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PROPER TIME AND PATH INTEGRAL REPRESENTATIONS FOR THE COMMUTATION FUNCTION

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Proper time and path integral representations for the commutation function

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Abstract

On the example of the quantized spinor field, interacting with arbitrary external electromagnetic field, the commutation function is studied. It is shown that a proper time representation is available in any dimensions. Using it, all the light cone singularities of the function are found explicitly, generalizing the Fock formula in four dimensions, and a path integral representation is constructed.

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I. INTRODUCTION

It is known that the solution of quantum field theory problems involves as a rule different singular functions, e.g. commutation functions, Green functions and so on. These functions are well studied for free fields (see for example [1]). Problems appear when an interaction is presented. In particular, it is important to study singular functions in external backgrounds such as external fields and curved spaces and in arbitrary dimensions. The latter may be important for multidimensional version of field theories, which are considered now in relation with the unification of all the interactions. One ought to say that the commutation functions play an important role in QFT with external backgrounds. In contrast with the case without external backgrounds, the perturbation theory, which takes into account a background exactly, uses complicated (matrix) propagators. Such propagators contain as component parts, besides the causal and anti-causal Green functions, the commutation functions as well [2,3]. Here we present proper time and path integral representations for the commutation function and for some related functions in external fields. On the basis of the proper time representation we study, in particular, light cone singularities of the function in arbitrary dimensions. Traditionally QED is a testing ground where new procedures and methods are worked through, not infrequently creating new ideas and more profound understanding of the structure of QFT. That is why we consider here the case of QED with an arbitrary external electromagnetic field, bearing in mind that the results can be extend to other theories and backgrounds.

Fock for the first time [4] introduced an integral over the proper time to present the regular on the light cone part of the commutation function $S(x, x')$ (in 3+1 dimensions) of the spinor fields, interacting with an external electromagnetic field $A_\mu(x)$,

$$S(x, x') = i[\psi(x), \bar\psi(x')]_+ \tag{1}$$

where $\psi(x)$ and $\bar\psi(x')$ are the electron-positron field operators. This function obeys the Dirac equation

$$\left(\hat{\mathcal{P}}_\nu \gamma^\nu - m\right) S(x, x') = 0, \quad \hat{\mathcal{P}}_\nu = i\partial_\nu - g A_\nu(x), \qquad (2)$$

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1),$$

and the initial condition

$$S(x, x')|_{x_0 = x_0'} = i\gamma^0 \delta(\mathbf{x} - \mathbf{x'}). \qquad (3)$$

The commutation function $S(x, x')$ is at the same time the propagation function of the Dirac equation, i.e. it connects solutions $\psi(x)$ of the equation in two different time instants,

$$\psi(x_0, \mathbf{x}) = -i \int S(x, x') \gamma^0 \psi(x_0', \mathbf{x'}) d\mathbf{x'}. \qquad (4)$$

Thus, the Cauchy problem can be solved by means of the function. Squaring the eq. (2) one gets

$$S(x, x') = \left(\hat{\mathcal{P}}_\nu \gamma^\nu + m\right) \Delta(x, x'), \qquad (5)$$

where the function $\Delta(x, x')$ obey the equation

$$\left[(\hat{\mathcal{P}}_\nu \gamma^\nu)^2 - m^2\right] \Delta(x, x') = 0, \qquad (6)$$

and the initial conditions

$$\Delta(x, x')|_{x_0 = x_0'} = 0, \quad \partial_0 \Delta(x, x')|_{x_0 = x_0'} = \delta(\mathbf{x} - \mathbf{x'}). \qquad (7)$$

Fock's solution of the equations (6), (7) reads

$$\Delta(x, x') = \epsilon(x_0 - x_0') \left[\Theta\left((x - x')^2\right) \Delta_R(x, x') + \frac{1}{2\pi} e^{ig\Lambda} \delta\left((x - x')^2\right) \right], \qquad (8)$$

$$\epsilon(x_0 - x_0') = \text{sign}(x_0 - x_0').$$

The function Λ is the line integral of the potentials,

$$\Lambda = -\int_{x'}^{x} A_\mu(\tilde{x}) d\tilde{x}^\mu; \qquad (9)$$

$\Delta_R(x, x')$ is the Riemann function, which is presented by means of a proper time integral

$$\Delta_R(x, x') = \int_{\Gamma_R} f(x, x', s) ds \qquad (10)$$

over the closed path Γ_R, (See Fig.1.) which is a clockwise circle around the point $s = 0$ with a small enough radius, inside of which the function $f(x, x', s)$ has not any singularities besides the essential singularity $s = 0$.

Fig.1.

The function $f(x, x', s)$ obeys the "Schrödinger equation"

$$i\frac{d}{ds} f(x, x', s) = \left[m^2 - (\hat{\mathcal{P}}_\nu \gamma^\nu)^2\right] f(x, x', s), \qquad (11)$$

and the boundary condition

$$\lim_{s \to +0} f(x, x', s) = i\delta(x - x'). \qquad (12)$$

One has to remark that the same function $f(x, x', s)$ appears in the Schwinger representation [5] for the causal Green function $S^c(x, x')$ (propagator) of the Dirac equation,

$$\left(\hat{\mathcal{P}}_\nu \gamma^\nu - m\right) S^c(x, x') = -\delta(x - x'). \qquad (13)$$

Namely,

$$S^c(x, x') = \left(\hat{\mathcal{P}}_\nu \gamma^\nu + m\right) \int_0^\infty f(x, x', s) ds, \qquad (14)$$

where at $s \to \infty$ one has to enter into the complex plane s, so that $\lim_{s \to \infty} f(x, x', s) = 0$. Extension of the Schwinger representation to the curved space case was made by DeWitt [2] and then, developing his technics, to the gauge theory [6].

The Schwinger representation for $S^c(x, x')$ and the Fock representation for $S(x, x')$ differ essentially in sense of possibilities of generalization. Thus, the Schwinger representation retains its form for any space-time dimensionality d. Moreover, the inverse operator S^c can be easily presented via an exponent by means of the Schwinger proper time representation (super-proper time representation [7]), so that the path integral representations follows [8,7]. At the same time the Fock representation has the specific form (8) for $\Delta(x, x')$ in $d = 3 + 1$.

Besides, the form (8), which separates the light cone singular part from the regular one, does not give any leading consideration to write a path integral for the commutation function, similar to one for the propagator.

Below we propose a proper time representation for the commutation function, which has an universal form in any dimensions. Using it, we find explicitly all the light cone singularities of the commutation function in arbitrary dimensions, generalizing the Fock's formula (8). Moreover, such a representation allows one to write a path integral for the commutation function. In the conclusion we present similar representations for some other singular functions of the Dirac equation on the basis of the results obtained.

II. PROPER TIME REPRESENTATION FOR THE COMMUTATION FUNCTION

Here we are going to write a proper time representation for the function $\Delta(x, x')$ from the formula (5) in arbitrary space-time dimensions $d \geq 2$. To this end we need to find first the behavior of the function $f(x, x', s)$ at $s \to 0$. We will use the equations (11), (12) in d-dimensions, where

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \mathrm{diag}(\underbrace{1, -1, \ldots, -1}_{d}).$$

Similar to Schwinger [5] we present $f(x, x's)$ as a matrix element of an evolution operator $U(s)$,

$$f(x, x', s) = i < x|U(s)|x' >, \tag{15}$$
$$U(s) = e^{-iHs}, \quad H = m^2 - (\Pi_\nu \gamma^\nu)^2,$$

where $|x>$ are eigenfunctions for some hermitian operators of coordinates X^μ, the corresponding canonically conjugate operators of momenta are P_μ, so that:

$$X^\mu|x> = x^\mu|x>, \quad <x|x'> = \delta(x - x'), \quad \int |x><x|dx = I,$$
$$[P_\mu, X^\nu]_- = -i\delta_\mu^\nu, \quad <x|P_\mu|x'> = -i\partial_\mu\delta(x - x'),$$

$$\Pi_\mu = -P_\mu - gA_\mu(X), \quad [\Pi_\mu, \Pi_\nu]_- = -igF_{\mu\nu}(X), \quad F_{\mu\nu}(X) = \partial_\mu A_\nu(X) - \partial_\nu A_\mu(X). \tag{16}$$

The matrix element obeys the conditions

$$\left(i\frac{\partial}{\partial x^\mu} - gA_\mu(x)\right) < x|U(s)|x' > = < x|\Pi_\mu U(s)|x' >,$$
$$\left(-i\frac{\partial}{\partial x'^\mu} - gA_\mu(x')\right) < x|U(s)|x' > = < x|U(s)\Pi_\mu|x' >. \tag{17}$$

Introducing the operators

$$X^\mu(s) = U^{-1}(s)X^\mu U(s), \quad \gamma^\mu(s) = U^{-1}(s)\gamma^\mu U(s), \quad \Pi_\mu(s) = U^{-1}(s)\Pi_\mu U(s),$$

we come to the equations

$$\frac{d}{ds}X^\mu(s) = i[H, X^\mu(s)]_- = 2\Pi^\mu(s),$$
$$\frac{d}{ds}\gamma^\mu(s) = i[H, \gamma^\mu(s)]_-, \quad \frac{d}{ds}\Pi_\mu(s) = i[H, \Pi_\mu(s)]_-. \tag{18}$$

Using the decomposition in powers of s, one can get for H:

$$H = -\frac{1}{4s}[X(s)X(s) - 2X(s)X(0) - X(0)X(0)] - i\frac{d}{2s} + O(1).$$

Then the solution of the equations (11), (12), (17) has a form

$$f(x, x', s)|_{s\to 0} = f_0(x, x', s)[1 + O(s)],$$
$$f_0(x, x', s) = \frac{1}{(4\pi s)^{d/2}}\exp\left\{-i\frac{\pi}{4}(d - 4) + ig\Lambda - \frac{i}{4s}(x - x')^2\right\}, \tag{19}$$

where Λ is the d-dimensional line integral (9). Thus, one can conclude that $f(x, x', s)$ has no singularities in a small enough neighbourhood of the point $s = 0$ (excluding this point). Based on this one can make a key observation in 3+1 dimensions. Namely, let us consider the Fock representation (5), (8) beyound the light cone $(x - x')^2 \neq 0$. In this case the relation holds

$$\Theta\left((x - x')^2\right)\int_{\Gamma_R} f(x, x', s)ds = \int_\Gamma f(x, x', s)ds, \tag{20}$$

where Γ (See Fig.2.) is a contour, which connects the points $s = +0$ and $s = e^{-i\pi}0$, and passes in the lower part of complex plane s in a small enough neighbourhood of the point $s = 0$ so that (19) is still valid inside the contour Γ.

Fig.2.

Then the function $\Delta(x,x')$ can be written in 3+1 dimensions at $(x-x')^2 \neq 0$ in the following form

$$\Delta(x,x') = \epsilon(x_0 - x_0') \int_\Gamma f(x,x',s)ds. \tag{21}$$

It turns out that (21) is valid on the light cone as well and, moreover, in any dimensions. Below we are going to prove this statement.

First of all one can remark that (19) implies

$$\lim_{s \to e^{-i\pi}0} f(x,x',s) = -i\delta(x-x'). \tag{22}$$

Together with (11), (12) and (22) this allows one to verify that the expression (21) obeys the equation (6) at $x_0 \neq x_0'$.

Now we have to study the behavior of the function (21) at $x_0 \to x_0'$. Using the representation (19) let us select all the light cone singularities in (21),

$$\Delta(x,x') = I_R(x,x') + \sum_{n=0}^{[d/2]-1} I^{(n)}(x,x'), \tag{23}$$

$$I_R(x,x') = \epsilon(x_0 - x_0') \int_\Gamma f_R(x,x',s)ds, \tag{24}$$

$$f_R(x,x',s) = f(x,x',s) - \sum_{n=0}^{[d/2]-1} f^{(n)}(x,x',s),$$

$$I^{(n)}(x,x') = \epsilon(x_0 - x_0') \int_\Gamma f^{(n)}(x,x',s)ds, \tag{25}$$

$$f^{(n)}(x,x',s) = \frac{1}{n!}\frac{d^n}{ds^n}\left[\frac{f(x,x',s)}{f_0(x,x',s)}\right]_{s=0} s^n f_0(x,x',s).$$

Here $I_R(x,x')$ is a regular on the light cone function, which is zero at $(x-x')^2 \leq 0$. All the singularities are concentrated in the functions $I^{(n)}(x,x')$. It is convenient to make a change of variables $t = s^{-1}$ in the expression (25) to present the latter in the following form

$$I^{(n)}(x,x') = \epsilon(x_0 - x_0')a^{(n)}(x,x')\left(\frac{d}{d\tau}\right)^{[\frac{d-2}{2}]-n} Y\left(\tau - (\mathbf{x}-\mathbf{x}')^2\right)\Big|_{\tau=(x_0-x_0')^2}, \tag{26}$$

$$a^{(n)}(x,x') = \pi^{-[\frac{d-1}{2}]}e^{ig\Lambda}(4i)^{-n}\frac{1}{4n!}\frac{d^n}{ds^n}\left[\frac{f(x,x',s)}{f_0(x,x',s)}\right]_{s=0},$$

where the function $Y\left(\tau - (\mathbf{x}-\mathbf{x}')^2\right)$ has different forms for even and for odd d. Namely, for even d,

$$Y\left(\tau - (\mathbf{x}-\mathbf{x}')^2\right) = Y_{even}\left(\tau - (\mathbf{x}-\mathbf{x}')^2\right) = 2\Theta\left(\tau - (\mathbf{x}-\mathbf{x}')^2\right)0, \tag{27}$$

and for odd d,

$$Y\left(\tau - (\mathbf{x}-\mathbf{x}')^2\right) = Y_{odd}\left(\tau - (\mathbf{x}-\mathbf{x}')^2\right)$$
$$= e^{i\frac{\pi}{4}}\int_{e^{i\pi}\infty}^\infty (4\pi t)^{\frac{1}{2}}\exp\left\{-\frac{i}{4\pi}\left[\tau - (\mathbf{x}-\mathbf{x}')^2\right]t\right\}dt. \tag{28}$$

Now one can see that for even d the function $I^{(\frac{d}{2}-1)}(x,x')$ can be expressed via the Θ-function (27), whereas the rest functions $I^{(n)}(x,x')$ are concentrated on the light cone. Thus, the function (23) is zero for points, which can not be causally connected, i.e. for $(x-x')^2 < 0$.

Consider the contribution of the distributions (27) and (28) to an integral with some continuous functions on the coordinates $(\mathbf{x}-\mathbf{x}')$ in the case $(x_0 - x_0') \to 0$. In this case the distributions are zero beyond the sphere of the radius $|x_0 - x_0'| + 0$, thus the formula (27) can be written in the form

$$Y_{even}\left(\tau - (\mathbf{x}-\mathbf{x}')^2\right) = 2V(\sqrt{\tau})\delta(\mathbf{x}-\mathbf{x}'), \tag{29}$$

where $V(r)$ is the volume of the $d-1$ sphere with the radius r,

$$V(r) = cr^{d-1}, \quad c = \pi^{\frac{d-1}{2}}\Gamma^{-1}\left(\frac{d+1}{2}\right), \tag{30}$$

and $\Gamma(x)$ is the gamma-function. The least power of $(x_0 - x_0')$ in the expression (23) comes from $I^{(0)}(x,x')$, the latter can be derived from (26), (29) and has the form $I^{(0)}(x,x') = e^{ig\Lambda}(x_0 - x_0')\delta(\mathbf{x}-\mathbf{x}')$. In case $d \geq 4$, the next power of $(x_0 - x_0')$ comes from the function $I^{(1)}(x,x') \sim e^{ig\Lambda}(x_0 - x_0')^3\delta(\mathbf{x}-\mathbf{x}')$. At $d=2$ the same power of $(x_0 - x_0')$ comes from the function $I_R(x,x')$ defined by eq. (24). Thus, we can write for any even d at $(x_0 - x_0') \to 0$:

$$\Delta(x,x')|_{x_0 \to x_0'} = e^{ig\Lambda}\left[x_0 - x_0' + O\left((x_0 - x_0')^3\right)\right]\delta(\mathbf{x}-\mathbf{x}'). \tag{31}$$

One can see that (31) is a continuous function of the time $(x_0 - x_0')$ together with its first derivatives, and obeys the equation (6) at $(x_0 - x_0') \to 0$, and the initial conditions (7).

The expression (28) at $(x_0 - x'_0) \to 0$ (d is odd) can be presented in the form

$$Y_{odd}\left(\tau - (\mathbf{x} - \mathbf{x}')^2\right) = B(\tau)\delta(\mathbf{x} - \mathbf{x}'), \qquad (32)$$

where

$$B(\tau) = \int_{V(r_0)} Y_{odd}(\tau - \mathbf{y}^2)d\mathbf{y}. \qquad (33)$$

The integration in (33) is going over the volume of $d - 1$ sphere with the radius $r_0 = |x_0 - x'_0| + 0$. The integral (33) is reducing to one over the radius r only,

$$B(\tau) = (d-1)c\int_0^{r_0} Y_{odd}(\tau - r^2)r^{d-2}dr, \qquad (34)$$

where c was defined in (30) and we remember that $r_0 > \tau$. The former can be calculated and presented in the form

$$B(\tau) = e^{i\frac{\pi d}{4}}\pi^{\frac{d-2}{2}}\int_{e^{i\pi}\infty}^{\infty} t^{-\frac{d}{2}}e^{-i\tau t}dt. \qquad (35)$$

Similar to the even case the least power of $(x_0 - x'_0)$ in (23) comes from the function $I^{(0)}(x, x')$. By means of (32) and (35) the latter can be written as

$$I^{(0)}(x, x') = \epsilon(x_0 - x'_0)\delta(\mathbf{x} - \mathbf{x}')\frac{e^{i\frac{3\pi}{4}}e^{ig\Lambda}}{4\sqrt{\pi}}\int_{e^{i\pi}\infty}^{\infty} t^{-\frac{3}{2}}e^{-i(x_0-x'_0)^2t}dt. \qquad (36)$$

Using the representation

$$t^{-\frac{1}{2}} = e^{-i\frac{\pi}{4}}\frac{2}{\sqrt{\pi}}\left(\int_0^{\sqrt{(x_0-x'_0)^2-0}} e^{iz^2t}dz + \int_{\sqrt{(x_0-x'_0)^2+0}}^{\infty} e^{iz^2t}dz\right)$$

in (36) and changing the order of the integration over t and z, we get finally: $I^{(0)}(x, x') = e^{ig\Lambda}(x_0 - x'_0)\delta(\mathbf{x} - \mathbf{x}')$. In the same manner one can verify that the next power of $(x_0 - x'_0)$ at $d \geq 5$ comes from the function $I^{(1)}(x, x') \sim e^{ig\Lambda}(x_0 - x'_0)^3\delta(\mathbf{x} - \mathbf{x}')$. In the case $d = 3$ the same power of $(x_0 - x'_0)$ comes from the function (24). That is why the same dependence (31) holds at any odd d.

Thus, we have shown that the function (21) obeys the equation (6) and the initial conditions (7) in any dimensions d. Then the commutation function can be written in an universal form in any dimensions (by means (5), (21)),

$$S(x, x') = \epsilon(x_0 - x'_0)\left(\hat{\mathcal{P}}_\nu\gamma^\nu + m\right)\int_\Gamma f(x, x', s)ds. \qquad (37)$$

Here we have used the initial conditions (7) to put $\epsilon(x_0 - x'_0)$ before the operator $\hat{\mathcal{P}}_\nu\gamma^\nu + m$.

It was already seen from (23) - (28) that the representation (21) is convenient to select the light cone singularities. In case of even d one can also get d-dimensional generalization of the Fock representation. To this end let us write the function $I_R(x, x')$ from (24) by means of an integral over the closed path Γ_R, defined in (10),

$$I_R(x, x') = \epsilon(x_0 - x'_0)\Theta\left((x - x')^2\right)\int_{\Gamma_R} f_R(x, x', s)ds. \qquad (38)$$

From $f_R(x, x', s)$ only the term $f^{(n)}(x, x', s)$ with $n = \frac{d}{2} - 1$ gives nonzero contribution, namely,

$$\Theta\left((x - x')^2\right)\int_{\Gamma_R} f^{(\frac{d}{2}-1)}(x, x', s)ds = \int_\Gamma f^{(\frac{d}{2}-1)}(x, x', s)ds.$$

That allows one to rewrite (23) for even d in the form

$$\Delta(x, x') = \epsilon(x_0 - x'_0)\left[\Theta\left((x - x')^2\right)\Delta_R(x, x')\right.$$
$$\left.+ \sum_{n=0}^{(d-4)/2} 2a^{(n)}(x, x')\left(\frac{d}{d\tau}\right)^{\frac{d-4}{2}-n}\delta\left(\tau - (\mathbf{x} - \mathbf{x}')^2\right)\Big|_{\tau=(x_0-x'_0)^2}\right], \qquad (39)$$

where $a^{(n)}(x, x')$ are defined in (26), and $\Delta_R(x, x')$ is d-dimensional Riemann function, defined by the integral (10). At $d = 4$ this expression coincides with the Fock's one (8).

The expression for the commutation function $i\left[\phi(x), \phi^\dagger(x')\right]_-$ of the scalar fields $\phi(x)$ and $\phi^\dagger(x')$ one can derive from the representation (21) for the function $\Delta(x, x')$, putting formally all the γ-matrices to zero. We do not also see any difficulties to extend the results obtained to the curved space and gauge theories using the Schwinger-DeWitt technics [2,6].

III. PATH INTEGRAL REPRESENTATION FOR THE COMMUTATION FUNCTION

Here we are going to discuss a path integral representation for the commutation function at $d = 4$. For our purpose, it is convenient to deal with the transformed by $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ function $\tilde{S}(x, x') = S(x, x')\gamma^5$, which obeys the properly transformed Dirac equation

$$\left(\hat{\mathcal{P}}_\nu \tilde{\gamma}^\nu - m\gamma^5\right) \tilde{S}(x,x') = 0, \tag{40}$$

and the initial condition

$$\tilde{S}(x,x')\Big|_{x_0 = x'_0} = -i\tilde{\gamma}^0 \delta(\mathbf{x} - \mathbf{x}'), \tag{41}$$

where $\hat{\mathcal{P}}_\nu = i\partial_\nu - gA_\nu(x)$, and $\tilde{\gamma}^\nu = \gamma^5\gamma^\nu$. The matrices $\tilde{\gamma}^\nu$ have the same commutation relations as initial ones γ^ν, $[\tilde{\gamma}^\mu, \tilde{\gamma}^\nu]_+ = 2\eta^{\mu\nu}$. For all the γ-matrices ($\tilde{\gamma}^5 = \gamma^5$) we have $[\tilde{\gamma}^m, \tilde{\gamma}^n]_+ = 2\eta^{mn}$, $m,n = \overline{0,3},5$; $\eta^{mn} = \text{diag}(1,-1,-1,-1,-1)$.

If one presents the function $\tilde{S}(x,y)$ in the form

$$\tilde{S}(x,x') = -\left(\hat{\mathcal{P}}_\nu \tilde{\gamma}^\nu - m\gamma^5\right) \tilde{\Delta}(x,x'), \tag{42}$$

then the function $\tilde{\Delta}$ obeys the equation

$$\left(\hat{\mathcal{P}}_\nu \tilde{\gamma}^\nu - m\gamma^5\right)^2 \tilde{\Delta}(x,x') = \left[(\hat{\mathcal{P}}_\nu\gamma^\nu)^2 - m^2\right] \tilde{\Delta}(x,x') = 0. \tag{43}$$

One can remark that according to the definition and to the eq. (6),(7) there is a relation $\tilde{\Delta}(x,x') = -\gamma^5\Delta(x,x')\gamma^5$, which allows on to conclude that the functions $\tilde{\Delta}(x,x')$ and $\Delta(x,x')$ obey the same initial conditions. Because they obey also the same equation they coincide. Thus, one can write, using the results obtained before (37),

$$\tilde{S}(x,x') = -\epsilon(x_0 - x'_0) \int_\Gamma \left(\hat{\mathcal{P}}_\nu \tilde{\gamma}^\nu - m\gamma^5\right) f(x,x',s)ds. \tag{44}$$

By means of the representation (15) for the function $f(x,x',s)$, where one can replace the operator $[(\Pi_\nu\gamma^\nu)^2 - m^2]$ by one $(\Pi_\nu\tilde{\gamma}^\nu - m\gamma^5)^2$, and introducing the operator $\left(\hat{\mathcal{P}}_\nu\tilde{\gamma}^\nu - m\gamma^5\right)$ under the sign of the matrix element, we get

$$\tilde{S}(x,x') = -i\,\epsilon(x_0 - x'_0) \int_\Gamma <x| \left(\Pi_\nu\tilde{\gamma}^\nu - m\gamma^5\right) \exp\left\{i\left(\Pi_\nu\tilde{\gamma}^\nu - m\gamma^5\right)^2\right\} |x'> ds. \tag{45}$$

The operator $(\Pi_\nu\tilde{\gamma}^\nu - m\gamma^5)$ can be presented via a Grassmannian integral,

$$\left(\Pi_\nu\tilde{\gamma}^\nu - m\gamma^5\right) = i\int e^{i\chi(\Pi_\nu\tilde{\gamma}^\nu - m\gamma^5)}d\chi,$$

where χ is a Grassmann variable, which anticommutes with γ matrices by the definition. Here and in what follow integrals over Grassmann variables are understood as Berezin's integrals [9]. Thus, the commutation function (45) takes the form

$$\tilde{S} = \tilde{S}(x_{out}, x_{in}) = \epsilon(x^0_{out}, x^0_{in}) \int_\Gamma ds \int \langle x_{out}|e^{-i\hat{\mathcal{H}}(s,\chi)}|x_{in}\rangle d\chi, \tag{46}$$

where

$$\hat{\mathcal{H}}(s,\chi) = s\left(m^2 - \Pi^2 + \frac{ig}{2}F_{\alpha\beta}\tilde{\gamma}^\alpha\tilde{\gamma}^\beta\right) + \left(\Pi_\nu\tilde{\gamma}^\nu - m\tilde{\gamma}^5\right)\chi. \tag{47}$$

Now one can present the matrix element entering in the expression (46) by means of a path integral. First, we write $\exp-i\hat{\mathcal{H}} = \left(\exp-i\hat{\mathcal{H}}/N\right)^N$, and then insert $(N-1)$ resolutions of identity $\int |x><x|dx = I$ between all the operators $\exp-i\hat{\mathcal{H}}/N$. Besides, we introduce N additional integrations over s and χ to transform then the ordinary integrals over these variables into the corresponding path-integrals,

$$\tilde{S} = \epsilon(x^0_{out}, x^0_{in}) \lim_{N\to\infty} \int_\Gamma ds_0 \int d\chi_0 dx_1...dx_{N-1}ds_1...ds_Nd\chi_1...d\chi_N$$
$$\times \prod_{k=1}^N \langle x_k|e^{-i\hat{\mathcal{H}}(s_k,\chi_k)\Delta\tau}|x_{k-1}\rangle\delta(s_k - s_{k-1})\delta(\chi_k - \chi_{k-1}), \tag{48}$$

where $\Delta\tau = 1/N$, $x_0 = x_{in}$, $x_N = x_{out}$. Bearing in mind the limiting process, one can calculate the matrix elements from (48) approximately,

$$\langle x_k|e^{-i\hat{\mathcal{H}}(s_k,\chi_k)\Delta\tau}|x_{k-1}\rangle \approx \langle x_k|1 - i\hat{\mathcal{H}}(s_k,\chi_k)\Delta\tau|x_{k-1}\rangle, \tag{49}$$

using the resolution of identity $\int |p><p|dp$, where

$$P_\mu|p> = p_\mu|p>, \quad <p|p'> = \delta^4(p - p'), \quad <x|p> = \frac{1}{(2\pi)^2}e^{ipx}.$$

In this connection it is important to notice that the operator $\hat{\mathcal{H}}(s_k,\chi_k)$ has originally the symmetric form in the operators $\hat{x}$ and $\hat{p}$. Indeed, the only one term in $\hat{\mathcal{H}}(s_k,\chi_k)$, which contains products of these operators is $[P_\alpha, A^\alpha(X)]_+$. One can verify that this is maximal symmetrized expression, which can be combined from entering operators (see remark in [10]). Thus, one can write $\hat{\mathcal{H}}(s,\chi) = \text{Sym}_{(\hat{x},\hat{p})} \mathcal{H}(s,\chi,\hat{x},\hat{p})$, where $\mathcal{H}(s,\chi,x,p)$ is the Weyl

symbol of the operator $\hat{\mathcal{H}}(s,\chi)$ in the sector of coordinates and momenta, $\mathcal{H}(s,\chi,x,p) = s\left(m^2 - \mathcal{P}^2 + \frac{ig}{2}F_{\alpha\beta}\tilde\gamma^\alpha\tilde\gamma^\beta\right) + (\mathcal{P}_\nu\tilde\gamma^\nu - m\gamma^5)\chi$, and $\mathcal{P}_\nu = -p_\nu - gA_\nu(x)$. That is a general statement [11], which can be easily checked in that concrete case by direct calculations, that the matrix elements (49) are expressed in terms of the Weyl symbols in the middle point $\bar{x}_k = (x_k + x_{k-1})/2$. Taking all that into account, one can see that in the limiting process the matrix elements (49) can be replaced by the expressions

$$\int \frac{dp_k}{(2\pi)^4}\exp i\left[p_k\frac{x_k-x_{k-1}}{\Delta\tau} - \mathcal{H}(s_k,\chi_k,\bar{x}_k,p_k)\right]\Delta\tau,\qquad(50)$$

which are non-commutative due to the γ-matrix structure and are situated in (48) so that the numbers k increase from the right to the left. For the two δ-functions, accompanying each matrix element (49) in the expression (48), we use the integral representations

$$\delta(s_k - s_{k-1})\delta(\chi_k - \chi_{k-1}) = \frac{i}{2\pi}\int e^{i[\pi_k(s_k-s_{k-1})+\nu_k(\chi_k-\chi_{k-1})]}d\pi_k\,d\nu_k,$$

where ν_k are odd variables. Then we attribute formally to γ-matrices, entering into (50), index k, and then we attribute to all quantities the "time" τ_k, according the index k they have, $\tau_k = k\Delta\tau$, so that $\tau\in[0,1]$. Introducing the T-product, which acts on γ-matrices, it is possible to gather all the expressions, entering in (48), in one exponent and deal then with the γ-matrices like with odd variables. Thus, we get for the right side of (48)

$$\tilde S = \epsilon(x_{out}^0,x_{in}^0)\mathrm{T}\int_\Gamma ds_0\int d\chi_0\int_{x_{in}}^{x_{out}}Dx\int Dp\int_{s_0}Ds\int_{\chi_0}D\chi\int D\pi\int D\nu\qquad(51)$$
$$\times\exp\left\{i\int_0^1\left[s\left(\mathcal{P}^2 - m^2 - \frac{ig}{2}F_{\alpha\beta}\tilde\gamma^\alpha\tilde\gamma^\beta\right) + \left(m\gamma^5 - \mathcal{P}_\nu\tilde\gamma^\nu\right)\chi + p\dot x + \pi\dot s + \nu\dot\chi\right]d\tau\right\},$$

where x,p,s,π, are even and χ,ν are odd trajectories, obeying the boundary conditions $x(0)=x_{\mathrm{in}}$, $x(1)=x_{\mathrm{out}}$, $s(0)=s_0$, $\chi(0)=\chi_0$. The operation of T-ordering acts on the γ-matrices, which suppose formally to depend on time τ. The expression (51) can be reduced to:

$$\tilde S = \epsilon(x_{out}^0,x_{in}^0)\int_\Gamma ds_0\int d\chi_0\int_{x_{in}}^{x_{out}}Dx\int Dp\int_{s_0}Ds\int_{\chi_0}D\chi\int D\pi\int D\nu\exp\left\{i\int_0^1\left[s\left(\mathcal{P}^2 - m^2\right.\right.\right.$$
$$\left.\left.\left.-\frac{ig}{2}F_{\alpha\beta}\frac{\delta_l}{\delta\rho_\alpha}\frac{\delta_l}{\delta\rho_\beta}\right) + \left(m\frac{\delta_l}{\delta\rho_5} - \mathcal{P}_\nu\frac{\delta_l}{\delta\rho_\nu}\right)\chi + p\dot x + \pi\dot s + \nu\dot\chi\right]d\tau\right\}\mathrm{T}\exp\int_0^1\rho_n(\tau)\tilde\gamma^n d\tau\bigg|_{\rho=0},$$

where five odd sources $\rho_n(\tau)$ are introduced, which anticommute with the γ-matrices by definition. One can present the quantity $\mathrm{T}\exp\int_0^1\rho_n(\tau)\tilde\gamma^n d\tau$ via a path integral over odd trajectories [7],

$$\mathrm{T}\exp\int_0^1\rho_n(\tau)\tilde\gamma^n d\tau = \exp\left(i\tilde\gamma^n\frac{\partial_l}{\partial\theta^n}\right)\int_{\psi(0)+\psi(1)=\theta}\exp\left[\int_0^1\left(\psi_n\dot\psi^n - 2i\rho_n\psi^n\right)d\tau\right.$$
$$\left.+ \psi_n(1)\psi^n(0)\right]\mathcal{D}\psi|_{\theta=0},\quad \mathcal{D}\psi = D\psi\left[\int_{\psi(0)+\psi(1)=0}D\psi\exp\left\{\int_0^1\psi_n\dot\psi^n d\tau\right\}\right]^{-1},\qquad(52)$$

where θ^n are odd variables, anticommuting with γ-matrices, and $\psi^n(\tau)$ are odd trajectories of integration, obeying the boundary conditions, which are pointed out below the signs of integration. Using (52) we get the Hamiltonian path integral representation for the commutation function:

$$\tilde S = \epsilon(x_{out}^0,x_{in}^0)\exp\left(i\tilde\gamma^n\frac{\partial_l}{\partial\theta^n}\right)\int_\Gamma ds_0\int d\chi_0\int_{s_0}Ds\int_{\chi_0}D\chi\int_{x_{in}}^{x_{out}}Dx\int Dp\int D\pi\int D\nu$$
$$\times\int_{\psi(0)+\psi(1)=\theta}\mathcal{D}\psi\exp\left\{i\int_0^1\left[s\left(\mathcal{P}^2 - m^2 + 2igeF_{\alpha\beta}\psi^\alpha\psi^\beta\right) + 2i\left(\mathcal{P}_\alpha\psi^\alpha - m\psi^5\right)\chi\right.\right.$$
$$\left.\left.-i\psi_n\dot\psi^n + p\dot x + \pi\dot s + \nu\dot\chi\right]d\tau + \psi_n(1)\psi^n(0)\right\}\bigg|_{\theta=0},$$

Integrating over momenta in the path integral, we get

$$\tilde S = \epsilon(x_{out}^0,x_{in}^0)\exp\left(i\tilde\gamma^n\frac{\partial_l}{\partial\theta^n}\right)\int_\Gamma de_0\int d\chi_0 G(e_0,\chi_0,x_{out}^0,x_{in}^0),\qquad(53)$$
$$G(e_0,\chi_0,x_{out}^0,x_{in}^0) = \int_{e_0}De\int_{\chi_0}D\chi\int_{x_{in}}^{x_{out}}Dx\int D\pi\int D\nu\int_{\psi(0)+\psi(1)=\theta}\mathcal{D}\psi$$
$$\times M(e)\exp\left\{i\int_0^1\left[-\frac{\dot x^2}{2e} - \frac{e}{2}m^2 - g\dot xA(x) + iegF_{\mu\nu}(x)\psi^\mu\psi^\nu + i\left(\frac{\dot x_\mu\psi^\mu}{e} - m\psi^5\right)\chi\right.\right.$$
$$\left.\left.-i\psi_n\dot\psi^n + \pi\dot e + \nu\dot\chi\right]d\tau + \psi_n(1)\psi^n(0)\right\}\bigg|_{\theta=0},\qquad(54)$$

where $M(e)$ is the integration measure,

$$M(e) = \int Dp\exp\left\{\frac{i}{2}\int_0^1 ep^2 d\tau\right\}.\qquad(55)$$

The exponent in the integrand (54) can be considered as an effective and non-degenerate Lagrangian action of a spinning particle in an external field. It consists of two principal parts. The first one, which unites two summand with the derivatives of e and χ, can be

treated as a gauge fixing term and corresponds to the gauge conditions $\dot{e} = \dot{\chi} = 0$. The rest part of the effective action, in fact, coincides with the gauge invariant action [12] of a spinning particle. One can interpret the pair e_0, χ_0 in the representation (53) as a super proper time.

Comparing the path integral representation (53) for the commutation function with one [7] for the Dirac propagator (causal Green function), one can remark that they are quite similar, one of the differences is in the contour of integration over s_0. Namely, the path integral representation for the Dirac propagator $\tilde{S}^c$ (transformed by γ^5) reads

$$\tilde{S}^c = \exp\left(i\tilde{\gamma}^n \frac{\partial_l}{\partial \theta^n}\right) \int_\Gamma de_0 \int d\chi_0 G(e_0, \chi_0, x_{out}^0, x_{in}^0) , \tag{56}$$

where the function $G(e_0, \chi_0, x_{out}^0, x_{in}^0)$ has the same form (54). Thus, in case of the commutation function the c-number component of the super proper time is complex in contrast with the case of the propagator.

IV. CONCLUSION

The results obtained for the commutation function allows one to get also similar proper time representation for some other singular functions. For example, it is easy to get for the retarded, $S^{ret}(x, x') = \Theta(x_0 - x_0')S(x, x')$, and advanced, $S^{adv}(x, x') = -\Theta(x_0' - x_0)S(x, x')$, functions the following representations, in which one has to understand $\Theta(0) = 1/2$,

$$S^{ret}(x, x') = \Theta(x_0 - x_0')\left(\hat{\mathcal{P}}_\nu \gamma^\nu + m\right) \int_\Gamma f(x, x', s)ds , \tag{57}$$

$$S^{adv}(x, x') = \Theta(x_0' - x_0)\left(\hat{\mathcal{P}}_\nu \gamma^\nu + m\right) \int_\Gamma f(x, x', s)ds . \tag{58}$$

Combining the Schwinger representation (14) for the causal Green function and the representation (37) for the commutation function, one can get proper time representations for positive and negative frequency functions $S^\mp(x, x')$. Namely, let us define them via the Schwinger representation of the causal Green function,

$$S^\mp(x, x') = \pm\Theta\left(\pm[x_0 - x_0']\right) S^c(x, x'). \tag{59}$$

Using the completeness relation

$$S(x, x') = S^-(x, x') + S^+(x, x') , \tag{60}$$

we get for any x, x'

$$S^\mp(x, x') = \Theta\left(\mp[x_0 - x_0']\right) S(x, x') \pm S^c(x, x'). \tag{61}$$

In this connection one ought to remark that there is a problem with the causal Green function definition in case of an arbitrary external field. From the one hand, there exists the Feynman definition, based on the definition of the inverse operator to the Dirac equation by means of the prescription $m^2 \to m^2 - i\epsilon$. On the other hand, in the perturbation theory there appears a field theoretical definition of the propagator in the form

$$S^c(x, x') = i < 0|T\psi(x)\bar{\psi}(x')|0 > . \tag{62}$$

In the absence of the external field or in fields of special form, which do not violate the vacuum stability (then the operators of the spinor fields in (62) have to be taken in the Furry representation), it is possible to verify that the Feynman causal Green function and the propagator (14) coincide. In the same case one can establish that the former function can be defined via the Schwinger proper time representation. In external fields, which violate the vacuum stability (create pairs from the vacuum), the situation is not so clear. In this case does not exist an unique vacuum for all the time instances. One has to distinguish the initial $|0, in >$ and final $|0, out >$ vacua [2,13,3]. In virtue of that, one has also to use different kinds of propagators in the perturbation theory,

$$S^c(x, x') = i\frac{< 0, out|T\psi(x)\bar{\psi}(x')|0, in >}{< 0, out|0, in >} , \tag{63}$$

$$S_{in}^c(x, x') = i < 0, in|T\psi(x)\bar{\psi}(x')|0, in > , \tag{64}$$

and the positive and negative frequency commutation functions,

$$S^-(x, x') = i\frac{< 0, out|\psi(x)\bar{\psi}(x')|0, in >}{< 0, out|0, in >} ,$$

$$S^+(x, x') = i\frac{< 0, out|\bar{\psi}(x')\psi(x)|0, in >}{< 0, out|0, in >} . \tag{65}$$

It was shown that in special cases of external fields, violating the vacuum stability, the Feynman causal Green function, presented by means of the Schwinger proper time integral, gives namely the propagator (63), whereas the propagator (64) demands a modification of the Schwinger contour in the proper time integration [14]. At the present time does not exist a proof of the equivalence between of the Feynman causal Green function and the propagator (63) for any external fields. Nevertheless, there is a strong believe that they are equivalent. If one excepts such an equivalence, then the positive and negative frequency commutation functions (65) have the representation (61) in arbitrary external fields.

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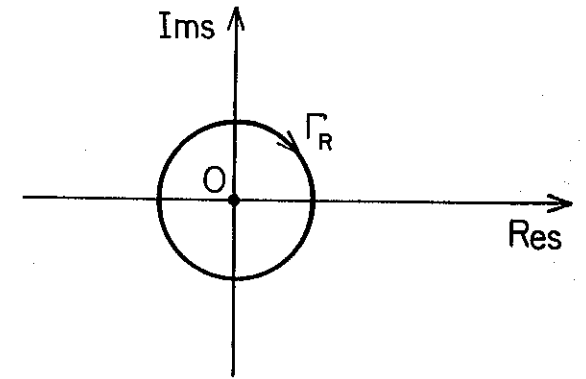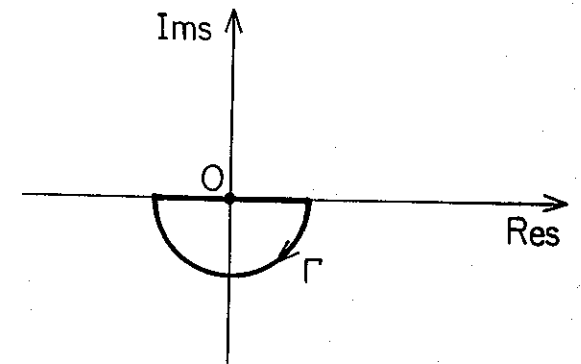
FIG.1.

FIG.2.