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ON THE INITIAL-VALUE PROBLEM FOR A CHIRAL GROSS-NEVEU SYSTEM

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Abstract

A time-dependent projection technique is used to treat the initial-value problem for self-interacting fermionic fields. On the basis of the general dynamics of the fields, we derive equations of kinetic type for the set of one-body dynamical variables. A nonperturbative mean-field expansion can be written for these equations. We treat this expansion in lowest order, which corresponds to the mean-field approximation, for an uniform system described by Chiral Gross-Neveu model. Literature static results are obtained such as dynamical mass generation due to chiral symmetry breaking and a phenomenon analogous to dimensional transmutation. The time evolution of the one-body dynamical variables initially displaced from equilibrium is discussed.

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1 Introduction

Over the last decade interest in the initial-value problem for quantum-field theoretical models stems mainly from two different areas. The study of the inflationary scenario of the early Universe which involves control of the time evolution of a driving scalar field [1]; and also by properties of hadronic matter manifest themselves through transient phenomena in globally off-equilibrium situations in high-energy collision [2]. In either of the two contexts nonperturbative methods must be employed. Any sufficiently microscopic model will involve a set of mutually interacting quantum fields, which can be thought of as interacting subsystems forming a large, possibly autonomous system. The quantum state of each of the different subsystems can be described in terms of density operator which in general will evolve nonunitarily on account of correlation effects involving different subsystem [3] [4]. The nonunitary effects will manifest themselves through the dynamical evolution of the eigenvalues of the subsystem density matrices, so that individual subsystems usually behave in a nonisoentropic manner [3].

The overwhelming complexity of such a picture is considerably reduced whenever one is able to find physical grounds to motivate a mean-field-like approximation which consists in assuming isoentropic subsystem evolution under effective, time-dependent Hamiltonian operators for each subsystem [4]. In this case the dynamics of the subsystem density matrix can be formulated in terms of a Liouville-von Neuman equation governed by an effective Hamiltonian , e.g., from the point of view of the functional field-theoretical Schrödinger picture, as proposed by Jackiw [4]. Unfortunately the resulting problem involves in general nonlinear Hamiltonians, and cannot be solved without further approximation. In the field-theoretical context, this has been implemented through the use of a Gaussian ansatz for the subsystem density functional in the framework of a time-dependent variational principle supplying the appropriate dynamical information.

It is not difficult to see that this last approximation amounts to a second mean-field approximation, now at microscopic level of the single-field, nonlinear, isoentropic effective dynamics. Actually, the Gaussian ansatz, having the form of a exponential of a quadratic form in the field operators, implies that many-point correlation functions can be factored in terms of two-point functions. This is well known in the context of the derivation of the Hartree-Fock approximation to the nonrelativistic many-body problem [5]. This factorization has been used by Chang [6] to implement the Gaussian approximation for the $\lambda\phi^4$ theory. The dynamics of the reduced two-point density becomes then itself isoentropic, since irreducible higher-order correlation effects are neglected.

The focus of this work is a reevaluation of this second mean-field approximation through a time-dependent projection approach developed earlier for the nonrelativistic nuclear many-body dynamics by Nemes and Toledo Piza [7]. This approach allows for the formulation of a mean-field expansion for the dynamics of the two-point correlation function from which one recovers the results of the Gaussian mean-field approximations in lowest order, i.e., this approach permit to include and to evaluate higher dynamical corrections effects to the simplest mean-field approximation. Moreover, the expansion is energy-conserving (for closed system) to all orders [8]. The resulting dynamical equations acquire the structure of kinetic equations which eliminate the isoentropic mean-field constraint describing the effective dynamics of a selected set of observables [7].

This approach was recently applied for the solution of the self-interacting $\lambda\phi^4$ theory in (1+1) dimensions [9]. Lin and Toledo Piza find that the mean-field approximation fails qualitatively and quantitatively in the description of certain field variables. These failures are partially corrected by the collisional terms. Motivated by success obtained in description of time evolution of a off-equilibrium uniform boson (scalar field) system beyond Gaussian mean-field approximation in quantum-field theoretical context, it becomes interesting to study the fermion case in this approach.

In this paper we develop an analogous formulation to treat the initial-value problem in the case of an off-equilibrium spatially uniform many-fermion system described by Chiral Gross-Neveu model [10]. On the basis of the general dynamics of the fields, we derive equations of kinetic type for the set of one-body variables in lowest order, which correspond to the mean-field approximation. The detailed consideration of collisional correlations is deferred to future work.

The outline of the paper is as follows. In Sec.II we obtain the dynamical equations which describe the time evolution of our (1+1) dimensional uniform fermion system. These

equations are the groundwork for the implementation of our projection technique. This technique and the approximation scheme are described in Sec.III. In Sec.IV we implement in the quantum-field theoretical context the projection technique and obtain in mean-field (isoentropic) approximation the dynamical equations which describe the effective dynamics of a off-equilibrium spatially uniform (1+1) dimensional self-interacting fermion system described by Chiral Gross-Neveu model [10]. In Sec.V we use the static solution of these equations in order to renormalize the theory, leading to the well-known effective potential obtained by Gross and Neveu using the $1/N$ expansion. In this same section, we show also that other static results which have been discussed in the literature such as dynamical mass generation due to chiral symmetry breaking and a phenomenon analogous to dimensional transmutation can be retrieved from this formulation in a mean-field approximation. Finally we study the time evolution of our system. Sec.VI is devoted to a final discussion and conclusions. Some points of a more technical nature are discussed in the Appendix.

2 Kinetics of a self-interacting fermionic field

In this section, we shall describe a formal treatment of the kinetics of a self-interacting quantum field. Although the procedure is quite general, we will adopt the specific context of a fermion field in (1+1) dimensions and assume spatial uniformity. We will illustrate all the relevant points of the approach and cut down inessential technical complications. Features of more general contexts are discussed in Ref. [11].

The idea of our approach is to focus on the time evolution of a set of simple observables. We argue that a large number of relevant physical observables are one-body operators. Consequently, the time evolution of observables which involve field bilinear forms such as $\bar{\psi}(x)\psi(x)$, $\psi(x)\psi(x)$, ... is desirable. These observables are kept under direct control when one works variationally using a Gaussian functional ansatz which will therefore be refered to as Gaussian observables. In order to keep as close as possible to the formulation appropriate for the many-body problem, we work instead with expressions which are bilinear in the creation and annihilation parts of the fields in momentum space with periodic boundary conditions in a spatial box of lenght L, defined in terms of an expansion mass parameter m.

We begin by expanding the Dirac field operators $\psi(x)$ and $\bar{\psi}(x)$ in Heisenberg picture as

$$\begin{aligned}\psi(x) &= \sum_{\mathbf{k}} \left(\frac{m}{k_0}\right)^{1/2} \left[b_{\mathbf{k},1}(x_0)u_1(\mathbf{k})\frac{e^{i\mathbf{kx}}}{\sqrt{L}} + b^{\dagger}_{\mathbf{k},2}(x_0)u_2(\mathbf{k})\frac{e^{-i\mathbf{kx}}}{\sqrt{L}}\right] \\ \bar{\psi}(x) &= \sum_{\mathbf{k}} \left(\frac{m}{k_0}\right)^{1/2} \left[b^{\dagger}_{\mathbf{k},1}(x_0)\bar{u}_1(\mathbf{k})\frac{e^{-i\mathbf{kx}}}{\sqrt{L}} + b_{\mathbf{k},2}(x_0)\bar{u}_2(\mathbf{k})\frac{e^{i\mathbf{kx}}}{\sqrt{L}}\right] ,\end{aligned} \tag{1}$$

where $b^{\dagger}_{\mathbf{k},1}$ and $b_{\mathbf{k},1}$ [$b^{\dagger}_{\mathbf{k},2}$ and $b_{\mathbf{k},2}$] are fermion creation and annihilation operators associated to positive[negative]-energy solution $u_1(\mathbf{k})$ [$u_2(\mathbf{k})$] of Dirac equation.

Canonical quantization demands that the creation and annihilation operators satisfy the standard anticommutation relations at equal times

$$\begin{aligned}\{b^{\dagger}_{\mathbf{k},\lambda}(x_0), b_{\mathbf{k}',\lambda'}(x'_0)\}_{x_0=x'_0} &= \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'} \quad \text{for} \quad \lambda,\lambda'=1,2 \\ \{b^{\dagger}_{\mathbf{k},\lambda}(x_0), b^{\dagger}_{\mathbf{k}',\lambda'}(x'_0)\}_{x_0=x'_0} &= \{b_{\mathbf{k},\lambda}(x_0), b_{\mathbf{k}',\lambda'}(x'_0)\}_{x_0=x'_0} = 0 \ .\end{aligned} \tag{2}$$

In Eq.(1) $\mathbf{x}$ is the spatial coordinate only and we use the notation

$$(k_0)^2 = (\mathbf{k})^2 + m^2 \quad \text{and} \quad kx = k_0x_0 - \mathbf{kx} \ .$$

At this point it is convenient to analyse the difference between spatially uniform and nonuniform systems. Uniform systems exhibit translational invariance (homogeneous system) and rotational invariance (isotropic system). In the case of (1+1) dimensions we say that uniform systems exhibit invariance under translation and under reflection (parity).

The state of the system (assumed spatially uniform) is given in terms of a many-body density operator $\mathcal{F}$ in the Heisenberg picture. $\mathcal{F}$ is therefore time independent, non-negative, Hermitian, and has unit trace. The corresponding mean values of the relevant bilinear forms of field operators are

$$\begin{aligned}R_{\mathbf{k},\lambda';\mathbf{k},\lambda}(x_0) &= Tr[(b^{\dagger}_{\mathbf{k},\lambda'}(x_0)b_{\mathbf{k},\lambda}(x_0))\mathcal{F}] \quad \text{for} \quad \lambda,\lambda'=1,2 \\ \Pi_{\mathbf{k},\lambda';\mathbf{k},\lambda}(x_0) &= Tr[(b_{\mathbf{k},\lambda'}(x_0)b_{\mathbf{k},\lambda}(x_0))\mathcal{F}] \quad \text{for} \quad \lambda,\lambda'=1,2 \ .\end{aligned} \tag{3}$$

The Hermitian matrix R and the antisymmetric matrix Π are the one-fermion density and pairing density respectively. Using these objects we can construct the extended one-body density [12]

$$\mathcal{R}_{\mathbf{k}}(x_0) = \begin{bmatrix} R_{\mathbf{k}}(x_0) & \Pi_{\mathbf{k}}(x_0) \\ -\Pi^*_{\mathbf{k}}(x_0) & I_2 - R_{\mathbf{k}}(x_0) \end{bmatrix} = \begin{bmatrix} \langle b^{\dagger}_{\mathbf{k},\lambda'}(x_0)b_{\mathbf{k},\lambda}(x_0)\rangle & \langle b_{-\mathbf{k},\lambda'}(x_0)b_{\mathbf{k},\lambda}(x_0)\rangle \\ \langle b^{\dagger}_{-\mathbf{k},\lambda'}(x_0)b^{\dagger}_{\mathbf{k},\lambda}(x_0)\rangle & \langle b_{\mathbf{k},\lambda'}(x_0)b^{\dagger}_{\mathbf{k},\lambda}(x_0)\rangle \end{bmatrix} \tag{4}$$

This object summarizes all information on the Gaussian observables and provides an adequate starting point for our kinetic treatment. The first step is standard and consists in reducing the extended one-body density to diagonal form. This can be achieved by subjecting the creation and annihilation operators to a canonical transformation of the Bogolyubov type with coefficients defined by the eigenvalue problem

$$\mathcal{X}^{\dagger}_{\mathbf{k}}(x_0)\mathcal{R}_{\mathbf{k}}(x_0)\mathcal{X}_{\mathbf{k}}(x_0) = \mathcal{Q}_{\mathbf{k}}(x_0) \ , \tag{5}$$

where the matrix $\mathcal{X}_{\mathbf{k}}(x_0)$ which diagonalizes $\mathcal{R}_{\mathbf{k}}(x_0)$ has the structure

$$\mathcal{X}_{\mathbf{k}} = \begin{bmatrix} X^*_{\mathbf{k}} & Y^*_{\mathbf{k}} \\ Y_{\mathbf{k}} & X_{\mathbf{k}} \end{bmatrix} \quad \text{and} \quad \mathcal{X}^{\dagger}_{\mathbf{k}} = \begin{bmatrix} X^T_{\mathbf{k}} & Y^{\dagger}_{\mathbf{k}} \\ Y^T_{\mathbf{k}} & X^{\dagger}_{\mathbf{k}} \end{bmatrix} \ . \tag{6}$$

$\mathcal{Q}_{\mathbf{k}}(x_0)$ is the extended density matrix in generalized natural orbitals basis (Bogolyubov quasi-particle basis)

$$Q_{\mathbf{k}}(x_0)=\begin{bmatrix}\nu_{\mathbf{k}}(x_0) & 0\\ 0 & I_2-\nu_{\mathbf{k}}(x_0)\end{bmatrix}=\begin{bmatrix}\langle\beta^\dagger_{\mathbf{k},\lambda'}(x_0)\beta_{\mathbf{k},\lambda}(x_0)\rangle & \langle\beta_{-\mathbf{k},\lambda'}(x_0)\beta_{\mathbf{k},\lambda}(x_0)\rangle\\ \langle\beta^\dagger_{-\mathbf{k},\lambda'}(x_0)\beta^\dagger_{\mathbf{k},\lambda}(x_0)\rangle & \langle\beta_{\mathbf{k},\lambda'}(x_0)\beta^\dagger_{\mathbf{k},\lambda}(x_0)\rangle\end{bmatrix}\ . \tag{7}$$

The matrix $\nu_{\mathbf{k}}(x_0)$ is diagonal with eigenvalues $\nu_{\mathbf{k},1}$ e $\nu_{\mathbf{k},2}$ which are quasi-fermion occupation numbers for the paired natural orbitals. Because of the assumed reflection symmetry one must have

$$\nu_{\mathbf{k},\lambda}(x_0)=\nu_{-\mathbf{k},\lambda}(x_0)\ . \tag{8}$$

The unitary conditions for $\mathcal{X}_{\mathbf{k}}(x_0)$ can be interpreted as orthogonality and completeness relations for the natural orbitals which read

$$\mathcal{X}^\dagger_{\mathbf{k}}\mathcal{X}_{\mathbf{k}}=I_4\quad\text{and}\quad\mathcal{X}_{\mathbf{k}}\mathcal{X}^\dagger_{\mathbf{k}}=I_4\ , \tag{9}$$

or in terms of $X_{\mathbf{k}}$ and $Y_{\mathbf{k}}$ matrix [see Eq.(6)]

$$\begin{aligned}&Y_{\mathbf{k}}Y^\dagger_{\mathbf{k}}+X_{\mathbf{k}}X^\dagger_{\mathbf{k}}=\mathbf{I}_2\quad,\quad Y_{\mathbf{k}}X^T_{\mathbf{k}}+X_{\mathbf{k}}Y^T_{\mathbf{k}}=\mathbf{0}_2\ ,\\ &Y^\dagger_{\mathbf{k}}Y_{\mathbf{k}}+X^T_{\mathbf{k}}X^*_{\mathbf{k}}=\mathbf{I}_2\quad,\quad Y^T_{\mathbf{k}}X^*_{\mathbf{k}}+X^\dagger_{\mathbf{k}}Y_{\mathbf{k}}=\mathbf{0}_2\ .\end{aligned} \tag{10}$$

Finally, from Eq.(6), we write the fermion operators $b^\dagger_{\mathbf{k},\lambda}(x_0)$ and $b_{\mathbf{k},\lambda}(x_0)$ in terms of the new quasi-fermion operators $\beta^\dagger_{\mathbf{k},\lambda}(x_0)$ and $\beta_{\mathbf{k},\lambda}(x_0)$ for $\lambda=1,2$

$$\begin{bmatrix}b_{\mathbf{k},1}\\ b_{\mathbf{k},2}\\ b^\dagger_{-\mathbf{k},1}\\ b^\dagger_{-\mathbf{k},2}\end{bmatrix}=\begin{bmatrix}X_{11} & X_{21} & Y^*_{11} & Y^*_{21}\\ X_{12} & X_{22} & Y^*_{12} & Y^*_{22}\\ Y_{11} & Y_{21} & X^*_{11} & X^*_{21}\\ Y_{12} & Y_{22} & X^*_{12} & X^*_{22}\end{bmatrix}\begin{bmatrix}\beta_{\mathbf{k},1}\\ \beta_{\mathbf{k},2}\\ \beta^\dagger_{-\mathbf{k},1}\\ \beta^\dagger_{-\mathbf{k},2}\end{bmatrix}\ . \tag{11}$$

With the help of Eq.(11) it is an easy task to express $\bar\psi(x)$ and $\psi(x)$ [Eq.(1)] in term of $\beta^\dagger_{\mathbf{k},\lambda}(x_0)$ and $\beta_{\mathbf{k},\lambda}(x_0)$ for $\lambda=1,2$. In doing so, one finds that the plane waves of $\bar\psi(x)$ and $\psi(x)$ are modified by a complex, moment-dependent redefinition of m involving the Bogolyubov parameters. The complex character of these parameters is actually crucial in dynamical situations, where the imaginary parts will allow for the description of time-odd (i.e., velocity-like) properties.

What we have achieved so far amounts to an expansion of the fields $\bar\psi(x)$ and $\psi(x)$ such that the mean values in $\mathcal{F}$ of Gaussian observables are parametrized in terms of the $X_{\lambda',\lambda}(\mathbf{k})$ and $Y_{\lambda',\lambda}(\mathbf{k})$ and of the occupation numbers $\nu_{\mathbf{k},\lambda}(x_0)=Tr\left(\beta^\dagger_{\mathbf{k},\lambda}\beta_{\mathbf{k},\lambda}\mathcal{F}\right)$ for $\lambda=1,2$. In general, all these quantities are time dependent under the Heisenberg dynamics of the field operators, and we now proceed to write the corresponding equations of motion.

The next step is to obtain the time evolution of the mean values of gaussian observables in the context of the initial-value problem, i.e., we want the equations of motion for the Bogolyubov parameters $X_{\lambda',\lambda}(\mathbf{k})$, $Y_{\lambda',\lambda}(\mathbf{k})$ and quasi-particle occupation numbers $\nu_{\mathbf{k},\lambda}$. Taking the time derivative of eigenvalue Eq.(5) and using the unitarity condition (9) we get

$$\mathcal{X}^\dagger_{\mathbf{k}}\dot{\mathcal{R}}_{\mathbf{k}}\mathcal{X}_{\mathbf{k}}=\dot{Q}_{\mathbf{k}}-\dot{\mathcal{X}}^\dagger_{\mathbf{k}}\mathcal{X}_{\mathbf{k}}Q_{\mathbf{k}}-Q_{\mathbf{k}}\mathcal{X}^\dagger_{\mathbf{k}}\dot{\mathcal{X}}_{\mathbf{k}}\ . \tag{12}$$

We now evaluate the left-hand side of this equation using the Heisenberg equation of motion to obtain

$$i\mathcal{X}^\dagger_{\mathbf{k}}\dot{\mathcal{R}}_{\mathbf{k}}\mathcal{X}_{\mathbf{k}}=\begin{bmatrix}Tr\left(\left[\beta^\dagger_{\mathbf{k},\lambda'}\beta_{\mathbf{k},\lambda},H\right]\mathcal{F}\right) & Tr\left(\left[\beta_{-\mathbf{k},\lambda'}\beta_{\mathbf{k},\lambda},H\right]\mathcal{F}\right)\\ Tr\left(\left[\beta^\dagger_{-\mathbf{k},\lambda'}\beta^\dagger_{\mathbf{k},\lambda},H\right]\mathcal{F}\right) & Tr\left(\left[\beta_{\mathbf{k},\lambda'}\beta^\dagger_{\mathbf{k},\lambda},H\right]\mathcal{F}\right)\end{bmatrix}\ . \tag{13}$$

The right-hand side of Eq.(12) can also be evaluated explicitly using Eq.(6) and (7) :

$$i\left(\dot{Q}_{\mathbf{k}}-\dot{\mathcal{X}}^\dagger_{\mathbf{k}}\mathcal{X}_{\mathbf{k}}Q_{\mathbf{k}}-Q_{\mathbf{k}}\mathcal{X}^\dagger_{\mathbf{k}}\dot{\mathcal{X}}_{\mathbf{k}}\right)=\begin{bmatrix}i\dot\nu_{\mathbf{k}}+[\nu_{\mathbf{k}},h^*_{\mathbf{k}}]_- & -g^*_{\mathbf{k}}+\{\nu_{\mathbf{k}},g^*_{\mathbf{k}}\}_+\\ -g_{\mathbf{k}}+\{\nu_{\mathbf{k}},g_{\mathbf{k}}\}_+ & -i\dot\nu_{\mathbf{k}}+[\nu_{\mathbf{k}},h_{\mathbf{k}}]_-\end{bmatrix}\ , \tag{14}$$

where the $h_{\mathbf{k}}$ and $g_{\mathbf{k}}$ matrix are given in terms of $X_{\mathbf{k}}$ and $Y_{\mathbf{k}}$ matrix

$$h_{\mathbf{k}} = -i(\dot{Y}_{\mathbf{k}}^{T} Y_{\mathbf{k}}^{*} + \dot{X}_{\mathbf{k}}^{\dagger} X_{\mathbf{k}})$$
$$g_{\mathbf{k}} = -i(\dot{Y}_{\mathbf{k}}^{T} X_{\mathbf{k}}^{*} + \dot{X}_{\mathbf{k}}^{\dagger} Y_{\mathbf{k}}) \ . \tag{15}$$

From Eqs.(13) and (14) we obtain (2x2) matrix dynamical equations which describe the time evolution of our (1+1) dimensional uniform fermion system. One immediately finds

$$\begin{aligned} i\dot{\nu}_{\mathbf{k}} + [\nu_{\mathbf{k}}, h_{\mathbf{k}}^{*}]_{-} &= Tr\left([\beta_{\mathbf{k},\lambda'}^{\dagger}\beta_{\mathbf{k},\lambda}, H]\mathcal{F}\right) \\ -\dot{g}_{\mathbf{k}}^{*} + \{\nu_{\mathbf{k}}, g_{\mathbf{k}}^{*}\}_{+} &= Tr\left([\beta_{-\mathbf{k},\lambda'}\beta_{\mathbf{k},\lambda}, H]\mathcal{F}\right) \ . \end{aligned} \tag{16}$$

Eqs.(16), together with the unitarity condition (10), determine the time rate of change of the Gaussian observables in terms of expectation values of appropriate commutators. They are, however, clearly not closed equations when the Hamiltonian H involves self-interacting fields or when the initial condition itself contain many-fermion correlation. In this cases, the time derivatives of the Gaussian observables are given in terms of traces which are in general not expressible in terms of the quantities themselves, i.e., the traces will in general contain many-fermion densities.

3 Projection technique and approximation scheme

In this section we introduce the time-dependent projection technique [7] which permit to obtain a closed approximation to the equations of motion (16). It developed earlier in the context of nonrelativistic nuclear many-body dynamics was recently applied in the quantum-field theoretical context to the self-interacting $\lambda\phi^4$ theory in (1+1) dimensions [9].It allows for the formulation of a mean-field expansion for the dynamics of the two-point correlation function from which one recovers the results of the Gaussian mean-field approximations in

lowest order. If carried to higher orders it allows for the inclusion and evaluation of higher dynamical correlation corrections to the simplest mean-field approximation.

In order to develop our treatment of the equations of motion (16) we begin by decomposing the full density $\mathcal{F}$ as

$$\mathcal{F} = \mathcal{F}_0(t) + \mathcal{F}'(t) \ , \tag{17}$$

where $\mathcal{F}_0(t)$ is a Gaussian ansatz which achieves a Hartree-Fock factorization of traces involving more than two field operators (Gaussian observables). This is what we refer to as the mean-field approximation of the equation of motion. $\mathcal{F}_0(t)$ is chosen as having the form of a exponential of a bilinear, Hermitian expression in the fields normalized to unit trace [5]. In the momentum basis, it reads

$$\mathcal{F}_0 = \frac{\exp\left[\sum_{(\mathbf{k}_1,\mathbf{k}_2)} A_{\mathbf{k}_1,\mathbf{k}_2} b_{\mathbf{k}_1}^{\dagger} b_{\mathbf{k}_2} + B_{\mathbf{k}_1,\mathbf{k}_2} b_{\mathbf{k}_1}^{\dagger} b_{\mathbf{k}_2}^{\dagger} + C_{\mathbf{k}_1,\mathbf{k}_2} b_{\mathbf{k}_1} b_{\mathbf{k}_2}\right]}{Tr\left\{\exp\left[\sum_{(\mathbf{k}_1,\mathbf{k}_2)} A_{\mathbf{k}_1,\mathbf{k}_2} b_{\mathbf{k}_1}^{\dagger} b_{\mathbf{k}_2} + B_{\mathbf{k}_1,\mathbf{k}_2} b_{\mathbf{k}_1}^{\dagger} b_{\mathbf{k}_2}^{\dagger} + C_{\mathbf{k}_1,\mathbf{k}_2} b_{\mathbf{k}_1} b_{\mathbf{k}_2}\right]\right\}} \ . \tag{18}$$

The parameters in Eq.(18) are fixed by requiring that mean values in $\mathcal{F}_0$ of expressions that are bilinear in the fields reproduce the corresponding $\mathcal{F}$ averages [see Eqs.(20) below]. $\mathcal{F}_0$ is a time-dependent object, which acquires a particularly simple form when expressed in terms of the Bogolyubov quasi-fermion operators

$$\mathcal{F}_0(t) = \prod_{\mathbf{k},\lambda} \left[\nu_{\mathbf{k},\lambda}\beta_{\mathbf{k},\lambda}^{\dagger}(x_0)\beta_{\mathbf{k},\lambda}(x_0) + (1-\nu_{\mathbf{k},\lambda})\beta_{\mathbf{k},\lambda}(x_0)\beta_{\mathbf{k},\lambda}^{\dagger}(x_0)\right] \ . \tag{19}$$

It has unit trace and the properties below

$$\begin{aligned} & Tr\left(\beta_a\mathcal{F}_0\right) = Tr\left(\beta_a\mathcal{F}\right) = Tr\left(\beta_a^{\dagger}\mathcal{F}_0\right) = Tr\left(\beta_a^{\dagger}\mathcal{F}\right) = 0 \ ; \\ & Tr\left(\beta_a\beta_b\mathcal{F}_0\right) = Tr\left(\beta_a\beta_b\mathcal{F}\right) = 0 \ ; \\ & Tr\left(\beta_a^{\dagger}\beta_b^{\dagger}\mathcal{F}_0\right) = Tr\left(\beta_a^{\dagger}\beta_b^{\dagger}\mathcal{F}\right) = 0 \ ; \\ & Tr\left(\beta_a^{\dagger}\beta_b\mathcal{F}_0\right) = Tr\left(\beta_a^{\dagger}\beta_b\mathcal{F}\right) = \nu_a\delta_{a,b} \ \text{and} \\ & Tr\left(\beta_a\beta_b^{\dagger}\mathcal{F}_0\right) = Tr\left(\beta_a\beta_b^{\dagger}\mathcal{F}\right) = (1-\nu_a)\delta_{a,b} \ . \end{aligned} \tag{20}$$

The "remainder" density $\mathcal{F}'(t)$, defined by Eq.(17), is a traceless, pure correlation density. As already remarked, a crucial point to observe is that $\mathcal{F}_0(t)$ can be written as a time-dependent projection of $\mathcal{F}$, i.e.,

$$\mathcal{F}_0(t) = \mathcal{P}(t)\mathcal{F} \quad \text{with} \quad \mathcal{P}(t)\mathcal{P}(t) = \mathcal{P}(t) \ . \tag{21}$$

$\mathcal{P}(t)$ is an operator acting on a linear space of densities, sometimes called super-space. Such operators are correspondingly sometimes called super-operators. In order to construct the projector $\mathcal{P}(t)$ we require that, in addition to Eq.(21), it satisfies

$$i\dot{\mathcal{F}}_0(t) = [\mathcal{P}(t), \mathcal{L}]\,\mathcal{F} = [\mathcal{F}_0(t), H] + \mathcal{P}(t)\,[H, \mathcal{F}] \ \ , \tag{22}$$

where $\mathcal{L}$ is the super-operator called Liouvillian which is defined as

$$\mathcal{L}\,\cdot = [H,\ \cdot\ ] \ , \tag{23}$$

$\mathcal{H}$ being the Hamiltonian of the field. Eq.(22) is the Heisenberg picture counterpart of the equation $\partial_t\,[\mathcal{P}(t)\mathcal{F}] = 0$ which has been used to define $\mathcal{P}(t)$ in the Schrödinger picture [8]. It is possible to prove that condition (21) and (22) make $\mathcal{P}(t)$ unique [3, 8, 9].

Once $\mathcal{P}(t)$ is obtained, the next step is to obtain a differential equation of $\mathcal{F}'(t)$. This follows in fact immediately from Eqs.(17), (21) and (22). It reads

$$i\,(\partial_t + \mathcal{P}(t)\mathcal{L})\,\mathcal{F}'(t) = (\mathcal{I} - \mathcal{P}(t))\,\mathcal{L}\mathcal{F}_0(t) \ . \tag{24}$$

This equation has the formal solution

$$\mathcal{F}'(t) = \mathcal{G}(t,0)\mathcal{F}'(0) - i\int_0^t dt'\,\mathcal{G}(t,t')\,(\mathcal{I} - \mathcal{P}(t'))\,\mathcal{L}\mathcal{F}_0(t') \ , \tag{25}$$

where the first term accounts for initial correlations (initial condition). The object $\mathcal{G}(t,t')$ is the time-ordered Green's function

$$\mathcal{G}(t,t') = T\left(\exp\left[i\int_{t'}^{t} d\tau\mathcal{P}(\tau)\mathcal{L}\right]\right) \ . \tag{26}$$

We see thus that $\mathcal{F}'(t)$, and therefore also $\mathcal{F}$ [see Eq.(17)], can be formally expressed in terms of $\mathcal{F}_0(t')$ (for $t' \leq t$) and of initial correlations $\mathcal{F}'(0)$. This allows us to express also the dynamical equations (16) as traces over functionals of $\mathcal{F}_0(t')$ and of the initial correlations. Since, on the other hand, the reduced density $\mathcal{F}_0(t')$ is expressed in terms of the one-fermion densities alone, we see that the resulting equations are now essentially closed equations. Note, however, that the complicated time dependence of the field operators is explicitly probed through the memory effects present in the expression (25) for $\mathcal{F}'(t)$. Approximations are therefore needed for the actual evaluation of this object.

A systematic expansion scheme for the memory effects has been discussed in Refs. [8, 9]. The lowest approximation which includes correlation contribuitions corresponds to replacing the full Heisenberg time-evolution of operators occurring in the collision integrals by a mean-field evolution governed by

$$H_0 = \mathcal{P}^\dagger(t)H \ .$$

Consistently with this approximation, $\mathcal{L}$ is replaced in (25) and (26) by $\mathcal{L}_0\,\cdot = [H_0,\ \cdot\ ]$. In this way correlation effects are treated to second order in H in the collision integrals.

An important feature of this scheme (valid also in higher orders of the expansion) is that the mean energy is conserved, namely

$$\frac{\partial}{\partial t}\langle H\rangle = 0$$

where

$$\langle H\rangle = Tr\ H\mathcal{F}_0(t) + Tr\ H\mathcal{F}'(t) \ .$$

The resulting scheme can be interpreted as follows. The dynamical evolution of the field is split into a pure mean-field part, related to the contributions to the dynamical equations involving the projected density $\mathcal{F}_0(t)$, plus correlation contributions, approximated by the contributions involving the adopted form for $\mathcal{F}'(t)$.

4 Effective dynamics for a uniform fermion system described by the Chiral Gross-Neveu model (CGNM) in mean-field approximation

We use the general expression obtained in the preceding section to discuss a uniform fermion system described by Chiral Gross-Neveu model (CGNM). We will consider only the lowest (mean-field) approximation in this paper, corresponding to $\mathcal{F}'(t) = 0$. Collisional correlations will be discussed elsewhere.

4.1 The CGNM Hamiltonian

The Hamiltonian density for the CGNM is given by

$$\mathcal{H}_{\text{CGNM}} = \sum_{i=1}^{N} \left\{ \bar{\psi}^i \left[-i\gamma_1 \partial_1 \right] \psi^i \right\} - \frac{g^2}{2} \left\{ \left[\sum_{i=1}^{N} \bar{\psi}^i \psi^i \right]^2 - \xi \left[\sum_{i=1}^{N} \bar{\psi}^i \gamma_5 \psi^i \right]^2 \right\} , \tag{27}$$

where ξ is a constant which indicates whether the model is invariant under discrete γ_5 transformation ($\xi = 0$) or under the Abelian chiral $U(1)$ group ($\xi = 1$).

In the form considered here, this is a massless fermion theory in (1+1) dimensions with quartic interaction. The model contains N species of fermions coupled symmetrically, where ψ^i is a complex Dirac spinor transforming as the fundamental representation of $SU(N)$ group. It is known that the actual symmetry of the theory is not $SU(N)$ but rather $O(2N)$ [13]. The transformations forming this group mix not only particles but also particles with antiparticles. This model is essentially equivalent to the Nambu-Jona-Lasinio model [14], except for the fact that in (1+1) dimensions it is renormalizable. Moreover, it is one of the very few know field theories which are assimptotically free. To leading order in $1/N$ expansion [10], the CGNM exhibits a number of interesting phenomena, like spontaneous symmetry breaking [15], dynamical fermion mass generation and dimensional transmutation. The model possess an infinite number of conservation laws, and as a consequence, the S-matrix may be computed exactly [16].

To obtain the time evolution of Bogolyubov parameters we have to obtain the CGNM Hamiltonian [see Eq.(16)]. From the Hamiltonian density (27) we can to evaluate the Hamiltonian of the system by integration over all one-dimensional space. This involves, in particular, choosing a representation for the γ-matrices. Here we have to be careful, since a bad choice of representation can spoil manifest translational invariance. In Appendix A (see also Ref.[14]) we give the representations for the γ-matrices that preserves the translational invariance of the system. We choose the Pauli-Dirac representations for the γ-matrices, namely

$$\gamma_0 = \sigma_3 \;\; ; \;\; \gamma_1 = i\sigma_2 \;\; \text{and} \;\; \gamma_5 = \gamma_0\gamma_1 = \sigma_1 \; . \tag{28}$$

In this representation the spinors $u_1(\mathrm{k})$ and $u_2(\mathrm{k})$ are given by

$$u_1(\mathrm{k}) = \left[\frac{(k_0+m)}{2m}\right]^{1/2} \begin{bmatrix} 1 \\ \\ \dfrac{\mathrm{k}}{(k_0+m)} \end{bmatrix} \; , \;\; u_2(\mathrm{k}) = \left[\frac{(k_0-m)}{2m}\right]^{1/2} \begin{bmatrix} 1 \\ \\ \dfrac{\mathrm{k}}{(k_0-m)} \end{bmatrix} \; . \tag{29}$$

Substituting in Hamiltonian density (27) the fields $\bar{\psi}^i$ and ψ^i given in (1) and using (28) and (29), we can calculate the CGNM Hamiltonian by integration over all one-dimensional space. The CGNM Hamiltonian is given in full in Appendix B.

4.2 Effective dynamics for a uniform fermion system described by the CGNM in mean-field approximation

We now evaluate the time evolution of a uniform fermion system described by the CGNM, when the chiral symmetry of the system is broken. The Bogolyubov transformation defined in (11) breaks both chiral and charge symmetries, but we restrict the following development to a special Bogolyubov transformation (to be called Nambu transformation) which breaks the chiral symmetry of our system only. The elements of this Nambu transformation, parametrize consistently with unitary conditions (10), are given by

$$\begin{aligned} X_{11} = X_{22} = \cos\varphi_{\mathbf{k}} \;\; &\text{and} \;\; X_{12} = X_{21} = 0 \\ Y_{12} = -Y_{21} = \sin\varphi_{\mathbf{k}} e^{i\gamma_{\mathbf{k}}} \;\; &\text{and} \;\; Y_{11} = Y_{22} = 0 \; . \end{aligned} \tag{30}$$

In the special case of a Nambu transformation, the elements of the $h_{\mathbf{k}}$ and $g_{\mathbf{k}}$ matrix [see Eq.(15)] are given by

$$\begin{aligned} h_{11} &= h_{22} = \dot{\gamma}_{\mathbf{k}} \sin^2 \varphi_{\mathbf{k}} \\ h_{12} &= h_{21} = g_{11} = g_{22} = 0 \\ g_{12} &= -g_{21} = [i\dot{\varphi}_{\mathbf{k}} - \dot{\gamma}_{\mathbf{k}} \sin\varphi_{\mathbf{k}} \cos\varphi_{\mathbf{k}}] e^{i\gamma_{\mathbf{k}}} \quad . \end{aligned} \tag{31}$$

On the other hand, in the mean-field approximation one finds that $\mathcal{F}_0(t)$ commutes with number operators

$$Tr\,\{H_{\mathrm{CGNM}}[\mathcal{F}_0, \beta^\dagger_{\mathbf{k},1}\beta_{\mathbf{k},1}]\} = Tr\,\{H_{\mathrm{CGNM}}[\mathcal{F}_0, \beta^\dagger_{\mathbf{k},2}\beta_{\mathbf{k},2}]\} = 0 \ , \tag{32}$$

while, due to the Nambu transformation (or charge conservation of the system), we obtain also

$$\begin{aligned} Tr\,\{[\beta^\dagger_{\mathbf{k},1}\beta_{\mathbf{k},2}, H_{\mathrm{CGNM}}]\mathcal{F}_0\} &= Tr\,\{[\beta^\dagger_{\mathbf{k},2}\beta_{\mathbf{k},1}, H_{\mathrm{CGNM}}]\mathcal{F}_0\} = 0 \\ Tr\,\{[\beta_{-\mathbf{k},1}\beta_{\mathbf{k},1}, H_{\mathrm{CGNM}}]\mathcal{F}_0\} &= Tr\,\{[\beta_{-\mathbf{k},2}\beta_{\mathbf{k},2}, H_{\mathrm{CGNM}}]\mathcal{F}_0\} = 0 \ . \end{aligned} \tag{33}$$

Substituting the elements of the $h_{\mathbf{k}}$ and $g_{\mathbf{k}}$ matrix given in (31) and the results (32) and (33) in the matrix dynamical equations (16), we obtain the equations which describe the time evolution of our system

$$\dot{\nu}_{\mathbf{k},1} = 0 \quad \text{and} \quad \dot{\nu}_{\mathbf{k},2} = 0 \tag{34}$$

$$[i\dot{\varphi}_{\mathbf{k}} + \dot{\gamma}_{\mathbf{k}} \sin\varphi_{\mathbf{k}} \cos\varphi_{\mathbf{k}}]\, e^{-i\gamma_{\mathbf{k}}} = \frac{Tr\,([\beta_{-\mathbf{k},1}\beta_{\mathbf{k},2}, H_{\mathrm{CGNM}}]\mathcal{F}_0)}{(1 - \nu_{\mathbf{k},1} - \nu_{\mathbf{k},2})} \ . \tag{35}$$

Equation (34) shows that the occupation numbers of the paired natural orbitals are constant, i.e., we recover the general isoentropic character of the mean-field approximation. The complex equation of motion (35) describes the time evolution of the Nambu parameters. Writing the CGNM Hamiltonian, given in Appendix B, in the Nambu basis using Eqs.(11) and (30), and substituting this Hamiltonian in Eq.(35), we obtain the explicit dynamical equation which describes the time evolution of the Nambu parameters. The calculation of traces is lengthy but straightforward. The resulting equation of motion is

$$\begin{aligned} i\dot{\varphi}_{\mathbf{k}} \ &+ \ \dot{\gamma}_{\mathbf{k}} \frac{\sin 2\varphi_{\mathbf{k}}}{2} = \frac{(\mathbf{k})^2}{k_0} \sin 2\varphi_{\mathbf{k}} - m\frac{|\mathbf{k}|}{k_0}\left[\sin^2\varphi_{\mathbf{k}} e^{-i\gamma_{\mathbf{k}}} - \cos^2\varphi_{\mathbf{k}} e^{i\gamma_{\mathbf{k}}}\right] + \\ &+ \ \left(\frac{g^2 m^2}{4\pi}\right)\frac{(\xi+1)}{k_0}\left[\sin 2\varphi_{\mathbf{k}} + \frac{|\mathbf{k}|}{m}\left(\sin^2\varphi_{\mathbf{k}} e^{-i\gamma_{\mathbf{k}}} - \cos^2\varphi_{\mathbf{k}} e^{i\gamma_{\mathbf{k}}}\right)\right](I_1 + I_2) \end{aligned} \tag{36}$$

where I_1 and I_2 are the divergent integrals below

$$\begin{aligned} I_1 &= \int \frac{d\mathbf{k}'}{k'_0} \cos 2\varphi_{\mathbf{k}'} (1 - \nu_{\mathbf{k}',1} - \nu_{\mathbf{k}',2}) \\ I_2 &= \int \frac{d\mathbf{k}'}{k'_0}\frac{|\mathbf{k}'|}{m} \sin 2\varphi_{\mathbf{k}'} \cos\gamma_{\mathbf{k}'} (1 - \nu_{\mathbf{k}',1} - \nu_{\mathbf{k}',2}) \ . \end{aligned} \tag{37}$$

We take the case $N = 1$ for simplicity. Splitting the complex equation (36) into real and imaginary parts we have

$$\begin{aligned} \dot{\varphi}_{\mathbf{k}} &= \sin\gamma_{\mathbf{k}} \frac{|\mathbf{k}|}{k_0}\left[m - \left(\frac{g^2 m}{4\pi}\right)(\xi+1)(I_1+I_2)\right] \\ \dot{\gamma}_{\mathbf{k}} \sin 2\varphi_{\mathbf{k}} &= \frac{2\sin 2\varphi_{\mathbf{k}}}{k_0}\left[\mathbf{k}^2 + \left(\frac{g^2 m^2}{4\pi}\right)(\xi+1)(I_1+I_2)\right] + \\ &+ \ 2\cos 2\varphi_{\mathbf{k}} \cos\gamma_{\mathbf{k}} \frac{|\mathbf{k}|}{k_0}\left[m - \left(\frac{g^2 m}{4\pi}\right)(\xi+1)(I_1+I_2)\right] \ . \end{aligned} \tag{38}$$

Finally, the mean-field energy is evaluated as

$$
\begin{aligned}
\langle H^{\text{M.F.}}_{\text{CGNM}}\rangle &= Tr\left[H_{\text{CGNM}}\mathcal{F}_0(t)\right] = \left(\frac{m^2}{2\pi}\right)(I_2 - I_3) - \left(\frac{g^2m^2}{8\pi^2}\right)\frac{(\xi+1)}{2}(I_1+I_2)^2 + \\
&- \left(\frac{g^2}{8\pi^2}\right)\frac{(\xi+1)}{2}\int d\mathbf{k}'(1+\nu_{\mathbf{k}',1}-\nu_{\mathbf{k}',2})\int d\mathbf{k}''(1-\nu_{\mathbf{k}'',1}+\nu_{\mathbf{k}'',2}) \quad ,
\end{aligned} \tag{39}
$$

where the Hamiltonian in the trace is in Nambu basis again. The divergent integrals I_1 and I_2 are given in (37) while I_3 is given below

$$
I_3 = \int \frac{d\mathbf{k}'}{k'_0}\left(\frac{\mathbf{k}'}{m}\right)^2 \cos 2\varphi_{\mathbf{k}}(1-\nu_{\mathbf{k}',1}-\nu_{\mathbf{k}',2}) \ . \tag{40}
$$

We can verify that all results above contain divergent integrals. Therefore a renormalization procedure is required.

5 Renormalization

In order to handle the infinities found above it is necessary to introduce a renormalization technique that will render physical quantities finite. In general, renormalization procedures consist in combining divergent terms with the bare mass and coupling constants of the theory to define finite (or renormalized) values of the mass and coupling constant. In other words, the bare mass and coupling constants are chosen to be cut-off dependent in a way that will cancel the divergent terms. In the present case, however the divergent integrals (37) and (40) involve the dynamical variables themselves in the integrand, so that their degree of divergence is not directly computable. In order to handle this situation we will use a self-consistent renormalization procedure inspired in Ref.[17].

5.1 Self-Consistent Renormalization

This technique involves consideration of the static solutions of the dynamical equations (38). They are determined by the solution to the equations

$$
\sin\gamma_{\mathbf{k}}|_{\text{eq}}\left[1-\left(\frac{g^2}{4\pi}\right)(\xi+1)(I_1+I_2)\right] = 0 \tag{41}
$$

$$
\tan 2\varphi_{\mathbf{k}}|_{\text{eq}} = \frac{-|\mathbf{k}|m\left[1-(g^2/4\pi)(\xi+1)(I_1+I_2)\right]}{\left[(\mathbf{k})^2+(g^2m^2/4\pi)(\xi+1)(I_1+I_2)\right]}\cos\gamma_{\mathbf{k}}|_{\text{eq}} \ . \tag{42}
$$

There are two possible solutions for equation (41) which correspond two possible phases of the system. They are

$$
\begin{aligned}
\text{symmetric phase} \ &: \ 1-\left(\frac{g^2}{4\pi}\right)(\xi+1)(I_1+I_2) = 0 \\
&\quad \text{with} \ \sin\gamma_{\mathbf{k}}|_{\text{eq}} \neq 0
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\text{broken phase} \ &: \ \sin\gamma_{\mathbf{k}}|_{\text{eq}} = 0 \\
&\quad \text{with} \ 1-\left(\frac{g^2}{4\pi}\right)(\xi+1)(I_1+I_2) \neq 0 \ .
\end{aligned} \tag{44}
$$

From Eq.(42), we see immediately that the first phase has as solution

$$
\tan 2\varphi_{\mathbf{k}}|_{\text{eq}} = 0 \Rightarrow \varphi_{\mathbf{k}}|_{\text{eq}} = n\pi \ \text{ for } \ n = 0, \pm 1, \pm 2, \ldots \ . \tag{45}
$$

If we substitute the solution (45) in the Nambu transformation [see Eqs.(11) and (30)], we obtain an identity transformation. Thus, this solution corresponds to the situation in which the chiral symmetry stays intact.

For the second phase we will solve a self-consistency problem. In order to proceed we introduce a regularizing momentum cut-off Λ and neglect contributions that vanish in the limit $\Lambda \to \infty$. The renormalized coupling constant for the CGNM can be obtained from the minimization of the CGNM vacuum energy density with respect to m, namely

$$
\frac{\delta}{\delta m}\left[Tr\left(H_{\text{CGNM}}\mathcal{F}_0^{\text{vacuum}}(t)\right)\right] = 0 \ , \tag{46}
$$

where H_{CGNM} is given in Appendix B. From this calculation we obtain (see also Ref.[18])

$$g^2 = \frac{4\pi}{(\xi+1)}\left[\ln\left(\frac{\Lambda^2}{m^2}\right)\right]^{-1} . \tag{47}$$

We assume that the integrals I_1 and I_2 have logarithmic divergence

$$\begin{aligned} I_1 &= a + b\ln\left(\frac{\Lambda^2}{m^2}\right) \\ I_2 &= c + d\ln\left(\frac{\Lambda^2}{m^2}\right) , \end{aligned} \tag{48}$$

where a, b, c and d are finite constants. Substituting (47) and the ansatz (48) in the static equation (42) we obtain for the second phase given in (44) that

$$\tan 2\varphi_{\mathbf{k}}|_{eq} = \frac{-(-1)^n m|\mathbf{k}|[1-(b+d)]}{[(\mathbf{k})^2 + m^2(b+d)]} , \tag{49}$$

where the divergence problem is controlled, since b and d are cut-off independent. Moreover, from (37), (48), and (49) we obtain the finite constants a, b, c and d (see Appendix C). From the self-consistency requirement we have $b=1$ while d remains arbitrary. Substituting this results into (49) we have the renormalized static equations which describe the second phase of our system in mean-field approximation

$$\begin{aligned} &\tan 2\varphi_{\mathbf{k}}|_{eq} = \frac{(-1)^n m|\mathbf{k}|d}{[\mathbf{k}^2 + (1+d)m^2]} \quad \text{with} \quad d \neq 0 \\ &\gamma_{\mathbf{k}}|_{eq} = n\pi \quad \text{for} \quad n = 0, \pm1, \pm2, \ldots \quad . \end{aligned} \tag{50}$$

We observe that our theory contains only one free parameter, say d. This is altogether reasonable since our starting point was a massless fermions theory which was determined by one dimensionless coupling constant g. We end up with a theory determined by one

free parameter d after the self-consistent renormalization procedure. What is needed is an interpretation of the parameter d. We begin by writing the fields $\psi(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$ given in (1) in the Nambu quasi-particle basis using (11), and (30). We find that the new Dirac spinors in this basis are given by

$$\begin{aligned} u_1'(\mathbf{k}) &= \cos\varphi_{\mathbf{k}}\, u_1(\mathbf{k}) + \sin\varphi_{\mathbf{k}}\, e^{i\gamma_{\mathbf{k}}}\, u_2(-\mathbf{k}) \\ u_2'(\mathbf{k}) &= \cos\varphi_{\mathbf{k}}\, u_2(\mathbf{k}) - \sin\varphi_{\mathbf{k}}\, e^{-i\gamma_{\mathbf{k}}}\, u_1(-\mathbf{k}) \ . \end{aligned} \tag{51}$$

From the renormalized static solution (50) we have that

$$\begin{aligned} \cos\varphi_{\mathbf{k}}|_{eq} &= \frac{1}{2\sqrt{k_0 x}}\left[(k_0 x + m|\mathbf{k}|d)^{1/2} + (k_0 x - m|\mathbf{k}|d)^{1/2}\right] \\ \sin\varphi_{\mathbf{k}}|_{eq} &= (-1)^n \frac{1}{2\sqrt{k_0 x}}\left[(k_0 x + m|\mathbf{k}|d)^{1/2} - (k_0 x - m|\mathbf{k}|d)^{1/2}\right] \end{aligned} \tag{52}$$

where $x = [k^2 + (1+d)^2 m^2]^{1/2}$. Substituting the solutions (52) and the particle spinors (29) in (51), we obtain the new spinors in quasi-particle basis. They are given as

$$u_1'(\mathbf{k}) = \left[\frac{(k_0^{ef} + m_{ef})}{2m_{ef}}\right]^{1/2} \begin{bmatrix} 1 \\ \frac{\mathbf{k}}{(k_0^{ef} + m_{ef})} \end{bmatrix} , \quad u_2'(\mathbf{k}) = \left[\frac{(k_0^{ef} - m_{ef})}{2m_{ef}}\right]^{1/2} \begin{bmatrix} 1 \\ \frac{\mathbf{k}}{(k_0^{ef} - m_{ef})} \end{bmatrix} \tag{53}$$

where

$$(k_0^{ef})^2 = (\mathbf{k})^2 + (m_{ef})^2 \quad \text{and} \quad m_{ef} = (1+d)m \ . \tag{54}$$

Comparing the spinors (53) in quasi-particle basis with the spinors (29) in particle basis, we see in Eq.(54) what amounts to a redefinition of the mass scale. The use of the Heisenberg equation in (13) leads thus to mass generation and the chiral symmetry breaking of the system

$(\varphi_{\mathbf{k}}|_{eq} \neq n\pi)$. Therefore, the renormalization procedure effectively replaces the dimensionless coupling constant g for a free parameter d associated to a mass scale [see Eq.(54)]. This is analogous to the phenomenon of dimensional transmutation found by Gross and Neveu [10] in the $1/N$ expansion. Finally, aside from the over-all mass scale (characterized by d) there are no free adjustable parameters.

Using Eqs.(38), (47), and (48), we finally rewrite the renormalized dynamical equations that describe the mean-field time evolution of this system in the broken chiral phase ($d \neq 0$ or $m_{ef} \neq m$) as

$$\begin{aligned}
\dot{\nu}_{\mathbf{k},1} &= 0 \quad \text{and} \quad \dot{\nu}_{\mathbf{k},2} = 0 \\
\dot{\varphi}_{\mathbf{k}} &= (-1) m d \frac{|\mathbf{k}|}{k_0} \sin\gamma_{\mathbf{k}} \\
\dot{\gamma}_{\mathbf{k}} \sin 2\varphi_{\mathbf{k}} &= \frac{2\sin 2\varphi_{\mathbf{k}}}{k_0}\left[(\mathbf{k})^2 + m^2(1+d)\right] + \\
&- 2md\frac{|\mathbf{k}|}{k_0}\cos 2\varphi_{\mathbf{k}} \cos\gamma_{\mathbf{k}} \ .
\end{aligned} \tag{55}$$

We next calculate the ground-state of our system in mean-field approximation. Taking $\nu_{\mathbf{k},1} = \nu_{\mathbf{k},2} = 0$ in (39) and evaluating the divergent integrals (37) and (40) from renormalization ansatz (47), and (48), we obtain

$$\begin{aligned}
\frac{\langle H^{\mathrm{M.F.}}_{\mathrm{CGNM}}\rangle^{\mathrm{vacuum}}}{L} &= -\left(\frac{1}{2\pi}\right)\left[1 + \left(\frac{g^2}{2\pi}\right)(\xi+1)\right]\Lambda^2 \\
&- \left(\frac{1}{2\pi}\right)\frac{(1+d)^2 m^2}{2}\left\{1 - \ln\left[\frac{(1+d)^2}{4}\right]\right\} \ ,
\end{aligned} \tag{56}$$

where we use the finite constants a, b and c given in Appendix C. The quadratically divergent term represents the vacuum energy density of this system. This can be verified by calculating the energy density to massless spinors instead of the mass spinors (29). We obtain

$$\left.\frac{\langle m=0|H^{\mathrm{M.F.}}_{\mathrm{CGNM}}|m=0\rangle}{L}\right|_{\mathrm{vacuum}} = -\left(\frac{\Lambda^2}{2\pi}\right)\left[1 + \left(\frac{g^2}{2\pi}\right)(\xi+1)\right] \ . \tag{57}$$

The presence of this divergent constant has no physical consequences, since only energy differences, not absolute energies, are mensurable. We shall follow usual practice of redefining the zero of the energy scale. Therefore, in terms of effective mass given by (54), the renormalized energy density to the ground-state of our system in broken chiral phase is given by

$$\frac{\langle H^{\mathrm{M.F.}}_{\mathrm{CGNM}}\rangle^{\mathrm{vacuum}}}{L} = -\left(\frac{m^2}{4\pi}\right)\left(\frac{m_{ef}}{m}\right)^2\left[1 + 2\ln 2 - \ln\left(\frac{m_{ef}}{m}\right)^2\right] \ . \tag{58}$$

In Fig.1 we see the renormalized energy density of the ground-state of our system as function of effective mass m_{ef}. This figure reproduces the well-know effective potential obtained in the case of $1/N$ expansion [10].

5.2 Time Evolution of One-Fermion Densities

In the renormalized dynamical equations (55) the quasi-particle occupations numbers $\nu_{\mathbf{k},1}$ and $\nu_{\mathbf{k},2}$ do not evolve in time, while the time evolution of the Nambu parameters $\varphi_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$ describe the time evolution of the Gaussian variables in terms of (30). In this section, we discuss numerical solutions of these nonlinear equations for given values of the renormalized parameter d.

First, we verify that $(d+1) = 0 \Longrightarrow m_{ef} = 0$ from (54). On the other hand, taking $g = 0$ in the Hamiltonian density (27) and calculating the kinetic equation (35) for a free fermion system we obtain

$$\begin{aligned}
\dot{\nu}_{\mathbf{k},1} &= 0 \quad \text{and} \quad \dot{\nu}_{\mathbf{k},2} = 0 \\
\dot{\varphi}_{\mathbf{k}} &= m\frac{|\mathbf{k}|}{k_0}\sin\gamma_{\mathbf{k}} \\
\dot{\gamma}_{\mathbf{k}} \sin 2\varphi_{\mathbf{k}} &= 2\frac{(\mathbf{k})^2}{k_0}\sin 2\varphi_{\mathbf{k}} + 2m\frac{|\mathbf{k}|}{k_0}\cos 2\varphi_{\mathbf{k}}\cos\gamma_{\mathbf{k}} \ .
\end{aligned} \tag{59}$$

Comparing (55) and (59) we verify that $(d+1) = 0$ corresponds to the free fermion system. Therefore, in the case of a free fermion system the minimum situation is massless ($m_{ef} = 0$). In Fig.2 we show the phase-space of the Nambu parameters for the (1+1) dimensional uniform free fermion system. We take $m = |\mathbf{k}| = \sqrt{2}$ and $(d+1) = 0$ to several initial-values $\varphi_{\mathbf{k}}^{in}$ and $\gamma_{\mathbf{k}}^{in}$. In this case, the equilibrium points are given by

$$\varphi_{\mathbf{k}}|_{eq} = \left(-\frac{\pi}{8} + n\frac{\pi}{2}\right) \quad \text{and} \quad \gamma_{\mathbf{k}}|_{eq} = 2n\pi \quad \text{for} \quad n = 0, \pm 1, \pm 2, ...$$

and

$$\varphi_{\mathbf{k}}|_{eq} = \left(+\frac{\pi}{8} + n\frac{\pi}{2}\right) \quad \text{and} \quad \gamma_{\mathbf{k}}|_{eq} = (2n+1)\pi \quad \text{for} \quad n = 0, \pm 1, \pm 2, ...$$

The phase-space trajectories exhibit two types of behavior. For initial-values $\varphi_{\mathbf{k}}^{in}$ and $\gamma_{\mathbf{k}}^{in}$ near the equilibrium-points the Nambu parameters oscillate around then, while for initial-values distant of equilibrium points the $\gamma_{\mathbf{k}}$ Nambu parameter varies monotonically in time.

From Eq.(50), we know that for $(d+1) = 1$ (or $d = 0$) the chiral symmetry of our system stays intact. In this case, the Nambu parameters not evolve in the time. For any values of the parameter d, except $(1+d) = 1$, the phase-space of Nambu parameters exhibit the same two types of behavior observed in Fig.2. On the other hand, the equilibrium points change as $(d+1)$ increase. For $(d+1) \to \infty$, and $m = |\mathbf{k}| = \sqrt{2}$, the equilibrium points go to

$$\varphi_{\mathbf{k}}|_{eq} = \left(+\frac{\pi}{8} + n\frac{\pi}{2}\right) \quad \text{and} \quad \gamma_{\mathbf{k}}|_{eq} = 2n\pi \quad \text{for} \quad n = 0, \pm 1, \pm 2, ...$$

and

$$\varphi_{\mathbf{k}}|_{eq} = \left(+\frac{3\pi}{8} + n\frac{\pi}{2}\right) \quad \text{and} \quad \gamma_{\mathbf{k}}|_{eq} = (2n+1)\pi \quad \text{for} \quad n = 0, \pm 1, \pm 2, ...$$

6 Discussion and conclusions

We have described a treatment of the initial-values problem in a quantum field theory of self-interacting fermions. Although the formalism is quite general, we have specilize it to

the treatment of a relativistic many-fermion system described by Chiral Gross-Neveu model (CGNM).

We have obtained the renormalized kinetic equations which describe the effective dynamics of the Gaussian variables in the mean-field approximation and in broken chiral symmetry phase to a relativistic uniform (1+1) dimensional fermion system described by CGNM. We use the static solution of these equations in order to renormalize the theory, leading to the well-known effective potential obtained by Gross and Neveu using the $1/N$ expansion. We show also that other static results discussed in the literature such as dynamical mass generation due to chiral symmetry breaking and a phenomenon analogous to dimensional transmutation can be retrieved from this formulation in a mean-field approximation. Finally, we discussed the time evolution of the Nambu parameters (Gaussian variables) of our system.

As a final comment we note that, unlike the situation found in connection with the $1/N$ expansion, the use of the Gaussian ansatz, Eq.(19), parametrized by the canonical transformation leading to the quasi-fermion basis, allows for the direct dynamical determination of the stable equilibrium situation of the system [see Eqs.(43) and (44)], including symmetry breaking and mass generation.

As an extension to this work we could take the quasi-particle occupations no-vanishing what corresponds to include finite matter density in our calculation. In this case, we have to take $\nu_{\mathbf{k},1} \neq 0$ and $\nu_{\mathbf{k},2} \neq 0$ for positive-energy states up to the Fermi momentum k_F. The dynamical equations themselves will consequently depend on k_F.

On the other hand, we can use the projection technique [7] to include and to evaluate higher dynamical correlations effects to the simplest mean-field approximation. In this case the occupation numbers are no longer constant, $\dot{\nu}_{\mathbf{k},\lambda} \neq 0$, and their time dependence affects the effective dynamics of the Gaussian variables (see reference [9]). A finite matter density calculation beyond the mean-field approximation allows one to study collisional observables such as transport coefficients [19]. Finally, we comment on the extension to non-uniform case. In this case the spatial dependence of the fields $\bar{\psi}(x)$ and $\psi(x)$ are expanded in the natural orbitals through the use of a non-homogeneous Bogolyubov transformation (see reference [9]).

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Appendix A : Uniform system and the representations for the γ^μ-matrices

An uniform system has to be translational and reflection invariant. The reflection invariance implies that the motion equations are direction independent, i.e., the motion equations are $|\mathbf{k}|$ dependent.

Let us define then the vacuum corresponding to the two solutions. Let

$$\psi^{(m_1)}(x) = \sum_{\mathbf{k}} \left(\frac{m_1}{Lk_0^{(m_1)}}\right)^{1/2} \left[b_{\mathbf{k},1}^{(m_1)} u_1^{(m_1)}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}} + b_{\mathbf{k},2}^{(m_1)\dagger} u_2^{(m_1)}(\mathbf{k}) e^{-i\mathbf{k}.\mathbf{x}}\right] \tag{60}$$

$$\bar{\psi}^{(m_1)}(x) = \sum_{\mathbf{k}} \left(\frac{m_1}{Lk_0^{(m_1)}}\right)^{1/2} \left[b_{\mathbf{k},1}^{(m_1)\dagger} \bar{u}_1^{(m_1)}(\mathbf{k}) e^{-i\mathbf{k}.\mathbf{x}} + b_{\mathbf{k},2}^{(m_1)} \bar{u}_2^{(m_1)}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}}\right] \tag{61}$$

be quantized fields satisfying the Dirac equation with mass m_1

$$(i\partial\!\!\!/ - m_1)\psi^{(m_1)}(x) = 0 \ , \tag{62}$$

and let

$$\psi^{(m_2)}(x) = \sum_{\mathbf{k}} \left(\frac{m_2}{Lk_0^{(m_2)}}\right)^{1/2} \left[b_{\mathbf{k},1}^{(m_2)} u_1^{(m_2)}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}} + b_{\mathbf{k},2}^{(m_2)\dagger} u_2^{(m_2)}(\mathbf{k}) e^{-i\mathbf{k}.\mathbf{x}}\right] \tag{63}$$

$$\bar{\psi}^{(m_2)}(x) = \sum_{\mathbf{k}} \left(\frac{m_2}{Lk_0^{(m_2)}}\right)^{1/2} \left[b_{\mathbf{k},1}^{(m_2)\dagger} \bar{u}_1^{(m_2)}(\mathbf{k}) e^{-i\mathbf{k}.\mathbf{x}} + b_{\mathbf{k},2}^{(m_2)} \bar{u}_2^{(m_2)}(\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}}\right] \tag{64}$$

be quantized fields satisfying the Dirac equation with mass m_2

$$(i\partial\!\!\!/ - m_2)\psi^{(m_2)}(x) = 0 \ . \tag{65}$$

By imposing a condition that

$$\psi^{(m_1)}(x) = \psi^{(m_2)}(x) \quad \text{for} \quad t = 0 \tag{66}$$

one can show that the operator sets $(a_{\mathbf{k}}^{(m_1)}, c_{\mathbf{k}}^{(m_1)})$ and $(a_{\mathbf{k}}^{(m_2)}, c_{\mathbf{k}}^{(m_2)})$ are related by canonical transformation given below

$$\begin{aligned} b_{\mathbf{k},1}^{(m_2)} &= \left\{\left(\frac{m_1}{m_2}\right)^{1/2} \left(\frac{k_0^{(m_2)}}{k_0^{(m_1)}}\right)^{1/2} \bar{u}_1^{(m_2)}(\mathbf{k}) u_1^{(m_1)}(\mathbf{k})\right\} b_{\mathbf{k},1}^{(m_1)} + \\ &+ \left\{\left(\frac{m_1}{m_2}\right)^{1/2} \left(\frac{k_0^{(m_2)}}{k_0^{(m_1)}}\right)^{1/2} \bar{u}_1^{(m_2)}(\mathbf{k}) u_2^{(m_1)}(-\mathbf{k})\right\} b_{-\mathbf{k},2}^{(m_1)\dagger} \end{aligned} \tag{67}$$

$$\begin{aligned} b_{\mathbf{k},2}^{(m_2)} &= -\left\{\left(\frac{m_1}{m_2}\right)^{1/2} \left(\frac{k_0^{(m_2)}}{k_0^{(m_1)}}\right)^{1/2} \bar{u}_2^{(m_1)}(\mathbf{k}) u_2^{(m_2)}(\mathbf{k})\right\} b_{\mathbf{k},2}^{(m_1)} + \\ &- \left\{\left(\frac{m_1}{m_2}\right)^{1/2} \left(\frac{k_0^{(m_2)}}{k_0^{(m_1)}}\right)^{1/2} \bar{u}_1^{(m_1)}(-\mathbf{k}) u_2^{(m_2)}(\mathbf{k})\right\} b_{-\mathbf{k},1}^{(m_1)\dagger} \ . \end{aligned} \tag{68}$$

In the case of uniform system, the coefficients of canonical transformation are invariant under $\mathbf{k} \to -\mathbf{k}$ transformation. Therefore, the conditions for a system to be uniform are given by

$$
\begin{aligned}
\bar{u}_1^{(m_2)}(\mathbf{k})u_1^{(m_1)}(\mathbf{k}) &= \bar{u}_1^{(m_2)}(-\mathbf{k})u_1^{(m_1)}(-\mathbf{k}) \\
\bar{u}_1^{(m_2)}(\mathbf{k})u_2^{(m_1)}(-\mathbf{k}) &= \bar{u}_1^{(m_2)}(-\mathbf{k})u_2^{(m_1)}(\mathbf{k}) \\
\bar{u}_2^{(m_1)}(\mathbf{k})u_2^{(m_2)}(\mathbf{k}) &= \bar{u}_2^{(m_1)}(-\mathbf{k})u_2^{(m_2)}(-\mathbf{k}) \\
\bar{u}_1^{(m_1)}(-\mathbf{k})u_2^{(m_2)}(\mathbf{k}) &= \bar{u}_1^{(m_1)}(\mathbf{k})u_2^{(m_2)}(-\mathbf{k}) \ .
\end{aligned}
\tag{69}
$$

We verify that only two representations for γ^μ-matrices satisfy the uniformity conditions (69). We have that only the Pauli-Dirac representation

$$
\gamma_0 = \sigma_3 \quad ; \quad \gamma_1 = i\sigma_2 \quad \text{e} \quad \gamma_5 = \gamma_0\gamma_1 = \sigma_1 \tag{70}
$$

and the representation below

$$
\gamma_0 = \sigma_3 \quad ; \quad \gamma_1 = i\sigma_1 \quad \text{e} \quad \gamma_5 = \gamma_0\gamma_1 = -\sigma_2 \ , \tag{71}
$$

provide uniform solutions. We choose the Pauli-Dirac representation (70) which has as uniform solution of Dirac equation the spinors given in (29). On the other hand, we must be careful because the spinors given below

$$
u_1(\mathbf{k}) = \left(\frac{k_0+m}{2m}\right)^{1/2} \begin{bmatrix} 1 \\ \\ \dfrac{\mathbf{k}}{(k_0+m)} \end{bmatrix} \quad u_2(\mathbf{k}) = \left(\frac{k_0+m}{2m}\right)^{1/2} \begin{bmatrix} \dfrac{\mathbf{k}}{(k_0+m)} \\ \\ 1 \end{bmatrix} \ , \tag{72}
$$

also are solution of Dirac equation for Pauli-Dirac representation, but they no satisfy the uniformity conditions (69).

Appendix B : The uniform CGNM Hamiltonian

In this appendix we show explicitly the Hamiltonian of Chiral Gross-Neveu model which preserve the reflection invariance of the system. In Pauli-Dirac representation (see Appendix A) the CGNM Hamiltonian is given by

$$
\begin{aligned}
&H_{\mathrm{CGNM}} = \\
&\sum_{i=1}^{N}\sum_{\mathbf{k}'} \frac{1}{[(\mathbf{k}'_i)^2+m^2]^{1/2}} \left\{ (\mathbf{k}'_i)^2 \left(b^\dagger_{\mathbf{k}'_i,1} b_{\mathbf{k}'_i,1} - b_{\mathbf{k}'_i,2} b^\dagger_{\mathbf{k}'_i,2} \right) - m|\mathbf{k}'_i| \left(b_{\mathbf{k}'_i,2} b_{-\mathbf{k}'_i,1} + b^\dagger_{\mathbf{k}'_i,1} b^\dagger_{-\mathbf{k}'_i,2} \right) \right\} + \\
+\ & \left(\frac{g^2m^2}{2L}\right) \sum_{i,j=1}^{N} \sum_{(\mathbf{k}',\mathbf{k}'',\mathbf{k}''',\mathbf{k}'''')} \left\{ \frac{1}{[(\mathbf{k}'_i)^2+m^2]^{1/4}[(\mathbf{k}''_i)^2+m^2]^{1/4}[(\mathbf{k}'''_j)^2+m^2]^{1/4}[(\mathbf{k}''''_j)^2+m^2]^{1/4}} \right\} \times \\
& \Big\{ b^\dagger_{\mathbf{k}'_i,1} b_{\mathbf{k}''_i,1} b^\dagger_{\mathbf{k}'''_j,1} b_{\mathbf{k}''''_j,1} \left[\xi \bar{u}_1(\mathbf{k}'_i)\gamma_5 u_1(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j)\gamma_5 u_1(\mathbf{k}''''_j) - \bar{u}_1(\mathbf{k}'_i) u_1(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j) u_1(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}'''_j,\mathbf{k}''_i+\mathbf{k}''''_j} \\
+\ & b^\dagger_{\mathbf{k}'_i,1} b_{\mathbf{k}''_i,1} b_{\mathbf{k}'''_j,2} b^\dagger_{\mathbf{k}''''_j,2} \left[\xi \bar{u}_1(\mathbf{k}'_i)\gamma_5 u_1(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j)\gamma_5 u_2(\mathbf{k}''''_j) - \bar{u}_1(\mathbf{k}'_i) u_1(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j) u_2(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}''''_j,\mathbf{k}''_i+\mathbf{k}'''_j} \\
+\ & b^\dagger_{\mathbf{k}'_i,1} b_{\mathbf{k}''_i,1} b^\dagger_{\mathbf{k}'''_j,1} b^\dagger_{\mathbf{k}''''_j,2} \left[\xi \bar{u}_1(\mathbf{k}'_i)\gamma_5 u_1(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j)\gamma_5 u_2(\mathbf{k}''''_j) - \bar{u}_1(\mathbf{k}'_i) u_1(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j) u_2(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}'''_j+\mathbf{k}''''_j,\mathbf{k}''_i} \\
+\ & b^\dagger_{\mathbf{k}'_i,1} b_{\mathbf{k}''_i,1} b_{\mathbf{k}'''_j,2} b_{\mathbf{k}''''_j,1} \left[\xi \bar{u}_1(\mathbf{k}'_i)\gamma_5 u_1(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j)\gamma_5 u_1(\mathbf{k}''''_j) - \bar{u}_1(\mathbf{k}'_i) u_1(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j) u_1(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i,\mathbf{k}''_i+\mathbf{k}'''_j+\mathbf{k}''''_j} \\
+\ & b_{\mathbf{k}'_i,2} b^\dagger_{\mathbf{k}''_i,2} b^\dagger_{\mathbf{k}'''_j,1} b_{\mathbf{k}''''_j,1} \left[\xi \bar{u}_2(\mathbf{k}'_i)\gamma_5 u_2(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j)\gamma_5 u_1(\mathbf{k}''''_j) - \bar{u}_2(\mathbf{k}'_i) u_2(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j) u_1(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}''''_j,\mathbf{k}''_i+\mathbf{k}'''_j} \\
+\ & b_{\mathbf{k}'_i,2} b^\dagger_{\mathbf{k}''_i,2} b_{\mathbf{k}'''_j,2} b^\dagger_{\mathbf{k}''''_j,2} \left[\xi \bar{u}_2(\mathbf{k}'_i)\gamma_5 u_2(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j)\gamma_5 u_2(\mathbf{k}''''_j) - \bar{u}_2(\mathbf{k}'_i) u_2(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j) u_2(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}'''_j,\mathbf{k}''_i+\mathbf{k}''''_j} \\
+\ & b_{\mathbf{k}'_i,2} b^\dagger_{\mathbf{k}''_i,2} b^\dagger_{\mathbf{k}'''_j,1} b^\dagger_{\mathbf{k}''''_j,2} \left[\xi \bar{u}_2(\mathbf{k}'_i)\gamma_5 u_2(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j)\gamma_5 u_2(\mathbf{k}''''_j) - \bar{u}_2(\mathbf{k}'_i) u_2(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j) u_2(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i,\mathbf{k}''_i+\mathbf{k}'''_j+\mathbf{k}''''_j} \\
+\ & b_{\mathbf{k}'_i,2} b^\dagger_{\mathbf{k}''_i,2} b_{\mathbf{k}'''_j,2} b_{\mathbf{k}''''_j,1} \left[\xi \bar{u}_2(\mathbf{k}'_i)\gamma_5 u_2(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j)\gamma_5 u_1(\mathbf{k}''''_j) - \bar{u}_2(\mathbf{k}'_i) u_2(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j) u_1(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}'''_j+\mathbf{k}''''_j,\mathbf{k}''_i} \\
+\ & b^\dagger_{\mathbf{k}'_i,1} b^\dagger_{\mathbf{k}''_i,2} b^\dagger_{\mathbf{k}'''_j,1} b_{\mathbf{k}''''_j,1} \left[\xi \bar{u}_1(\mathbf{k}'_i)\gamma_5 u_2(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j)\gamma_5 u_1(\mathbf{k}''''_j) - \bar{u}_1(\mathbf{k}'_i) u_2(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j) u_1(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}''_i,\mathbf{k}'''_j+\mathbf{k}''''_j} \\
+\ & b^\dagger_{\mathbf{k}'_i,1} b^\dagger_{\mathbf{k}''_i,2} b_{\mathbf{k}'''_j,2} b^\dagger_{\mathbf{k}''''_j,2} \left[\xi \bar{u}_1(\mathbf{k}'_i)\gamma_5 u_2(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j)\gamma_5 u_2(\mathbf{k}''''_j) - \bar{u}_1(\mathbf{k}'_i) u_2(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j) u_2(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}''_i+\mathbf{k}''''_j,\mathbf{k}'''_j} \\
+\ & b^\dagger_{\mathbf{k}'_i,1} b^\dagger_{\mathbf{k}''_i,2} b^\dagger_{\mathbf{k}'''_j,1} b^\dagger_{\mathbf{k}''''_j,2} \left[\xi \bar{u}_1(\mathbf{k}'_i)\gamma_5 u_2(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j)\gamma_5 u_2(\mathbf{k}''''_j) - \bar{u}_1(\mathbf{k}'_i) u_2(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j) u_2(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}''_i+\mathbf{k}'''_j+\mathbf{k}''''_j,0} \\
+\ & b^\dagger_{\mathbf{k}'_i,1} b^\dagger_{\mathbf{k}''_i,2} b_{\mathbf{k}'''_j,2} b_{\mathbf{k}''''_j,1} \left[\xi \bar{u}_1(\mathbf{k}'_i)\gamma_5 u_2(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j)\gamma_5 u_1(\mathbf{k}''''_j) - \bar{u}_1(\mathbf{k}'_i) u_2(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j) u_1(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}''_i,\mathbf{k}'''_j+\mathbf{k}''''_j} \\
+\ & b_{\mathbf{k}'_i,2} b_{\mathbf{k}''_i,1} b^\dagger_{\mathbf{k}'''_j,1} b_{\mathbf{k}''''_j,1} \left[\xi \bar{u}_2(\mathbf{k}'_i)\gamma_5 u_1(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j)\gamma_5 u_1(\mathbf{k}''''_j) - \bar{u}_2(\mathbf{k}'_i) u_1(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j) u_1(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}''_i+\mathbf{k}''''_j,\mathbf{k}'''_j} \\
+\ & b_{\mathbf{k}'_i,2} b_{\mathbf{k}''_i,1} b_{\mathbf{k}'''_j,2} b^\dagger_{\mathbf{k}''''_j,2} \left[\xi \bar{u}_2(\mathbf{k}'_i)\gamma_5 u_1(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j)\gamma_5 u_2(\mathbf{k}''''_j) - \bar{u}_2(\mathbf{k}'_i) u_1(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j) u_2(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}''_i+\mathbf{k}'''_j,\mathbf{k}''''_j} \\
+\ & b_{\mathbf{k}'_i,2} b_{\mathbf{k}''_i,1} b^\dagger_{\mathbf{k}'''_j,1} b^\dagger_{\mathbf{k}''''_j,2} \left[\xi \bar{u}_2(\mathbf{k}'_i)\gamma_5 u_1(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j)\gamma_5 u_2(\mathbf{k}''''_j) - \bar{u}_2(\mathbf{k}'_i) u_1(\mathbf{k}''_i) \bar{u}_1(\mathbf{k}'''_j) u_2(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}''_i,\mathbf{k}'''_j+\mathbf{k}''''_j} \\
+\ & b_{\mathbf{k}'_i,2} b_{\mathbf{k}''_i,1} b_{\mathbf{k}'''_j,2} b_{\mathbf{k}''''_j,1} \left[\xi \bar{u}_2(\mathbf{k}'_i)\gamma_5 u_1(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j)\gamma_5 u_1(\mathbf{k}''''_j) - \bar{u}_2(\mathbf{k}'_i) u_1(\mathbf{k}''_i) \bar{u}_2(\mathbf{k}'''_j) u_1(\mathbf{k}''''_j) \right] \delta_{\mathbf{k}'_i+\mathbf{k}''_i+\mathbf{k}'''_j+\mathbf{k}''''_j,0} \Big\} .
\end{aligned}
\tag{73}
$$

The CGNM Hamiltonian above is in particle basis. In dynamical equations (35) we have to use the CGNM Hamiltonian in Nambu quasi-particle basis. This task is realised from Nambu transformation defined in (11), and (30).

Appendix C : Self-consistent renormalization

From (49) we obtain that

$$\begin{aligned} \sin 2\varphi_{\mathbf{k}}|_{eq} &= \frac{-(-1)^n m|\mathbf{k}|[1-(b+d)]}{k_0[k^2+m^2(b+d)^2]^{1/2}} \\ \cos 2\varphi_{\mathbf{k}}|_{eq} &= \frac{[k^2+m^2(b+d)]}{k_0[k^2+m^2(b+d)^2]^{1/2}}. \end{aligned} \tag{74}$$

Substituting (74) in I_1 and I_2 given in (37) and making the integration follow that

$$I_1 = \int_{-\Lambda}^{+\Lambda} \frac{dk}{(k^2+m^2)} \frac{[k^2+m^2(b+d)]}{[k^2+m^2(b+d)^2]} = a + b\ln\left(\frac{\Lambda^2}{m^2}\right)$$

$$\begin{aligned} a &= \frac{2d}{[(1+d)^2-1]^{1/2}} \arctan[(1+d)^2-1]^{1/2} + \ln\left[\frac{4}{(1+d)^2}\right] \quad \text{and } b = 1 \\ &\quad \text{when } (b+d)^2 > 1 \\ a &= \frac{d}{[1-(1+d)^2]^{1/2}} \ln\left[\frac{1+\{1-(1+d)^2\}^{1/2}}{1-\{1-(1+d)^2\}^{1/2}}\right] + \ln\left[\frac{4}{(1+d)^2}\right] \quad \text{and } b = 1 \\ &\quad \text{when } (b+d)^2 < 1 \end{aligned} \tag{75}$$

$$I_2 = 2(b+d-1)\int_{-\Lambda}^{\Lambda} \frac{dk}{(k^2+m^2)} \frac{k^2}{[k^2+m^2(b+d)^2]^{1/2}} = c + d\ln\left(\frac{\Lambda^2}{m^2}\right)$$

$$\begin{aligned} c &= \frac{-2d}{[(1+d)^2-1]^{1/2}} \arctan[(1+d)^2-1]^{1/2} + d\ln\left[\frac{4}{(1+d)^2}\right] \quad \text{and } d = d \\ &\quad \text{when } (b+d)^2 > 1 \\ c &= \frac{-d}{[1-(1+d)^2]^{1/2}} \ln\left[\frac{1+\{1-(1+d)^2\}^{1/2}}{1-\{1-(1+d)^2\}^{1/2}}\right] + d\ln\left[\frac{4}{(1+d)^2}\right] \quad \text{and } d = d \\ &\quad \text{when } (b+d)^2 < 1 \end{aligned} \tag{76}$$

We verify that the ansatz (48) is self-consistent. From (75) and (76) we have the values of finite constants a, b and c. We must observe that d stays arbitrary in this calculation.

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Figure Captions

Figure 1 - The mean-field effective potential for ground-state of an uniform fermion system described by the CGNM in broken chiral symmetry phase as function of effective mass m_{ef}.

Figure 2 - Phase-space of Nambu parameters for $|\mathbf{k}| = m = \sqrt{2}$ in the case of free fermion system $[(d+1=0)$ or $m_{ef}=0]$. The various curves correspond the following initial-values off-equilibrium
Dotted line : $\varphi_{\mathbf{k}}^{\rm in} = (-0.2 + n\pi/2)$, $\gamma_{\mathbf{k}}^{\rm in} = 2n\pi$ and $\varphi_{\mathbf{k}}^{\rm in} = (+0.2 + n\pi/2)$, $\gamma_{\mathbf{k}}^{\rm in} = (2n+1)\pi$,
Dashed line : $\varphi_{\mathbf{k}}^{\rm in} = (-0.1 + n\pi/2)$, $\gamma_{\mathbf{k}}^{\rm in} = 2n\pi$ and $\varphi_{\mathbf{k}}^{\rm in} = (+0.1 + n\pi/2)$, $\gamma_{\mathbf{k}}^{\rm in} = (2n+1)\pi$,
Solid line : $\varphi_{\mathbf{k}}^{\rm in} = (-\pi/4 + n\pi/2)$, $\gamma_{\mathbf{k}}^{\rm in} = 0$ and $\varphi_{\mathbf{k}}^{\rm in} = (0 + n\pi/2)$, $\gamma_{\mathbf{k}}^{\rm in} = 0$,
Dot-Dashed line : $\varphi_{\mathbf{k}}^{\rm in} = (\pi/8 + n\pi/2)$, $\gamma_{\mathbf{k}}^{\rm in} = 0$.

Figure 1

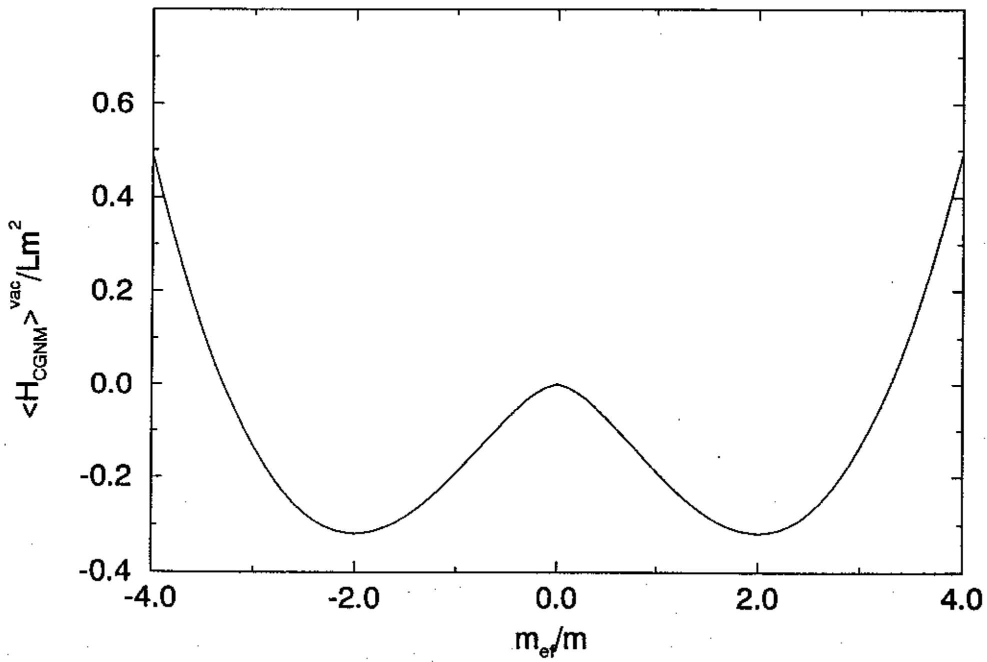

Figure 2