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On the Nonrelativistic Limit of the φ^4 Theory in 2+1 Dimensions*

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Abstract

We study the nonrelativistic limit of the quantum theory of a real scalar field with quartic self-interaction. The two body scattering amplitude is written in such way as to separate the contributions of high and low energy intermediary states. From this result and the two loop computation of the self energy correction, we determine an effective nonrelativistic action.

I. INTRODUCTION

It is generally believed that nonrelativistic field theories can be obtained from corresponding relativistic ones as appropriated limits for low momenta. Earlier attempts to the quest of a reliable scheme of nonrelativistic approximation have been based on canonical transformations over the Lagrangians[1]. More recently, there has been proposals of construction of effective Lagrangians by amending the nonrelativistic theories with other interaction terms, representing the effect of the integration of the relativistic degrees of freedom, inspired in the renormalization group spirit[2]. In any case, the goodness of any conceivable approximation will certainly rely on how and how much of the high energy part of the Hilbert space is considered to influence the low energy sector of the theory.

Nonrelativistic field theories in $2+1$ dimensions present many interesting aspects. The simplest of them, $\lambda\phi^4$, shows scale anomaly[3] while the addition of a coupling with a Chern-Simons gauge field has been suggested as a field-theoretical formulation of the Aharonov-

Bohm effect[4] and as an effective theory for the fractional quantum Hall effect[5].

In this paper we discuss the nonrelativistic limit of the relativistic theory of a real scalar field with quartic self-interaction in $2+1$ dimensions. This limit is not trivial. In the corresponding nonrelativistic model[3], the self-energy vanishes identically while in the relativistic theory the lowest order (2 loops) correction is logarithmically divergent. On the other hand, the four point function is logarithmically divergent in the nonrelativistic model while it is finite in the relativistic case.

To better illustrate our procedure, we begin by discussing the two particle scattering. In Sec. 2, we present the model and calculate the 1PI four-point function to one loop order in an approximation, for low external momenta, such that it is possible to know the part of the Hilbert space each contribution comes from. We obtain the leading correction to the dominant nonrelativistic particle–particle scattering amplitude, which coincides with the small $\mid \vec{p} \mid$ expansion of the exact 1–loop amplitude. The two-point function is calculated to two loop order in Sec.3 where we discuss the renormalization of the theory. We present, in Sec.4, a nonrelativistic reduction scheme for the 2-particle scattering amplitude, compare our results with those obtained from the corresponding nonrelativistic model and derive an effective nonrelativistic Lagrangian that accounts for the results up to order $|\vec{p}|^2/m^2$.

II. PARTICLE-PARTICLE AMPLITUDE

We consider a real self-interacting scalar field in $2+1$ dimensions whose Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \mathcal{L}_{c.t.}, \tag{2.1}$$

where $\mathcal{L}_{c.t.}$ is the counterterm Lagrangian needed to fix the mass m, the field intensity and the coupling λ at their renormalized values. Our units are such that $\hbar = c = 1$ and the Minkowski metric has signature (1, -1, -1). The mass dimension of the scalar relativistic field is $1/2$ and that of λ is 1. The quartic self interaction is super-renormalizable and the

degree of superficial divergence of a graph G is given by $d(G) = 3 - \frac{N}{2} - V$, where N and V are the numbers of external legs and vertices respectively. By assuming a Wick ordering prescription in (2.1), the only divergences are those arising from the two loop self-energy diagram.

We are going to calculate the two body amplitude to one loop order in an approximation, for low external momenta, which separates the contributions coming from the low (L) and high (H) energy intermediary states through the introduction of an intermediate cutoff Λ in the $|\vec{k}|$ -integration of the loop momenta. No cutoff is introduced in the integration over k^o. Precisely, the procedure consists of the following steps. Firstly, we use the Feynman identity

$$\frac{1}{ab^n} = \int_0^1 dx \frac{nx^{n-1}}{[(b-a)x + a]^{n+1}} \tag{2.2}$$

and make the necessary change of variables, to put the integrand in a symmetric form. We then integrate over k^0, over the angular part of $\vec{k}$ and perform the parametric integral. The remaining integration over $|\vec{k}|$ is then divided into two parts, corresponding to the low and high energy contributions of the loop integration, by introducing an intermediate cutoff Λ such that $|\vec{p}| << \Lambda << m$ and $\frac{|\vec{p}|}{\Lambda} \simeq \frac{\Lambda}{m}$. In the low energy sector, $0 \leq |\vec{k}| < \Lambda$, we approximate the integrand by expanding it in powers of $\frac{|\vec{p}|}{m}$ and $\frac{|\vec{k}|}{m}$. In the high energy part, $\Lambda < |\vec{k}| < \Lambda_0 (\to \infty)$, relativistic virtual modes are involved and only $\frac{|\vec{p}|}{m}$ can be considered a small quantity. Keeping all contributions up to order η^2 , where $\eta \simeq \frac{|\vec{p}|}{m}$ ($\simeq \frac{|\vec{p}|^2}{\Lambda^2} \simeq \frac{\Lambda^2}{m^2}$), we are able to evaluate the amplitude up to order $\frac{|\vec{p}|^2}{m^2}$, separating the contributions that come from low and high loop momenta. It should be noticed that, since in our prescription the integration over k^0 is unrestricted, locality in time is guaranteed.

It should be remarked that the use of Feynman's parameterization (2.2) is not essential. After the k^0–integration, one could introduce the intermediate cutoff Λ, proceed the approximations for the L and H parts of the loop integration as outlined above, and perform the angular and radial integrations. The possible differences appearing in the outcome of these alternative calculations are physically irrelevant as we shall see later.

The 1PI four-point function to one loop order is given diagrammatically in Fig. 1. Notice

that the last two diagrams do not appear in the nonrelativistic theory where propagation is only forward in time.

Let us initially concentrate in the s-channel amplitude which is given by

$$A_s(p_1,p_2,m,\lambda) = -\frac{\lambda^2}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{(k^2-m^2+i\epsilon)((p_1+p_2-k)^2-m^2+i\epsilon)}. \tag{2.3}$$

Taking the external momenta on shell and working in the center of mass (CM) frame (that is, $\vec{p}_1 = -\vec{p}_2 = \vec{p}$, $\vec{p}_1{}' = -\vec{p}_2{}' = \vec{p}\,'$ and $p_1^0 = p_2^0 = p_1{}'^0 = p_2'^0 = w_p = \sqrt{m^2+\vec{p}^2}$), this amplitude can be written, after performing the k^0 integration using the Cauchy-Goursat theorem and the trivial angular integration, as

$$A_s(|\,\vec{p}\,|,\lambda,m) = \frac{\lambda^2}{32\pi}\int_0^{\Lambda_0^2} d(\vec{k}^2)\frac{1}{w_k}\frac{1}{\vec{p}^2-\vec{k}^2+i\epsilon} \tag{2.4}$$

where $w_k = \sqrt{\vec{k}^2+m^2}$ and Λ_0 is an ultraviolet cutoff that can be made infinity at any time since the graph is finite. The above integral can be calculated exactly and the result expanded for small $\vec{p}^2$ but, as said before, we introduce an intermediate cutoff Λ to distinguish the contributions of low and high momenta.

The low $\vec{k}^2$ contribution to A_s, the integration from 0 to Λ^2, is calculated using the approximation

$$w_k^{-1} = \frac{1}{m}\left[1-\frac{\vec{k}^2}{2m^2}+\frac{3}{8}\frac{\vec{k}^4}{m^4}-\ldots\right], \tag{2.5}$$

and one gets, retaining terms up to $\mathcal{O}(\eta^2)$,

$$A_s^{(L)} \simeq \frac{\lambda^2}{32\pi m}\left\{-\left(1-\frac{\vec{p}^2}{2m^2}\right)\left[\ln\left(\frac{\Lambda^2}{\vec{p}^2}\right)+i\pi\right]+\frac{\vec{p}^2}{\Lambda^2}+\frac{\vec{p}^4}{2\Lambda^4}+\frac{\Lambda^2}{2m^2}-\frac{3\Lambda^4}{16m^4}\right\}. \tag{2.6}$$

Hereafter, the symbol $\simeq$ indicates that the expression which follows holds up to order η^2.

To obtain the high $\vec{k}^2$ contribution, integrating from Λ^2 to Λ_0^2, equation (2.5) can no longer be used but we can still simplify the integrand by taking

$$\frac{1}{\vec{p}^2-\vec{k}^2} = -\frac{1}{\vec{k}^2}\left[1+\frac{\vec{p}^2}{\vec{k}^2}+\frac{\vec{p}^4}{\vec{k}^4}+\ldots\right], \tag{2.7}$$

resulting, to the same η^2 order,

$$A_s^{(H)} \simeq \frac{\lambda^2}{32\pi m}\left\{\left(1-\frac{\vec{p}^2}{2m^2}\right)\ln\left(\frac{\Lambda^2}{4m^2}\right)-\frac{\vec{p}^2}{2m^2}-\frac{\vec{p}^2}{\Lambda^2}-\frac{\vec{p}^4}{2\Lambda^4}-\frac{\Lambda^2}{2m^2}+\frac{3\Lambda^4}{16m^4}\right\}. \qquad (2.8)$$

Although we have made distinct approximations (2.5) and (2.7) in the integrands of the low and high contributions, there exist an exact cancellation of the Λ dependent terms in $A_s^{(L)}$ and $A_s^{(H)}$, order by order in η, as it should. Notice that if we had not used the Feynman's trick or had chosen a distinct routing of the external momenta through the graph, the L- and H-parts of the s-channel amplitude would differ from (2.6) and (2.8) only in the coefficients of the $(|\vec{p}|/\Lambda)^n$ terms.

The nonrelativistic limit of the amplitude (2.4), to subleading order, is given by

$$A_s(|\vec{p}|,\lambda,m) \simeq -\frac{\lambda^2}{32\pi m}\left\{\left(1-\frac{\vec{p}^2}{2m^2}\right)\left[\ln\left(\frac{4m^2}{\vec{p}^2}\right)+i\pi\right]+\frac{\vec{p}^2}{2m^2}\right\}. \qquad (2.9)$$

This is the same result one obtains by evaluating (2.4), without introducing the cutoff Λ, and expanding around $\vec{p}^2=0$.

The t-channel amplitude, in the CM frame with external momenta on shell, can be written after using Feynman identity (2.2), making the substitution $k \to k+qx$ ($q = p-p'$ being the transferred momentum), evaluating the k^0 and the angular integrations and integrating in the Feynman parameter, as

$$A_t(|\vec{p}|,\lambda,m,\theta) = -\frac{\lambda^2}{32\pi}\int_0^{\Lambda_0^2} d(\vec{k}^2)\frac{1}{w_k(\vec{k}^2+\vec{q}^2/4+m^2)}, \qquad (2.10)$$

where $\vec{q}^2 = 2\vec{p}^2(1-\cos\theta)$ and θ is the scattering angle. Proceeding as before, separating the low and high loop momentum contributions to A_t, we get,

$$A_t^{(L)} \simeq -\frac{\lambda^2}{32\pi m}\left\{\frac{\Lambda^2}{m^2}-\frac{3\Lambda^4}{4m^4}\right\} \qquad (2.11)$$

and

$$A_t^{(H)} \simeq -\frac{\lambda^2}{32\pi m}\left\{2-\frac{\vec{q}^2}{6m^2}-\frac{\Lambda^2}{m^2}+\frac{3\Lambda^4}{4m^4}\right\}. \qquad (2.12)$$

We clearly see that the resultant amplitude, given by

$$A_t(|\vec{p}|,\lambda,m,\theta) = -\frac{\lambda^2}{32\pi m}\left\{2-\frac{\vec{p}^2}{3m^2}(1-\cos\theta)\right\}, \qquad (2.13)$$

comes entirely from the high-energy states in the Hilbert space while the s amplitude comes from both low and high momenta. Certainly, this should be expected for the nonrelativistic limit of a diagram that does not even exist in the nonrelativistic theory.

The nonrelativistic limit of the u-channel amplitude is obtained from (2.13) by taking $\theta \to \theta - \pi$, that is $A_u(|\vec{p}|, \lambda, m, \theta) = A_t(|\vec{p}|, \lambda, m, \theta - \pi)$, which corresponds to the exchange of the two particles in the final state. The sum of these two contributions is independent of the scattering angle and given by

$$A_t + A_u \simeq -\frac{\lambda^2}{32\pi m}\left\{4 - \frac{2\vec{p}^2}{3m^2}\right\}. \tag{2.14}$$

This is the parcel of the one loop amplitude that comes from the diagrams that involve virtual pair creation and annihilation only allowed in the relativistic theory. It arises exclusively from the high $\vec{k}^2$ integration but its contribution is greater than the subleading order of (2.9). Although transitions involving relativistic modes have very low probabilities, the summation of the contributions of all virtual relativistic momenta turn out to be significant.

Adding the tree amplitude to (2.9) and (2.14), we obtain the scattering amplitude to one loop order, in our nonrelativistic approximation, as

$$A_{(1)}(|\vec{p}|, \lambda, m) \simeq \lambda - \frac{\lambda^2}{32\pi m}\left\{\left(1 - \frac{\vec{p}^2}{2m^2}\right)\left[\ln\left(\frac{4m^2}{\vec{p}^2}\right) + i\pi\right] + 4 - \frac{\vec{p}^2}{6m^2}\right\} \tag{2.15}$$

We see that the leading correction to the dominant term of the low momenta, nonrelativistic, scattering comes from the high $\vec{k}^2$ loop integrations of t- and u- channels, that are intrinsically relativistic. Notice, again, that the amplitude is finite, though $m >> |\vec{p}|$.

The particle-particle amplitude to one loop order can be exactly calculated in an arbitrary frame and is given by

$$\begin{aligned} A_{(1)} = \lambda - \frac{\lambda^2}{32\pi m}\Bigg\{ & \frac{1}{\sqrt{(p_1+p_2)^2/4m^2}}\left[\ln\left(\frac{\sqrt{(p_1+p_2)^2/4m^2}+1}{\sqrt{(p_1+p_2)^2/4m^2}-1}\right) + i\pi\right] \\ & + \left[\frac{1}{\sqrt{(p_1-p_1')^2/4m^2}}\ln\left(\frac{1+\sqrt{(p_1-p_1')^2/4m^2}}{1-\sqrt{(p_1-p_1')^2/4m^2}}\right) + (p_1' \to p_2')\right]\Bigg\}, \end{aligned} \tag{2.16}$$

where p_i and p_i' are the external on shell incoming and outgoing momenta. Taking this expression in the CM frame and expanding for small $|\vec{p}_i|^2$, it reduces to (2.15), showing

that our approximation reproduces the small momentum expansion of the exact result up to order $\vec{p}^2/m^2$. Therefore, although apparently unnecessary in the ϕ^4 theory where one has exact results, our approximation procedure can be applied with confidence in other theories where, eventually, exact analytical calculations can not be done. It should also be noticed that the separation of the contributions of low and high energy intermediate states to the amplitude can be effectuated in the general situation but, for simplicity, we made it in the CM frame.

III. PARTICLE SELF-ENERGY

With normal ordering imposed to the Lagrangian (2.1), the first nonvanishing contribution for the two point function comes from the two loop diagram of Fig.2. This is the unique primitive divergent diagram of $\lambda\phi^4$ theory in $2+1$ dimensions. This type of diagram and, actually, all the self-energy insertions are not allowed in the nonrelativistic theory where the propagation is always forward in time. There, the physical mass is the natural parameter and the full propagator coincides with the free one. We shall calculate this two loop self-energy exactly and also by the same procedure of separating low and a high internal momentum contributions, as we did before. In this case, however, the ultraviolet cutoff Λ_0 can not be made infinity before a subtraction because the graph is logarithmically divergent. We shall see that both the finite part and the divergent one come from the region of integration where both loop momenta are high.

Before trying to calculate any of the k or l integrations we completely disentangle these variables by proceeding as follows. We apply (2.2) to the k-loop and make the substitution $k \to k + lx$. We then repeat (2.2), rescale the variables such that $k \to k/\sqrt{y}$ and $l \to l/\sqrt{C}$ and perform the translation $l \to l + \frac{(1-y)}{\sqrt{C}}p$ to obtain

$$\hat{\Sigma}_{(2)}\,(p, m, \lambda, \Lambda_0) = \frac{i\lambda^2}{192\pi^6}\int_0^1 dx \int_0^1 dy \frac{y}{[yC]^{3/2}} \int d^3k d^3l \frac{1}{\left[k^2 + l^2 + Dp^2 - m^2 + i\epsilon\right]^3} \qquad (3.1)$$

where $C = C(x,y) = yx(1-x) + (1-y)$ and $D = D(x,y) = (1-y)\,(1-(1-y)/C(x,y))$. It can be shown that these functions satisfy $0 \leq C \leq 1$ and $0 \leq D \leq 1/9$ for all $0 \leq x, y \leq 1$.

Owning to the form of the integrand of (3.1), the (k^0, l^0) integration can be exactly done using polar coordinate with $r = \sqrt{k^{0\,2} + l^{0\,2}}$ and $\alpha = \arctan\left(\frac{l^0}{k^0}\right)$ resulting, after trivial angular integration, in

$$\hat{\Sigma}_{(2)}\,(p, m, \lambda, \Lambda_0) = \frac{i\lambda^2}{384\pi^3}\int_0^1 dx \int_0^1 dy \frac{y}{[yC]^{3/2}} F(p^2, m^2, x, y, \Lambda_0) \tag{3.2}$$

where

$$F(p^2, m^2, x, y, \Lambda_0) = \int_0^{\Lambda_0^2} d(\vec{k}^2) \int_0^{\Lambda_0^2} d(\vec{l}^2) \frac{1}{\left[\vec{k}^2 + \vec{l}^2 + m^2 - p^2 D(x,y) - i\epsilon\right]^2}\,. \tag{3.3}$$

The $\vec{k}^2$ and $\vec{l}^2$ integrations above can be exactly performed giving, for $\Lambda_0 >> m$,

$$F(p^2, m^2, x, y, \Lambda_0) = \ln\left(\frac{\Lambda_0^2}{m^2}\right) - \ln\left[2\left(1 - \frac{p^2}{m^2} D(x,y)\right)\right]. \tag{3.4}$$

To see from where this contribution comes, as we did before, we introduce the same intermediate cutoff Λ^2 for both $\vec{k}^2$ and $\vec{l}^2$ integrations dividing the $(\vec{k}^2, \vec{l}^2)$ quadrant into four parts denoted, in a self-explained notation, as $L-L$, $L-H$, $H-L$ and $H-H$. We then obtain the contributions for F coming from each of these parts as

$$F_{L-L} \simeq \frac{\Lambda^4}{b^2 m^4} \tag{3.5}$$

$$F_{L-H} = F_{H-L} \simeq \frac{\Lambda^2}{bm^2} - \frac{3\Lambda^4}{2b^2 m^4} \tag{3.6}$$

$$F_{H-H} \simeq -\frac{2\Lambda^2}{bm^2} + \frac{2\Lambda^4}{b^2 m^4} - \ln 2 - \ln b + \ln\left(\frac{\Lambda_0^2}{m^2}\right), \tag{3.7}$$

where $b(p^2/m^2, x, y) = 1 - D(x,y)\,p^2/m^2 - i\epsilon$. This clearly shows that $\hat{\Sigma}_{(2)}$ comes entirely from the high, relativistic, virtual states, as it is expected.

Inserting (3.4) in (3.2), the cutoff regulated two loop self-energy, is given by

$$\hat{\Sigma}_{(2)}\,(p, m, \lambda, \Lambda_0) = \frac{i\lambda^2}{192\pi^2}\left[\ln\left(\frac{\Lambda_0^2}{m^2}\right) - E\left(\frac{p^2}{m^2}\right)\right], \tag{3.8}$$

where the function $E(z)$ is defined by

$$E(z) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \frac{y \ln\left(2\left[1 - zD(x,y)\right]\right)}{[yC(x,y)]^{3/2}}. \tag{3.9}$$

Notice that, the procedure described just above (3.1) can be readily extended to any dimension yielding a much easier computation of the "sunset"graph of Fig.2, in the general case. Also, if one treats (3.1) by dimensional regularization, one gets

$$\hat{\Sigma}^{(\dim)}_{(2)}(p,m,\lambda,d) = \frac{i\lambda^2}{192\pi^2}\left[\frac{1}{3-d} + (\ln 2 - \gamma) - E\left(\frac{p^2}{m^2}\right)\right], \tag{3.10}$$

so that by making a minimal subtraction, the finite part obtained differs from that of (3.8) by a constant term, as it should.

The mass and wave function renormalization program can now be implemented . The full propagator is given by $G_R(p^2) = i(p^2 - m^2 - i\Sigma)^{-1}$, where $\Sigma = \hat{\Sigma} + i(Z-1)(p^2-m^2) - i\delta m^2 Z$. Particle interpretation of the theory requires that the complete propagator G_R has a pole of residue i at $p^2 = m^2$ which implies that

$$\delta m^2 = m^2\left(\frac{\lambda^2}{192\pi^2 m^2}\right)\left[\ln\left(\frac{\Lambda_0^2}{m^2}\right) - E(1)\right] \tag{3.11}$$

and

$$Z = 1 + \left(\frac{\lambda^2}{192\pi^2 m^2}\right)\left.\frac{\partial E}{\partial z}\right|_{z=1}, \tag{3.12}$$

up to λ^2 order. As we saw in the last section, the four-point function is finite so that no coupling constant renormalization is necessary.

IV. NONRELATIVISTIC REDUCTION

The approximation we have used, introducing an intermediate cutoff Λ , not only permits the identification of the origin (in the Hilbert space) of each contribution, but it also allows the construction of a nonrelativistic reduction scheme at the level of the Green's functions.

Adding separately the low and high energy contributions of each channel to the scattering amplitude, in an arbitrary reference frame but for external nonrelativistic particles on the mass shell, one obtains (up to order $\frac{|\vec{p}|^2}{m^2}$)

$$A^{(L)}_{(1)} \simeq \lambda + \frac{\lambda^2}{32\pi m}\left\{-\left(1 - \frac{(\vec{p}_1-\vec{p}_2)^2}{8m^2}\right)\left[\ln\left(\frac{4\Lambda^2}{(\vec{p}_1-\vec{p}_2)^2}\right) + i\pi\right] + \frac{(\vec{p}_1-\vec{p}_2)^2}{4\Lambda^2} + \frac{(\vec{p}_1-\vec{p}_2)^4}{8\Lambda^4} - \frac{3\Lambda^2}{2m^2} + \frac{21\Lambda^4}{16m^4}\right\} \tag{4.1}$$

and

$$A^{(H)}_{(1)} \simeq \frac{\lambda^2}{32\pi m}\left\{\left(1-\frac{(\vec{p}_1-\vec{p}_2)^2}{8m^2}\right)\ln\left(\frac{\Lambda^2}{4m^2}\right)-\frac{(\vec{p}_1-\vec{p}_2)^2}{4\Lambda^2}\right.$$
$$\left.-\frac{(\vec{p}_1-\vec{p}_2)^4}{8\Lambda^4}+\frac{3\Lambda^2}{2m^2}-\frac{21\Lambda^4}{16m^4}-4+\frac{(\vec{p}_1-\vec{p}_2)^2}{24m^2}\right\}. \qquad (4.2)$$

One should naturally expects that the low $|\vec{k}|$ contribution expresses, to some extent, the scattering amplitude obtained from a nonrelativistic (NR) theory although the arbitrariness in the introduction of the intermediate cutoff prevents any straightforward identification.

We must note that these amplitudes were calculated from a relativistic theory in which the states are normalized as $\langle \vec{p}\,'|\vec{p}\rangle = 2w_p\delta^3(\vec{p}\,'-\vec{p})$. On the other hand, the usual normalization of states in a nonrelativistic theory does not have the $2w$ factor. Thus, for the purpose of comparison, it is necessary to multiply our results by

$$(16w_{p1}w_{p2}w_{p1'}w_{p2'})^{-1/2} = \frac{1}{4m^2}\left(1-\frac{\vec{p}_1^{\,2}+\vec{p}_2^{\,2}}{2m^2}+...\right) \qquad (4.3)$$

Let us initially analyze the above expressions for the amplitude up to the dominant order of the 1-loop correction, that is, let us consider

$$A^{(L)} = \frac{A^{(L)}_{(1)}}{4m^2} = \frac{\lambda}{4m^2}-\frac{\lambda^2}{128\pi m^3}\left[\ln\left(\frac{4\Lambda^2}{(\vec{p}_1-\vec{p}_2)^2}\right)+i\pi\right] \qquad (4.4)$$

and

$$A^{(H)} = \frac{A^{(H)}_{(1)}}{4m^2} = \frac{\lambda^2}{128\pi m^3}\ln\left(\frac{\Lambda^2}{4m^2}\right). \qquad (4.5)$$

One can see that equation (4.4) coincides with the result from the nonrelativistic theory specified by the Lagrangian density

$$\mathcal{L}^{NR} = \phi^\dagger\left(i\partial_t+\frac{\nabla^2}{2m}\right)\phi-\frac{v_0}{4}(\phi^\dagger\phi)^2, \qquad (4.6)$$

with $v_o = \lambda/4m^2$ (compare with equation (2.13) of ref. 3), if Λ is reinterpreted as a genuine ultraviolet cutoff. Such an interpretation, however, can only be sustained after performing a nonrelativistic reduction procedure as follows. First, notice that, neglecting terms of order $\eta \simeq |\vec{p}|/m$ or higher in the self-energy, one gets $\hat{\Sigma}_{(2)L-L}= 0$ $(=\hat{\Sigma}_{(2)L-H})$ showing that

in this case the low energy contribution for $\hat{\Sigma}_{(2)}$ vanishes identically. This agrees with the nonrelativistic result where there is no radiative correction at all to the propagator. We then fix the parameter m and promote Λ to be the ultraviolet cutoff of the reduced nonrelativistic theory. This last step is the fundamental reinterpretation required for our reduction process. It produces an unrenormalized logarithmic divergent four-point function as one has in the nonrelativistic theory (4.6).

The above nonrelativistic reduction of the leading term of the L-contribution to the two particle scattering amplitude is equivalent to the $m \to \infty$ limit effectuated on the classical Lagrangian[3,6] and it is also reproduced by making a Foldy-Wouthuysen transformation in the free part of the Lagrangian (2.1)[7]. An interesting aspect of this reduction procedure is that the contribution of high energy states appears providing the necessary counterterm to make the amplitude finite instead of logarithmic divergent. The divergence produced in the low energy contribution (4.4) would be naturally compensated by the high part, if the full, relativistic, theory were considered. One can, in this way, better understand the renormalization of the nonrelativistic model of ref. 3.

It is worthwhile to mention that, one could naively think that the divergence of the 2-particle amplitude calculated with the nonrelativistic theory is due to the complete exclusion of the propagation backwards in time. In fact, if one splits the relativistic propagator as a sum of its positive (particle) and negative (antiparticle) frequency parts,

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{1}{2w_k}\frac{i}{k^0 - w_k + i\epsilon} + \frac{1}{-2w_k}\frac{i}{k^0 + w_k + i\epsilon} \quad , \tag{4.7}$$

the relativistic CM s-channel amplitude can be written as $A_s = A_s^+ + A_s^-$ where

$$A_s^{\pm} = \frac{\lambda^2}{32\pi}\int_0^{\Lambda_0^2} d(\vec{k}^2)\frac{1}{2w_k^2}\frac{1}{w_p \mp w_k + i\epsilon} \quad ,$$

with the superscripts $+$ and $-$ denoting the contributions of the particle-particle and the antiparticle-antiparticle propagations, respectively. As one immediately sees, both of these parcels are finite when $\Lambda_0 \to \infty$. Naturally, one can introduce an auxiliary cutoff to separate the L- and the H- contributions for each part, and doing so, one finds that the particle

propagation contribution is dominant, as expected.

Let us now examine the sub-dominant order. Disregarding constant terms which can be absorbed into a coupling constant renormalization, the low energy part is

$$A^{(L)} = \frac{\lambda}{4m^2} - \frac{\lambda}{4m^2}\frac{(\vec{p}_1+\vec{p}_2)^2+(\vec{p}_1-\vec{p}_2)^2}{4m^2} - \frac{\lambda^2}{128\pi m^3}\left[1-\frac{2(\vec{p}_1+\vec{p}_2)^2+3(\vec{p}_1-\vec{p}_2)^2}{8m^2}\right]\left[\ln\left(\frac{4\Lambda^2}{(\vec{p}_1-\vec{p}_2)^2}\right)+i\pi\right]. \tag{4.8}$$

To reproduce the new terms appearing in this expression, we add to (4.6) the effective interaction Lagrangian

$$\mathcal{L}^{NR}_{int} = \frac{v_1}{4}\left(\phi^\dagger\frac{(\nabla^2\phi^\dagger)}{m^2}\phi^2+\frac{(\vec{\nabla}\phi^\dagger)^2}{m^2}\phi^2\right)+\frac{v_2}{4}\left(\phi^\dagger\frac{(\nabla^2\phi^\dagger)}{m^2}\phi^2-\frac{(\vec{\nabla}\phi^\dagger)^2}{m^2}\phi^2\right) \tag{4.9}$$

which is the more general, dimension 6, quadrilinear local nonrelativistic interaction. For the calculation of the contributions arising from these new vertices we will have to introduce ultraviolet cutoffs. It is easily verified that the polynomial part of the result is cutoff dependent and this freedom can be used to adjust it to match the polynomial part of (4.8). For that reason, we restrict the discussion to the non-polynomial part of the additional contribution which, in one loop order and up to $\mathcal{O}(\vec{p}^2/m^2)$, is (again, disregarding constant terms)

$$-\frac{m}{8\pi}v_0\left(v_1\frac{(\vec{p}_1+\vec{p}_2)^2}{m^2}+v_2\frac{(\vec{p}_1-\vec{p}_2)^2}{m^2}\right)\left[\ln\left(\frac{4\Lambda^2}{(\vec{p}_1-\vec{p}_2)^2}\right)+i\pi\right]. \tag{4.10}$$

Comparing with (4.8) we find

$$v_1 = -\frac{\lambda}{16m^2} \qquad \text{and} \qquad v_2 = -\frac{3\lambda}{32m^2}, \tag{4.11}$$

which fixes the effective nonrelativistic Lagrangian up to the order $\vec{p}^2/m^2$. As before, the high energy part furnishes theo counterterm needed to make the amplitude finite.

V. CONCLUSIONS

In this work we discussed the nonrelativistic limit of the 2+1 dimensional ϕ^4 theory by means of a scheme which separates the contributions from the high and low virtual momenta

in the loop integrations. This method provides a systematic way of extracting different orders in $|\vec{p}|/m$ in the nonrelativistic approximation and can be applied to more general situations. Proceeding along these lines, we were able to derive an effective Lagrangian which, up to order $|\vec{p}^2|/m^2$, correctly reproduces the nonrelativistic limit. The interaction Lagrangian so obtained is equivalent, in the leading order, to the quantum mechanical delta function potential. The new terms arising in the subleading order, however, can not be interpreted in terms of a two body potential.

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FIGURES

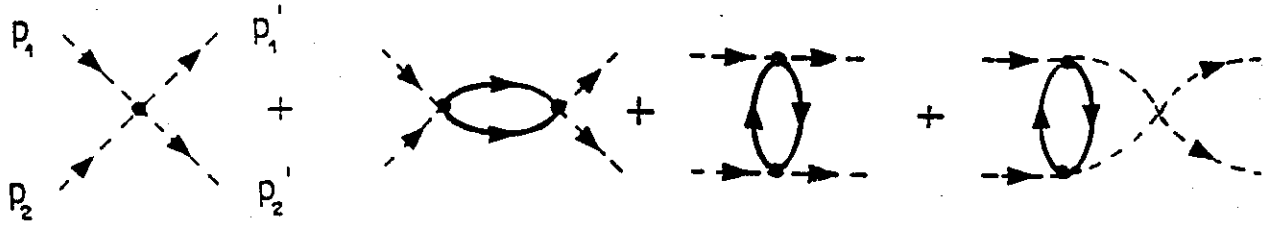

Fig 1: The 1PI four point function to one loop order.

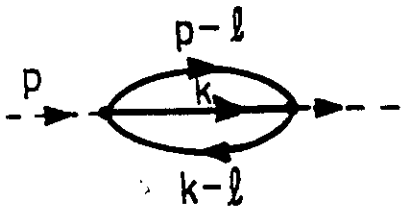

Fig 2: Two loop self–energy.