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NON-EXISTENCE OF SUPEREXPONENTIAL SOLUTIONS IN A SYSTEM OF DELAY DIFFERENTIAL EQUATIONS MODELING A TWO-NEURON NETWORK

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Abstract

We show that a two-neuron network model with delay satisfies conditions presented in [9] for the nonexistence of superexponential solutions. One implication is that the notions of weak and strong oscillations are equivalent for this system, and that solutions are either damped to an equilibrium point or asymptotically periodic.

1 Introduction

The dynamics of neural network models composed of graded-response units has been investigated recently in order to detect instabilities induced by the presence of non-zero inter-unit transmission times (see references in [1]).

In this respect, understanding the behavior of small networks composed of a few neurons has played an important role, as the behavior of these systems can be more easily analyzed than that of large networks composed of an arbitrary number of neurons.

In our previous work, we have studied the dynamics of a self-exciting single neuron with delay [1]. It was shown that, in that system, trajectories of almost all initial conditions converged to equilibrium points, and that nonconverging solutions were asymptotically periodic. The results on the convergence relied on the monotonicity of the system [2], and the ones on the asymptotic behavior of non-converging solutions were based on a Poincaré-Bendixson theorem combined with the non-existence of superexponential solutions established for scalar delay-differential equations (DDEs) with monotone feedback [3].

In a companion paper we have examined how monotonicity results can be applied to the study of two-neuron networks [4]. In this paper, we study the extension of results concerning the behavior of solutions that do not converge to a stable equilibrium point. This analysis requires the extension of a number of technical results valid for scalar DDEs to the case of systems, which is the main goal of this report.

2 The model

The dynamics of two neurons connected to each other by weights W and W' and delay $A > 0$, are determined by the following system of delay differential equations (DDEs)[4]:

$$\begin{cases} \epsilon \frac{dx}{dt}(t) = -x(t) + W\sigma_\alpha(y(t-A)) , \\ \epsilon' \frac{dy}{dt}(t) = -y(t) + W'\sigma_{\alpha'}(x(t-A)) , \end{cases} \tag{1}$$

where $\epsilon > 0$, $\epsilon' > 0$ and:

$$\sigma_\alpha(a) = \tanh(\alpha a) = \frac{e^{\alpha a} - e^{-\alpha a}}{e^{\alpha a} + e^{-\alpha a}} \tag{2}$$

For an initial condition $\Phi = (\phi_1, \phi_2)$ in $S = C([-A, 0], \mathbb{R}^2)$, there exists a unique solution of Eq. (1), denoted by $z(t, \Phi) = (x(t, \Phi), y(t, \Phi))$, such that $x(\theta, \Phi) = \phi_1(\theta)$ and $y(\theta, \Phi) = \phi_2(\theta)$ for $-A \leq \theta \leq 0$ and $z(t, \Phi)$ satisfies Eq. (1) for $t \geq 0$. For such a solution of the DDE, we note $z_t(\Phi) = (x_t(\Phi), y_t(\Phi))$ the element of S defined by $x_t(\Phi)(\theta) = x(t+\theta, \Phi)$, and $y_t(\Phi)(\theta) = y(t+\theta, \Phi)$, for $\theta \in [-A, 0]$. z_t is the differentiable semi-flow generated by Eq. (1) in the space S.

The constant function $z(t) = 0$ for $t \geq -A$ is a solution of system (1). It represents an equilibrium point. We are interested in the trajectories that eventually tend to this equilibrium. In ODEs, solutions tending to an equilibrium point can be eventually approximated by an appropriate exponential function obtained from linearizing the system. In DDEs, such

solutions may decay faster than exponential functions, and in this case they are referred as superexponential or small solutions. Their existence is an important issue as it can modify the way the DDE can be analyzed. For example, for some scalar DDEs that do not admit such solutions, it is possible to construct a discrete Lyapunov function which counts the number of zeros of solutions [5, 6, 7], and to derive a Morse decomposition of the attractor [8].

In this paper we show that system (1) does not admit superexponential solutions. To this end we prove that it satisfies the sufficient conditions given in [9] for the non-existence of such solutions. These conditions are referred to as hypotheses (H1) through (H5) in [9], and we shall use the same notation in this paper.

3 Monotonicity

We now introduce the following conditions, definitions and terminology.

From here on, we suppose that the parameters satisfy the following conditions:

Condition (C). *Positive feedback:* $\alpha > 0$, $W > 0$, $\alpha' > 0$ *and* $W' > 0$.

For $\Phi = (\phi_1, \phi_2)$ and $\Psi = (\psi_1, \psi_2)$ in S, we say that Φ is larger (resp. strictly larger) than Ψ, noted $\Phi \geq \Psi$ (resp. $\Phi > \Psi$), if $\phi_1(\theta) \geq \psi_1(\theta)$ (resp. $\phi_1(\theta) > \psi_1(\theta)$) and $\phi_2(\theta) \geq \psi_2(\theta)$ (resp. $\phi_2(\theta) > \psi_2(\theta)$) for all θ in $[-A, 0]$.

We rewrite system (1) as

$$\frac{dz}{dt} = F(z_t) , \tag{3}$$

where F is the map from $S = C([-A, 0], \mathbb{R}^2)$ to $\mathbb{R}^2$ defined as

$$F(\Phi) = \begin{pmatrix} (-\phi_1(0) + W\sigma_\alpha(\phi_2(-A)))/\epsilon \\ (-\phi_2(0) + W\sigma_{\alpha'}(\phi_1(-A)))/\epsilon' \end{pmatrix} \text{ for } \Phi = (\phi_1, \phi_2) \in S \,. \tag{4}$$

F satisfies the following hypothesis.

Hypothesis (H1). *F is continuous on its domain, sends bounded sets of S into bounded sets of $\mathbb{R}^2$, and is such that system (1) has one and only one solution starting from any given data $\Phi \in S$.*

For every pair $\Phi = (\phi_1, \phi_2)$, $\Psi = (\psi_1, \psi_2)$ in S such that $\Phi \leq \Psi$ and $\Phi(0) = \Psi(0)$, we have $F(\Phi) \leq F(\Psi)$.

The system also satisfies the following hypothesis.

Hypothesis (H2). *System (1) is strongly monotone, in the sense that it verifies (H1) and for $\Phi \leq \Psi$ and $\Phi \neq \Psi$, there is $t_1 \geq 0$ such that $z_t(\Phi) \ll z_t(\Psi)$ for all $t \geq t_1$.*

Proof. Under the positive feedback condition (C), system (1) is an irreducible cooperative system. Irreducibility means that there is a directed path connecting each neuron to the other, and cooperativity arises from the fact that the neurons are mutually exciting [10]. Moreover, the right-hand-side of system (1) transforms bounded sets of S into bounded sets. Thus theorem 2.5 in [2] (see also [11]) implies that system (1) generates a strongly monotone semi-flow, that is:

$$\text{If } \Phi \geq \Phi' \text{ and } \Phi \neq \Phi' \text{ then } z_t(\Phi) > z_t(\Phi') \text{ for } t \geq 2A. \tag{5}$$

□

4 Linearization

By linearizing system (1) in the neighborhood of 0, and looking for exponential solutions of the form $\exp(\lambda t)$ we obtain the following characteristic equation [12]:

$$\Delta(\lambda) = (\lambda + 1/\epsilon)(\lambda + 1/\epsilon') - l_1(e^{\lambda \cdot})l_2(e^{\lambda \cdot}) \,, \tag{6}$$

where $l_1(\phi) = \frac{\alpha W}{\epsilon}\phi(-A)$ and $l_2(\phi) = \frac{\alpha' W'}{\epsilon'}\phi(-A)$ are positive linear functionals on $C([-A, 0], \mathbb{R})$.

We assume that $A/\epsilon > \theta_m$ and $A/\epsilon' > \theta'_m$ where θ_m and θ'_m are the solutions of $\theta_m \exp(\theta_m) = 1/(\alpha W)$ and $\theta'_m \exp(\theta'_m) = 1/(\alpha' W')$. Then it follows that $l_1(\theta e^{-\theta/\epsilon}) = -\alpha W \frac{A}{\epsilon} e^{A/\epsilon} < -1$ and $l_2(\theta e^{-\theta/\epsilon'}) = -\alpha' W' \frac{A}{\epsilon'} e^{A/\epsilon'} < -1$.

Hence from Lemma 3 ([9], p.273) we obtain that system (1) satisfies the following hypothesis.

Hypothesis (H3). *Assume that $A/\epsilon > \theta_m$ and $A/\epsilon' > \theta'_m$, then the characteristic equation (6) has one and only one real root. This root has multiplicity one.*

5 Damped oscillations

Before going any further we need to define the notions of weak and strong oscillations. We take the same definitions as in [9], adapted for the case of a two-variable system.

Definition 1. *Let $a : [t_0, +\infty) \to \mathbb{R}$ be a continuous function. we say that a is strictly oscillatory if for every $T \geq t_0$ there exists $T' \geq T$ and $T'' \geq T$ such that $a(T') \times a(T'') < 0$.*

Definition 2. *Weak oscillations. A solution $z(t) = (x(t), y(t))$ of system (1) is weakly oscillating if, for all T, there exists $T' \geq T$ such that $x(T') \times y(T') < 0$.*

Definition 3. *Strong oscillations. A solution $z(t) = (x(t), y(t))$ of system (1) is strongly oscillating if at least one of its components, x or y, is strictly oscillating in the sense of definition 1.*

A solution $z = (x, y)$ of (1) is weakly oscillating and not strongly oscillating if and only if there is $T > 0$ such that $x(t).y(t) < 0$ for all $t > T$. We show that system (1) satisfies the following hypothesis.

Hypothesis (H4). *Let z be a solution of system (1) such that z oscillates weakly and does not oscillate strongly, then $z(t) \to 0$, as $t \to +\infty$.*

Proof. Let $d_1 = 1/\epsilon$, $d_2 = 1/\epsilon'$, $M_1(\phi) = \frac{W}{\epsilon}\sigma_\alpha(\phi(-A))$, $M_2(\phi) = \frac{W'}{\epsilon'}\sigma_{\alpha'}(\phi(-A))$, where $\phi \in C([-A, 0], \mathbb{R})$. We have $\text{sign}(\phi).M_i(\phi) \geq 0$ $(i \in \{1, 2\})$, when ϕ has a constant sign. Furthermore we have $(d_1 x + m_1(x))(d_2 x + m_2(x)) > 0$ for $x \neq 0$, where $m_1(x) = \frac{W}{\epsilon}\sigma_\alpha(x)$ and $m_2(x) = \frac{W'}{\epsilon'}\sigma_{\alpha'}(x)$. Thus, Lemma 4 ([9], p.275) shows that system (1) satisfies the hypothesis (H4). □

6 Nonexistence of superexponential solutions

System (1) can be re-written as:

$$\frac{dz}{dt} = L z_t + f(z_t), \tag{7}$$

where $L : S \to \mathbb{R}^2$ is the linear map defined by $L(\Phi) = M\Phi(-A) + N\Phi(0)$, where M and N are the 2×2 matrices defined as:

$$M = \begin{pmatrix} 0 & \frac{\alpha W}{\epsilon} \\ \frac{\alpha' W'}{\epsilon'} & 0 \end{pmatrix}, \quad \text{and} \quad N = \begin{pmatrix} \frac{-1}{\epsilon} & 0 \\ 0 & \frac{-1}{\epsilon'} \end{pmatrix}, \tag{8}$$

and $f : S \to \mathbb{R}^2$ is the map defined by

$$f(\Phi) = \begin{pmatrix} \frac{W}{\epsilon}[\sigma_\alpha(\phi_2(-A)) - \alpha\phi_2(-A)] \\ \frac{W'}{\epsilon'}[\sigma_{\alpha'}(\phi_1(-A)) - \alpha'\phi_1(-A)] \end{pmatrix}. \tag{9}$$

The linear part L satisfies the following hypothesis:

Hypothesis (H5'). *$L(\Phi) = M\Phi(-A) + \int_{-r^+}^{0} d\eta(\theta)\Phi(\theta)$, where M is nonsingular, and $\int_{-r^+}^{0} d\eta(\theta)\Phi(\theta) = N\Phi(0)$.*

The nonlinear part f satisfies the following hypothesis.

Hypothesis (H5"). *For the nonlinear part f of system (1), there exists a function g defined from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $g(u)/u \to 0$ as $u \to 0$ and $||f(\Phi)|| \leq g(||\Phi(-A)||)$, where for $X = (x, y) \in \mathbb{R}^2$, we note $||X|| = |x| + |y|$.*

Proof.

$$\begin{aligned} ||f(\Phi)|| &= \tfrac{W}{\epsilon}|\sigma_\alpha(\phi_2(-A)) - \alpha\phi_2(-A)| + \tfrac{W'}{\epsilon'}|\sigma_{\alpha'}(\phi_1(-A)) - \alpha'\phi_1(-A)| \\ &= \tfrac{W}{\epsilon}|\psi_\alpha(\phi_2(-A))| + \tfrac{W'}{\epsilon'}|\psi_{\alpha'}(\phi_1(-A))| \end{aligned} \tag{10}$$

where $\psi_\beta : \mathbb{R} \to \mathbb{R}$ is defined by: $\psi_\beta(u) = \sigma_\beta(u) - \beta u$. ψ_β is a strictly decreasing odd function for $\beta > 0$, so that $|\psi_\beta(u)| = -\psi_\beta(|u|)$. Thus we have:

$$||f(\Phi)|| = -\tfrac{W}{\epsilon}\psi_\alpha(|\phi_2(-A)|) - \tfrac{W'}{\epsilon'}\psi_{\alpha'}(|\phi_1(-A)|). \tag{11}$$

$-\psi_\beta$ is an increasing function $(\beta > 0)$, and we have $|\phi_i(-A)| \leq |\phi_1(-A)| + |\phi_2(-A)| = ||\Phi(-A)||$, for $i \in \{1, 2\}$, so that:

$$\begin{aligned} ||f(\Phi)|| &\leq -(\tfrac{W}{\epsilon}\psi_\alpha(||\Phi(-A)||) + \tfrac{W'}{\epsilon'}\psi_{\alpha'}(||\Phi(-A)||) \\ ||f(\Phi)|| &\leq g(||\Phi(-A)||) \end{aligned} \tag{12}$$

where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(u) = -(\frac{W}{\epsilon}\psi_\alpha(u) + \frac{W'}{\epsilon'}\psi_{\alpha'}(u))$. As we have $\psi_\beta(u)/u \to 0$ as $u \to 0$, the same property holds for g, that is, $g(u)/u \to 0$ as $u \to 0$. □

Thus proposition 5 ([9], p.278) implies that system (1) satisfies the following hypothesis

Hypothesis (H5). *System (1) does not have any solution z such that each component of z is $\neq 0$ for each t large enough and $z(t) \to 0$ faster than any exponential as $t \to +\infty$.*

7 Strong Oscillations

Property. *Assume the characteristic equation of (1) at 0 has no root with a zero real part. If $z(t) = (x(t), y(t))$ is a weakly oscillating solution of (1), then both $x(t)$ and $y(t)$ are oscillating, in the sense that they have zeros at arbitrarily large times.*

Proof. From the second inequality in (12), and the fact that g is an increasing function we deduce that: $||f(\Phi)|| \le g(||\Phi||_\infty)$, where $||\Phi||_\infty$ represents the supremum norm of Φ.

A Taylor expansion in the neighborhood of 0, shows that: $g(u) = \mathcal{O}(u^3)$. So that for Φ close to zero (in S), we have: $||f(\Phi)|| \le \mathcal{O}(||\Phi||^3_\infty)$.

Finally, theorem 3 ([9], p.281) yields that weakly oscillating solutions of (1) are strongly oscillating, that is, at least one of the components $x(t)$ or $y(t)$ is oscillating. For the special case of (1) this implies that both are oscillating. □

8 Global behavior

Let $\epsilon = \epsilon'$ and $A = 1$ in (1), then the characteristic equation of the linear part L is :

$$\Delta(\lambda) = (\lambda + 1/\epsilon)^2 - \frac{\alpha\alpha' W W'}{\epsilon^2} e^{-2\lambda} = 0 \,. \tag{13}$$

Under the positive feedback condition $\alpha\alpha' W W' > 0$, so that the characteristic equation is equivalent to:

$$\begin{aligned} \Delta^*(\mu) &= (\mu + \gamma)^2 - (\gamma K e^{-\mu})^2 = 0 \,, \\ \Delta^*(\mu) &= (\mu + \gamma - \gamma K e^{-\mu})(\mu + \gamma + \gamma K e^{-\mu}) = 0 \,, \end{aligned} \tag{14}$$

where $\mu = \frac{\lambda}{2}$, $\gamma = \frac{1}{2\epsilon}$ and $K = \sqrt{\alpha\alpha' W W'}$. Thus, $\Delta^*(\mu)$ is the product of the characteristic equations with positive and negative feedbacks.

Damped oscillations. For $K < 1$, all the roots of the characteristic equation have strictly negative real parts, therefore, 0 is a locally asymptotically stable equilibrium. In fact, this point is globally asymptotically stable. Hence, all oscillating solutions are damped to 0.

Non-existence of homoclinic orbits through 0 for $K > 1$

For $K > 1$, let $\mu = a + ib$ be a root of (14), with $b \ge 0$. We have:

$$\begin{cases} a + \gamma + \eta\gamma K e^{-a}\cos(b) = 0 \,, \\ b - \eta\gamma K e^{-a}\sin(b) = 0 \,, \end{cases} \tag{15}$$

where $\eta^2 = 1$ (if $\eta = +1$ (resp. $\eta = -1$), μ is a root of the negative (resp. positive) feedback characteristic equation).

We rewrite (15) as:

$$\begin{cases} -\eta\frac{a+\gamma}{\gamma K} e^a = \cos(b) \,, \\ \eta\frac{b}{\gamma K} e^a = \sin(b) \,. \end{cases} \tag{16}$$

By adding the square of the two equations we obtain:

$$\frac{e^{2a}}{(\gamma K)^2}(b^2 + (a + \gamma)^2) = 1 \,. \tag{17}$$

So that:

$$b = \sqrt{\gamma^2 K^2 e^{-2a} - (a + \gamma)^2} \,. \tag{18}$$

Note that there is $a_c(\gamma, K) > 0$ such that $\gamma K e^{-a} > |a+\gamma|$, for all $a < a_c$.

For $a < a_c$, we define $f(a) = \gamma^2 K^2 e^{-2a} - (a+\gamma)^2$, then the derivative of f is given by:

$$f'(a) = -2(\gamma^2 K^2 e^{-2a} + a + \gamma) . \tag{19}$$

This quantity reaches its maximal value at $a_0 = \mathrm{Log}(\sqrt{2}\gamma K)$, and

$$f'(a_0) = -2(1/2 + \gamma + \mathrm{Log}(\sqrt{2}\gamma K)) \tag{20}$$

As long as $K \geq \frac{e^{-(\gamma+1/2)}}{\sqrt{2}\gamma}$, we have $f'(a_0) \leq 0$, and therefore f is decreasing. Note that as $K > 1$, the above inequality is satisfied for γ larger than the root of $\sqrt{2}\gamma - e^{-(\gamma+1/2)} = 0$. Finally we have the following result:

Roots of the characteristic equation (13). Assume that the characteristic equation (13) has no root with zero real part. For $K > \max(1, \frac{e^{-(\gamma+1/2)}}{\sqrt{2}\gamma})$, we have:

$$\mathrm{Sup}\{\mathcal{I}(\lambda) \text{ such that } \Delta(\lambda) = 0 \text{ and } \mathcal{R}(\lambda) > 0\} < \mathrm{Inf}\{\mathcal{I}(\lambda) \text{ such that } \Delta(\lambda) = 0 \text{ and } \mathcal{R}(\lambda) < 0\}$$

where $\mathcal{R}(\lambda)$ and $\mathcal{I}(\lambda)$ represent the real and the imaginary parts of λ respectively.

9 Conclusion

The above result, combined with the fact that the equation (1) does not admit superexponential solutions tending to 0, and the fact that the number of zeros of solutions is non-increasing along time [3, 13, 14] shows that for $K > \max(1, \frac{e^{-(\gamma+1/2)}}{\sqrt{2}\gamma})$, there can be no homoclinic orbits through 0 [5, 6, 7]. Combined with the Poincaré-Bendixson theorem shown in [3], the non-existence of homoclinic orbits shows that oscillating solutions are either damped to 0 or asymptotically periodic.

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