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GENERALIZED CORRELATION FUNCTIONS FOR THE $\lambda\phi^4$ MODEL

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Pág. 1-18

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Abstract

N-point correlation functions for the $\lambda\phi^4$ model are built over a gaussian approximation at zero and finite temperature. This is done by introducing different small amplitude external sources corresponding to perturbations to different parameters of the model like, for example, the mass, "current" and condensate $\bar{\phi}(\mathrm{x}, t)$. It is then possible to calculate the response of the system. This calculation is performed on both phases of the potential taking into account corrections to the "mean field" approximation for the quantum fluctuations. Final expressions, mainly for the two and four point Green's functions, are calculated analytically or semi-analytically. These functions correspond to response functions of classical and quantum parts of the field to the corresponding external perturbations. Renormalization is performed on both phases of the potential.

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1 Introduction

The calculation of n- point correlation functions is the aim of many theoretical efforts since they are directly related to important observables, specially the 2 and 4 point ones. In functional quantum field theory, they can be calculated from the functional generator with the introduction of an external source in terms of which the effective action is obtained and with relation to which the generator is derived resulting the wanted Green's function, see, for composite operators in the static case, [1]. This is not exactly the general idea which will be develop, but rather a RPA type approximation. In fact, Green's functions can be obtained by calculating the response function of a system to external probes. This is directly related to the fundamental idea of quantum physics that the measurement of observables is performed in such a way to perturb the system externally. The physical response of a system to external perturbations $\hat{O}$ is characterized by correlation functions at different (or equal) times, t_i, of the form $< \psi|O_2(t_2)O_1(t_1)|\psi >$ [2].

The $\lambda\phi^4$ model has been extensively studied since "it seems to be" a simple model and, in principle, it can be used as a probe to theoretical approachs. However, there are renormalization problems which make it intriguing [3]. We are particularly interested on the time dependent gaussian approach to the $\lambda\phi^4$ which is non perturbative and has been well developped. It has allowed a large range of interesting results [4, 5, 6, 7, 8, 9, 10].

The aim of this article is to exhibit a method for the calculation of generalized correlation functions going beyond the Hatree Bogoliubov (gaussian) approach at zero and non zero temperatures. This approximation can also be thougth as variational and it keeps more non-linearities than perturbative methods corresponding to an infinite sum of "cactus type" diagrams. It is exact in the free field limit as well as in the limits of $N \to \infty$ and $N = 1$ of an $O(N)$ invariant model [4]. We explore both the symmetric($\bar{\phi} =< \phi >= 0$) and asymmetric phases ($\bar{\phi} \neq 0$) calculating general response functions at zero and finite temperature to several general external sources corresponding to perturbations to the parameters of the model like mass, "current" and condensate. One of the four- point correlation functions is directly related to the T- matrix, which has been calculated in the symmetric phase in [10]. It is even possible to define " three- point correlation functions " for this model in the asymmetric phase; their meaning will be clarified in the text. The coupling constant renormalization is performed on both phases and the bare coupling constant has to go to zero while the cut- off is sent to infinity in order to keep the renormalized coupling constant finite in agreement with [3, 10]. In section 2 we present the basis of the time dependent variational method for zero and non zero temperature case. In section 3 the calculation of the generalized correlation functions is performed and in the following section the renormalization as well as some properties are shown. The conclusions are discussed in section 5.

2 Time dependent Hartree Bogoliubov approach at finite temperature

The lagrangian for a scalar field ϕ with bare mass m_0^2 and quartic coupling constant b is given by

$$\mathcal{L}(\mathbf{x}) = \frac{1}{2}\left\{\partial_\mu\phi(\mathbf{x})\partial^\mu\phi(\mathbf{x}) - m_0^2\phi(\mathbf{x})^2 - \frac{b}{12}\phi(\mathbf{x})^4\right\} \tag{1}$$

Thus the corresponding hamiltonian reads

$$H = \frac{1}{2}\left(\pi^2(\mathbf{x}) + (\nabla\phi)^2 + m_0^2\phi^2(\mathbf{x}) + \frac{b}{12}\phi^4(\mathbf{x})\right) \tag{2}$$

where the action of the canonical momentum operator, π, in the functional Schrodinger representation is $-i\delta/\delta\phi(\mathbf{x})$.

The functional state is the density matrix, ρ, which trial fom is given by

$$\begin{aligned}\rho[\phi_1,\phi_2] = \; & Nexp\left\{-\frac{1}{2}\int d\mathbf{x}d\mathbf{y}\,(\delta\phi_1(\mathbf{x})A(\mathbf{x},\mathbf{y})\delta\phi_1(\mathbf{y}) + \delta\phi_2(\mathbf{x})A^*(\mathbf{x},\mathbf{y})\delta\phi_2(\mathbf{y}) + \right.\\ & \left. 2\delta\phi_1(\mathbf{x})B(\mathbf{x},\mathbf{y})\delta\phi_2(\mathbf{y})) + i\int d\mathbf{x}\bar{\pi}(\mathbf{x})\,(\phi_1(\mathbf{x}) - \phi_2(\mathbf{x}))\right\}\end{aligned} \tag{3}$$

Where $\delta\phi_i(\mathbf{x},t) = \phi_i(\mathbf{x}) - \bar{\phi}(\mathbf{x},t)$, the normalization is N, the variational parameters are the condensate $\bar{\phi}(\mathbf{x}) =< \hat{\phi} >$ and its conjugated variable $\bar{\pi}(\mathbf{x},t) =< \pi >$; quantum fluctuations are represented by the other parameters, being A symmetric and B hermitian (considered to be real) because the density matrix itself is hermitian. B is the amount of mixing of states in ρ, its non zero value means that the system is not at zero temperature, it can be thermalized or not.

In the gaussian approximation at zero temperature the density matrix is composed by pure states and it can be written in terms of the wavefunctionals Ψ so that:

$$\rho[\phi_1,\phi_2] = \Psi^*[\phi_1]\,\Psi[\phi_2] \tag{4}$$

The mean value of an operator $\hat{O}$ was taken to be

$$< \hat{O} >= \int \mathcal{D}[\phi_i]\,\hat{O}\rho[\phi_1,\phi_2] \tag{5}$$

The density matrix obbeys the Liouville equation which, for non explicit time dependence, reads

$$i\frac{\partial\rho}{\partial t} = [H,\rho]\,. \tag{6}$$

In the following we use some mean values, for instance:

$$\begin{aligned}&< \phi(\mathbf{x})_i\phi(\mathbf{x})_j >= \frac{1}{2(A_R+B_R)_{i,j}}(\mathbf{x},b x) + \frac{b}{2}\bar{\phi}_i\bar{\phi}_j\\ &< \pi(\mathbf{x})_i\pi(\mathbf{x})_j >= \frac{1}{2}(A_R-B_R)_{i,j}(\mathbf{x},\mathbf{x}) + \bar{\pi}_i(\mathbf{x})\bar{\pi}_j(\mathbf{x}) + \left(A_I.\frac{1}{2(A_R+B_R)}.A_I\right)_{i,j}(\mathbf{x},\mathbf{x})\\ &< \phi_i(\mathbf{x})\pi_j(\mathbf{x}) + \pi_i(\mathbf{x})\phi_j(\mathbf{x}) >= -A_I.\frac{1}{2(A_R+B_R)}_{i,j}(\mathbf{x},\mathbf{x}) - \frac{1}{2(A_R+B_R)}.A_{I i,j}(\mathbf{x},\mathbf{x}) +\\ &-\bar{\phi}_i(\mathbf{x})\bar{\pi}_j(\mathbf{x}) - \bar{\pi}_i(\mathbf{x})\bar{\phi}_j(\mathbf{x})\end{aligned} \tag{7}$$

In order to keep the Liouville equation, we deduce the equations for the real and imaginary parts of the variational parameters. They read (repeated index mean integration in that variable)

$$\begin{aligned}&\partial_t A_R(\mathbf{x},\mathbf{y}) = (A_I(\mathbf{x},\mathbf{z})A_R(\mathbf{z},\mathbf{y}) + A_R(\mathbf{x},\mathbf{z})A_I(\mathbf{z},\mathbf{y}))\\ &\partial_t A_I(\mathbf{x},\mathbf{y}) = -\left(-A_I^2(\mathbf{x},\mathbf{y}) - B_R^2(\mathbf{x},\mathbf{y}) + A_R^2(\mathbf{x},\mathbf{y}) - \Gamma(\mathbf{x},\mathbf{y})\right)\\ &\partial_t B_R(\mathbf{x},\mathbf{y}) = 2\,(A_I B_R + B_R A_I)\\ &\partial_t\bar{\phi}(\mathbf{x}) = -\bar{\pi}(\mathbf{x})\\ &\partial_t\bar{\pi}(\mathbf{x}) = \Gamma(\mathbf{x},\mathbf{y})\bar{\phi}(\mathbf{y}) - \frac{b}{3}\bar{\phi}^2(\mathbf{x})\end{aligned} \tag{8}$$

Where ∂_t means time derivative and $\Gamma(\mathbf{x},\mathbf{y}) = -\Delta + \left(m_0^2 + \frac{b}{2}G(\mathbf{x},\mathbf{x}) + \frac{b}{2}\bar{\phi}(\mathbf{x})^2\right)\delta(\mathbf{x}-\mathbf{y})$. In this approximation the interaction term $b\phi^4$ becomes quadratic, i.e., it contributes to a self consistent mass. This procedure yields the same equations of mouvement as those obtained in [8] using the Balian Vénéroni method [11] where one has to consider also a prescription for the observable itself and then to perform variations with relations to the parameter of this new variational functional.

In order to keep contact to usual gaussian approach at zero temperature or not [8, 9], we identify the following variables:

$$\begin{aligned}A_R &= G^{-1}/2\\ A_I &= 2\Sigma\\ B_R &= G^{\frac{-1}{2}}\xi^1 G^{\frac{-1}{2}}\end{aligned} \tag{9}$$

For a translational invariant system, we perform a Fourier transformation like

$$D(\mathbf{x},\mathbf{y}) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\,D(k,k')\exp(i\mathbf{k}.\mathbf{x} - i\mathbf{k}'.\mathbf{y}) \tag{10}$$

Now we can rewrite the equations in the following form in which the variables are simultaneously diagonalized

$$
\begin{aligned}
\dot{G} &= 2G_k\Sigma_{k'} + 2G_{k'}\Sigma_k \\
\dot{\Sigma} &= -2\Sigma_k\Sigma_{k'} + \frac{1-\xi^2}{8}G_k^{-1}G_{k'}^{-1} - 2\left(k^2 + m_0^2 + \frac{b}{2}H(x,x) + \frac{b}{2}\bar{\phi}\right) \\
\ddot{\bar{\phi}}_k &= -\left(k^2 + m_0^2 + \frac{b}{6}\bar{\phi}^2 + \frac{b}{2}H(x,x)\right)\bar{\phi}_k
\end{aligned} \tag{11}
$$

Where

$$
H(\mathbf{x},\mathbf{x}) = \int d^3p \frac{G(\mathbf{p})}{1-\xi} \tag{12}
$$

With the above considerations we get $\dot{\xi} = 0$ for the third equation of (8), concernig B_R. This means that given a thermal state at an initial time, it keeps its thermal characteristics. These equations, at the limit $\xi = T = 0$, correspond to the exact equations of motion of the $O(N)$ model in the limit of large N and when $N = 1$ with a scale transformaion [4].

There is no exact analytical solution but several numerical approachs have been worked out for zero and non zero temperature case [5, 12, 6, 7, 8, 9].

Taking the static limit $\dot{G} = \dot{\Sigma} = \Sigma$ we get the GAP equations for non zero temperature cases, where the departure of this last state is done by the ξ parameter. We will adopt the prescription $\xi = (cosh\beta\omega_k)^{-1}$ and $A_R = \omega_k coth(\beta\omega_k)$, where β is the inverse of the temperature, remembering that we are using natural units such that the Boltzmann constant is 1 ($k_B = 1$) and $\omega_k = \sqrt{k^2+\mu^2}$ with the physical mass μ^2. Then we arrive at an equilibrium finite temperature configuration. Still in this case the Green's function $< \phi^2 >$, in the symmetric phase, can be written as

$$
< \phi^2 >_k = \frac{1 + coth\left(\frac{\beta\omega}{2}\right)}{\omega} = \frac{1}{\omega_k}\left(\frac{1}{2} + f(k)\right) \tag{13}
$$

where $f(k)$ is the Bose- Einstein occupation number. To make contact with zero temperature case, ($\xi = 0$), we have $2\omega_k = G_k^{-1}$. The above expression diverges and the renormalization will be done in section 4 for the zero temperature limit.

3 External sources and response functions

N-point Green's functions may be related to the quantum response functions of the system to external sources. We will introduce, in a general way, several sources as supplementary terms in the equation of motion. This corresponds to consider the same terms in the action which yields the Liouville equation

given in [11, 8]. The small amplitude external source terms are given by

$$
\begin{aligned}
\delta S_{ext} = & \int dt \int d\mathbf{x}d\mathbf{y} \left\{ \phi(\mathbf{x})K(\mathbf{x},\mathbf{y})\phi(\mathbf{y}) + (\phi(\mathbf{x})L(\mathbf{x},\mathbf{y})\pi(\mathbf{y}) + \pi(\mathbf{x})L(\mathbf{x},\mathbf{y})\phi(\mathbf{y})) + \right. \\
& \left. \pi(\mathbf{x})M(\mathbf{x},\mathbf{y})\pi(\mathbf{y}) \right\} + \int dt \int d\mathbf{x} \left\{ \phi(\mathbf{x})J(\mathbf{x}) + \bar{\pi}(\mathbf{x})J_\pi(\mathbf{x}) \right\}
\end{aligned} \tag{14}
$$

The sources K, L, M, J_π and J correspond to perturbations in the mass, current and condensate. They cause dynamical deviations from their values on the minimum of the potential for the variational parameters $G, \Sigma, \bar{\pi}$ and $\bar{\phi}$, but not on the ξ parameter which remains constant in the prescription we have adopted. The L term comes from a Bogoliubov transformation and its issues will be analysed bellow.

For very small amplitude perturbations the system goes away from the state of minimum of the potential which is characterized by G_0, ξ and $\bar{\phi}_0$. Then, we consider the equations of motion (11) around the state of minimum with $G(\mathbf{x},t) = G_0(\mathbf{x}) + \delta G(t,\mathbf{x})$, $\Sigma(\mathbf{x},t) = \delta\Sigma(\mathbf{x})$, $\bar{\phi}(t) = \bar{\phi}_0 + \delta\phi(t)$ and ξ constant . We obtain, in momentum space, (to the first order in deviations and/ or sources) the following equations

$$
\begin{aligned}
\delta\ddot{G}_{k,k'} = & \ \delta G_{k,k'}(\epsilon_k + \epsilon_{k'})^2(1-\xi^2) + \frac{b}{4}\left(\frac{1}{\epsilon_k} + \frac{1}{\epsilon_{k'}}\right)(\delta G + 2\bar{\phi}_0\delta\phi) - K\left(\frac{1}{\epsilon_k} + \frac{1}{\epsilon_{k'}}\right) + (\epsilon_k + \epsilon_{k'})M(1-\xi) \\
& + \dot{L}\left(\frac{(1+\xi)}{\epsilon_k} + \frac{(1+\xi)}{\epsilon_{k'}}\right) \\
\delta\ddot{\bar{\phi}} = & -\frac{b}{2}\bar{\phi}_0\delta G - (q^2 + \mu^2)\delta\bar{\phi} - J - 2K\bar{\phi}_0 + \dot{J}_\pi + 2\bar{\phi}_0\dot{L}
\end{aligned} \tag{15}
$$

With $\epsilon_k = \sqrt{k^2+\mu^2}$ (being $k' = k + q$) and $\mu^2 = m_0^2 + \frac{b}{2}H + \frac{b}{2}\bar{\phi}_0^2$.

For an adiabatic branchement of these local sources of small amplitudes ϵ_i we will suppose the general plane wave form (for infinite medium)

$$
\begin{aligned}
&< \mathbf{x}, t|K|\mathbf{y}, t' > = \epsilon_K e^{-i\mathbf{q}.(\mathbf{x}-\mathbf{y}))} e^{-i(\omega+i\eta)t} \\
&< \mathbf{x}, t|L|\mathbf{y}, t' > = \epsilon_L e^{-i\mathbf{q}.(\mathbf{x}-\mathbf{y}))} e^{-i(\omega+i\eta)t} \\
&< \mathbf{x}, t|M|\mathbf{y}, t' > = \epsilon_M e^{-i\mathbf{q}.(\mathbf{x}-\mathbf{y}))} e^{-i(\omega+i\eta)t} \\
&J(\mathbf{x}, t) = \epsilon_J e^{-i\mathbf{q}.\mathbf{x}} e^{-i(\omega+i\eta)t} \\
&J_\pi(\mathbf{x}, t) = \epsilon_{J_\pi} e^{-i\mathbf{q}.\mathbf{x}} e^{-i(\omega+i\eta)t}
\end{aligned} \tag{16}
$$

With momentum and energy transfer equal to $\mathbf{q}$ and ω. Some of these sources may produce similar effects to the interaction with other particles at a time $t = 0$, this will be discussed elsewhere.

The induced responses of the field and fluctuations G are parametrized in such a way to have the same time dependence of the sources

$$\delta G(\mathbf{x},\mathbf{y},t) = \alpha e^{-i\mathbf{q}(\mathbf{x}-\mathbf{y})} e^{-i(\omega+i\eta)t}$$
$$\delta\bar{\phi}(\mathbf{x},t) = \beta e^{-i\mathbf{q}\mathbf{x}} e^{-i(\omega+i\eta)t} \tag{17}$$

where $\alpha = \sum_k < \mathbf{k}|\delta G/(1-\xi)|\mathbf{k}' >$ and, for an homogeneous configuration, of the condensate $\beta = \delta\bar{\phi}$. With these plane waves prescriptions one can rewrite the equations of motion. They yield two coupled equations for α and β:

$$\beta = \frac{-\frac{b}{2}\bar{\phi}_0\alpha - \epsilon_J - 2\epsilon_K\bar{\phi}_0 - i(\omega+i\eta)(\epsilon_{J_\pi} + 2\bar{\phi}_0\epsilon_L)}{q^2 + m_0^2 + \frac{b}{2}(H(x,x) + \bar{\phi}_0^2) - (\omega+i\eta)^2}$$
$$\delta G_{kk'} = \frac{1}{(\omega+i\eta)^2 - (1-\xi^2)(\epsilon_k + \epsilon_{k+q})^2}\left\{\left(\frac{1}{\epsilon_k} + \frac{1}{\epsilon_{k+q}}\right)\left(\frac{b}{4}\alpha + \frac{b}{2}\bar{\phi}_0\beta - \epsilon_K - i\epsilon_L(\omega+i\eta)\right)\right. \tag{18}$$
$$\left. + \frac{\epsilon_M}{2}(1-\xi)(\epsilon_k + \epsilon_{k+q})\right\}$$

Integrating the second expression in $\mathbf{k}$ for $\delta G_{k,k'}$ we can solve the system of the two equations.

Next, we define the "mean field" propagator

$$J_0 = \frac{1}{q^2 + \mu^2 - (\omega+i\eta)^2} \tag{19}$$

and the mean field general polarizabilities (or transition amplitude for the Π_0 case)

$$\Pi_0^\xi(\omega,q) = -2\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{(1-\xi)^{-1}}{(\omega+i\eta)^2 - (1-\xi^2)(\epsilon_k + \epsilon_{k+q})^2}\left(\frac{1}{\epsilon_k} + \frac{1}{\epsilon_{k+q}}\right)$$
$$\Pi_0'(\omega,q) = -2\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{(\omega+i\eta)^2 - (1-\xi^2)(\epsilon_k + \epsilon_{k+q})^2}\left(\frac{1}{\epsilon_k} + \frac{1}{\epsilon_{k+q}}\right) \tag{20}$$
$$\Pi_3'(\omega,q) = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\epsilon_k + \epsilon_{k+q}}{(\omega+i\eta)^2 - (1-\xi^2)(\epsilon_k + \epsilon_{k+q})^2}$$

n the zero temperature case $\xi = 0$ these integrals simplify a lot and will be analysed in the next section and in appendix B. At finite temperature these functions can be written in terms of $\beta\omega_k$ through the substitution for ξ indicated in the previous section.

We can note that the mean field propagator differs from the bare one by the physical mass μ^2, which in the latter case would be the bare mass m_0^2. In appendix A we show that the Π_0 polarizability at zero temperature has the same form as the real part of the covariant Feynman polarizability for the first order in perturbation theory over the tree level approximation and an imaginary part with an opposite sign. This means that if one changes the bare mass by the gaussian renormalized mass μ^2 on the one loop polarizability we get the above integral for $\Pi_0(\xi = 0)$ different by a factor 2, by definition. This is completely in agreement with the underlying ideas of the mean field approach

described in section 2 for the quantum fluctuations, where the mass μ^2 plays the role of a mean field. This Π_0 function, as expected, has a logarithmic divergence while Π_3 diverges quadratically. This second function comes from a perturbation on the "kinetic term" (π^2), but it cannot be necessarily associated to time dependent behavior or divergences as we can see from the mean value $< \pi^2 >$. In both cases a cut off will be introduced in order to regularize the divergences, for which the expressions are written in appendix B.

The final expressions for the generalized one particle correlation functions in the asymmetric phase are obtained by the derivation of the induced fluctuation of the classical field with relation to the one point- sources

$$S(q,\omega) \equiv \frac{\delta\beta}{\delta\epsilon_J} = -J_0\frac{1 + \frac{b}{4}\Pi_0^\xi(\omega,q)}{1 + \frac{b}{4}\Pi_0^\xi(\omega,q)\left(1 - b\bar{\phi}_0^2 J_0\right)}$$
$$S_\pi(q,\omega) \equiv \frac{\delta\beta}{\delta\epsilon_{J_\pi}} = -i\omega J_0\frac{1 + \frac{b}{4}\Pi_0'(\omega,q)}{1 + \frac{b}{4}\Pi_0^\xi(\omega,q)\left(1 - b\bar{\phi}_0^2 J_0\right)} \tag{21}$$

Where this last correlation function, at zero temperature, reduces to $S_\pi(q,\omega) = i\omega S(q,\omega)$.

For the dynamical polarizabilities (two particle functions) one considers the variation of the fluctuation with relation to its (two point) perturbation, for which one obtains

$$\Pi(\omega,q) \equiv \frac{\delta\alpha}{\delta\epsilon_K} = \frac{\Pi_0^\xi(\omega,q)\left(1 + b\bar{\phi}^2 J_0\right)}{1 + \frac{b}{4}\Pi_0^\xi(\omega,q)\left(1 - bJ_0\bar{\phi}^2\right)}$$
$$\Pi_\pi(\omega,q) \equiv \frac{\delta\alpha}{\delta\epsilon_M} = \frac{\Pi_3'(\omega,q)}{1 + \frac{b}{4}\Pi_0^\xi(\omega,q)\left(1 - bJ_0\bar{\phi}^2\right)} \tag{22}$$
$$\Pi_{\phi\pi}(\omega,q) \equiv \frac{\delta\alpha}{\delta\epsilon_L} = i\omega\frac{\Pi_0'(\omega,q)\left(1 + b\bar{\phi}^2 J_0\right)}{1 + \frac{b}{4}\Pi_0^\xi(\omega,q)\left(1 - bJ_0\bar{\phi}^2\right)}$$

Where this last function at zero temperature reduces to $\Pi_{\phi\pi} = i\omega\Pi$.

Since classical and quantum parts are "coupled", we notice that in the asymmetric phase it is possible to define "three point correlation functions", such as for example:

$$S_G(q,\omega) \equiv \frac{\delta\beta}{\delta\epsilon_K} = -\frac{2J_0\bar{\phi}_0}{1 + \frac{b}{4}\Pi_0^\xi(\omega,q)\left(1 - bJ_0\bar{\phi}_0^2\right)} \tag{23}$$

And also

$$\Pi_\phi(\omega,q) \equiv \frac{\delta\alpha}{\delta\epsilon_J} = \frac{\frac{b}{2}\bar{\phi}_0\Pi_0^\xi(\omega,q)J_0}{1 + \frac{b}{4}\Pi_0^\xi(\omega,q)\left(1 - bJ_0\bar{\phi}^2\right)}$$
$$\Pi_{J\pi}(\omega,q) \equiv \frac{\delta\alpha}{\delta\epsilon_{J_\pi}} = -i\omega\frac{\frac{b}{2}\bar{\phi}_0\Pi_0'(\omega,q)J_0}{1 + \frac{b}{4}\Pi_0^\xi(\omega,q)\left(1 - bJ_0\bar{\phi}^2\right)} \tag{24}$$

Where this last function at zero temperature reduces to $\Pi_{J\pi}(\omega,q) = -i\omega\Pi_\phi(\omega,q)$.

These "three point functions" are zero in the symmetric phase and directly proportional to $\bar{\phi}_0$ (it is interesting to remember that in the mean field approximation in the asymmetric vacuum one has

$\bar{\phi}_0^2 = 3\mu^2/b$). In fact, they represent the fact that both parts of the field, quantum and classical, are coupled" and "interact" between each other.

The zero temperature symmetric phase case for the polarizability $\Pi(\omega, q)$ has been studied in [10]. n this case we obtain:

$$
\begin{aligned}
S(q,\omega) &= -J_0 = \frac{1}{\omega^2 - q^2 - \mu^2} \\
\Pi(\omega,q) &= \frac{\Pi_0(\omega,q)}{1+\frac{b}{4}\Pi_0(\omega,q)}
\end{aligned} \tag{25}
$$

n this case $S(\omega, q)$ reduces to the mean field propagator.

Expressions (21,22) show corrections to the gaussian case for the 2 and 4 points correlation funcions. This is clear if we look at the resulting un-renormalized mass, as the inverse of the propagator n the static and homogeneous case, following (21)

$$
m_{res}^2 = \mu^2 \frac{1+\frac{b}{4}\overline{\Pi}_0^{\xi}(q,\omega\to 0)}{1+\frac{b}{4}\Pi^{\xi}(q,\omega\to 0)} \tag{26}
$$

Vhere $\overline{\Pi}_0$ is defined as

$$
\overline{\Pi}_0(\omega,q) = \Pi_0(\omega,q)\left(1 - b\bar{\phi}^2 J_0\right) \tag{27}
$$

n the next section we will perform the renormalization for the zero temperature case ($\xi = 0$) and hen we will see that the divergence in Π_0 present in the above expression is absorbed in the coupling onstant redefinition like $b\Pi_0 \to \lambda_R \Pi_0^R$.

The curves corresponding to the functions Π_0 and Π_3 at zero temperature are showed in figures , 2 and 3 for the values $m = 100MeV$, $q = 1fm^{-1}$, $b = 6$ and $\Lambda = 10fm^{-1}$ in function of ω. As xpected, the imaginary parts of these functions become non zero at the threshold, for which point he real parts also show a change of the behavior. It is important to note that there are divergences n the numerator of some expressions, like for Π_π. This means that these expressions are not (re) ormalized. For some of then, this "normalization" just coincides with the renormalization of the oupling constant, for others we have to consider a more general expression, as for example that of he T-matrix (done in [10]) to make it finite. This problem is addressed in the next section.

4 Renormalization

n the asymmetric phase, Π at zero temperature may be written as

$$
\Pi(\omega,q) = \frac{\Pi_0(\omega,q)\frac{q^2+\bar{\mu}^2-\omega^2}{q^2+\bar{\mu}^2-\omega^2}}{\frac{q^2+\mu^2-\omega^2}{q^2+\bar{\mu}^2-\omega^2}+\frac{b}{4}\Pi_0(\omega,q)} \tag{28}
$$

Where one has defined $\bar{\mu}^2 = \mu^2 - b\bar{\phi}_0^2$, that, with the relationship ($\bar{\phi}_0^2 = 3\mu^2/b$) from the asymmetric phase, becomes $\bar{\mu}^2 = -2\mu^2$ and also in this phase $\tilde{\mu} = 4\mu^2$. At the symmetric phase we have $\bar{\mu}^2 = \tilde{\mu}^2 = \mu^2$. Logarithmic divergence of function $\Pi_0(\omega, q)$ may be eliminated with coupling constant renormalization. Thus we rewrite the denominator (28) as

$$
\frac{q^2+\mu^2-\omega^2}{q^2-2\mu^2-\omega^2} + \frac{\lambda_R}{16\pi^2}\left\{\sqrt{1-\frac{4\mu^2}{\omega^2-q^2}}Log\left(\frac{\sqrt{1-\frac{4\mu^2}{\omega^2-q^2}}+1}{\sqrt{1-\frac{4\mu^2}{\omega^2-q^2}}-1}\right) - Log\left(\frac{2\mu^2}{e\bar{m}^2}\right)\right\} \tag{29}
$$

Where $\bar{m}^2$ is a scale factor. One has defined the renormalized coupling constant in asymmetric phase in the limit

$$
\frac{1}{\lambda_R} = -\frac{2}{b} + \frac{1}{16\pi^2}Log\frac{4\Lambda^2}{e^2\bar{m}^2} \tag{30}
$$

In the symmetric phase the flot of λ_R has been already calculated in [10] and is given by

$$
\frac{1}{\lambda_R} = \frac{2}{b} + \frac{1}{16\pi^2}Log\frac{4\Lambda^2}{e\bar{m}^2} \tag{31}
$$

It corresponds to the limit $\bar{\phi} = 0$ in equation (29), i.e., $\bar{\mu} = \mu$. We note that when the cut off goes to infinite the bare coupling constant must go to zero in order to keep a finite renormalized coupling constant. In the symmetric (asymmetric) phase it goes to zero from the negative (positive) value. This agrees with Kerman and Lin's calculations [10] and also with that of [3].

After these remarks, we can note that the numerator renormalization of $\Pi_0(\omega, q)$ comes directly because the reintroduction of α in equation (18) results $b.\Pi_0(\omega, q)$ which has been made finite.

It is worth to emphasize the agreement with the results of [10] by establishing the relationship of our function $\Pi(\omega, q)$ in the symmetric phase and the T- matrix from that cited paper. The denominator is clearly the same (Kerman and Lin call it Δ^+), and the difference comes from a normalization factor which is not present in our calculation but which can be obtained if we replace the expression for α in equation (18) as stated above. We also consider a factor γ, so that $\delta G(\omega) = \gamma\delta(\mathbf{k}' - \mathbf{k} - \mathbf{q}) + \delta G_{k,k'}(\omega)$. The above substitution results the T matrix without the factor $h(\mathbf{q}, \mathbf{k}+\mathbf{k}')$ given in [10]. It is interesting to remember that the two particle correlation function Π was obtained by considering a mass (mean field) perturbation.

4.1 Properties

All the n- point correlation functions at zero temperature have similar properties since the denominators which give the analytical structure are equal to

$$1/J_0 = w^2 - q^2 - \mu^2$$
$$I_0 = 1 + \frac{b}{4}\Pi_0(\omega, q)\left(1 - b\bar{\phi}_0^2 J_0\right) \tag{32}$$

At finite temperature these analytical properties are modified because the presence of the factor $(1-\xi)^2$, but no analytical solution were obtained, remaining this study for a future publication.

Here we use the unrenormalized expression. This causes no problem since, due the behavior of the renormalized coupling constant, this model can be regarded as an effective theory where the cut-off is fixed according to the physical content and scale of the system. Otherwise we could appeal to the "precarious renormalization" suggested by [3]. Besides that, expressions (30) and (31) relate renormalized and bare coupling constants to each other in both phases of the potential. As pointed out before, the complete renormalization with the cutoff sent to infinity induces a trivial model, although non trivial effects may be present [3].

The imaginary part of a Green's function come from unitarity requirements and it is associated to real transitions. The value of the frequency above which they become non zero is the threshold given by $\omega_t^2 = 4\mu^2 + q^2$. According to the above formula Π_0 and Π_3 (at zero temperature) tend to a non zero constant value for large ω values.

The poles of polarizabilities correspond to bound states. In order to study their existence, we consider the renormalized dispersion relation which reads

$$\frac{1}{\lambda_R}\frac{q^2+\mu^2-\omega^2}{q^2+\bar{\mu}^2-\omega^2} = -\frac{1}{16\pi^2}\left(\sqrt{1-\frac{4\mu^2}{\omega^2-q^2}}Log\left(\frac{\sqrt{1-\frac{4\mu^2}{\omega^2-q^2}}+1}{\sqrt{1-\frac{4\mu^2}{\omega^2-q^2}}-1}\right) - Log\left(\frac{2\mu^2}{e\bar{m}^2}\right)\right) \tag{33}$$

The right hand side function is always positive and small and this expression may have a solution for some values of λ_R [3, 10]. In the asymmetric phase ($\bar{\mu} = -2\mu$) it is possible to have bound states for some values of $\lambda_R > 0$ although for this phase the potential would not be stable [3, 10]. However we have to keep in mind the fact that its non zero value (of b or λ_R) is directly related to the cut- off and the last fixes the former.

In the limit of $q \to \infty$, the renormalized denominator of the polarizability $\Pi(q,\omega)$ in expression (22) goes to infinity and thus the dynamical scattering amplitude goes to zero. This also happens when the frequency tends to infinity ($\omega \to \infty$).

In the asymmetric phase one can rewrite the expressions of $S(q,\omega)$ and $\Pi(q,\omega)$ as

$$S(q,\omega) = J_0\left(\frac{1+\frac{b}{4}\Pi_0(\omega,q)}{1+\frac{b}{4}\overline{\Pi}_0(\omega,q)}\right)$$
$$\Pi(\omega,q) = \frac{\tilde{\Pi}_0(\omega,q)}{1+\frac{b}{4}\overline{\Pi}_0(\omega,q)} \tag{34}$$

Where we have defined

$$\tilde{\Pi}_0(\omega,q) = \Pi_0(\omega,q)\left(1 + b\bar{\phi}^2 J_0\right) \tag{35}$$

We see that the presence of the condensate "re-normalizes" the bare polarizability $\Pi_0(\omega,q)$ through $\overline{\Pi}_0$ and $\tilde{\Pi}_0$. This causes important effects as we have noticed in the renormalization.

It is also possible to establish several relationships among the non renormalized generalized correlation functions. For example:

$$S \equiv \frac{\delta\bar{\phi}}{\epsilon_J} = -J_0\left(1 - b\frac{\delta G}{\delta\epsilon_K} + \frac{b\Pi_0}{1+\frac{b\overline{\Pi}_0}{4}}\right) \tag{36}$$

At the threshold of the imaginary part (when $\omega_t^2 = 4\mu^2 + q^2$) the polarizability in symmetric phase is exactly equal to $\Pi(\omega = \omega_s) = 4/b$.

4.2 Time evolution and Bogoliubov transformation

Response functions to perturbations L and J_π are proportional to $i\omega$ and may diverge for large frequencies. This proportionality at zero temperature reduces to a more general result which is a consequence of the following property:

$$\Pi_{\phi\pi}(q,\omega) = -\partial_t \delta G(q,\omega)$$
$$S_\pi(q,\omega) = -\partial_t \delta\bar{\phi}(q,\omega) \tag{37}$$

Thus, to find $\Pi_{\pi\phi}$ and S_π means to explicite the temporal evolution respectively of the 4- point and 2- point correlation functions. It clearly corresponds to a Bogoliubov transformation of the following form (and correspondent transformations for the field $\hat{\phi}$ and $\hat{\pi}$):

$$b_i = f_+ a_i + f_- a_i^\dagger$$
$$b_i = f_-^* a_i + f_+^* a_i^\dagger \tag{38}$$

where f_{sign} are complex numbers which values are respectively given by (for the case of the first expression of (37))

$$f_+ = \frac{1}{2\omega} - \frac{1}{2} - i\left(\frac{\sqrt{\omega}L}{4\mu^2} + \frac{L}{\sqrt{4\omega}}\right)$$
$$f_- = \frac{1}{2\omega} + \frac{1}{2} + i\left(\frac{\sqrt{\omega}L}{4\mu^2} - \frac{L}{\sqrt{4\omega}}\right) \quad (39)$$

We have considered again that the source amplitude (ϵ_L) is infinitesimal and also neglected terms which would come from the quartic interaction of the model in the transformation of the order of $\phi^3\pi\epsilon_L$. These other terms would introduce another contribution to the $\Pi_{\phi\pi}$ function. Conjugated with this, one would introduce other sources which would allow us to study particle production through, for example, six- point correlation functions. This remarks are valid for the S_π result which also correspond to a time varying correlation function. We note that this time dependence of the one and two particle correlation function do not show additional ultra violet divergences, although they may exist [9].

5 Conclusion

The calculation of generalized n- point correlation functions for the $\lambda\phi^4$ model taking into account corrections to the gaussian variational approach has been performed. These functions were obtained as response functions to different external sources corresponding to several parameters which determine the static and dynamical configurations. The calculation has been performed in a self consistent way in both phases of the potential. We have noticed the possibility of defining 3- point correlation functions in the asymmetric phase. Logarithmic ultra violet divergences were eliminated with coupling constant renormalization. In the symmetric phase the bare coupling constant tend to zero from the negative side while in the asymmetric phase it goes to zero from the positive side, keeping at this level of approximation, in 1+3 dimensions, the triviality of the model. A perturbation to $< \pi\pi >$ term in the action introduces quadratic ultra- violet divergences. However, this quadratic divergence does not come necessarily from temporal evolution.

"Current" perturbations, which corresponds to perform a Bogoliubov transformation produces imaginary responses, and they yield the time dependence respectivelly of the one and two particle Green's functions.

The dynamical polarizability $\Pi(\omega, q)$ indicates that there may be bound states in the asymmetric

phase of the potential depending on the renormalized coupling constant value which, in fact, depends on the choice of the cut off or mass parameter scale $\bar{m}$, since this model is an effective one and the cut- off has to be chosen to fix a scale if one does not adopt the precarious renormalization [3, 10]. The definition of sum rules at finite temperature for these bosonic systems would be very usefull to the application of this formalism to phenomenological examples, either in ultrarelativistic heavy ion Physics, condensed Helium-4, etc.

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Appendix A: "Free" and mean field polarizabilities

We have calculated the retarded response functions, i.e., backward in time propagation. In particular the polarizability in the gaussian approximation, $\Pi_0(\omega, q)$, in this non covariant formalism, can be directly related to the covariant one at the one loop level perturbation theory, $\Pi_0^F(\omega, q)$ [13], unless for the fact that in the first we have the renormalized mass while in the second we use the bare mass. This has been discussed in section 3.

In this work we have used the non covariant form of the "free" polarizability given by:

$$\Pi_0(\omega, q) = -\int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{(\omega + i\eta)^2 - (\epsilon_k + \epsilon_{k+q})^2}\left(\frac{1}{\epsilon_k} + \frac{1}{\epsilon_{k+q}}\right) \quad (A.1)$$

The covariant form, corresponding to the one loop two particle scattering amplitude, in terms of the four- vectors ($p = (\omega_0, \mathbf{k}), Q = (\omega, \mathbf{q})$) is given by:

$$\Pi_0^F(Q) = i\int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m_0^2 + i\epsilon)} \cdot \frac{1}{((p+Q)^2 - m_0^2 + i\epsilon)} \quad (A.2)$$

By performing the $d\omega_0$ integral in the above expression we found that these two expressions are related by:

$$\frac{1}{2}\Pi_0^F = \Re e\Pi_0^R - i\Im m\Pi_0^R \quad (A.3)$$

Being only the sign of imaginary part different. This can be verified simply by integrating out the time variable in the expression of Π_0^F and substituting the bare mass by the renormalized one.

Appendix B: Analytical integrations of the functions Π_0 and Π_3

The Π_0 integral in function of a cut-off Λ results

$$\begin{aligned} \Re e\Pi_0(q,\omega,\Lambda) &= -\frac{1}{8\pi^2}\left(1 - ln\left(\frac{\mu^2}{\Lambda^2}\right) - \sqrt{1-\frac{4\mu^2}{Q^2}}ln\left|\frac{\sqrt{1-\frac{4\mu^2}{Q^2}}+1}{\sqrt{1-\frac{4\mu^2}{Q^2}}-1}\right|\right) \\ \Im m\Pi_0(q,\omega) &= -\frac{1}{8\pi}\sqrt{\frac{Q^2-4\mu^2}{Q^2}} \qquad (\omega^2 \geq q^2+4\mu^2) \end{aligned} \tag{B.1}$$

Where $Q^2 = \omega^2 - q^2$.

In the Π_3 case we obtain semi analytical formulae for the general expression

$$\begin{aligned} \Re e\Pi_3(q,\omega,\Lambda) &= \frac{1}{q(2\pi)^2}\int_0^\Lambda kdk\left\{\sqrt{(k+q)^2+\mu^2} - \sqrt{(k-q)^2+\mu^2} + \right. \\ &+\frac{\omega_k}{2}Log\left|\frac{\omega^2-(\omega_k+\sqrt{(k-q)^2+\mu^2})^2}{\omega^2-(\omega_k+\sqrt{(k+q)^2+\mu^2})^2}\right| \\ &\left.+\frac{\omega}{2}\left[Log\left|\frac{\omega+\omega_k+\sqrt{(k-q)^2+\mu^2})}{\omega+\omega_k+\sqrt{(k+q)^2+\mu^2}}\right| - Log\left|\frac{\omega-\omega_k-\sqrt{(k-q)^2+\mu^2})}{\omega-\omega_k-\sqrt{(k+q)^2+\mu^2}}\right|\right]\right\} \\ \Im m\Pi_3(q=0,\omega) &= -\frac{1}{\pi^2}\frac{\omega}{16}\sqrt{\frac{\omega^2}{4}-\mu^2} \end{aligned} \tag{B.2}$$

For some limits we get analytical expressions for the real part of Π_3 like when $\omega \to 0$ and $\mu \to 0$

$$\begin{aligned} lim_{\omega\to 0}\Pi_3 &= -\frac{1}{4\pi^2}\left(\frac{\Lambda}{2}\sqrt{\Lambda^2+\mu^2} - \frac{\mu^2}{2}ln\left|\frac{\Lambda+\sqrt{\Lambda^2+\mu^2}}{\mu}\right|\right) \\ lim_{\mu\to 0}\Pi_3 &= \frac{1}{\pi^2}\left(-\frac{\Lambda^2}{8} + \frac{\omega^2}{32}ln\left|\frac{\omega^2-4\Lambda^2}{\omega^2}\right|\right) \end{aligned} \tag{B.3}$$

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Figure Captions

Figure 1: Real (solid line) and imaginary (dotted line) parts of function $\Pi_0(\omega, q)$ in function of ω.

Figure 2: Real part of function $\Pi_3(\omega, q)$, in function of ω.

Figure 3: Imaginary part of function $\Pi_3(\omega, q)$, in function of ω.

Fig. 1

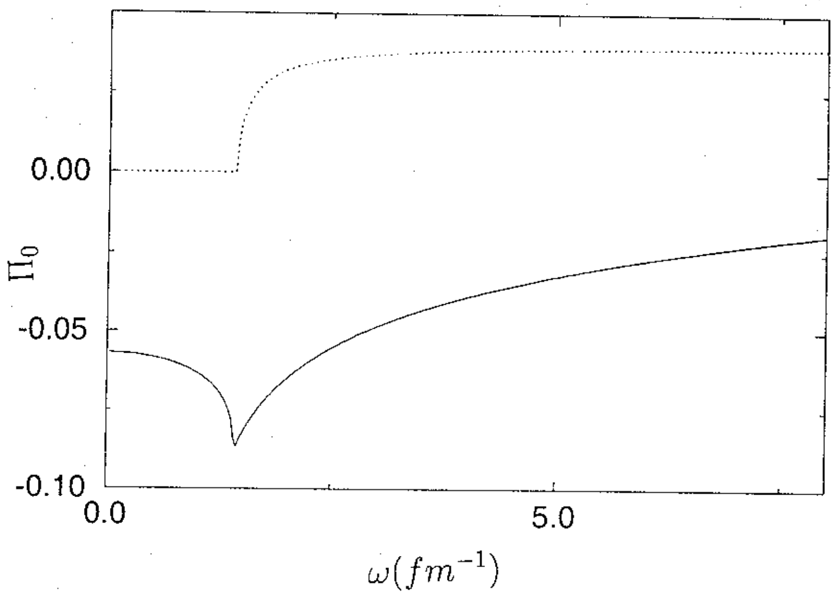

Fig. 2

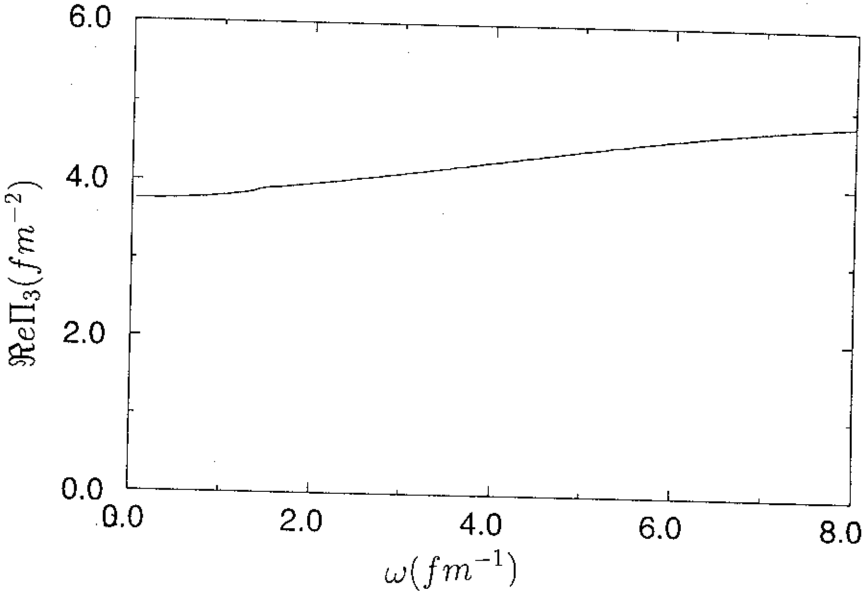

Fig. 3