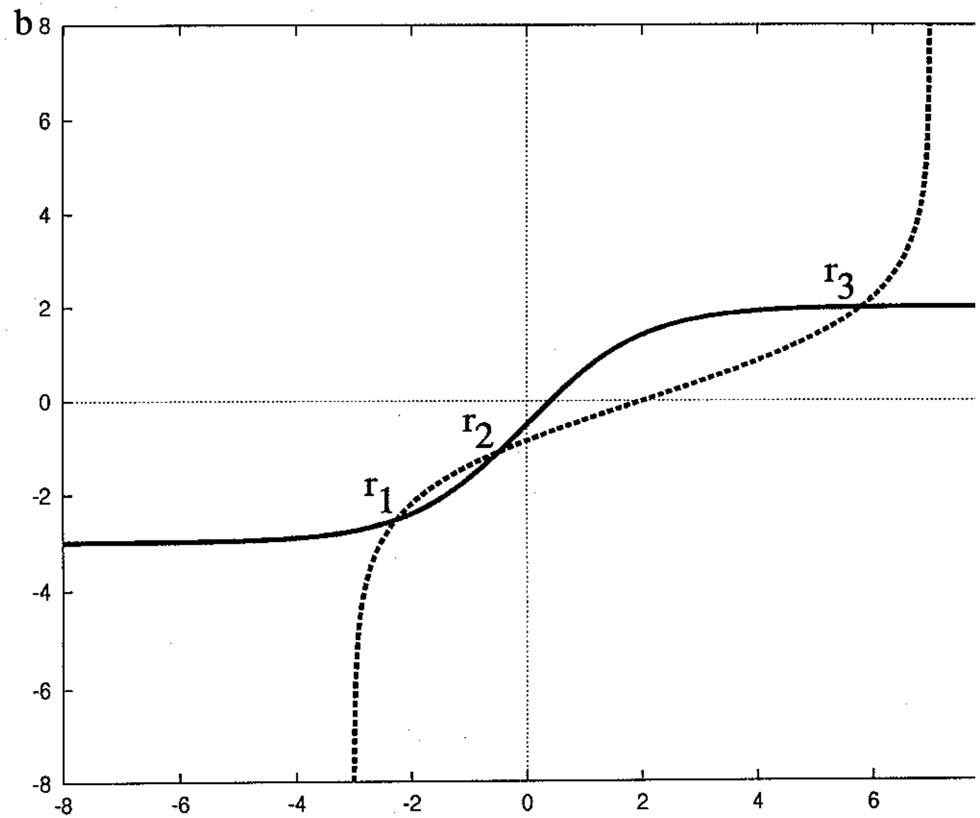

Figure 3

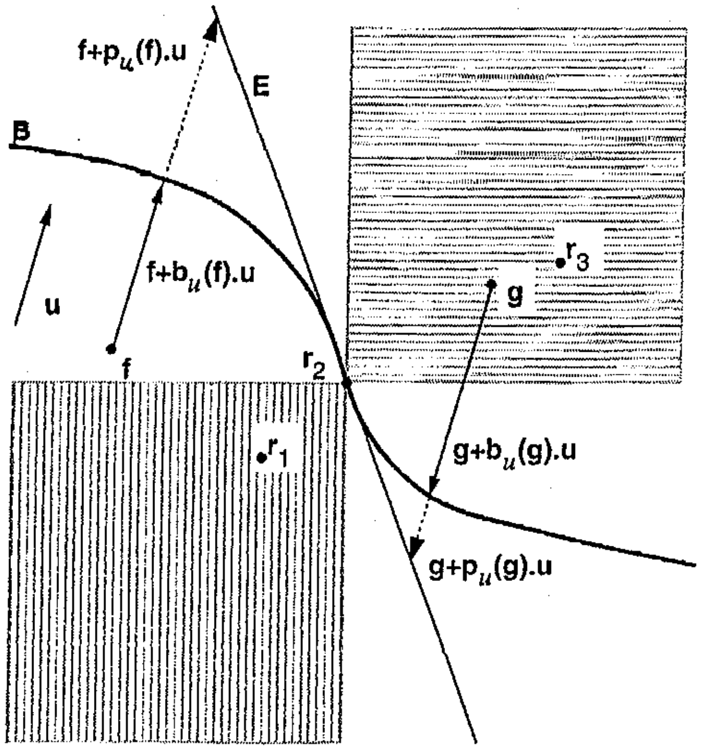

Figure 4

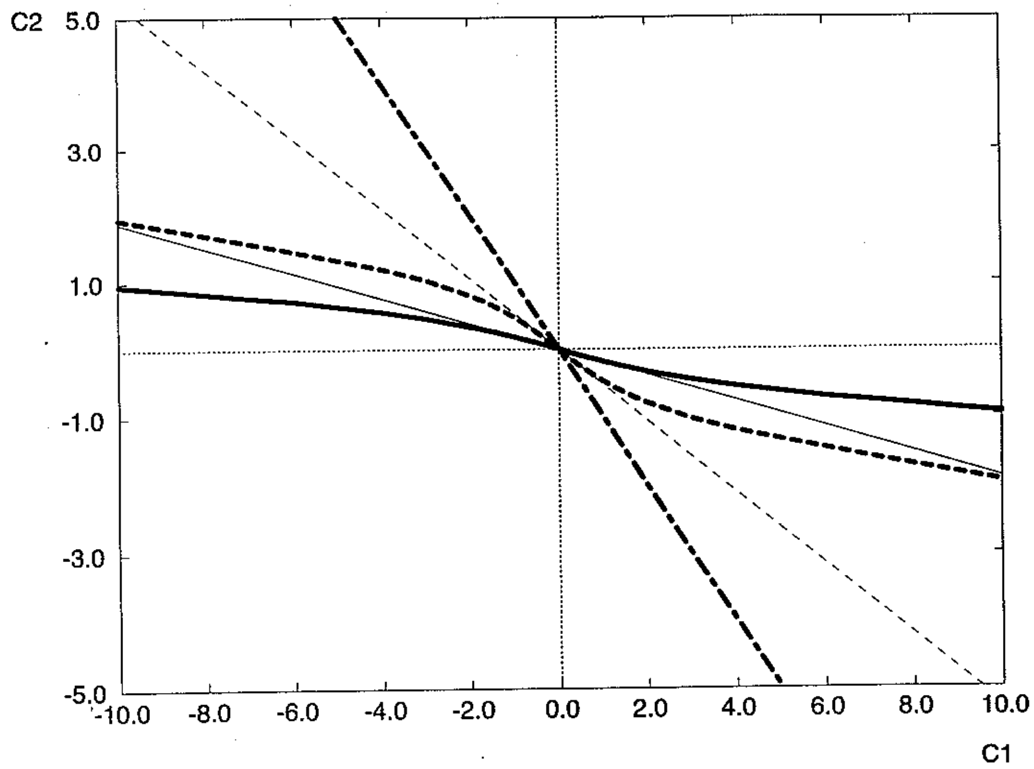

Figure 5