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VACUUM INSTABILITY IN EXTERNAL FIELDS

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Vacuum instability in external fields

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Abstract

We study particles creation from the vacuum by external electric fields, in particular, by fields, which are acting for a finite time, in the frame of QED in arbitrary space-time dimensions. In all the cases special sets of exact solutions of Dirac equation (IN- and OUT- solutions) are constructed. Using them, characteristics of the effect are calculated. The time and dimensional analysis of the vacuum instability is presented. It is shown that the distributions of particles created by quasiconstant electric fields can be written in a form which has a thermal character and seams to be universal, i.e. is valid for any theory with quasiconstant external fields. Its application, for example, to the particles creation in external constant gravitational field reproduces the Hawking temperature exactly.

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I. INTRODUCTION

The effect of particles creation from vacuum by an external field (vacuum instability in an external field) ranks among the most intriguing nonlinear phenomena in quantum theory. Its consideration is theoretically important, since it requires one to go beyond the scope of the perturbation theory, and its experimental observation would verify the validity of the theory in the superstrong field domain. The study of the effect began, in fact, first in 3 + 1-dimensional QED in connection with the so-called Klein [1] paradox, which revealed the possibility of electron penetration through an arbitrary high barrier formed by an external field. Then Schwinger [2] found the vacuum-to-vacuum transition probability in a constant electric field. It became clear that the effect can actually be observed as soon as the external field strength approaches the characteristic value (critical field) $E_c = m^2c^3/|e|\hbar \simeq 1,3 \cdot 10^{16}\ V/cm$. Although does not exist a real possibility to create such fields under laboratory conditions, they can exist in astrophysics, where the characteristic values of electromagnetic fields near and gravitational fields near black holes are enormous. One can also mention that Coulomb fields of superheavy nuclei can create the electron-positron pairs. General considerations, concrete calculations and detailed bibliography regarding the vacuum instability in QED can be found in [3-6]. Particles creation by external gravitational fields [5,7,8] and non-Abelian gauge fields [9] can also be considered in analogy with the electrodynamics. There are also various problems in modern quantum theory which are closely related to the vacuum instability in question, for example, phase transitions in field theories, the problem of boundary conditions or topology influence on the vacuum, the problem of consistent vacuum construction in QCD, string theories, multiple particles creation, and so on [5,7,10–13].

In spite of the particles creation effect in external fields was calculated in numerous papers, there are still some problems which are interesting to study and discuss. In the present paper we are going to focus our attention on the time scenario of the process and to consider it in arbitrary dimensions of space-time to be able analyze its dependence on the dimensions. To fulfill the first part of the program we consider special external fields which act effectively during a finite time and then compare results with ones in a constant field. In fact, such a consideration plays also the role of a regularization and helps to solve divergency problems which appear in constant external fields. The dimensional analysis may be interesting in relation with the study of multidimensional versions of field theories and gravity. A particular interest can be also in lower dimensions, e.g. in $2+1$ dimensions. Field theoretical models in such dimensions [14] attract in the last years a great attention due to various reasons: e.g. nontrivial topological properties, and especially the possibility of the existence of particles with fractional spins and exotic statistics (anyons), having probably applications to fractional Hall effect, high-T_c superconductivity and so on [15].

For calculations we are using the general approach, which was elaborated in frame of the field theory for such kind of problems [16–18,6]. According to it all the information about the processes of particles scattering and creation by an external field (in zeroth order with respect to the radiative corrections) can be extracted from special complete sets of exact solutions of the relativistic wave equations in the external field (IN- and OUT- solutions). A complete collection of exact solutions of such equations in $3+1$ QED is presented in the book [19], in particular, IN- and OUT-solutions and related bibliography can be found in [6]. That is why in the beginning we analyze and classify exact solutions of the Dirac equation in uniform external electric fields in arbitrary space-time dimensions. IN- and OUT-solutions are presented explicitly for T-constant, adiabatic, and constant electric fields. Probabilities of particles scattering, pairs creation, vacuum-to-vacuum probability and mean numbers of particles created are calculated in three types of electric fields mentioned above in arbitrary dimensions. The full consideration in the case of the T-constant field, which is most important for the time analysis, has not any $3+1$-dimensional analog and is presented explicitly for the first time. In spite of some of the formulas in two other cases have $3+1$-dimensional analog, their d-dimensional generalization appears to be not trivial. Moreover, some of these formulas were not presented even in $3+1$-dimensional case, i.e. the total mean numbers of particles created and vacuum-to-vacuum probability in the adiabatic field.

Using general expressions obtained in electric fields, which act for a finite time, we study the effect behavior at small and big times. Thus, in particular, we can estimate a stabilization time of the process, the time of a pair formation, and give a quasiclassical interpretation of a pair creation. Besides, this analysis allows one to select and estimate consistently time divergences, which appear in the constant fields. Comparing results obtained in different time configurations of the electric field, we estimate the role of switching on and off effects in the vacuum instability.

We analyze how the effect of the vacuum instability depends on the space dimensions and on the possible boundary conditions and on a non trivial topology.

We consider a possibility to add an uniform magnetic field to the electric one and calculate the effect. It turns out that one can formulate universal rules to generalize all the formulas obtained in the pure electric field to the case when the magnetic field is included as well. Its influence on the vacuum instability is studied.

Finally, it is shown, taking into account the vacuum level shift, that the distributions of particles created by the quasiconstant electric fields can be written in a form, which has a thermal character and seams to be universal, i.e. is valid for any theory with a quasiconstant external fields. Its application, for example, to the particles creation in external constant gravitational field, reproduces the Hawking temperature exactly.

II. GENERAL CONSIDERATION IN AN UNIFORM ELECTRIC FIELD

The d-dimensional Dirac equation in an external electromagnetic field with potentials $A_\mu(x)$ has the form (further $\hbar = c = 1$)

$$(P_\mu \gamma^\mu - m)\psi(x) = 0 \,, \quad P_\mu = i\partial_\mu - eA_\mu(x) \,, \tag{1}$$

where $\psi(x)$ is a $2^{[\frac{d}{2}]}$-component column, γ^μ are γ-matrices in d dimensions [20],

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \mathrm{diag}(\underbrace{1,-1,-1,\ldots}_{d}), \quad d = D+1,$$

and $x = (x^\mu) = (x^0, \mathbf{x})$, $\mathbf{x} = (x^i)$, $\mu = 0, 1, \ldots, D$, $i = 1, \ldots, D$.

As usual, it is convenient to present $\psi(x)$ in the form

$$\psi(x) = (P_\mu \gamma^\mu + m)\,\phi(x)\,. \tag{2}$$

Then the functions ϕ have to obey the squared Dirac equation in d dimensions,

$$\left(P^2 - m^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)\phi(x) = 0\,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,,\ \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\,. \tag{3}$$

Let us consider the field $F_{\mu\nu}$ with only one nonzero invariant $I = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, which supposes to be negative $I < 0$. In this case there exists a reference frame where only the components F_{0i} of the field differ from zero. That corresponds to a pure electric field, which is a particular case of external fields, violating the vacuum stability (creating particles). Let this electric field be uniform. It can be nonstationary, but with a constant direction in the space. Then one can always direct it along the axis x^D. Thus,

$$F_{0i} = (0, \ldots, 0, E(x^0)),\ F_{ik} = 0. \tag{4}$$

For such a field we will use the following potentials: $A_0 = A_1 = \ldots = A_{D-1} = 0$, $A_D = A_D(x^0)$. The constant uniform electric field is of a special interest, because QED with such external field (as with any free external field) can be considered as exact QED (without external fields) with some special initial states of the electromagnetic field [21,22,6], which provide the corresponding nonzero mean values of the electromagnetic field. Sometimes an alternating electric field can also be treated as a slight nonuniform free field, which is stipulated by some specific external conditions: existence of a waveguide [23], interference of two coherent waves [24] and so on. However, the study of the constant field shows that there appear divergences related to the infinite action time of the field. More correct consideration demands a regularization in time, for instance, one can consider a field, which acts only a finite time T, being constant within this interval. Such an approach allows also to avoid problems with the definition of IN- and OUT-states in nonswitching external fields at $x^0 \to \pm\infty$. Another possibility is to consider an alternating field, which switches on and off adiabatically at $x^0 \to \pm\infty$, and is quasiconstant at finite times. In the next section we are going to consider all the possibilities mentioned to study the time scenario of the particles creation.

Solutions of the equation (3) in the field (4) can be written in the form

$$\phi_{\mathbf{p},s,r}(x) = \phi_{\mathbf{p},s}(x^0)\exp\{i\mathbf{p}\mathbf{x}\}v_{s,\{r\}},\ \ r = (r_1, \ldots, r_{[\frac{d}{2}]-1}),\ s = \pm 1,\ r_j = \pm 1, \tag{5}$$

where $v_{s,\{r\}}$ are some constant orthonormal spinors, $v^\dagger_{s,\{r\}}v_{s,\{r'\}} = \delta_{r,r'}$. The eq.(3) allows one to subject these spinors to some supplementary conditions,

$$\begin{aligned} &S_\pm v_{\mp 1,\{r\}} = 0,\ S_\pm = \frac{1}{2}(1 \pm \gamma^0\gamma^D),\ \mathrm{rank}\ S_\pm = J_{(d)} = 2^{[\frac{d}{2}]-1}; \\ &R_\pm v_{s,\{\mp 1,\bar{r}\}} = 0,\ \ R_\pm = \frac{1}{2}(1 \pm \frac{i\gamma\mathbf{p}_\perp}{|\mathbf{p}_\perp|}),\quad \mathrm{rank}R_\pm = \frac{1}{2}J_{(d)},\ \text{ if } d > 3, \\ &\bar{r} = (r_2, \ldots, r_{[\frac{d}{2}]-1}),\ p^a_\perp = p^a,\ a = 1, \ldots, D-1,\ p^D_\perp = 0\,. \end{aligned} \tag{6}$$

If $d \le 3$ the quantum numbers r do not appear and for $d = 2$ the perpendicular components of the momenta are absent.

Taking into account the conditions (6), one can write an equation for the functions $\phi_{\mathbf{p},s}(x^0)$,

$$\left[\frac{d^2}{dx_0^2} + (p_D - eA_D(x^0))^2 + \mathbf{p}_\perp^2 + m^2 + iseE(x^0)\right]\phi_{\mathbf{p},s}(x^0) = 0\,. \tag{7}$$

A formal transition to the spinless case, which corresponds to use the Klein-Gordon equation instead of the Dirac one, can be done by putting $s = 0$ in (7) and $v_{s,\{r\}} = 1$ in (5).

The eq.(7) has two independent solutions at fixed $\mathbf{p}$ and s. Thus, an additional quantum number ζ appears, $\zeta = \pm$. Combining two independent solutions, which correspond to different ζ , one can construct two complete sets of solutions ${}_\zeta\phi_{\mathbf{p},s}(x^0)$ and ${}^\zeta\phi_{\mathbf{p},s}(x^0)$, obeying the following conditions

$$\begin{aligned} &i\frac{d}{dx^0}\,{}_\zeta\phi_{\mathbf{p},s}(x^0) = {}_\zeta\mathcal{E}_\mathbf{p}\,{}_\zeta\phi_{\mathbf{p},s}(x^0),\ \ \mathrm{sign}\,{}_\zeta\mathcal{E}_\mathbf{p} = \zeta,\ \ x^0 \to -\infty\,, \\ &i\frac{d}{dx^0}\,{}^\zeta\phi_{\mathbf{p},s}(x^0) = {}^\zeta\mathcal{E}_\mathbf{p}\,{}^\zeta\phi_{\mathbf{p},s}(x^0),\ \ \mathrm{sign}\,{}^\zeta\mathcal{E}_\mathbf{p} = \zeta,\ \ x^0 \to +\infty\,. \end{aligned} \tag{8}$$

They provide in turn the following behavior

$$H_{o.p.}(x^0)\,{}_\zeta\psi_{\mathbf{p},s,r}(x) = {}_\zeta\mathcal{E}_{\mathbf{p}}\,{}_\zeta\psi_{\mathbf{p},s,r}(x), \quad \text{sign}\,{}_\zeta\mathcal{E}_{\mathbf{p}} = \zeta,\ x^0 \to -\infty\,,$$
$$H_{o.p.}(x^0)\,{}^\zeta\psi_{\mathbf{p},s,r}(x) = {}^\zeta\mathcal{E}_{\mathbf{p}}\,{}^\zeta\psi_{\mathbf{p},s,r}(x), \ \text{sign}\,{}^\zeta\mathcal{E}_{\mathbf{p}} = \zeta,\ x^0 \to +\infty\,, \tag{9}$$

of the corresponding Dirac equation solutions, ${}_\zeta\psi_{\mathbf{p},s,r}(x) = (\gamma P + m)\ {}_\zeta\phi_{\mathbf{p},s,r}(x)$ and ${}^\zeta\psi_{\mathbf{p},s,r}(x) = (\gamma P + m)\ {}^\zeta\phi_{\mathbf{p},s,r}(x)$. In the eq. (9) $H_{o.p.} = \gamma^0(m + \gamma\mathbf{P})$ is one-particle Dirac Hamiltonian, and $\mathcal{E}$ are quasi-energies. The solutions ${}_\pm\psi_{\mathbf{p},s,r}(x)$ describe particle (+) and antiparticle (-) in the initial time-instant whereas ${}^\pm\psi_{\mathbf{p},s,r}(x)$ describe particle (+) and antiparticle (-) in the final time-instant [17,6].

One can see that the solutions with different s and fixed ζ, $\mathbf{p}$, r are dependent, for example,

$${}_\zeta\psi_{\mathbf{p},s,r}(x) = \frac{m + ib_{\mathbf{p},r}}{{}_\zeta a_{\mathbf{p},-s}}\,{}_\zeta\psi_{\mathbf{p},-s,r}(x), \tag{10}$$

where $b_{\mathbf{p},r} = r_1|\mathbf{p}_\perp|$ if $d > 3$, $b_{\mathbf{p},r} = p_1$ if $d = 3$, $b_{\mathbf{p},r} = 0$ if $d = 2$, and ${}_\zeta a_{\mathbf{p},s}$ are some coefficients. To see how (10) appears one can use (6), (7), (8) and the following consequence of two latter,

$$\left[i\frac{d}{dx^0} + s(p_D - eA_D(x^0))\right]{}_\zeta\phi_{\mathbf{p},s}(x^0) = {}_\zeta a_{\mathbf{p},s}\,{}_\zeta\phi_{\mathbf{p},-s}(x^0)\,, \quad {}_\zeta a_{\mathbf{p},-1}\,{}_\zeta a_{\mathbf{p},+1} = m^2 + \mathbf{p}_\perp^2\,.$$

Similar relation holds for ${}^\zeta\psi_{\mathbf{p},s,r}(x)$. The eq.(10) means, in fact, that the spin projections of a particle (+) and an antiparticle (-) can take on only $J_{(d)}$ values. Taking that into account, one can only use the following independent solutions,

$${}_\pm\psi_{\mathbf{p},r}(x) = (\gamma P + m)\,{}_\pm\phi_{\mathbf{p},\pm1,r}(x)\,, \quad {}^\pm\psi_{\mathbf{p},r}(x) = (\gamma P + m)\,{}^\pm\phi_{\mathbf{p},\mp1,r}(x)\,. \tag{11}$$

Further, we are going to calculate different matrix elements between the solutions (11) by means of the conventional time independent Dirac scalar product $(\psi,\psi') = \int\bar\psi(x)\gamma^0\psi'(x)d\mathbf{x}$. In the case under consideration, due to the above mentioned properties (6),(7) of the spinors ${}_\zeta\phi_{\mathbf{p},s,r}(x)$ and ${}^\zeta\phi_{\mathbf{p},s,r}(x)$, the scalar product can be reduced to a form which is convenient for calculation and, in particular, does not content γ matrices at all,

$$({}^{\mp}_{+}\psi_{\mathbf{p},r}, {}^{\mp}_{+}\psi_{\mathbf{p}',r'}) = i(2\pi)^D\delta_{r,r'}\delta(\mathbf{p}-\mathbf{p}')\,{}^{\mp}_{+}\phi^*_{\mathbf{p},+1}(x^0)\,\overleftrightarrow{\partial_0}\,\left(i\partial_0 + p_D - eA_D(x^0)\right){}^{\mp}_{+}\phi_{\mathbf{p},+1}(x^0),$$

$$({}^{\pm}_{-}\psi_{\mathbf{p},r}, {}^{\pm}_{-}\psi_{\mathbf{p}',r'}) = i(2\pi)^D\delta_{r,r'}\delta(\mathbf{p}-\mathbf{p}')\,{}^{\pm}_{-}\phi^*_{\mathbf{p},-1}(x^0)\,\overleftrightarrow{\partial_0}\,\left(i\partial_0 - p_D + eA_D(x^0)\right){}^{\pm}_{-}\phi_{\mathbf{p},-1}(x^0),$$
$$({}^{\mp}_{+}\psi_{\mathbf{p},r}, {}^{\pm}_{-}\psi_{\mathbf{p}',r'}) = i(2\pi)^D\delta_{r,r'}\delta(\mathbf{p}-\mathbf{p}')(m - ib_{\mathbf{p},r})\,{}^{\mp}_{+}\phi^*_{\mathbf{p},+1}(x^0)\,\overleftrightarrow{\partial_0}\,{}^{\pm}_{-}\phi_{\mathbf{p},-1}(x^0), \tag{12}$$

where $\overleftrightarrow{\partial_0} = \overrightarrow{\partial_0} - \overleftarrow{\partial_0}$. (The right side of (12) reproduces the corresponding Klein-Gordon scalar product if one puts formally $\zeta[i\partial_0 \pm (p_D - eA_D(x^0))] = m - ib = 1$.)

One can see from (8) and (12) that the solutions (11) can be normalized to obey the orthonormality relations,

$$({}_\zeta\psi_{\mathbf{p},r}, {}_{\zeta'}\psi_{\mathbf{p}',r'}) = \delta_{\zeta,\zeta'}\delta_{r,r'}\delta(\mathbf{p}-\mathbf{p}'), \quad \left({}^\zeta\psi_{\mathbf{p},r}\ {}^{\zeta'}\psi_{\mathbf{p}',r'}\right) = \delta_{\zeta,\zeta'}\delta_{r,r'}\delta(\mathbf{p}-\mathbf{p}'). \tag{13}$$

Moreover, each set of solutions ${}_\zeta\psi_{\mathbf{p},r}(x)$ and ${}^\zeta\psi_{\mathbf{p},r}(x)$ forms a complete system, thus, we are dealing with the so-called IN- and OUT-sets of solutions correspondingly [17,18,6].

Using (12), one can find decomposition coefficients $G\left({}_\zeta|{}^{\zeta'}\right)$ of the OUT-solutions in the IN-solutions,

$${}^\zeta\psi(x) = {}_+\psi(x)G\left({}_+|{}^\zeta\right) + {}_-\psi(x)G\left({}_-|{}^\zeta\right)\,. \tag{14}$$

The matrices $G\left({}_\zeta|{}^{\zeta'}\right)$ obey the following relations,

$$G\left({}_\zeta|{}^+\right)G\left({}_\zeta|{}^+\right)^\dagger + \kappa G\left({}_\zeta|{}^-\right)G\left({}_\zeta|{}^-\right)^\dagger = (\zeta\mathbf{I})^{\frac{1-\kappa}{2}},$$
$$G\left({}_+|{}^+\right)G\left({}_+|{}^+\right)^\dagger + \kappa G\left({}_+|{}^-\right)G\left({}_-|{}^-\right)^\dagger = 0\,, \tag{15}$$

where $\mathbf{I}$ is the unit matrix and $\kappa = \pm1$ for fermions and bosons respectively. Relations (15) can be derived from the conditions (13). Due to eq.(12) we can easily see that the matrices $G\left({}_\zeta|{}^{\zeta'}\right)$ are diagonal,

$$G\left({}_\zeta|{}^{\zeta'}\right)_{\mathbf{p},r,\mathbf{p}',r'} = \delta_{r,r'}\delta(\mathbf{p}-\mathbf{p}')\,g\left({}_\zeta|{}^{\zeta'}\right). \tag{16}$$

All the information about the processes of particles creation, annihilation, and scattering in an external field (without radiative corrections) one can extract from the matrices $G\left({}_\zeta|{}^{\zeta'}\right)$ because they define a canonical transformation between IN and OUT creation and annihilation operators in the generalized Furry representation [17,18,6],

$$a^\dagger(out) = a^\dagger(in)G\left({}_+|^+\right) + b(in)G\left({}_-|^+\right),$$
$$b(out) = a^\dagger(in)G\left({}_+|^-\right) + b(in)G\left({}_-|^-\right). \tag{17}$$

Here $a_n^\dagger(in)$, $b_n^\dagger(in)$, $a_n(in)$, $b_n(in)$ are creation and annihilation operators of IN-particles and antiparticles respectively and $a_n^\dagger(out)$,$b_n^\dagger(out)$, $a_n(out)$, $b_n(out)$ are ones of OUT-particles and antiparticles, n presents momentum $\mathbf{p}$ and spin projections r. For example, the mean numbers of particles created (which are also equal to the numbers of pairs created) by the external field from the IN-vacuum $|0, in>$ with a given momentum $\mathbf{p}$ and spin projections r is

$$N_{\mathbf{p},r} = <0, in|a^\dagger_{\mathbf{p},r}(out)a_{\mathbf{p},r}(out)|0, in> = \left|g\left({}_-|^+\right)\right|^2. \tag{18}$$

Here the standard volume regularization was used, so that $\delta(\mathbf{p}-\mathbf{p}') \to \delta_{\mathbf{p},\mathbf{p}'}$. The probabilities of a particle scattering and of a pair creation have the following forms respectively

$$P(+|+)_{\mathbf{p},r,\mathbf{p}',r'} = |<0, out|a_{\mathbf{p},r}(out)a^\dagger_{\mathbf{p}',r'}(in)|0, in>|^2 = \delta_{r,r'}\delta_{\mathbf{p},\mathbf{p}'}\frac{1}{1-\kappa N_{\mathbf{p},r}}P_v\,, \tag{19}$$

$$P(-+|0)_{\mathbf{p},r,\mathbf{p}',r'} = |<0, out|b_{\mathbf{p},r}(out)a_{\mathbf{p}',r'}(out)|0, in>|^2 = \delta_{r,r'}\delta_{\mathbf{p},\mathbf{p}'}\frac{N_{\mathbf{p},r}}{1-\kappa N_{\mathbf{p},r}}P_v\,, \tag{20}$$

where $|0, out>$ is the OUT vacuum and

$$P_v = |<0, out|0, in>|^2 = \exp\left\{\kappa\sum_{\mathbf{p},r}\ln\left(1-\kappa N_{\mathbf{p},r}\right)\right\}, \tag{21}$$

is the probability for a vacuum to remain a vacuum. The probabilities of a antiparticle scattering and a pair annihilation are described by the same expressions $P(+|+)$ and $P(-+|0)$ respectively.

Thus, to be able to calculate the quantities (18)-(21), in the case under consideration, one has to find solutions of the ordinary differential equation (7), which is in fact Schrödinger equation for a linear oscillator with time-dependent frequency. However, one can make some general conclusions, which do not depend on the concrete time dependence of the electric field in eq.(7). First of all, the matrices $G\left({}_\zeta|^{\zeta'}\right)$ are diagonal in all the quantum numbers introduced. Second, the quantum numbers r do not enter in the eq.(7) and, due to the

structure of the scalar product (12), the matrices can depend on r via a phase only. That is why all the probabilities and the mean numbers do not depend on r, so that in the fermionic case the total (summed over all r) probabilities and the mean numbers are $J_{(d)}$ times more than the correspondent differential quantities. For example, the total number of particles created with a given momentum $\mathbf{p}$ is

$$N_{\mathbf{p}} = \sum_r N_{\mathbf{p},r} = J_{(d)}N_{\mathbf{p},r}. \tag{22}$$

Finally, it is clear that due to the structure of the eq. (7) and the scalar product (12) the dimensionality d enter in the differential probabilities and mean values via the combination $\mathbf{p}_\perp^2$ only.

III. T-CONSTANT, ADIABATIC, AND CONSTANT ELECTRIC FIELDS

A. T-constant field

To analyze the time dependence of the particles creation effects let us consider the field (4) with $E(x^0)$ having the form

$$E(x^0) = \begin{cases} 0, & x^0 \in I \\ E, & x^0 \in II \\ 0, & x^0 \in III\,, \end{cases} \tag{23}$$

where the time intervals are: $I = (-\infty, t_1)$, $II = [t_1, t_2]$, $III = (t_2, +\infty)$, $t_2 - t_1 = T$, $t_2 = -t_1$, and $eE > 0$ is chosen. Thus, in fact, we consider a constant electric field E, which is acting a finite time T. Further we will call it T-constant field. The corresponding potential $A_D(x^0)$ can be chosen in the form

$$A_D(x^0) = \begin{cases} Et_1, & x^0 \in I \\ Ex^0, & x^0 \in II \\ Et_2, & x^0 \in III\,. \end{cases} \tag{24}$$

In each interval I, II, III the equation (7) has two independent solutions, which are correspondingly in I: $\exp\{-ip_0(t_1)x^0\}$ and $\exp\{+ip_0(t_1)x^0\}$, in II: $D_{\nu-\frac{1+\chi}{2}}[(1-i)\xi]$ and $D_{-\nu-\frac{1-\chi}{2}}[(1+i)\xi]$, and in III: $\exp\{-ip_0(t_2)x^0\}$ and $\exp\{+ip_0(t_2)x^0\}$, where $D_\nu(z)$ are Weber parabolic cylinder functions (WPC-functions) [25], and

$$\nu=\frac{i\lambda}{2},\quad \lambda=\frac{m^2+\mathbf{p}_\perp^2}{eE},\quad \xi(x^0)=\frac{eEx^0-p_D}{\sqrt{eE}},\quad p_0(x^0)=\sqrt{m^2+\mathbf{p}_\perp^2+(p_D-eA_D(x^0))^2}\ .$$

Using them and conditions (8), one can construct IN- and OUT-solutions ${}_-\psi_{\mathbf{p},r}(x)$ and ${}^+\psi_{\mathbf{p},r}(x)$ (see Sect.II). The corresponding expressions for ${}_-\phi_{\mathbf{p},-1}(x^0)$ and ${}^+\phi_{\mathbf{p},-1}(x^0)$ are of the form

$$\begin{aligned}&{}_-\phi_{\mathbf{p},-1}(x^0) \\ &=C_1\begin{cases}\exp\{+ip_0(t_1)(x^0-t_1)\}, & I\\ a_1D_\nu[(1-i)\xi]+a_2D_{-\nu-1}[(1+i)\xi], & II\\ g({}^+|_-)\exp\{-ip_0(t_2)(x^0-t_2)\}+g({}^-|_-)\exp\{+ip_0(t_2)(x^0-t_2)\}, & III\,;\end{cases}\end{aligned}\tag{25}$$

$$\begin{aligned}&{}^+\phi_{\mathbf{p},-1}(x^0) \\ &=C_2\begin{cases}g({}_+|^+)\exp\{-ip_0(t_1)(x^0-t_1)\}+g({}_-|^+)\exp\{+ip_0(t_1)(x^0-t_1)\}, & I\\ a_1'D_\nu[(1-i)\xi]+a_2'D_{-\nu-1}[(1+i)\xi], & II\\ \exp\{-ip_0(t_2)(x^0-t_2)\}\,, & III\,,\end{cases}\end{aligned}\tag{26}$$

where the normalization constants are

$$C_i=(2\pi)^{-D/2}(2p_0(t_i)q_i)^{-1/2},\ q_i=p_0(t_i)-(-1)^i(p_D-eA_D(t_i)),\ i=1,2.$$

To provide the continuity of the solutions in the time instants t_1 and t_2 one has to impose the following conditions

$${}^+_-\phi_{\mathbf{p},-1}(t_i+0)={}^+_-\phi_{\mathbf{p},-1}(t_i-0),\quad \frac{d}{dx^0}{}^+_-\phi_{\mathbf{p},-1}(t_i+0)=\frac{d}{dx^0}{}^+_-\phi_{\mathbf{p},-1}(t_i-0)\ ,$$

which allow one to define step by step all the coefficients a_i, a_i', and $g({}_\pm|^+)$, $g({}^\pm|_-)$. The first ones are

$$a_i=-(-1)^i\frac{p_0(t_1)f_i^{(+)}(t_1)}{M\sqrt{2eE}}\ ,\quad a_i'=(-1)^i\frac{p_0(t_2)f_i^{(-)}(t_2)}{M\sqrt{2eE}}\ ,$$

where

$$M=D_\nu(z)\frac{d}{dz}D_{-\nu-1}(iz)-D_{-\nu-1}(iz)\frac{d}{dz}D_\nu(z)=\exp\{-(\nu+1)\frac{i\pi}{2}\}$$

is the Wronskian determinant [25], and

$$f_1^{(\pm)}(x^0)=\left(1\pm\frac{i\partial_0}{p_0(x^0)}\right)D_{-\nu-1}[(1+i)\xi]\,,\quad f_2^{(\pm)}(x^0)=\left(1\pm\frac{i\partial_0}{p_0(x^0)}\right)D_\nu[(1-i)\xi]\,.$$

They can be used to define the latter coefficients. From those we need to know explicitly only $g({}_-|^+)$ and $g({}^+|_-)$, which are

$$\begin{aligned}g\left({}_-|^+\right)&=\exp\{(\nu+1)\frac{i\pi}{2}\}\left(\frac{p_0(t_1)q_1p_0(t_2)}{8eEq_2}\right)^{\frac12}\left[f_2^{(-)}(t_2)f_1^{(-)}(t_1)-f_1^{(-)}(t_2)f_2^{(-)}(t_1)\right],\\ g\left({}^+|_-\right)&=\exp\{(\nu+1)\frac{i\pi}{2}\}\left(\frac{p_0(t_1)q_2p_0(t_2)}{8eEq_1}\right)^{\frac12}\left[f_2^{(+)}(t_2)f_1^{(+)}(t_1)-f_1^{(+)}(t_2)f_2^{(+)}(t_1)\right].\end{aligned}\tag{27}$$

One can see that the coefficients (27) obey the properties

$$g\left({}^+|_-\right)\Big|_{p_D\to-p_D}=-g\left({}_-|^+\right),\quad g\left({}^+|_-\right)=g\left({}_-|^+\right)^*.\tag{28}$$

The first one can be verified directly, whereas the second one is easy to derive comparing representations of the scalar product (12) in the time instants t_1 and t_2. Thus, one can conclude that $|g({}_-|^+)|$ is an even function of the momentum p_D.

To calculate the probabilities and the mean numbers according to the formulas (18-21) we need really to know only the coefficients $g({}_-|^+)$. Comparing (14) and (25), (26), we conclude that the relations (16) hold and we have expression for the mean numbers of pairs created in the form (18), in which $g({}_-|^+)$ is defined by (27). In fact, the mean numbers $N_{\mathbf{p},r}$ define all the probabilities via the formulas (19-21). As it was shown above this function is even in all the momenta $\mathbf{p}$, including p_D and does not depend on the spin quantum number r. Using the recipe presented in Sect.II, it is easy to get an explicit form $N_\mathbf{p}$ for the the bosonic case from the fermionic one (18) and (27).

Now we are going to analyze the dependence of all the characteristics on the time T and on the momenta. One has to remark that the dependence of the longitudinal momentum p_D is of a special interest. This dependence is essentially correlated with the T-dependence.

One can see, e.g. from the four-dimensional case [16,6], that in the constant field ($T=\infty$) all the characteristics do not depend on the momentum p_D. This is a source of some kind divergences if one is interested in the total characteristics, which suppose integration over p_D. Due to the reasons mentioned above it is enough to analyze only the quantity $N_{\mathbf{p},r}$. As to the momentum p_D, one can restrict itself only by p_D positive or p_D negative. Remark that the momenta p_D enter in all the formulas via two dimensionless parameters ξ_1 and ξ_2 only,

$$\xi_1=\xi(-\frac{T}{2})=\frac{1}{\sqrt{eE}}(-eE\frac{T}{2}-p_D),\ \xi_2=\xi(+\frac{T}{2})=\frac{1}{\sqrt{eE}}(+eE\frac{T}{2}-p_D),$$

which in turn appear in the WPC-functions. Thus, in fact, one needs to analyze the dependence of the latter on ξ. It is convenient to consider the region $-\sqrt{eE}\frac{T}{2}\le\xi_1<+\infty$, $\xi_2\ge\sqrt{eE}\frac{T}{2}$, which corresponds to p_D negative, $0\le -p_D<+\infty$, in particular, there always $\xi_2>\xi_1$.

In the region $\xi_1\ge K$, where K is a given number $K>>1+\lambda$ (in terms of the momentum this region corresponds to $|p_D|\ge eE\frac{T}{2}+K\sqrt{eE}$) one can use the asymptotic expansion of WPC- functions [25],

$$D_\nu(z)=z^\nu\exp\left\{-z^2/4\right\}\left(\sum_{n=0}^{N}\frac{(-\frac{1}{2}\nu)_n(\frac{1}{2}-\frac{1}{2}\nu)_n}{n!(-\frac{1}{2}z^2)^n}+O(|z|^{-2(N+1)})\right),\ |\arg z|<\frac{3}{4}\pi, \qquad (29)$$

to conclude that at any T the behavior of the mean numbers (18), (27) is

$$N_{\mathbf{p},r}=O\left(\left[\frac{\lambda}{\xi_1^2}\right]^3\right). \qquad (30)$$

For small T, $T<<\frac{1}{\sqrt{eE}}$, and $|p_D|<<eE\frac{T}{2}$, one can get

$$N_{\mathbf{p},r}=\frac{eET^2}{4\lambda+eET^2}\left[1+O(\sqrt{eE}T)\right]. \qquad (31)$$

At $T<<\frac{\sqrt{\lambda}}{\sqrt{eE}}$ and $T<<\frac{1}{\sqrt{eE\lambda}}$ the form (31) reduces to $N_{\mathbf{p},r}=eET^2/(4\lambda)$ and coincides with one which can be derived in the frame of perturbation theory with respect to the external field.

The most important for the time divergences is the region of big T, namely, let us consider $T>>\frac{1}{\sqrt{eE}}(1+\lambda)$. In this case ξ_2 is always big and positive $\xi_2>>1+\lambda$, so that the

asymptotic expansion (29) can be used for any given momentum p_D. As to the parameter ξ_1, the whole region $-\sqrt{eE}\frac{T}{2}\le\xi_1<+\infty$ can be divided in three ones:

$$a)\ -\sqrt{eE}\frac{T}{2}\le\xi_1\le -K;\quad b)\ -K<\xi_1<K;\quad c)\ \xi_1\ge K\ .$$

The mean numbers $N_{\mathbf{p},r}$ were estimated in the region c) before, see (30). In the region a) one can use some relations between the WPC-functions [25], for example, $D_\nu(z)=\exp\{i\pi\nu\}D_\nu(-z)-\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\exp\frac{i\pi\nu}{2}D_{-\nu-1}(-iz)$, and the asymptotic expansion (29). Then one finds

$$N_{\mathbf{p},r}=e^{-\pi\lambda}\left[1+O\left(\left[\frac{1+\lambda}{\xi_1}\right]^3\right)+O\left(\left[\frac{1+\lambda}{\xi_2}\right]^3\right)\right],\ -\sqrt{eE}\frac{T}{2}\le\xi_1\le -K. \qquad (32)$$

The latter expression allows one to consider the limit $T\to\infty$ at any given $\mathbf{p}$. In this limit the mean numbers take a simple form

$$N_{\mathbf{p},r}=e^{-\pi\lambda}\,. \qquad (33)$$

Thus, when the electric field is acting long enough, the mean numbers of particles created in a given quantum state are stabilized and coincide with expressions which were obtained in the constant electric field in 3+1 QED [16]. (The stabilization (32) was first remarked in [26] for particles created with zero momenta, using a finite action electric field in 3+1 QED.) One can also estimate a characteristic time of such a stabilization. To this end one can see that $\frac{1+\lambda}{\sqrt{eE}T}$ is a small parameter in the decomposition (32) in case $|p_D|<<eE\frac{T}{2}$. If $T>>T_0$, $T_0=\frac{1+\lambda}{\sqrt{eE}}$ then the mean numbers are stabilized and T_0 is that characteristic time.

The intermediate region b) does not allows one to use an asymptotic expansion of WPC-functions to analyze ξ_1 dependence. However, one can make some necessary conclusions about its contribution in integrals over the momenta. For example, due to the Fermi statistics $N_{\mathbf{p},r}$ is always smaller then unity, that is why the integral over the momentum p_D in the region b) is less then $2\sqrt{eE}K$ and is not essential in comparison with the same integral in the region a) at $T\to\infty$.

Using these considerations, one can now estimate the sum over the longitudinal momentum p_D of $N_{\mathbf{p},r}$, which is the mean number of particles created with all possible momenta p_D. To do this we go over to the integral, $\sum_{p_D} \to \frac{L}{2\pi}\int dp_D$, where L is the length in the direction x^D. As was shown above, at $T >> \frac{1}{\sqrt{eE}}(1+\lambda)$, $N_{\mathbf{p},r}$ is quasiconstant in the area a), the asymptotic in the area c) has the form (30) and the contribution to the integral form the area b) is less then $\sqrt{eE}KL/\pi$. Then one can conclude

$$N_{\mathbf{p}_\perp,r} = \frac{L}{2\pi}\int_{-\infty}^{+\infty} N_{\mathbf{p},r}dp_D = \frac{\sqrt{eE}L}{2\pi}\left[\sqrt{eE}Te^{-\pi\lambda} + O(K)\right]. \tag{34}$$

Thus, when $T >> \frac{1}{\sqrt{eE}}K >> \frac{1}{\sqrt{eE}}(1+\lambda)$ we can effectively replace the integral over p_D by eET and write

$$N_{\mathbf{p}_\perp,r} = \Delta_{long}e^{-\pi\lambda},\ \ \Delta_{long} = \frac{1}{2\pi}eELT\,. \tag{35}$$

The factor Δ_{long} can be can interpreted as the total number of states with the longitudinal momenta p_D of particles created.

It turns out that the expressions for $N_{\mathbf{p},r}$ and $N_{\mathbf{p}_\perp,r}$ at big T for scalar particles coincide with ones for spinor particles.

To get the total number N of particles created one can sum over the spin projections, using eq.(22), and then over the transversal momenta, the latter sum can be easily transformed into an integral,

$$N = \sum_{\mathbf{p}} N_{\mathbf{p}} = \frac{V_{(d-1)}}{(2\pi)^{d-1}}\int d\mathbf{p}N_{\mathbf{p}}, \tag{36}$$

where $V_{(d-1)}$ is (d-1)-dimensional spatial volume. Thus, on gets

$$N = J_{(d)}^{\frac{1+\kappa}{2}}\frac{V_{(d-1)}Tm^d}{(2\pi)^{d-1}}\left(\frac{E}{E_c}\right)^{\frac{d}{2}}\exp\left\{-\pi\frac{E_c}{E}\right\},\ \ J_{(d)} = 2^{[\frac{d}{2}]-1}, \tag{37}$$

where $E_c = m^2/e$ is the critical field strength. As one can see the velocity of particles creation is constant at big T.

The vacuum-to-vacuum transition probability (21) can be calculated, using both kinds of regularization, with respect to the volume and to the time. Thus, we get the many-dimensional analog of the well-known Schwinger formula [2],

$$P_v = \exp\left\{-\mu N\right\},\ \ \mu = \sum_{n=0}^{\infty}\frac{(-1)^{(1-\kappa)\frac{n}{2}}}{(n+1)^{\frac{d}{2}}}\exp\left\{-n\pi\frac{E_c}{E}\right\}. \tag{38}$$

As to the Schwinger result, it was, in fact, obtained from the constant field consideration by means of a regularization. In such a way the space-time volume VT appeared in his formula.

B. Adiabatic field

Let us consider an alternating uniform electric field (4), where the function $E(x^0)$ has the following form

$$E(x^0) = E\cosh^{-2}\left(\frac{x^0}{\alpha}\right). \tag{39}$$

Such a field switches on and off adiabatically at $x^0 \to \pm\infty$, and is quasiconstant at finite times. We will call it adiabatic field. The corresponding nonzero potential is

$$A_D(x^0) = \alpha E\tanh\frac{x^0}{\alpha}. \tag{40}$$

In this case solutions of eq.(7) can be written in terms of hypergeometric functions $F(a,b;c;y)$ [25], for example,

$$\begin{aligned}
&{}_\zeta\phi_{\mathbf{p},+1}(x^0) = {}_\zeta Ce^{-i\omega_- x^0}\left(1+e^{\frac{2x^0}{\alpha}}\right)^{\frac{i\alpha}{2}(\omega_- - \omega_+)}{}_\zeta u(x^0), \\
&{}_+u(x^0) = F(a,b;c;y),\ \ {}_-u(x^0) = y^{1-c}F(a-c+1,b-c+1;2-c;y), \\
&a = \frac{i\alpha}{2}(2eE\alpha + \omega_+ - \omega_-),\ \ b = 1 + \frac{i\alpha}{2}(-2eE\alpha + \omega_+ - \omega_-), \\
&c = 1 - i\alpha\omega_-,\ \ y = \frac{1}{2}(1+\tanh\frac{x^0}{\alpha}),\ \ \omega_\pm = \sqrt{m^2 + \mathbf{p}_\perp^2 + (p_D \mp eE\alpha)^2},
\end{aligned} \tag{41}$$

where ${}_\zeta C$ are some normalization constants. Considering the asymptotic of the functions (41) at $x^0 \to -\infty$, (in this case $F(a,b;c;y) = 1$ [25]), one can verify that the relations (8) hold and ${}_\zeta\mathcal{E}_{\mathbf{p}} = \zeta\omega_-$. Moreover, the solutions at $x^0 \to -\infty$ describe free particles. By analogy one can construct solutions ${}^\zeta\phi_{\mathbf{p},+1}(x^0)$, which describe free particles with energies ${}^\zeta\mathcal{E}_{\mathbf{p}} = \zeta\omega_+$ at $x^0 \to +\infty$. Their asymptotic form at $x^0 \to +\infty$ is ${}^\zeta\phi_{\mathbf{p},+1}(x^0) = {}^\zeta C\exp(-i\zeta\omega_+ x^0)$. To calculate the coefficients $G({}_+|{}^-)$ from (14) it is enough to know the corresponding asymptotic ${}_+\phi_{\mathbf{p},+1}(x^0)$ and ${}^-\phi_{\mathbf{p},+1}(x^0)$, let say at $x^0 \to +\infty$, and normalization constants ${}^-C = (2\pi)^{-D/2}\left(2\omega_+(\omega_+ - p_D + eE\alpha)\right)^{-1/2}$, ${}_+C = (2\pi)^{-D/2}\left(2\omega_-(\omega_- + p_D + eE\alpha)\right)^{-1/2}$. Thus, we get the mean numbers of fermions created,

$$N_{\mathbf{p},r} = \frac{\sinh\left[\frac{\pi\alpha}{2}(2eE\alpha + \omega_- - \omega_+)\right]\sinh\left[\frac{\pi\alpha}{2}(2eE\alpha + \omega_+ - \omega_-)\right]}{\sinh(\pi\alpha\omega_+)\sinh(\pi\alpha\omega_-)}\,. \tag{42}$$

In 3+1 QED the corresponding formula was found first in [27]. For scalar particles it has a different form

$$N_{\mathbf{p},r} = \frac{\cosh^2\left[\pi\sqrt{(eE\alpha^2)^2 - \frac{1}{4}}\right] + \sinh^2\left[\frac{\pi\alpha}{2}(\omega_+ - \omega_-)\right]}{\sinh(\pi\alpha\omega_+)\sinh(\pi\alpha\omega_-)}\,. \tag{43}$$

Let us consider α-dependence of these expressions. For small α, $\alpha << \frac{1}{eE}\sqrt{m^2 + \mathbf{p}^2}$, when the potential is changed sharply, we get for fermions

$$N_{\mathbf{p},r} = \frac{(\pi eE\alpha^2)^2\left(1 - \frac{p_D^2}{m^2 + \mathbf{p}^2}\right)}{\sinh^2(\pi\alpha\sqrt{m^2 + \mathbf{p}^2})}\,, \tag{44}$$

and for bosons

$$N_{\mathbf{p},r} = \frac{(\pi eE\alpha^2)^2\left[(eE\alpha^2)^2 + \frac{p_D^2}{m^2 + \mathbf{p}^2}\right]}{\sinh^2(\pi\alpha\sqrt{m^2 + \mathbf{p}^2})}\,. \tag{45}$$

Small α in the case under consideration correspond in a sense to small T of the T-constant field. Thus, we have to compare the expressions (31) and (44). One can see that they are quite different, so that the effects of switching on and off are essential at small times.

Further let us consider big α only, $\alpha >> \frac{1}{\sqrt{eE}}(1 + \sqrt{\lambda})$. Then the mean numbers for fermions and bosons have the same form,

$$N_{\mathbf{p},r} = \exp\left\{-\pi\alpha(\omega_+ + \omega_- - 2eE\alpha)\right\}. \tag{46}$$

Let us take small longitudinal momenta $|p_D| << eE\alpha$, then

$$N_{\mathbf{p},r} = \exp\left\{-\pi\lambda\left[1 + \left(\frac{p_D}{eE\alpha}\right)^2\right]\right\}. \tag{47}$$

Considering the limit $\alpha \to \infty$, one gets the formula (33). That means that the effects of switching on and off are not essential at big times and small longitudinal momenta. For big

longitudinal momenta $|p_D| >> eE\alpha$, the mean numbers of particles created are exponentially small,

$$N_{\mathbf{p},r} = \exp\left\{-2\pi\alpha(|p_D| - eE\alpha)\right\}. \tag{48}$$

Let us find the total numbers of particles created with all the longitudinal momenta at any fixed $\mathbf{p}_\perp$, r. Passing from the summation over p_D to the corresponding integration, we get

$$N_{\mathbf{p}_\perp,r} = \frac{LeE\alpha}{2\pi\sqrt{\lambda}}e^{-\pi\lambda}\,. \tag{49}$$

Comparison with the formula (35) shows that the adiabatic field at big times (big α, $\alpha >> \frac{1}{\sqrt{eE}}(1 + \sqrt{\lambda})$ and fixed $\mathbf{p}_\perp$, r is equivalent to the T-constant field at $T >> \frac{1}{\sqrt{eE}}(1 + \lambda)$ with the identification $\alpha = \sqrt{\lambda}T$. To do summation over all transversal momenta, it is convenient to use the representation

$$\frac{1}{\sqrt{\lambda}} = 2\int_0^\infty \exp(-\pi\lambda s^2)ds.$$

Then the total number N of the particles created reads

$$N = J_{(d)}^{\frac{1+\kappa}{2}}\frac{V_{(d-1)}\alpha\delta m^d}{(2\pi)^{d-1}}\left(\frac{E}{E_c}\right)^{\frac{d}{2}}\exp\left\{-\pi\frac{E_c}{E}\right\}, \tag{50}$$

where

$$\delta = \int_0^\infty dt t^{-\frac{1}{2}}(t+1)^{-\frac{d-2}{2}}\exp(-t\pi\frac{E_c}{E}) = \sqrt{\pi}\Psi\left(\frac{1}{2}, -\frac{d-2}{2}; \pi\frac{E_c}{E}\right)$$

is expressed via the confluent hypergeometric function [25]. The vacuum-to-vacuum transition probability P_v has the form

$$P_v = \exp\{-\mu N\}, \quad \mu = \sum_{n=0}^{\infty}\frac{(-1)^{(1-\kappa)\frac{n}{2}}\epsilon_{n+1}}{(n+1)^{\frac{d}{2}}}\exp\left\{-n\pi\frac{E_c}{E}\right\}, \tag{51}$$
$$\epsilon_n = \delta^{-1}\sqrt{\pi}\Psi\left(\frac{1}{2}, -\frac{d-2}{2}; n\pi\frac{E_c}{E}\right).$$

If $E/E_c << 1$ one can use an asymptotic of Ψ-function [25], $\Psi\left(\frac{1}{2}, -\frac{d-2}{2}; n\pi\frac{E_c}{E}\right) = \frac{1}{\sqrt{\pi n}}\sqrt{\frac{E}{E_c}} + O\left(\left[\frac{E}{E_c}\right]^{-3/2}\right)$. Then $\delta = \sqrt{\frac{E}{E_c}}$, $\epsilon_n = n^{-\frac{1}{2}}$ and $\mu = 1$. In this case the adiabatic field is equivalent to T-constant field with the identification $\alpha = T\sqrt{\frac{E_c}{E}}$. At strong fields $E \sim E_c$ all the terms with different ϵ_n contribute to the sum in (51) and the expression for P_v differs essentially from one for the T-constant field.

C. Constant field

Here we consider constant uniform electric field (4). In this case $E(x^0) = E$ and potential $A_D = Ex^0$. Solutions of the eq.(7) in such a field can be found in the form

$$_{+}^{-}\phi_{\mathbf{p},s}(x^0) = CD_{\nu-\frac{1+s}{2}}(\pm(1-i)\xi), \quad ^{+}_{-}\phi_{\mathbf{p},s}(x^0) = CD_{-\nu-\frac{1-s}{2}}(\pm(1+i)\xi). \quad (52)$$

Using an asymptotic expansion of WPC-functions (29), one can get asymptotic of the quasi-energies,

$$_{\zeta}\mathcal{E}_{\mathbf{p}} = \zeta|eEx^0 - p_D|, \quad ^{\zeta}\mathcal{E}_{\mathbf{p}} = \zeta(eEx^0 - p_D) \ ,$$

so that IN and OUT-solutions can be constructed from (52) by means of eq. (11). The same asymptotic expansion (29) allows one to calculate the normalization constants, $C = (2\pi)^{-D/2}(2eE)^{-1/2}\exp\{-\pi\lambda/8\}$ for spinor case and $C = (2\pi)^{-D/2}(2eE)^{-1/4}\exp\{-\pi\lambda/8\}$ for scalar one. Straightforward calculations, similar to ones where made in the two previous cases, lead to the expression (33) for the mean numbers of particles created. It does not depend on the dimensionality of the space and coincides with the result which was derived in [16] for 3+1 QED. In that paper the authors used quasiclassical considerations to advocate the classification of the solutions (52). The constant character of the field does not allow one to treat consistently time divergences, so that they got over them "by hand", using also quasiclassical considerations. Now one can see that the consideration of the T-constant field gives a possibility both to ground all the results obtained from the constant field solutions, solving consistent the problem of the time divergences and to go beyond the scope of the constant field to analyze the time scenario of the process.

D. Inclusion of a magnetic field

In the same manner as before one can consider a more general case when a constant uniform magnetic field is included, provided the invariant I is negative. In fact, in $d > 3$ there are $[\frac{d}{2}] - 1$ independent invariant parameters H_j, $j = 1, 2, \ldots, H_{[\frac{d}{2}]-1}$ of the magnetic

field, that corresponds to a possibility to construct $[\frac{d}{2}]$ invariants of the electromagnetic field. In convenient reference frame the magnetic part of the field tensor $F_{\mu\nu}$ is presented by the components, $F^{\perp}_{\mu\nu} = \sum_{j=1}^{[\frac{d}{2}]-1} H_j(\delta^{j+1}_{\mu}\delta^{j}_{\nu} - \delta^{j+1}_{\nu}\delta^{j}_{\mu})$. One can always select solutions of the squared Dirac equation (3) as eigenfunctions for all independent nonzero terms, which describe the interaction of intrinsic magnetic moment of a particle with the external magnetic field. In this case the matrices $G\left({}_{\zeta}|^{\zeta'}\right)$ are diagonal and one can construct them using the correspondent expressions in the pure electric field. Namely, for $d > 3$ one has to make there a replacement

$$|\mathbf{p}_{\perp}|^2 \to \sum_{j=1}^{[\frac{d}{2}]-1} \omega_j + \omega_0, \quad \omega_0 = \begin{cases} 0, & \text{d is even} \\ p^2_{d-2}, & \text{d is odd ,} \end{cases} \quad (53)$$

$$\omega_j = \begin{cases} |eH_j|(2n_j + 1 - r_j), \ n_j = 0, 1, \ldots \ , & H_j \neq 0 \\ p_j^2 + p_{j+1}^2 \ , & H_j = 0 \end{cases} .$$

In the presence of the magnetic field some momenta p_j have to be replaced by the discrete quantum numbers n_j. The number of these momenta p_j corresponds to the number of nonzero parameters H_j. The magnetic field lifts the degeneracy in spin projections in all the characteristics of the particles creation effect.

We present here explicit formulas in presence of the magnetic field for the total characteristics N and P_v in case of the T-constant field at big T,

$$N = J_{(d)}^{\frac{1+\kappa}{2}} \frac{V_{(d-1)}Tm^2\beta(1)}{2^{(d-1)}\pi^{d/2}} \frac{E}{E_c} \exp\left\{-\pi\frac{E_c}{E}\right\},$$

$$P_v = \exp\{-\mu N\}, \ \mu = \sum_{n=0}^{\infty} \frac{(-1)^{(1-\kappa)\frac{n}{2}}\beta(n+1)}{(n+1)\beta(1)} \exp\left\{-n\pi\frac{E_c}{E}\right\} , \quad (54)$$

where

$$\beta(n) = \prod_{j=1}^{(d-2)/2} \left\{ \frac{eH_j}{\sinh(n\pi H_j/E)} [\cosh(n\pi H_j/E)]^{\frac{1+\kappa}{2}} \right\} , \ d \text{ is even ,}$$

$$\beta(n) = \left(\frac{m^2 E}{n\pi E_c}\right)^{\frac{1}{2}} \prod_{j=1}^{(d-3)/2} \left\{ \frac{eH_j}{\sinh(n\pi H_j/E)} [\cosh(n\pi H_j/E)]^{\frac{1+\kappa}{2}} \right\} , \ d \text{ is odd .}$$

The corresponding formulas for $3+1$-dimensional case where first written in [16,3], and, in fact, can be derived easily from the calculations of Schwinger [2]. They follow from (54) at $d = 4$.

IV. DISCUSSION

A. Time and space-dimensional analysis

The calculations and analysis presented in Sect. III for fields, which are effectively acting a finite time, allows one to study both pairs formation in time and the role of switching on and off effects. Besides, due to the fact that these calculations are made in arbitrary dimensions of the Minkowski space-time, one gets a possibility to analyze the influence of the dimensionality on the vacuum instability.

Studying the T-constant field, one can see that the stabilization of the mean numbers of particles created with given $\mathbf{p}, r$ in the form (33) for the longitudinal momenta $|p_D| << eE\frac{T}{2}$ comes at $T >> T_0$, where $T_0 = \frac{1}{\sqrt{eE}}(1+\lambda)$. The characteristic time T_0 can be called stabilization time. At the same time $N_{\mathbf{p},r}$ for the big longitudinal momenta $|p_D| >> eE\frac{T}{2}$ decrease according to the power low (30).

The stabilization of the mean numbers with given $\mathbf{p}, r$ in the adiabatic field in the same form (33) comes for the longitudinal momenta $|p_D| << eE\alpha$ at $\alpha >> \alpha_0$, $\alpha_0 = \frac{1}{\sqrt{eE}}(1+\sqrt{\lambda})$. For big $|p_D| >> eE\alpha$ the mean numbers are exponentially small (48). For big α the adiabatic field varies slowly and coincides nearly with the constant one in the time interval $|x^0| \leq \alpha$. Then α_0 is a characteristic time of the stabilization in this field. Thus, the stabilization time α_0 in the adiabatic field differs from the corresponding time T_0 in the T-constant field. Thus, one can believe that the stabilization process depends of the effects of switching on and off. In the case $E/E_c < 1$, which corresponds to $\lambda > 1$, one can see the stabilization comes quicker for adiabatic field than for the T-constant one ($\alpha_0 < T_0$), i.e. the adiabatic form of switching on and off affects less the quantum system than the instantaneous one in the T-constant field. If $E/E_c \geq 1$, there exists a domain of the transversal momenta $\mathbf{p}_\perp$

where $\lambda \leq 1$. In this case the stabilization times in both cases are the same, $\alpha_0 \sim T_0 \sim \frac{1}{\sqrt{eE}}$, so that for any E the relation $\alpha_0 \leq T_0$ holds.

Thus, one can conclude, that in some cases T-constant and adiabatic electric fields act on the vacuum similar. However, the momentum dependence of the mean numbers $N_{\mathbf{p},r}$ differs essentially at big momenta for both fields. That is related to the effects of switching on and off. To estimate the role of the effects of switching on and off on the whole it is convenient to compare total characteristics. First, let us compare the total mean numbers with all the longitudinal momenta, namely, compare the formulas (35) and (49). In this case the effective action of both kind of fields is the same if to identify α with $\sqrt{\lambda}T$. In spite of this identification of α and T is different for different λ (for different $\mathbf{p}_\perp$), one can use it in a domain $\mathbf{p}_\perp$ of the transversal momentum, $|\Delta\mathbf{p}_\perp| << \sqrt{m^2+\mathbf{p}_\perp^2}$. As to the total numbers (37) and (50), they coincide if one accepts the identification $\alpha = T\delta^{-1}$. However, this identification provides only the coincidence of P_v for both cases (38) and (51) if $E/E_c << 1$ (then $\delta = \sqrt{\frac{E}{E_c}}$). In this case the coefficients μ in (38) and (51) are the same. One can conclude that the effects of switching on and off are not essential for $E/E_c << 1$ and for big $T >> \frac{1}{m}\left(\frac{E_c}{E}\right)^{3/2}$, or for big α respectively. In case of strong fields, $E/E_c \geq 1$, these effects appear to be essential and one has to take into account the back reaction of particles created for more realistic external field definition (e.g. see [5,7]).

The stabilization of the mean numbers of particles created with given $\mathbf{p}, r$ at $T >> T_0$ can be interpret in the following way: In the T-constant electric field in the finite time instant $\frac{T}{2}$ the pairs are created with equal for particles and antiparticles quantum number $|p_D| < eE\frac{T}{2}$. This corresponds to the region $0 < \pi^D(\frac{T}{2}) < eET$ of the observed kinetic momenta, $\pi^D(\frac{T}{2}) = -(p_D - eA_D(\frac{T}{2})) = -p_D + eE\frac{T}{2}$ of a particle in each pair (direction of antiparticle kinetic momenta is opposite). At $T >> T_0$ the effects of switching on and off are already not essential. That is why the probabilities of pairs creation do not depend on the time instants t, $-\frac{T}{2} < t < \frac{T}{2}$. One can think that at this time instant the particles in the pairs are materialized with almost zero longitudinal kinetic momenta at any given

$\mathbf{p}_\perp$, i.e. with the energies $\sqrt{m^2+\mathbf{p}_\perp^2}$. Then the electric field accelerates them until the end of its action. Let a particle was created in a time instant t. An expression for the kinetic momentum of such a particle in the final time instant (which is equal to its expression in the time instant when the field switches off) can be found solving the classical equation of motion $\frac{d\pi^D}{dx^0}=eE$, so that $\pi^D(\frac{T}{2})=eE\left(\frac{T}{2}-t\right)$. Thus, a particle, which was discovered with the quantum number p_D in the final time instant, was created in the time instant $t=\frac{p_D}{eE}$. Then the integration over the longitudinal momenta p_D is equivalent to one over the time t, $\int dp_D = eET$. This conclusion coincides with one derived in course of the strict quantum analysis presented in Subsect. IIIA. According to the same interpretation, for particles with relatively nonzero mean numbers, the maximum value of the kinetic momentum $\pi^D(\frac{T}{2})$ is eET that corresponds to the particles, which were born in the initial $t=-\frac{T}{2}$ time instant, whereas its minimal value is 0 and corresponds to the particles, which were born in the final $t=\frac{T}{2}$) time instant. This conclusion coincides also with one derived from quantum consideration in Subsect. IIIA.

In the conclusion of the time analysis, one can remark that the time T_0, which was introduced by us as the stabilization time, was interpreted in some papers as the time of a pair creation [16,26]. However, we have seen that in the adiabatic field the stabilization time α_0 is different, thus T_0 is not an universal characteristics, and depends of the field form. In this connection one can propose another characteristic time, which a pair formation in a quasiconstant electric fields. Indeed, as we have mentioned above, all the results in the T-constant and adiabatic fields are comparable if $E/E_c << 1$. In this case the adiabatic form of the field is disturbing quantum system less than the T-constant one. Here $\alpha_0 << T_0$ and $\alpha_0 \approx T_0^f = \frac{\sqrt{\lambda}}{\sqrt{eE}}$. Since the adiabatic field is quasiconstant for the time interval T_0^f and it is big enough for the stabilization, one can interpret T_0^f as the time of a pair formation. One can extrapolate this interpretation of T_0^f for any field strength E. A quasiclassical consideration confirms this interpretation. Thus, a virtual particle with initial zero energy gets in the electric field for the time T_0^f the energy $\sqrt{m^2+\mathbf{p}_\perp^2}$ necessary for the materialization. It is

easily to see that the time T_0^f is always either less than the stabilization times T_0, α_0 or equal to them. Some other consideration related to the time T_0^f see in the next subsection.

Turning to the dimensional analysis, one can see that the increase of degrees of freedom due to the increase of the dimensionality of the space-time itself and due to the related increase of the spinning space dimension $J_{(d)}$ affects essentially the total numbers of particles created in the unit of the volume and the probability for vacuum to remain a vacuum. Thus, the increase of spinning degrees of freedom leads to an increase of N and P_v at any ratio E/E_c. In particular, in $d>3$ the numbers of fermions created is greater than one of bosons. The increase of spatial dimensions leads to decrease of the total numbers of particles created in the unit of the volume and the probability for a vacuum to remain a vacuum at $E/E_c<1$ and their increase at $E/E_c>1$.

The presence of walls or of a nontrivial topology affects the spectrum of particles created. If the length L_i of the space in the direction of an axis x^i is restricted by the walls that leads to the quantization of the corresponding momentum $|p_i|=\frac{2\pi n}{L_i}$, $n=1,2,\dots$. At $L_i \sim \frac{1}{m}$ the dependence of the mean numbers on the boundary conditions is essential. At $L_i << \frac{1}{\sqrt{eE}}$ the mean numbers $N_{\mathbf{p},r}$ in the quasistationary fields are very small for any strength E. In this connection one can treat $L_0=\frac{1}{m}[1+(\frac{E_c}{E})^{1/2}]$ as a characteristic dimension of the system, for which the boundary conditions are essential. At $E/E_c \geq 1$ it is the Compton wave length. It is interesting to remark that the stabilization times T_0, α_0 coincide with L_0 at $E/E_c=1$.

Imposing periodic conditions in the direction of an axis x^i (that corresponds, in particular, to the torus topology), one gets for the momentum $|p_i|=\frac{2\pi n}{L_i}$, $n=0,1,2,\dots$. Then at $L_i << \frac{1}{\sqrt{eE}}$ only particles with $p_i=0$ can be created. If the electric field has the same direction, then the total number N and the probability P_v do not depend on time T, since this dependence arises in course of a summation over the longitudinal momenta. It is interesting that the presence of the magnetic field acts as a dimensional reduction. Indeed, in the strong magnetic field with some $H_j >> E$ the lowest energy level of a boson can not be less than $|eH_j|$, whereas for a fermion it can. That means that the strong magnetic field acts on bosons as some walls and on fermions as the presence of the torus topology. Thus, one can see that if some of the magnetic fields are strong enough, then the corresponding spin projection become frozen and total characteristics, like total mean numbers decrease. These dimensional effects may be relevant to the matter creation at early universe.

B. Relation between the vacuum instability in external electromagnetic and gravitational fields

It is interesting to compare particles creation in external electromagnetic fields and in external fields of different nature, for example, in external gravitational fields. To this end one can use results obtained in the quasiconstant electric fields and in the static gravitational fields. The latter problem was considered first by Hawking [8] who, in particular, calculated the mean numbers of particles created by static gravitational field of a black hole with mass M in a specific thermal environment,

$$N_n = \left[\exp\left\{2\pi\frac{\omega}{g_{(H)}}\right\} + \kappa\right]^{-1}, \tag{55}$$

where ω is the energy of a particle created, which supposes to be dependent on a complete set of quantum numbers n, $g_{(H)} = \frac{GM}{r_g^2}$, where r_g is the gravitational radius, so that $g_{(H)}$ is free falling acceleration at this radius. This spectrum was interpreted as Planck one with the temperature $\theta_{(H)} = \frac{g_{(H)}}{2\pi k_B}$ (k_B is the Boltzmann constant). As before $\kappa = +1$ for fermions and $\kappa = -1$ for bosons. It is also known [28] that an observer, which is moving with a constant acceleration $g_{(R)}$ (with respect to its proper time), will register in the Minkowski vacuum some particles (Rindler particles). The mean numbers of Rindler bosons have the same Planck form (55) (with $\kappa = -1$), where one has to replace $g_{(H)}$ by $g_{(R)}$, so that the correspondent temperature is $\theta_{(R)} = \frac{g_{(R)}}{2\pi k_B}$. One can find many other examples when the particles creation in external gravitation fields (and due to a nontrivial topology) can be described by means of an effective temperature [5,7](see also references in the recent publications [29]). On the other hand the distributions obtained in external electromagnetic

fields have not the thermal form at a first glance. Nevertheless, there were attempts to find close relations between the distributions in both cases, moreover, to derive the Hawking distribution from one in external electromagnetic field.

In the paper [30] the distribution (33) at $\mathbf{p}_\perp = 0$ was interpreted as the Boltzmann one for particles in the ground state with the energy m and the effective temperature $\theta_{(E)} = \frac{2eE}{\pi m}$. The same temperature follows from some other consideration [31] for the same restricted case. Unfortunately, such an interpretation does not allow one to include other states with nonzero momenta in the consideration.

In the papers [32] they did not introduce an effective temperature direct in the electrodynamical case but tried to find a relation between both distributions, in particular, to extract the Hawking temperature from the electrodynamical formulas. We are going to repeat briefly here this consideration , using some new details, which came from the results of the present paper. As was established, a particle with given momenta is created in a time instant with the energy $\omega = \sqrt{m^2 + \mathbf{p}_\perp^2}$, which corresponds to zero longitudinal kinetic momentum at this time instant. Thus, namely this expression plays the role of the total energy of the particle in the time instant of creation. Then we can compare equations of motion for a classical particle in the constant electric field $d\pi/dx^0 = e\mathbf{E}$ with ones in the static gravitational field $d\pi/dx^0 = \omega\mathbf{g}$. In the latter ω is the total energy of the test particle, and $\mathbf{g}$ is the three-dimensional gravitational field strength vector. Although these equations are formally similar, there is a fundamental difference between them: the electromagnetic coupling constant e of a charged particle is not affected by its motion, while the coupling to the gravitational field is proportional to the total energy of the test particle. The latter property is a direct consequence of the equivalence principle. Let us formally replace the electric field strength E by a quantity that characterizes the gravitational field strength $\mathbf{g}$ and exploit the equivalence principle by considering, that the coupling of the particle to the field is proportional to the energy of the former, which also allows us to replace e by ω. The expression that arises from (33) as a result of these replacements can be interpreted as mean numbers of particles created by the corresponding gravitational field and have the form of the Boltzmann distribution with characteristic temperature $\theta' = \frac{g}{\pi k_B}$. If g is the gravitational field strength at the horizon of the black hole $g_{(H)}$, then θ' is only 2 time more than the Hawking temperature $\theta_{(H)}$. In spite of an explicit progress achieved on this way in the comparison of both distributions some questions remain. For example, why the temperature derived by means of the equivalence principle from the electrodynamical distribution differs by the factor 2 from the Hawking one? Is there a thermal interpretation of the electrodynamical distribution (33) or some universal form of particles created spectrum which is valid in both cases? Below we propose some interpretation of the electrodynamical formulas which pretends to answer these questions. We go beyond the classical consideration, taking into account properties of the physical vacuum in the time dependent external field.

First, one can remark that due to the time dependence of the potential $A_D(x^0)$, which defines the quasiconstant electric field, the level of the vacuum energy is changed with time. Thus, one has to calculate carefully the difference between energies of the system in the initial (vacuum) and final (with particles) states. Let us do the calculations in the case of fermions and in zero order with respect to radiative corrections in the T-constant electric field. In this case the correspondent Hamiltonian has the form

$$H(x^0) = \int \bar{\Psi}(x) H_{o.p.} \Psi(x) dx\,, \tag{56}$$

where $H_{o.p.}$ was defined in (9), and $\bar{\Psi}(x)$, $\Psi(x)$ are electron-positron field operators in the generalized Furry picture [17,18,6]. Being written in terms of IN- and OUT- operators of creation and annihilation at $x^0 \to \mp\infty$ respectively, the Hamiltonian $H(x^0)$ has the diagonal forms,

$$\begin{aligned} H(x^0) &= \sum_{\mathbf{p},r} p_0(t_1)\left[a^\dagger_{\mathbf{p},r}(in)a_{\mathbf{p},r}(in) + b^\dagger_{\mathbf{p},r}(in)b_{\mathbf{p},r}(in) - 1\right], \qquad x^0 \to -\infty, \\ H(x^0) &= \sum_{\mathbf{p},r} p_0(t_2)\left[a^\dagger_{\mathbf{p},r}(out)a_{\mathbf{p},r}(out) + b^\dagger_{\mathbf{p},r}(out)b_{\mathbf{p},r}(out) - 1\right], \quad x^0 \to +\infty, \end{aligned} \tag{57}$$

where, as before, $t_2 = -t_1 = \frac{T}{2}$, and $p_0(t_i) = \sqrt{m^2 + \mathbf{p}_\perp^2 + (\pi_D(t_i))^2}$ is the energy of a particle in the initial and final time instants t_i (the longitudinal momenta $\pi_D(t_i) = p_D - eA_D(t_i)$ in the T-constant field have the form $\pi_D(\pm\frac{T}{2}) = p_D \mp eE\frac{T}{2}$. Let us consider the variation of

the total energy of the system, which goes over from the initial vacuum state $|0, in>$ to the final state $|N, out>$ with pairs created in all the levels possible for the T-constant field,

$$|N, out> = \prod_{\mathbf{p}_\perp, r, |p_D| < eET/2} a^\dagger_{\mathbf{p},r}(out) b^\dagger_{\mathbf{p},r}(out)|0, out>.$$

Then one can formally write the energy of the initial state as

$$\mathcal{E}_1 = -\sum_{\mathbf{p},r} p_0(t_1),$$

and of the final state as

$$\mathcal{E}_2 = \sum_{\mathbf{p}_\perp, r}\left[\sum_{|p_D|<eET/2} 2p_0(t_2) - \sum_{p_D} p_0(t_2)\right].$$

Thus

$$\Delta\mathcal{E} = \mathcal{E}_2 - \mathcal{E}_1 = \sum_{\mathbf{p}_\perp, r}\left(\sum_{|p_D|<eET/2}[p_0(t_2) + p_0(t_1)] + \Delta\mathcal{E}_{vac}\right), \tag{58}$$

where

$$\Delta\mathcal{E}_{vac} = \sum_{|p_D|>eET/2}[p_0(t_1) - p_0(t_2)] \tag{59}$$

is the shift of the vacuum energy related to the levels with given $\mathbf{p}_\perp, r$, in which no pairs appear. We are going to analyze this shift only. That is why we do not discuss here regularization problems of the total sum (58) (that can be done, using, for example, the methods described in [11]). One can see that

$$\sum_{|p_D|<eET/2}[p_0(t_1) - p_0(t_2)] = 0,$$

since $p_0(t_1) - p_0(t_2)$ is an odd function of p_D. That allows one to extend the summation in (59) over all the longitudinal momenta. The vacuum before the time instant t_1 was free and therefore symmetric with respect to the longitudinal kinetic momentum $\pi_D(t_1) = \pi_D = p_D + eE\frac{T}{2}$. Replacing the summation over p_D by one over π_D, one can therefore treat the corresponding improper integral in sense of its principal value. Thus

$$\Delta\mathcal{E}_{vac} = \frac{L}{2\pi}\lim_{M\to\infty}\int_{-M}^{M}\left(\sqrt{m^2+\mathbf{p}_\perp^2+\pi_D^2}-\sqrt{m^2+\mathbf{p}_\perp^2+(\pi_D-eET)^2}\right)d\pi_D$$
$$= -\frac{L}{2\pi}(eET)^2 = -\frac{L}{2\pi}\left[\pi_D(t_2)-\pi_D(t_1)\right]^2. \quad (60)$$

Since the number of states with given p_D, in which particles can be created, is equal to $\frac{1}{2\pi}eELT$, see (35), then the shift (60) can be rewritten in the form

$$\Delta\mathcal{E}_{vac} = \sum_{|p_D|<eET/2}\Delta\epsilon_{vac},\ \ \Delta\epsilon_{vac} = -eET = -|\pi_D(t_2)-\pi_D(t_1)|. \quad (61)$$

Thus,

$$\Delta\mathcal{E} = \sum_{\mathbf{p}_\perp,r}\ \sum_{|p_D|<eET/2}\Delta\epsilon,\ \ \Delta\epsilon = p_0(t_2)+p_0(t_1)+\Delta\epsilon_{vac}\,, \quad (62)$$

where $\Delta\epsilon$ can be interpreted as a work which the external field accomplishes for the creation of a pair in a given state. It contains a contribution $\Delta\epsilon_{vac}$ which takes into account a shift of the vacuum energy in those states which remain vacuum ones. The corresponding work with respect to a particle will be denoted by ω, so that,

$$\omega = \frac{1}{2}\Delta\epsilon$$
$$= \frac{1}{2}\left[\sqrt{m^2+\mathbf{p}_\perp^2+(\pi_D(t_2))^2}+\sqrt{m^2+\mathbf{p}_\perp^2+(\pi_D(t_1))^2}-|\pi_D(t_2)-\pi_D(t_1)|\right]. \quad (63)$$

Now we remark that due to the conditions of the stabilization $T >> T_0$, $|p_D| < eE\frac{T}{2}$, under which all the results for the T-constant field were obtained, we can write

$$\omega = \frac{1}{4}\lambda eE\left(\frac{1}{|\pi_D(t_2)|}+\frac{1}{|\pi_D(t_1)|}\right) = \frac{\lambda}{T} = \frac{\lambda eE}{2p_0(t_2)},\ \ \lambda = \frac{m^2+\mathbf{p}_\perp^2}{eE}. \quad (64)$$

Then the spectrum (33) can be rewritten in the following form [1]

$$N_{\mathbf{p},r} = \exp\left\{-2\pi\frac{\omega}{\frac{\hbar}{c}g}\right\}, \quad (65)$$

where the quantity g can be written in several equivalent forms

$$g = \frac{ceE}{2}\left(\frac{1}{|\pi_D(t_2)|}+\frac{1}{|\pi_D(t_1)|}\right) = \frac{2c}{T} = \frac{ceE}{p_0(t_2)}. \quad (66)$$

[1] we have restored $\hbar$ and c here for convenience of the reader

The latter of them allows one to treat g as the classical acceleration of a particle in the electric field in the final time instant $t_2 = \frac{T}{2}$, if the action time of the field is big enough, so that the corresponding velocities are near c. Formally this is valid under the quantum condition of stabilization. The distribution (65) is, in fact, the Boltzmann one with the temperature $\theta = \frac{\hbar g}{2\pi c k_B}$ having literally the Hawking form. Thus, if one identifies the work ω, we have introduced, with the energy of a particle in the formula (55), then the distributions in electrodynamical and gravitational cases have the same thermal structure. Let us discuss now the possible origin of the differences in the electrodynamical and gravitational formulas. First of all, the formula (55) is derived in the formalism of the stationary scattering theory, where it is not necessary to take into account a shift of the vacuum level separately. In this case the energy of a particle created may coincide with the correspond work of the field. Second, the different form of the thermal distributions (Boltzmann, Planck) can be connected with principally different situation in both cases. In the electrodynamical case we deal in fact with pure states, whereas in the gravitational problems a density matrix is arisen necessary due to the horizon of events formation. At $\omega/g << 1$ the Planck spectrum coincides with the Boltzmann one. In this case one can believe the form (65) for the spectrum of particles created is universal and applicable to any theory with quasiconstant external fields.

The form (65) can be useful to describe situations in constant fields, where it is convenient to avoid the consideration of the time evolution, as we have seen comparing it with the gravitational cases. Another form of the distribution (65) with $2\omega = \Delta\epsilon$ from (63) (or (62)) and with the acceleration g from (66), can be useful in problems with explicit time dependence.

The universality of the formula (65) can be examine also in the case of the adiabatic electric field, $\alpha >> \alpha_0$, considered in Sect.IIIB. To apply it to the latter case one needs to put $t_{2,1} \to \pm\infty$, then $\pi_D(t_2) = p_D - eE\alpha$, $\pi_D(t_1) = p_D + eE\alpha$ and $p_0(t_2) = \omega_+$, $p_0(t_1) = \omega_-$. In this case according to (61) $\Delta\epsilon_{vac} = -2eE\alpha$ and the distribution (46) follows.

One can remark in this connection that in case $E/E_c << 1$ ($\alpha_0 << T_0$) the formulas (64) (and, therefore, (63)) in the adiabatic field are valid at the condition $t_2 = -t_1 = \frac{T}{2} >> \alpha_0 \sim T_0^f$, which are weaker than in the case of the T-constant field. That allows one to interpret the time T_0^f, which already had appeared before in Subsect.A as a pair formation time, from another point of view. Considering the formula (63), one can see that at $T >> T_0^f$ the $\Delta\epsilon$ is less than $2\sqrt{m^2 + \mathbf{p}^2}$, due to the essential contribution of the vacuum shift $\Delta\epsilon_{vac}$. That means the work of the external field to produce a pair is less than one, which could be expected from the perturbation theory, where no vacuum change is taken into account.

The consideration presented was made only for fermions. However, if one believes that the quantity $\Delta\epsilon_{vac}$ can be taken in the form (61) for bosons as well, then the distribution (65) holds also in the scalar case. A consistent analysis for charged boson is more complicated and needs to take into account possible condensate formation and its evolution in an external field (see, for example, [5,33]).

Finally, we believe that the formulas derived and the time-dimensional analysis presented can be also useful to describe some collective effects in the frame of quantum field theory, for instance, to describe multiple particles creation by means of the external field approach [13] or in string models with external field [11,12,34].

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