UNIVERSIDADE DE SÃO PAULO

INSTITUTO DE FÍSICA
CAIXA POSTAL 66318
05389-970 SÃO PAULO - SP
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TRANSITION LAYER EQUATIONS FOR POSITIVE-FEEDBACK DELAYED EQUATIONS

C. Grotta-Ragazzo[1], **C.P. Malta**[2], **K. Pakdaman**[3], **O. Arino**[4] and **J.-F. Vibert**[3]

[1]Instituto de Matemática e Estatística, Universidade de São Paulo
CP 66281, 05389-970 São Paulo, BRASIL

[2]Instituto de Física, Universidade de São Paulo

[3]B3E, INSERM U 444, ISARS, UPMC
Falcuté de Médecine Saint-Antoine
27, rue Chaligny, 75571 Paris Cedex 12, FRANCE

[4]Laboratoire de Mathématiques Appliquées
Université de Pau et des Pays de l'Adour
IPRA, URA CNRS 1204
Avenue de l'Université, 64000 Pau, FRANCE

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C. Grotta-Ragazzo[1], C.P. Malta[2], K. Pakdaman[3], O. Arino[4] and J.-F. Vibert[3]

1) Instituto de Matemática e Estatística
Universidade de São Paulo
CP 66281, 05389-970 São Paulo, BRASIL

2) Instituto de Física
Universidade de São Paulo
CP 66318, 05389-970 São Paulo, BRASIL

3) B3E, INSERM U 444, ISARS, UPMC
Faculté de Médecine Saint-Antoine
27, rue Chaligny
75571 Paris Cedex 12 FRANCE

4) Laboratoire de Mathématiques Appliquées
Université de Pau et des Pays de l'Adour
IPRA, URA CNRS 1204
Avenue de l'Université, 64000 Pau, FRANCE

Abstract

We consider the scalar delayed differential equation $\epsilon\dot{x}(t) = -x(t) + f(x(t-1))$, where $\epsilon > 0$ and f verifies $df/dx > 0$ and some other conditions. This equation has three equilibria $-\gamma_1$, 0, and γ_3. In the study of the singular limit $\epsilon \to 0$ a crucial role is played by the so called transition layer equation related to the above equation. In this case the transition layer equation is given by $\dot{y}(t) = -y(t) + f(y(t+r))$, where $r > 0$. We prove that there is a special value of r for which the transition layer equation has a solution such that $y(t) \to -\gamma_1$ as $t \to -\infty$, and $y(t) \to \gamma_3$ as $t \to \infty$.

Key words: delayed differential equation, singular perturbation, transition layer.

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1 A Theorem

Let us consider the following family of scalar advanced differential equations

$$\dot{y}(t) = -y(t) + f(y(t+r)), \tag{1}$$

where r is a positive parameter. We assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and verifies the following hypotheses:

(H1) $f(0) = 0$, $f(-\gamma_1) = -\gamma_1$, $f(\gamma_2) = \gamma_2$, where $\gamma_1 > 0$, $\gamma_2 > 0$, and $f(x) \neq 0$ for $x \in (-\gamma_1, 0) \cup (0, \gamma_2)$;

(H2)

$$\frac{df}{dx}(x) \geq 0, \quad \text{and} \quad \frac{df}{dx}(0) > 1.$$

Our goal in this paper is to prove the following theorem.

Theorem 1 *There exists $r_* > 0$ such that equation (1) with $r = r_*$ has a solution $\phi : \mathbb{R} \to \mathbb{R}$ with the following properties:*

$$\frac{d\phi}{dt}(t) \geq 0, \quad for \quad t \in \mathbb{R}, \quad \phi(0) = 0,$$
$$\lim_{t\to-\infty} \phi(t) \to -\gamma_1, \quad \lim_{t\to\infty} \phi(t) \to \gamma_2.$$

There also exists $r_{**} > 0$ such that equation (1) with $r = r_{**}$ has a solution $\chi : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\frac{d\chi}{dt}(t) \leq 0, \quad for \quad t \in \mathbb{R}, \quad \chi(0) = 0,$$
$$\lim_{t\to-\infty} \chi(t) \to \gamma_2, \quad \lim_{t\to\infty} \chi(t) \to -\gamma_1.$$

Moreover, if f is an odd function then $r_* = r_{**}$ and $\phi(t) = -\chi(t)$.

In order to prove theorem 1 it is sufficient to show the existence of r_* and ϕ. The existence of r_{**} and χ is a consequence of this result applied to equation (1) after the change of variables $y \to -y$. If f is odd then $\phi(t) = -\chi(t)$ is a consequence of the symmetry of equation (1) with respect to the change of variables $y \to -y$.

The proof of the above theorem will be made in several steps. In section 2 we consider a family of auxiliary problems defined in compact sets $[-L, L]$ of the real line. We show that these problems have solutions ϕ_L, r_L for all L. In section 3 we show that r_L is uniformly bounded with respect to L from above and below and that there is a sequence L_n, $n = 1, 2, \ldots$, of values of L such that $\phi_{L_n}, r_{L_n} \to \phi, r_*$, in compact subsets of $\mathbb{R}$, as $n \to \infty$. In section 4 we show that the function ϕ obtained in section 3 has the properties in theorem 1.

2 A family of approximating problems

We start this section with some definitions. For $L > 0$, let C_L be the Banach space defined by

$$C_L \stackrel{def}{=} \{z : [-L, L] \to \mathbb{R} \mid z \quad \text{continuous}\}, \qquad \|z\|_L = \sup_{|t|\le L} |z(t)| \quad .$$

Let Λ_L be the following subset of C_L (endowed with the induced topology)

$$\Lambda_L \stackrel{def}{=} \{z \in C_L \mid z(0) = 0,\ t \le t' \Rightarrow z(t) \le z(t'),\ -\gamma_1 \le z(t) \le \gamma_2\} \quad .$$

Proposition 1 *The set Λ_L has the following properties:*

- *i) it is bounded,*
- *ii) it is closed,*
- *iii) it is convex.*

These properties can be easily verified.

Let X be the set of functions given by

$$\begin{aligned} X \stackrel{def}{=} & \{z : \mathbb{R} \to \mathbb{R} \mid z \quad \text{continuous for } t \in \mathbb{R}, \text{nondecreasing for } \ t < 0, \\ & \text{strictly increasing for } t > 0, z(0) < 0, \\ & \lim_{t\to-\infty} z(t) = -\gamma_1, \text{ and } \lim_{t\to\infty} z(t) = \gamma_2\} \quad . \end{aligned}$$

We endow X with the metric $d(x,z) = \sup_{t\in\mathbb{R}}|x(t) - z(t)|$. Denoting the restriction of a function $z : \mathbb{R} \to \mathbb{R}$ to the interval $[-L, L]$ by $z|_L$ we define the set X_L as

$$X_L \stackrel{def}{=} \{z : \mathbb{R} \to \mathbb{R} \mid z|_L \in \Lambda_L, \quad z(t) = -\gamma_1,\ t < -L, \quad z(t) = \gamma_2,\ t > L\} \quad .$$

We endow X_L with the the metric $d(x,z) = \sup_{|t|\le L}|x(t) - z(t)|$. Notice that every function in X_L is an extension to $\mathbb{R}$ of a function in Λ_L, originally defined on the interval $[-L, L]$. We denote this extension mapping by $\overline{\Gamma} : \Lambda_L \to X_L$. We define a mapping $\underline{A}_L : X_L \to X$ by

$$\underline{A}_L z(t) \stackrel{def}{=} e^{-t} \int_{-\infty}^{t} e^s f(z(s))ds = \int_{-\infty}^{0} e^s f(z(s+t))ds \quad .$$

It is not hard to verify that $\underline{A}_L z$ indeed belongs to X. For each $z \in X$ there exists a unique $r(z) \in \mathbb{R}$, $r(z) > 0$, such that $z(r(z)) = 0$. For a fixed L, the composed function $r \circ \underline{A}_L : X_L \to \mathbb{R}_+$ satisfies the following bounds, independently of z.

Proposition 2 *For a given L and any $z \in X_L$ we have*

$$\frac{e^{-L}\gamma_1}{\gamma_2} + 1 \le e^{r\circ\underline{A}_L(z)} \le \frac{\gamma_1}{\gamma_2} + e^L .$$

Proof. In the following, in order to simplify the notation, we will write $r \circ \underline{A}_L(z)$ just as r. The definition of $\underline{A}_L$ implies that

$$\int_{-\infty}^{r} e^s f(z(s))ds = 0 \tag{2}$$

If $r \le L$ then the upper bound for r is trivial. So, let us assume that $r > L$. Using that $-f(z(s)) \le \gamma_1$ for $s \le 0$ and that $f(z(s)) \ge 0$ for $s \ge 0$, equation (2) implies

$$\gamma_1 = \gamma_1 \int_{-\infty}^{0} e^s ds \ge -\int_{-\infty}^{0} e^s f(z(s))ds = \int_{0}^{r} e^s f(z(s))ds \ge \int_{L}^{r} e^s f(z(s))ds = \gamma_2[e^r - e^L] \quad .$$

This inequality implies the upper bound for r. Equation (2) implies that

$$e^{-L}\gamma_1 - \int_{-L}^{0} e^s f(z(s))ds = \int_{0}^{r} e^s f(z(s))ds \quad . \tag{3}$$

The lower bound for r comes from the following inequality obtained from equation (3)

$$e^{-L}\gamma_1 \le \int_{0}^{r} e^s f(z(s))ds \le \gamma_2(e^r - 1) \quad .$$

□

We define the set X_* as

$$\begin{aligned} X_* \stackrel{def}{=} & \{z : \mathbb{R} \to \mathbb{R} \mid z \quad \text{continuous for } t \in \mathbb{R}\ , \text{nondecreasing for } \ t < 0, \\ & \text{strictly increasing for } t > 0, \\ & \lim_{t\to-\infty} z(t) = -\gamma_1, \text{ and } \lim_{t\to\infty} z(t) = \gamma_2\} \quad . \end{aligned}$$

We endow X_* with the metric $d(x,z) = \sup_{t\in\mathbb{R}}|x(t) - z(t)|$. We define the mapping $T_r : X \to X_*$ as $T_r z(t) = z(t + r(z))$ and the restriction mapping $\underline{\Gamma} : X_* \to \Lambda_L$. Finally, we define a mapping $A_L : \Lambda_L \to \Lambda_L$ as

$$A_L = \underline{\Gamma} \circ T_r \circ \underline{A}_L \circ \overline{\Gamma} \quad . \tag{4}$$

It is easy to check that $A_L z : [-L, L] \to \mathbb{R}$ is: continuous, bounded, nondecreasing, and satisfies $A_L z(0) = 0$. So, $A_L z$ indeed belongs to Λ_L. A more explicit way to write A_L is

$$A_L z(t) \stackrel{def}{=} e^{-t-r} \int_{-\infty}^{t+r} e^s f_L(z(s))ds \quad , \tag{5}$$

where

$$\begin{aligned} & f_L(z(s)) = -\gamma_1 && \textit{for} \quad s < -L, \\ & f_L(z(s)) = \gamma_2 && \textit{for} \quad s > L, \\ & f_L(z(s)) = f(z(s)) && \textit{for} \quad |s| \le L \\ & r \quad \text{is an abbreviation of} && r \circ \underline{A}_L(z). \end{aligned}$$

Proposition 3 *The mapping $A_L : \Lambda_L \to \Lambda_L$ is continuous.*

Proof. In order to prove the theorem we have to show that the four mappings in the definition (4) of A_L are continuous. It is easy to show that $\overline{\Gamma}$ and $\underline{\Gamma}$ are continuous. The continuity of $\underline{A}_L$ is proved in the following way. As f is continuously differentiable there exists a constant μ such that

$$|f(x) - f(y)| \leq \mu|x - y| \qquad \text{for} \qquad -\gamma_1 \leq x \leq \gamma_2, \quad -\gamma_1 \leq y \leq \gamma_2 \quad .$$

Thus, if $x \in X_L$, $z \in X_L$, satisfy $d(x, z) < \delta$ then

$$\begin{aligned} |\underline{A}_L x(t) - \underline{A}_L z(t)| &= \left| \int_0^\infty e^{-s}[f(x(t-s)) - f(z(t-s))]ds \right| \\ &\leq \mu\delta \int_0^\infty e^{-s} ds = \mu\delta \quad . \end{aligned}$$

This inequality implies the continuity of $\underline{A}_L$. The continuity of T_r is a more difficult point, because r itself is a function of the point $z \in X_L$ to which we apply T_r. Let us denote by z, z' two points in X and by r and r' their respective zeroes ($z(r) = 0, z'(r') = 0$), or, equivalently, the values of the function r at z and z' ($r(z) = r$, $r(z') = r'$). We want to show that for any given $z \in X$ and $\epsilon > 0$ there exists a $\delta > 0$ such that $d(z, z') < \delta$ implies that

$$d(T_{r'}z', T_r z) = \sup_{t \in \mathbb{R}} |T_{r'}z'(t) - T_r z(t)| < \epsilon \quad ,$$

where we used the notation $T_r z(t) = z(t + r)$ and $T_{r'}z'(t) = z'(t + r')$. Let $\delta < \epsilon/2$. Since

$$\begin{aligned} |T_{r'}z'(t) - T_r z(t)| &= |T_{r'}z'(t) - T_{r'}z(t) + T_{r'}z(t) - T_r z(t)| \\ &\leq |T_{r'}z'(t) - T_{r'}z(t)| + |T_{r'}z(t) - T_r z(t)|, \end{aligned}$$

and $|T_{r'}z'(t) - T_{r'}z(t)| = |z'(t+r') - z(t+r')| < \delta < \epsilon/2$ for any $t \in \mathbb{R}$, we just have to show that it is possible to further decrease $\delta > 0$ such that the following inequality becomes true

$$\sup_{t \in \mathbb{R}} |T_{r'}z(t) - T_r z(t)| = \sup_{t \in \mathbb{R}} |z(t + r') - z(t + r)| = \sup_{t \in \mathbb{R}} |z(t + r' - r) - z(t)| < \epsilon/2.$$

The continuity, monotonicity, and boundness of z imply that z is uniformly continuous. So, it is possible to find the desired δ if we show that the function $z \to r(z)$ is continuous, namely, that for any given $z \in X$ and $\bar{\epsilon} > 0$ there exists a $\bar{\delta} > 0$ such that $d(z', z) < \bar{\delta}$ implies $|r' - r| < \bar{\epsilon}$. In order to prove this we set $\epsilon_1 \stackrel{def}{=} \min\{\bar{\epsilon}, r/2\}$. Notice that z is strictly increasing in the interval $(r - 2\epsilon_1, r + 2\epsilon_1)$, because z is strictly increasing in $(0, \infty)$. Now, we define $\bar{\delta} \stackrel{def}{=} \min\{|z(r - \epsilon_1)|, |z(r + \epsilon_1)\} > 0$. The definitions of ϵ_1 and $\bar{\delta}$ imply that: if $|r - r'| \geq \epsilon_1$ then $d(z, z') \geq |z(r') - z'(r')| = |z(r')| \geq \bar{\delta}$. So, if $d(z, z') < \bar{\delta}$ then $|r - r'| < \epsilon_1 \leq \bar{\epsilon}$, which proves that $z \to r(z)$ is continuous and ends the proof of the proposition. □

Proposition 4 *The mapping A_L is completely continuous, namely, A_L is continuous and maps bounded sets to compact sets (see [2] section 2.2).*

Proof. Since $\Lambda_L \subset C_L$ is bounded and $A_L : \Lambda_L \to \Lambda_L$ is continuous by proposition 3, then, in order to prove that A_L is completely continuous, it is enough to show that the range of

A_L is compact. This is a consequence of the Arzela-Ascoli's theorem if we show that there exists a constant K', independent of $z \in \Lambda_L$, such that

$$|A_L z(t) - A_L z(t')| \leq K'|t - t'| \quad \text{for all} \quad |t| \leq L, |t'| \leq L.$$

The definition of A_L and the fact that $r(z) > 0$ imply that the above inequality is true if there exists a constant K, independent of $z \in X_L$, such that

$$|\underline{A}_L z(t) - \underline{A}_L z(t')| \leq K|t - t'| \quad \text{for all} \quad t > -L, t' > -L. \tag{6}$$

For $|t| < L$, $\underline{A}_L z$ is differentiable and

$$\frac{d}{dt}\underline{A}_L z(t) = -\underline{A}_L z(t) + f(z(t)),$$

which implies

$$\left|\frac{d}{dt}\underline{A}_L z(t)\right| \leq |\underline{A}_L z(t)| + |f(z(t))| \leq 2\max\{\gamma_1, \gamma_2\} \quad . \tag{7}$$

For $t > L$, $\underline{A}_L z$ is explicitly given by

$$\underline{A}_L(t)z = e^{-t}\left\{-e^{-L}\gamma_1 + \int_{-L}^{L} e^s f(z(s))ds + \gamma_2(e^t - e^L)\right\} = e^{-t}\left\{\underline{A}_L z(L) + \gamma_2(e^t - e^L)\right\},$$

which implies that $\underline{A}_L z$ is differentiable and

$$\left|\frac{d}{dt}\underline{A}_L z(t)\right| \leq e^{-t}|\underline{A}_L z(L)| + \gamma_2 \leq 2\gamma_2 \quad . \tag{8}$$

Inequalities (7) and (8) and the continuity of $\underline{A}_L z$ at $t = L$ imply that inequality (6) is true. □

The following proposition is an immediate consequence of the definition of Λ_L.

Proposition 5 *The null function $\underline{0} \in \Lambda_L$ is not a fixed point of A_L.*

Finally, propositions 1, 4 and 5, and the Schauder fixed point theorem (see for instance [2], section 2.2), imply the following lemma.

Lemma 1 *The mapping $A_L : \Lambda_L \to \Lambda_L$ has a fixed point ϕ_L different from $\underline{0}$.*

3 Uniform bounds

Let us denote by r_L the shift that appears in the definition of A_L (5) and that is related to the fixed point ϕ_L given by lemma 1. Our goal in this section is to find bounds, independently of L, for r_L and for the derivative of ϕ_L.

From the definition of A_L (equation (5)), for $|t| \leq L$, we have

$$\phi_L(t) = e^{-t-r_L}\int_{-\infty}^{t+r_L} e^s f(\phi_{L*}(s))ds \quad , \tag{9}$$

where:

$$\phi_{L*}(s) = \phi_L(s) \quad for \quad |s| \le L,$$
$$\phi_{L*}(s) = -\gamma_1 \quad for \quad s < -L,$$
$$\phi_{L*}(s) = \gamma_2 \quad for \quad s > L.$$

Using that $\phi_L(0) = 0$, we can rewrite (9) as

$$\phi_L(t) = e^{-t-r_L} \int_{r_L}^{t+r_L} e^s f(\phi_{L*}(s))ds \quad . \tag{10}$$

We shall find an upper bound for r_L in several steps.

Proposition 6 *There exists $M_1 > 0$ such that if $L > M_1$ then $r_L < L$.*

Proof. Let us assume that $r_L \ge L$. Then, from (10), we obtain that for $t \in [0, L]$

$$\phi_L(t) = e^{-t-r_L}\gamma_2 \int_{r_L}^{t+r_L} e^s ds = \gamma_2(1 - e^{-t}) \quad . \tag{11}$$

Now, using (11), the facts that $|f(z)| \ge |z|$ for $-\gamma_1 \le z \le \gamma_2$, and $\phi_L(0) = 0$, we get

$$\begin{aligned} \gamma_1 &\ge -\int_{-\infty}^{0} e^s f(\phi_{L*}(s))ds = \int_0^{r_L} e^s f(\phi_{L*}(s))ds \\ &\ge \int_0^{r_L} e^s \phi_{L*}(s)ds \ge \int_0^L e^s \gamma_2(1 - e^{-s})ds = \gamma_2[e^L - 1) - L]. \end{aligned}$$

This inequality holds if, and only if, $L \le M_1$, where M_1 is the positive root of

$$\frac{\gamma_1}{\gamma_2} + 1 = e^{M_1} - M_1 \quad .$$

Therefore, if $L > M_1$ then $r_L < L$. □

Proposition 7 *For $L > M_1$ the following two inequalities are true:*

$$\frac{\phi_L(r_L)}{f(\phi_L(r_L))} \ge 1 - e^{-r_L} \quad , \tag{12}$$
$$\frac{\gamma_1}{g(r_L)} \ge \phi_L(r_L) \quad , \tag{13}$$

where $g(r_L) = e^{r_L} - 1 - r_L$.

Proof. From (10), proposition 6 and $0 \le t \le r_L$ we obtain

$$\begin{aligned} \phi_L(t) &= e^{-t-r_L} \int_{r_L}^{t+r_L} e^s f(\phi_{L*}(s))ds \\ &\ge e^{-t-r_L} \int_{r_L}^{t+r_L} e^s f(\phi_{L*}(r_L))ds = f(\phi_L(r_L))[1 - e^{-t}] \quad . \end{aligned} \tag{14}$$

For $t = r_L$ this inequality gives (12). From inequality (14), proposition 6, and $\phi_L(0) = 0$, we obtain

$$\begin{aligned} \gamma_1 &\ge -\int_{-\infty}^{0} e^s f(\phi_{L*}(s))ds = \int_0^{r_L} e^s f(\phi_L(s))ds \\ &\ge \int_0^{r_L} e^s \phi_L(s)ds \ge \int_0^{r_L} e^s f(\phi_L(r_L))(1 - e^{-s})ds \\ &= f(\phi_L(r_L))[e^{r_L} - 1 - r_L] \ge \phi_L(r_L)g(r_L) \quad . \end{aligned}$$

□

The fact that f is continuously differentiable, $f(0) = 0$, and $\frac{df}{dz}(0) = \nu > 1$, imply that there exists $b > 0$ such that

$$\frac{f(z)}{z} > \frac{\nu + 1}{2} \quad \text{for} \quad 0 \le z \le b \quad . \tag{15}$$

The function g appearing in proposition 7 has the following properties:

$$g(0) = 0, \qquad \frac{dg}{dr}(r) > 0 \quad \text{for} \quad r > 0, \quad \lim_{r \to \infty} g(r) = \infty \quad .$$

Therefore, there exists a unique r_* such that $g(r_*) = \gamma_1/b$ and $g(r_L) > \gamma_1/b$, for $r_L > r_*$. This and inequality (13) imply that

$$\phi_L(r_L) \le \frac{\gamma_1}{g(r_L)} < b, \qquad \text{if} \quad r_L > r_* \quad . \tag{16}$$

Now, let r_{**} be the only positive root of

$$\frac{2}{\nu + 1} = 1 - e^{-r_{**}} \quad .$$

This implies that

$$\frac{2}{\nu + 1} < 1 - e^{-r} \qquad \text{if} \quad r_L > r_{**} \quad . \tag{17}$$

Lemma 2 *Let $\overline{r} = \max\{r_*, r_{**}\}$ and $L > M_1$. Then $r_L \le \overline{r}$ independently of L.*

Proof. Let us argue by contradiction assuming that $r_L > \overline{r}$. This and inequality (16) imply that $\phi_L(r_L) < b$. Using (12) and (17) (since $r_L > \overline{r}$) we obtain

$$\frac{\phi_L(r_L)}{f(\phi_L(r_L))} \ge 1 - e^{-r_L} > \frac{2}{\nu + 1} \quad .$$

But this inequality, and the fact that $\phi_L(r_L) < b$, contradict inequality (15). Therefore $r_L \le \overline{r}$. □

Lemma 3 *Let $L > M_1$ and*

$$\underline{r} \stackrel{def}{=} \frac{\gamma_1 e^{-\overline{r}}}{\gamma_1 + \gamma_2} > 0 \quad .$$

Then $\underline{r} \le r_L$, independently of L.

Proof. Since $r_L < L$ (proposition 6) the function ϕ_L is differentiable for $t \in [-L, 0]$. Differentiating expression (9) and using that $f(z) \le z$ for $z \in [-\gamma_1, 0]$ we obtain that, for $t \in [-L, 0]$,

$$\begin{aligned} \dot\phi_L(t) &= -\phi_L(t) + f(\phi_L(t + r_L)) \\ &\le -f(\phi_L(t)) + f(\phi_L(t + r_L)) = \frac{d}{dt}\int_t^{t+r_L} f(\phi_L(s))ds \quad . \end{aligned}$$

Integrating this inequality in the interval $[-L, 0]$, we obtain

$$-\phi_L(-L) \le \int_0^{r_L} f(\phi_L(s))ds - \int_{-L}^{-L+r_L} f(\phi_L(s))ds \le (\gamma_1 + \gamma_2)r_L \quad . \tag{18}$$

Equation (9), the fact that $\phi_L(s) < 0$ for $s < 0$, that $r_L < L$, and lemma 2 imply that

$$\phi_L(-L) = e^{L-r_L}\{-\gamma_1 e^{-L} + \int_{-L}^{-L+r_L} e^s f(\phi_L(s))ds\} \le -e^{-r_L}\gamma_1 \le -e^{-\bar r}\gamma_1 \quad .$$

This and inequality (18) imply the inequality in the lemma. □

Lemma 4 *There exist infinite sequences L_n, r_n, ϕ_n, $n = 1, 2, \ldots$, with $L_n \to \infty$ as $n \to \infty$, such that the limits*

$$r_n \to r > 0, \quad \textit{and} \quad \phi_n \to \phi \quad \textit{as} \quad n \to \infty$$

converge. Moreover, ϕ_n converges uniformly, on compact intervals, to a function ϕ having the following properties:

- *it is continuously differentiable and nondecreasing;*
- *$\phi(0) = 0$;*
- *$-\gamma_1 \le \phi(t) \le \gamma_2$ for $t \in \mathbb{R}$;*
- *it is a solution of the transition layer equation (1).*

Also, $\dot\phi_n$ converges to $\dot\phi$ uniformly on compact intervals.

Proof. Let $L = L_1, L_2, L_3, \ldots$ be an infinite sequence of values of L and r_{L_k}, ϕ_{L_k}, $k = 1, 2, \ldots$ be their corresponding sequences of r_L and ϕ_L. Propositions 2 and 3 imply that the sequence r_{L_k} is bounded from above and below by positive numbers. The sequence ϕ_{L_k} is bounded, $-\gamma_1 \le \phi_{L_k}(t) \le \gamma_2$, $|t| \le k$, and it is equicontinuous (the equicontinuity is a consequence of estimates (7) and (8) that are independent of L and are also valid for $\dot\phi_L$). The remainder of the proof of this lemma involves standard limiting arguments for sub-sequences of ϕ_{L_k} and r_{L_k} using Arzela-Ascoli's theorem and the fact that ϕ_{L_k}, r_{L_k} satisfy the integral identity (9).
□

4 The nontriviality of ϕ

Our goal in this section is to show that the function ϕ obtained in lemma 4 is nontrivial. This is a consequence of the following lemma.

Lemma 5 *There exists $M > 0$ such that at least one of the following inequalities hold:*

$$i) \quad \phi(M) > 0, \qquad ii) \quad \phi(-M) < 0 \quad .$$

We make the following claims:

Claim 1:if $\phi(-M) < 0$ then $\phi(r) > 0$.
Suppose this is false. Then $\phi(t) = 0$ for $t \in [0, r]$, because ϕ is nondecreasing. But this contradicts the fact that ϕ is a solution of equation (1) (lemma 4). Indeed, in this case the theorem of uniqueness of backward continuation of solutions of (1) would imply $\phi(t) = 0$ for all $t < 0$, which is false.

Claim 2: if $\phi(b) > 0$, for some $b > 0$, then $\phi(t) < 0$, for $t < 0$.
In order to show this, let $t_* = \sup\{t|\phi(t) = 0\} \ge 0$. Using that ϕ is a solution of equation (1) we get $\dot\phi(t_*) = f(\phi(t_* + r)) > 0$. Thus, $\phi(t) < 0$ for $t < t_*$, because ϕ is nondecreasing. This and the fact that $\phi(0) = 0$ imply our claim.

Before proving this lemma let us use it to finish the proof of theorem 1. A consequence of lemma 5 and the two claims above is that

$$\phi(-t)\phi(t) < 0 \qquad \text{for all} \quad t \in \mathbb{R}. \tag{19}$$

This, the bounds $-\gamma_1 \le \phi(t) \le \gamma_2$, $t \in \mathbb{R}$, and the integral equation satisfied by ϕ,

$$\phi(t) = \int_{-\infty}^0 e^s f(\phi(s + t + r))ds \quad , \tag{20}$$

imply the limits in the statement of theorem 1, namely

$$\lim_{t\to-\infty} \phi(t) \to -\gamma_1, \quad \lim_{t\to\infty} \phi(t) \to \gamma_2 \quad .$$

Indeed, using that ϕ is nondecreasing we conclude that the limits $\lim_{t\to\pm\infty} |\phi(t)| \stackrel{def}{=} |\phi(\pm\infty)|$ exist and are bounded by $\max\{\gamma_1, \gamma_2\}$. So, we can take limits on both sides of equation (20) to conclude that $\phi(\pm\infty) = f(\phi(\pm\infty))$. This, inequalities $-\gamma \le \phi(-\infty) < 0$ and $0 < \phi(\infty) \le \gamma_2$, and hypothesis (H1) on f (see section 1) imply the above limits.

The only thing remaining in order to complete the proof of theorem 1 is to prove lemma 5. This proof is the content of the rest of the paper.

Let us assume that lemma 5 is false. Then for any $K > 0$

$$||\phi_n||_K = \sup_{|t| \le K \le L_n} |\phi_n(t)| \to 0 \qquad \text{as} \qquad n \to \infty \quad .$$

Let N_K be such that $L_n + r_n \ge K$ for $n > N_K$. For $n > N_K$ we define a sequence of functions $x_n : I_K \to \mathbb{R}$, $I_K \stackrel{def}{=} [-K, K]$, as

$$x_n(t) = \frac{\phi_n(t)}{||\phi_n||_K} \quad .$$

Notice that $||x_n||_K = 1$ and at least one of the identities $x_n(-K) = -1$ or $x_n(+K) = 1$ is true. The function ϕ_n is differentiable for $t \in (-L, L-r)$. Differentiating expression (9) we find that in this interval ϕ_n satisfies

$$\dot{\phi}_n(t) = -\phi_n(t) + f(\phi_n(t + r_n)) \quad .$$

This implies that $x_n : I_K \to \mathbb{R}$, $n > N_K$, are differentiable, satisfy $\dot{x}_n(t) \geq 0$, and also

$$\dot{x}_n(t) = -x_n(t) + \nu x_n(t + r_n) + R(||\phi_n||_K, x_n(t + r_n)), \tag{21}$$

where R is a continuous function such that $R(0, x) = 0$ and, for $\xi \neq 0$,

$$R(\xi, x) \stackrel{def}{=} -\nu x + \frac{f(\xi x)}{\xi} \qquad \text{with} \quad \frac{df}{dx}(0) = \nu > 1 \quad .$$

Equation (21), the definition of R, and $||x_n||_K = 1$ imply that $||\dot{x}_n||_K$ are uniformly bounded for $n > N_K$. This, $||x_n||_K = 1$, the uniform boundness of r_n, and Arzela-Ascoli's theorem imply that there exist sub-sequences r_{n_j}, x_{n_j} that converge to r and x, respectively, as $j \to \infty$. This, equation (21), and the fact that $R(||\phi_n||_K, x(t + r_n)) \to 0$ as $n \to \infty$ uniformly for $|t| \leq K$ (because $||\phi_n|| \to 0$ as $n \to \infty$), imply that x is continuously differentiable and satisfies the linear equation

$$\dot{x}(t) = -x(t) + \nu x(t + r), \qquad for \quad t \in [-K, K - r]. \tag{22}$$

The properties of x_n easily imply that $||x||_K = 1$, $\dot{x} \geq 0$, and $x(0) = 0$. These properties, the fact that x is a solution of equation (22), and an argument similar to the one that lead us to statement (19), imply that $x(-t)x(t) < 0$, for $t \neq 0$, and $\dot{x}(0) > 0$. Let us define the function

$$y(t) \stackrel{def}{=} -x(-t), \qquad t \in [-K, K] \quad .$$

This function satisfies the equation

$$\dot{y}(t) = +y(t) - \nu y(t - r), \qquad for \quad t \in [-K + r, K], \tag{23}$$

and has the following properties:

$$\dot{y} \geq 0, \tag{24}$$
$$y(0) = 0, \tag{25}$$
$$y(-t)y(t) < 0 \quad \text{for} \quad t \neq 0 \quad . \tag{26}$$

The following lemma contradicts our assumption that $K > 0$ can be chosen arbitrarily large, thus proving lemma 5.

Lemma 6 *There exists $M > 0$ such that if $K > M$ then any solution $y : [-K, K] \to IR$ of equation (23) cannot simultaneously satisfy properties (24), (25), and (26).*

In order to prove this lemma we need some definitions from the theory of linear delayed differential equations (see [2], [1]). The characteristic equation related to equation (23) is

$$P(\lambda) \stackrel{def}{=} \lambda - 1 + \nu e^{-r\lambda} = 0 \quad . \tag{27}$$

All the roots of the characteristic equation are on the left hand side of a vertical straight line (c) in the complex plane. The fundamental solution ξ of equation (23) is defined as the one that satisfies $\xi(t) = 0$ for $t < 0$, and $\xi(0) = 1$. For $0 \leq t \leq r$ it is explicitly given by $\xi(t) = e^t$. The Laplace transform of ξ can be easily written in terms of P as

$$\hat{\xi}(u) \stackrel{def}{=} \int_0^\infty e^{-ut}\xi(t)dt = \frac{1}{P(u)} \quad . \tag{28}$$

The function $\hat{\xi}$ is defined for u complex and is analytic on the left hand side of the line (c). Using the inverse integral for the Laplace transform (see [2], [1]) one can show that the fundamental solution has the following integral representation in terms of P

$$\xi(t) = \int_{(c)} \frac{1}{P(\lambda)} d\lambda \quad . \tag{29}$$

Let

$$\eta \stackrel{def}{=} \max\{Re\lambda | P(\lambda) = 0\} \quad . \tag{30}$$

There is at most one pair of complex conjugate roots $\lambda_1, \overline{\lambda_1}$ of (27) (and only one root when λ_1 is real) such that $\mathrm{Re}\lambda_1 = \eta$. In the case that λ_1 is not real, then $\lambda_1, \overline{\lambda_1}$ are simple roots of the characteristic equation (27). Let

$$\eta' \stackrel{def}{=} \max\{Re\lambda | P(\lambda) = 0, \quad \lambda \neq \lambda_1, \quad \lambda \neq \overline{\lambda_1}\} < \eta \quad . \tag{31}$$

Using (29) it can be shown [2], [1] that if $\eta < 0$ then there exist $0 < a < -\eta$, and $b > 0$, such that

$$|\xi(t)| < be^{-at}, \qquad t > 0 \quad . \tag{32}$$

If $\eta \geq 0$ and $\lambda_1 = \eta + \omega i$, $\omega \neq 0$, then there exist constants $a \neq 0$, $b > 0$, $c \in [0, 2\pi)$, and $d \in (\eta', \eta)$, such that

$$|\xi(t) - ae^{\eta t}\cos(\omega t + c)| \leq be^{td}, \qquad t \geq 0 \quad . \tag{33}$$

This estimate is a consequence of formula (29) and the residue theorem (see [1] p. 116, ex.1). For $-K + r \leq t' < t \leq K$ the following "variation of constants formula" (see [2], [1]) is valid

$$y(t) = y(t')\xi(t - t') - \nu \int_{-r}^{0} \xi(t - t' - s - r)y(t' + s)ds \quad . \tag{34}$$

Using this formula and the above properties of ξ we will prove lemma 6. In order to simplify the exposition we break the proof into three propositions.

Proposition 8 *Assume that η defined in (30) satisfies $\eta < 0$ and that equation (23) has a solution y satisfying (25) (26) and such that $\dot{y} \geq 0$ for $t \in [0, r]$. Then there is $M_1 > 0$ such that $y(K) < y(r)$ for all $K > M_1$. In particular y cannot satisfy (24) if $K > M_1$.*

Proof. The variation of constants formula (34) with $t' = r$ and inequality (32) imply

$$\begin{aligned} y(t) &\leq y(r)\left\{|\xi(t-r)| + \nu\int_{-r}^{0}|\xi(t-r-s-r)|ds\right\} \\ &\leq y(r)be^{-a(t-r)}\left\{1+\nu\int_{-r}^{0}e^{a(s+r)}ds\right\} \\ &= y(r)be^{-a(t-r)}\left\{1+\frac{\nu}{a}(e^{ar}-1)\right\}, \end{aligned}$$

where $y(r) > 0$. Now, there is M_1 such that

$$be^{-a(K-r)}\left\{1+\frac{\nu}{a}(e^{ar}-1)\right\} < be^{-a(M_1-r)}\left\{1+\frac{\nu}{a}(e^{ar}-1)\right\} = 1$$

for all $K > M_1$. This implies that $y(K) < y(r)$. This proves the proposition. □

Proposition 9 *Assume that η defined in (30) satisfies $\eta \geq 0$ and that $\lambda_1 = \eta + \omega i$ with $\omega > 0$. Moreover, assume that equation (23) has a solution y satisfying (25) and such that $y(t) < 0$ for $t \in [-K, 0)$. Then there is $M_2 > 0$ such that for all $K > M_2$ there is $t \in (0, K]$ such that $y(t) < 0$. In particular, y cannot satisfy (26) if $K > M_2$.*

Proof. In this case equation (33) implies that

$$|e^{-\eta t}\xi(t) - a\cos(\omega t + c)| \leq be^{-(\eta-d)t} \quad .$$

This equation, the fact that $\eta - d > 0$, and $\xi(t) > 0$ for $t \in [0, r]$, imply that there exists a $t = t_* > r$ such that $\xi(t_*) = 0$ and $\xi(t) > 0$ for $t \in [0, t_*)$. We claim that

$$\xi(t_*) = 0 \quad \Longrightarrow \quad \xi(t) < 0 \quad \text{for} \quad t \in (t_*, t_* + r) \quad . \tag{35}$$

Indeed, ξ satisfies equation (23) implying that $\dot{\xi}(t_*) = -\nu\xi(t_* - r) < 0$. Therefore, $\xi(t)$ is negative in some interval (t_*, δ). If $\xi(\delta) = 0$ and $\delta < t_* + r$ then $\dot{\xi}(\delta) = -\nu\xi(\delta - r) < 0$, which is absurd. So, $\delta \geq t_* + r$ and $\xi(t) < 0$ for $t \in (t_*, t_* + r)$.

Now, let us take $M_2 = t_* + r$ and $K > M_2$. The variation of constants formula (34) with $t' = 0$ and $t = t_* + r$ implies

$$y(t_* + r) = -\nu\int_{-r}^{0}\xi(t_* - s)y(s)ds \quad .$$

Using that $y(s) < 0$ for $s < 0$ and $\xi(t) < 0$ for $t \in (t_*, t_* + r)$, we obtain that $y(t_* + r) < 0$. □

Proposition 10 *Assume that η defined in (30) satisfies $\eta \geq 0$ and that $\lambda_1 = \eta$. Moreover, assume that equation (23) has a solution y satisfying (24) and (25). Then there is $M_3 > 0$ such that $y(-r) \geq 0$ for all $K > M_3$. In particular y cannot satisfy (26) if $K > M_3$.*

Proof.

Let $\zeta : [0, \infty) \to \mathbb{R}$ be the function defined as

$$\zeta(t) \stackrel{def}{=} \xi(t) - \nu\int_{-r}^{0}\xi(t-s-r)ds \tag{36}$$

Supose that there exists $\bar{t} > 0$ such that

$$\zeta(\bar{t}) \leq 0, \quad . \tag{37}$$

Let us take $M_3 = \bar{t} + 2r$ and $K > M_3$. The variation of constants formula (34) with $t' = -\bar{t} - r$ and $t = -r$ implies

$$y(-r) = y(-\bar{t}-r)\xi(\bar{t}) - \nu\int_{-r}^{0}\xi(\bar{t}-s-r)y(-\bar{t}-r+s)ds \quad . \tag{38}$$

Using that y is nondecreasing and that $y(0) = 0$ we obtain that $y(-\bar{t}-r+s) \leq y(-\bar{t}-r) \leq 0$ for $s \in [-r, 0]$. This, equation (38), and (37) imply that

$$y(-r) \geq y(-\bar{t}-r)\left\{\xi(\bar{t}) - \nu\int_{-r}^{0}\xi(\bar{t}-s-r)ds\right\} = y(-\bar{t}-r)\zeta(\bar{t}) \geq 0 \quad .$$

Therefore, in order to finish the proof of this proposition we just have to show that there exists $\bar{t}$ such that (37) is true. This is done in the following.

Let us define the function $\hat{\zeta} : (\eta, \infty) \to \mathbb{R}$ as

$$\hat{\zeta}(u) \stackrel{def}{=} \int_{0}^{\infty}e^{-ut}\zeta(t)dt \quad . \tag{39}$$

This definition and definition (36) of ζ imply that

$$\begin{aligned} \hat{\zeta}(u) &= \int_{0}^{\infty}e^{-ut}\zeta(t)dt \\ &= \int_{0}^{\infty}e^{-ut}\xi(t)dt - \nu\int_{0}^{\infty}\int_{-r}^{0}e^{-ut}\xi(t-s-r)dsdt \\ &= \hat{\xi}(u) - \nu\int_{-r}^{0}\int_{0}^{\infty}e^{-ut}\xi(t-s-r)dtds \\ &= \hat{\xi}(u) - \nu\int_{-r}^{0}e^{-u(s+r)}\int_{-s-r}^{\infty}e^{-ut'}\xi(t')dt'ds \\ &= \hat{\xi}(u)\left\{1 - \frac{\nu(1-e^{-ur})}{u}\right\} = \frac{\hat{\xi}(u)}{u}[u - \nu + \nu e^{-ur}], \end{aligned} \tag{40}$$

where $\hat{\xi}(u)$, $u \in (\eta, \infty)$, is the Laplace transform of ξ restricted to the infinite interval (η, ∞). Equations (28) and (40) imply that

$$\hat{\zeta}(u) = \frac{\hat{\xi}(u)}{u}[u - \nu + \nu e^{-ur}] = \frac{1}{uP(u)}[u - \nu + \nu e^{-ur}]. \tag{41}$$

Notice that

$$P(u) > 0 \qquad \text{for} \quad u > \eta \quad , \tag{42}$$

because η is the largest real root of $P(u) = 0$ and $P(u) \to \infty$ as $u \to \infty$. Using that $P(\eta) = 0$, P is continuous, and $\nu > 1$, we obtain that there is an $\epsilon > 0$ such that

$$u - \nu + \nu e^{-ur} = P(u) - \nu + 1 < 0 \qquad \text{for} \quad u \in (\eta, \eta + \epsilon] \quad . \tag{43}$$

Combining equations (41), (42), and (43) we obtain that $\hat{\zeta}(\eta + \epsilon) < 0$. This and the definition (39) of $\hat{\zeta}$ imply that $\zeta(t)$ must be negative on some interval, which implies the existence of $\bar{t}$ as stated in (37). □

Propositions 8, 9, and 10, exhausts all the possibilities for η. Therefore lemma 6 is proved and so is theorem 1. □

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