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SIEGEL SUPERPARTICLE, HIGHER ORDER FERMIONIC CONSTRAINTS, AND PATH INTEGRALS

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Siegel superparticle, higher order fermionic constraints, and path integrals

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Abstract

We study Siegel superparticle moving in $R^{4|4}$ flat superspace. Canonical quantization is accomplished yielding the massless Wess-Zumino model as an effective field theory. Path integral representation for the corresponding superpropagator is constructed and proven to involve the Siegel action in a gauge fixed form. It is shown that higher order fermionic constraints intrinsic in the theory, though being a consequence of others in $d = 4$, make a crucial contribution into the path integral.

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1 Introduction

During the past decade a great deal of interest has been attracted to the investigation of different aspects of quantization of relativistic particles. The extensive researches were mainly focused in two directions. The first line began with the works [1,2] where a pseudoclassical model (PM) for the Dirac particle in four dimensions was constructed, studied, and quantized. The chief goal was to see how the quantum mechanics of a spin-$\frac{1}{2}$ particle, which may also be treated as a free spinor field theory, can be reconstructed in the course of first quantization of Grassmann classical mechanics (pseudoclassical mechanics). A number of papers generalizing the results to various spins and divers dimensions appeared [3–7], which can generally be regarded as answering the questions: Is it possible to treat any field

theory (at least a free field theory) as the result of first quantization of a classical model or a PM? Does there exist a regular method to construct such a model for a given field theory? In the relation to the latter point it is worth mentioning that the construction of a path integral representation for a propagator of a field theory may serve as a heuristic method to find such a PM. The effectiveness of this approach was demonstrated for the scalar and spinor particles [8] and then used to construct a new minimal PM for spinning particles in odd dimensions [7] (for the related works on path integral quantization of the models see also Refs. 9,10).

Another line of research originated from the superstring theory where the problem of consistent covariant quantization proved to have its analog in more simple mechanics examples. Of prime interest (for motivations and a comprehensive list of references see Ref. 11) seems to be the Casalbuoni-Brink–Schwarz (CBS) superparticle [12] which involves an infinitive ghost tower when $BRST$ quantized. As is known, the puzzle of covariant quantization of the model can be addressed in two respects [11] (for an alternative twistor-harmonic approach see Ref. 13): The problem of quantizing the infinitely reducible first–class constraints; The problem of quantizing the infinitely reducible second–class ones. It is the Siegel superparticle [14], later referred to as AB–superparticle, which allows the studying of the former in an independent way. Since the AB–formulation involves only first–class constraints of the CBS theory, the two models are not equivalent [15]. However, in Ref. 16 further modifications including higher order fermionic constraints (ABC–, $ABCD$–superparticles) have been proposed and shown [17] to be equivalent to the CBS model. As the new formulations possess first–class constraints only, the second–class constraints problem intrinsic in the CBS theory can be avoided.

Quantization of the $10d$ $ABCD$–superparticle both by BV and BFV methods has been accomplished in a series of work [18–20]. In particular, cohomologies of the $BRST$ operator have been evaluated [19] giving the ten-dimensional super Yang-Mills multiplet in the result. It is to be mentioned, however, that in both approaches the expression for the effective action involves an infinite tower of ghost variables and, hence, looks formal.

In the present paper we study the AB–superparticle in $R^{4|4}$ flat superspace. As shown below, this dimension is unique in the sense that the analog of the higher order fermionic C–constraint of the $10d$ $ABC-$ $-,ABCD--$formulations (which removes negative norm states from quantum spectrum [16,17]) automatically holds in the $4d$ AB–model. Although it is not of direct relevance to superstring theory (since beyond of critical

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dimension), quantization of the theory suggests an instructive example of generalizing the results of the theory of spinning particles to supersymmetric area. By now, only a few articles treating a precise relation between (world volume) supersymmetric mechanics and corresponding field theory superpropagators are known [21,22].

There is one more motivation for this work. In a recent paper [23] a recipe how to supplement infinitely reducible first class constraints up to a constraint system of finite stage of reducibility has been proposed. It may suggest an efficient way to cure the infinite ghost tower problem intrinsic in the *BRST* quantized Siegel theory. We expect that the knowledge of propagators of the first quantized theory will provide a considerable simplification in the forthcoming path integral test of this approach.

The work is organized as follows. In Sec. 2 Hamiltonian analysis of the model is presented. Complete constraint system is found showing that the analog of the C–constraint of the $10d$ $ABC-$, $ABCD$–formulations is a consequence of others in $d = 4$. Light-cone quantization of the model is examined in Sec. 3. Quantum spectrum proves to contain for essentially different states, the corresponding helicities being $(-\frac{1}{2}, 0, 0, \frac{1}{2})$. In Sec. 4 in the course of Dirac quantization we reproduce the results of Sec. 3 in a covariant fashion. An effective field theory corresponding to the first quantized $4d$ Siegel superparticle proves to be the massless Wess-Zumino model in the component form. Structure of the superfield representation realized in the quantum theory is discussed in Sec. 5. Quantum states of the first quantized Siegel superparticle are shown to form a reducible representation of the super Poincaré group which contains superhelicities 0 and $-1/2$. Path integral representation for propagator of the first quantized theory is constructed in Sec. 6 and proven to involve the Siegel action in a gauge fixed form. It is shown that higher order fermionic constraints, though being a consequence of others in $d = 4$, make a crucial contribution into the path integral.

2 Canonical formalism

As originally formulated in the first order formalism the $4d$ Siegel superparticle action is [14]

$$S = \int d\tau \{ p_m (\dot{x}^m + i\theta\sigma^m\dot{\bar{\theta}} - i\dot{\theta}\sigma^m\bar{\theta} + i\psi\sigma^m\bar{\rho} - i\rho\sigma^m\dot{\bar{\psi}}) - \frac{ep^2}{2} - \rho^\alpha\dot{\theta}_\alpha - \bar{\rho}_{\dot\alpha}\dot{\bar{\theta}}^{\dot\alpha} \},$$
(1)

and can be put into Lagrangian form by removing p^n with the use of its equation of motion

$$S = \int d\tau \{ \frac{1}{2e}(\dot{x}^m + i\theta\sigma^m\dot{\bar{\theta}} - i\dot{\theta}\sigma^m\bar{\theta} + i\psi\sigma^m\bar{\rho} - i\rho\sigma^m\dot{\bar{\psi}})^2 - \rho^\alpha\dot{\theta}_\alpha - \bar{\rho}_{\dot\alpha}\dot{\bar{\theta}}^{\dot\alpha} \}.$$
(2)

The model is invariant under rigid supersymmetry transformations in the standard realization

$$\delta\theta = \epsilon, \qquad \delta\bar{\theta} = \bar{\epsilon}, \qquad \delta x^n = i\theta\sigma^n\bar{\epsilon} - i\epsilon\sigma^n\bar{\theta},$$
(3)

and with respect to local reparametrizations and κ-symmetry,

$$\delta_\alpha\theta = \alpha\dot{\theta}, \qquad \delta_\alpha\bar{\theta} = \alpha\dot{\bar{\theta}}, \qquad \delta_\alpha x^n = \alpha\dot{x}^n,$$
$$\delta_\alpha\rho = \alpha\dot{\rho}, \qquad \delta_\alpha\bar{\rho} = \alpha\dot{\bar{\rho}}, \qquad \delta_\alpha e = (\alpha e)^{\cdot},$$
$$\delta_\alpha\psi = (\alpha\psi)^{\cdot}, \quad \delta_\alpha\bar{\psi} = (\alpha\bar{\psi})^{\cdot};$$
(4a)

$$\delta_\kappa\theta = -ie^{-1}\Pi_n\sigma^n\bar{\kappa}, \qquad \delta_\kappa\bar{\theta} = ie^{-1}\Pi_n\kappa\sigma^n,$$
$$\delta_\kappa x^n = i\delta\theta\sigma^n\bar{\theta} - i\theta\sigma^n\delta\bar{\theta} - i\kappa\sigma^n\bar{\rho} + i\rho\sigma^n\bar{\kappa},$$
$$\delta_\kappa e = 4\dot{\theta}\kappa + 4\bar{\kappa}\dot{\bar{\theta}}, \qquad \delta_\kappa\psi = \dot{\kappa},$$
$$\delta_\kappa\bar{\psi} = \dot{\bar{\kappa}},$$
(4b)

where we denoted $\Pi^m = \dot{x}^m + i\theta\sigma^m\dot{\bar{\theta}} - i\dot{\theta}\sigma^m\bar{\theta} + i\psi\sigma^m\bar{\rho} - i\rho\sigma^m\dot{\bar{\psi}}$.

The meaning of the variables entering into the action (2) is quite different and deserves to be mentioned here. The coordinates $(x^m, \theta^\alpha, \bar{\theta}_{\dot\alpha})$ parametrize the standard $R^{4|4}$ superspace. The variables e and $(\psi^\alpha, \bar{\psi}_{\dot\alpha})$ prove to be gauge fields for the local $\alpha-$ and κ–symmetries respectively. The pair $(\rho^\alpha, \bar{\rho}_{\dot\alpha})$ provides the terms corresponding to (mixed) covariant propagator for fermions and, as shown below, allows one to accomplish covariant quantization without lose of Lorentz covariance.

Passing to the Hamiltonian formalism one finds primary constraints in the problem[1]

$$p_\theta{}^\alpha - i(p_n\sigma^n\bar{\theta})^\alpha - \rho^\alpha = 0, \quad p_{\rho\alpha} = 0,$$
$$p_{\bar{\theta}\dot\alpha} + i(\theta\sigma^n p_n)_{\dot\alpha} - \bar{\rho}_{\dot\alpha} = 0, \quad p_{\bar{\rho}}{}^{\dot\alpha} = 0,$$
(5a)

$$p_e = 0, \qquad p_{\psi\alpha} = 0, \qquad p_{\bar{\psi}}{}^{\dot\alpha} = 0,$$
(5b)

where $(p, p_\theta, p_{\bar\theta}, p_\psi, p_{\bar\psi}, p_\rho, p_{\bar\rho}, p_e)$ are momenta canonically conjugate to the variables $(x, \theta, \bar{\theta}, \psi, \bar{\psi}, \rho, \bar{\rho}, e)$ respectively. The total Hamiltonian has the form

$$H^{(1)} = p_e\lambda_e + p_{\psi\alpha}\lambda_\psi{}^\alpha + p_{\bar{\psi}}{}^{\dot\alpha}\lambda_{\bar{\psi}\dot\alpha} + p_{\rho\alpha}\lambda_\rho{}^\alpha + p_{\bar{\rho}}{}^{\dot\alpha}\lambda_{\bar{\rho}\dot\alpha}$$

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[1] We define momenta conjugate to fermi variables to be right derivatives of the Lagrangian with respect to velocities. This corresponds to the following choice of the Poisson bracket $\{\theta^\alpha, p_{\theta\beta}\} = \delta^\alpha{}_\beta$, $\{\bar{\theta}_{\dot\alpha}, p_{\bar\theta}{}^{\dot\beta}\} = \delta_{\dot\alpha}{}^{\dot\beta}$ and the position of momenta and velocities in the Hamiltonian as specified below in Eq. (6).

$$+(p_{\bar\theta} + i\theta\sigma^n p_n - \bar\rho)^{\dot\alpha}\lambda_{\bar\theta\dot\alpha} + (p_\theta - ip_n\sigma^n\bar\theta - \rho)_\alpha\lambda_\theta{}^\alpha$$
$$+e\frac{p^2}{2} - i\psi\sigma^n\bar\rho p_n + i\rho\sigma^n\bar\psi p_n, \qquad (6)$$

where $\lambda_{...}$ denote Lagrange multipliers corresponding to the primary constraints. The consistency conditions for the primary constraints imply the secondary ones

$$p^2 = 0, \qquad p_n\sigma^n\bar\rho = 0, \qquad \rho\sigma^n p_n = 0, \qquad (7)$$

and determine some of the Lagrange multipliers,

$$\begin{aligned} \lambda_\theta &= -ip_n\sigma^n\bar\psi, & \lambda_{\bar\theta} &= i\psi\sigma^n p_n, \\ \lambda_\rho &= -2ip_n\sigma^n\lambda_{\bar\theta} \approx 0, & \lambda_{\bar\rho} &= 2i\lambda_\theta\sigma^n p_n \approx 0. \end{aligned} \qquad (8)$$

No tertiary constraints arise at the next stage of the Dirac procedure, the remaining Lagrange multipliers being unfixed.

Thus, the complete constraint set of the model is given by Eqs. (5),(7) and it is convenient to rewrite the latter in the equivalent form

$$p^2 = 0, \qquad p_m\tilde\sigma^m p_\theta = 0, \qquad p_m\sigma^m p_{\bar\theta} = 0. \qquad (9)$$

The constraints (5a) are second–class and allow one to omit the pairs (ρ, p_ρ), $(\bar\rho, p_{\bar\rho})$ after introducing the associated Dirac bracket

$$\begin{aligned} \{A, B\}_D = \{A, B\} &- \{A, p_\rho\}\{p_\theta - ip_n\sigma^n\bar\theta - \rho, B\} \\ &- 2i\{A, p_\rho\}\sigma^n p_n\{p_{\bar\rho}, B\} + \{A, p_\theta - ip_n\sigma^n\bar\theta - \rho\}\{p_\rho, B\} \\ &- 2i\{A, p_{\bar\rho}\}\sigma^n p_n\{p_\rho, B\} - \{A, p_{\bar\rho}\}\{p_{\bar\theta} + i\theta\sigma^n p_n - \bar\rho, B\} \\ &+ \{A, p_{\bar\theta} + i\theta\sigma^n p_n - \bar\rho\}\{p_{\bar\rho}, B\}. \end{aligned} \qquad (10)$$

The Dirac brackets for the remaining variables prove to coincide with the Poisson ones. The first–class constraints (5b) admit the covariant gauge

$$e = 1, \qquad \psi = 0, \qquad \bar\psi = 0, \qquad (11)$$

which yields

$$\lambda_e = 0, \qquad \lambda_\psi = 0, \qquad \lambda_{\bar\psi} = 0, \qquad (12)$$

and implies that the canonical pairs (e, p_e), (ψ, p_ψ), $(\bar\psi, p_{\bar\psi})$ can be eliminated from the consideration.

Thus, after the partial phase space reduction, there remain only (x, p), (θ, p_θ), $(\bar\theta, p_{\bar\theta})$ variables being subject to the first–class constraints (9). The total Hamiltonian vanishes on the constraint surface in the full agreement with the reparametrization invariance of the model.

Some remarks are in order. First, by making use of a shift of the variables $\rho \to \rho + i\sigma^n\bar\theta p_n$, $\bar\rho \to \bar\rho - i\theta\sigma^n p_n$, $e \to e + 2(\psi\theta + \bar\theta\bar\psi)$ the original Lagrangian (1) can be simplified to

$$S = \int d\tau\{p_m(\dot x^m + i\psi\sigma^m\bar\rho - i\rho\sigma^m\bar\psi) - \frac{ep^2}{2} - \rho^\alpha\dot\theta_\alpha - \bar\rho_{\dot\alpha}\dot{\bar\theta}^{\dot\alpha}\}, \qquad (13)$$

or eliminating p_n

$$S = \int d\tau\{\frac{1}{2e}(\dot x^m + i\psi\sigma^m\bar\rho - i\rho\sigma^m\bar\psi)^2 - \rho^\alpha\dot\theta_\alpha - \bar\rho_{\dot\alpha}\dot{\bar\theta}^{\dot\alpha}\}. \qquad (14)$$

In contrast to the theory (2), the formulation (14) proves to possess a more simple and, in particular, off-shell closed algebra of local symmetries[2] (see also Ref. 24)

$$\begin{aligned} \delta_\kappa\theta &= -ie^{-1}\Pi_n\sigma^n\bar\kappa, & \delta_\kappa\bar\theta &= ie^{-1}\Pi_n\kappa\sigma^n, \\ \delta_\kappa x^n &= i\rho\sigma^n\bar\kappa - i\kappa\sigma^n\bar\rho, & \delta_\kappa\psi &= \dot\kappa, \\ \delta_\kappa\bar\psi &= \dot{\bar\kappa}, \end{aligned} \qquad (15a)$$

$$\delta_b\theta = ib_n\sigma^n\dot{\bar\rho}, \qquad \delta_b\bar\theta = -ib_n\dot\rho\sigma^n, \qquad (15b)$$

where $\Pi^m = \dot x^m + i\psi\sigma^m\bar\rho - i\rho\sigma^m\bar\psi$. The rigid (on-shell) supersymmetry is realized in the theory in a nonstandard way

$$\begin{aligned} \delta\theta &= \epsilon, & \delta\bar\theta &= \bar\epsilon, & \delta x^n &= i\epsilon\sigma^n\bar\theta - i\theta\sigma^n\bar\epsilon \\ \delta\rho &= -ie^{-1}\Pi_n\sigma^n\bar\epsilon, & \delta\bar\rho &= ie^{-1}\Pi_n\epsilon\sigma^n, \\ \delta e &= -2(\psi\epsilon + \bar\epsilon\bar\psi), \end{aligned} \qquad (16)$$

and can be closed off-shell by applying the standard technique [25]. It is sufficient to introduce an auxiliary vector variable D^n which transforms as $\delta_\kappa D^n = i(\dot{\bar\theta} + ie^{-1}(\Pi_n + D_n)\sigma^n\bar\psi)\sigma^m\bar\epsilon - i\epsilon\sigma^m(\dot{\bar\theta} - ie^{-1}(\Pi_n + D_n)\psi\sigma^n)$, exchange Π^m with $\Pi^m + D^m$ in Eq. (16) and add the trivial term $-\frac{1}{2e}D^2$ to the action (14). As was shown in Ref. 24, the model (14) admits an interesting geometric formulation that appeals to new extensions of the Poincaré superalgebra in $d = 3, 4, 6, 10$ (in this relation see also Ref. 20). Beautiful enough, it is this form of the Siegel action which appears in the path integral when constructing a path integral representation for superpropagator of the first quantized theory (see Sec. 6).

Second, from Eq. (9) it follows

$$p_\theta^2 = 0, \qquad p_{\bar\theta}^2 = 0. \qquad (17)$$

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[2]The transformation with a local parameter b^n is trivial on-shell. It is the only trivial symmetry needed to close the algebra. The closure of the algebra (4) is known to require an infinite number of trivial symmetries.

This means that the C-constraint of the $10d$ $ABC-$,$ABCD-$superparticles, which removes negative norm states from quantum spectrum, automatically holds in four dimensions. In particular, one can check that the model

$$S = \int d\tau \{ p_m(\dot{x}^m + i\psi\sigma^m\bar{\rho} - i\rho\sigma^m\bar{\psi}) - \frac{e}{2}p^2 - \rho\dot{\theta} - \bar{\rho}\dot{\bar{\theta}}$$
$$+ \lambda(\sigma + \tilde{\sigma})(\rho - i\sigma^n\bar{\theta}p_n)^2(\bar{\rho} + i\theta\sigma^n p_n)^2 \},$$
(18)

with $\lambda, \sigma, \tilde{\sigma}$ the new fermionic variables, is physically equivalent to the original theory (1). It seems surprising, but the higher order fermionic constraints (17), although being a consequence of others in $d = 4$, play an important role in quantum theory and make a crucial contribution into the path integral.

3 Ligh-cone quantization

A covariant gauge to Eq. (9) is known to be problematic in the original phase space. Before turning to covariant quantization, it is worth considering the problem in the light-cone framework. This analysis proves to suggest a correct choice of a superfield wave function when accomplishing covariant quantization in the next section.

On the constraint surface $p^2 = 0$ only half of the fermionic constraints (9) are linearly independent. Assuming the standard light-cone condition $p^+ \neq 0$, one finds them to be

$$p_{\theta 2} - \frac{p^1 + ip^2}{\sqrt{2}p^+}p_{\theta 1} = 0, \qquad p_{\bar{\theta}}^{\dot{1}} + \frac{p^1 - ip^2}{\sqrt{2}p^+}p_{\bar{\theta}}^{\dot{2}} = 0.$$
(19)

After imposing the conventional light-cone gauge in the sector of fermi variables[3]

$$\theta\sigma^+ = 0, \qquad \sigma^+\bar{\theta} = 0, \qquad$$
(20a)

or

$$\theta^2 = 0, \qquad \bar{\theta}_{\dot{1}} = 0,$$
(20b)

there remain only $(x^n, p_n), (\theta^1, p_{\theta 1}), (\bar{\theta}_{\dot{2}}, p_{\bar{\theta}}^{\dot{2}})$ variables[4] subject to usual canonical commutation relations, provided the Dirac bracket associated with Eqs. (19),(20b) has been introduced. The gauge fixed action is

$$S = \int d\tau \{ p_m\dot{x}^m + p_\theta\dot{\theta} + p_{\bar{\theta}}\dot{\bar{\theta}} - \frac{1}{2}p^2 \}.$$
(21)

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[3]We refrain from imposing a gauge to the bosonic first class constraint $p^2 = 0$ in order to maintain the explicit connection between the Pauli-Lubanski vector and the momentum vector in Eq. (30) below.

[4]In what follows we omit the indices carried by the fermi variables.

In passing to quantum description, a representation space F for the fermi operators may be chosen to be a linear span[5] of four vectors

$$|0\rangle, \qquad |p\rangle, \qquad |\bar{p}\rangle, \qquad |p, \bar{p}\rangle,$$
(22)

with the action of the operators being defined like that of creation and annihilation operators

$$\hat{\theta}|0\rangle = 0, \qquad \hat{\theta}|p\rangle = i|0\rangle, \qquad \hat{\theta}|\bar{p}\rangle = 0, \qquad \hat{\theta}|p, \bar{p}\rangle = i|\bar{p}\rangle,$$
$$\hat{\bar{\theta}}|0\rangle = 0, \qquad \hat{\bar{\theta}}|p\rangle = 0, \qquad \hat{\bar{\theta}}|\bar{p}\rangle = i|0\rangle, \qquad \hat{\bar{\theta}}|p, \bar{p}\rangle = -i|p\rangle,$$
$$\hat{p}_\theta|0\rangle = |p\rangle, \qquad \hat{p}_\theta|p\rangle = 0, \qquad \hat{p}_\theta|\bar{p}\rangle = |p, \bar{p}\rangle, \qquad \hat{p}_\theta|p, \bar{p}\rangle = 0,$$
$$\hat{p}_{\bar{\theta}}|0\rangle = |\bar{p}\rangle, \qquad \hat{p}_{\bar{\theta}}|p\rangle = -|p, \bar{p}\rangle, \qquad \hat{p}_{\bar{\theta}}|\bar{p}\rangle = 0, \qquad \hat{p}_{\bar{\theta}}|p, \bar{p}\rangle = 0.$$
(23)

The total Hilbert space is defined to be a tensor product of the F and the space of squire integrable functions in which $\hat{x}^n$ and $\hat{p}_n$ act in the usual coordinate representation. An arbitrary state is

$$|\psi\rangle = (|0\rangle a + |p\rangle b + |\bar{p}\rangle c + |p, \bar{p}\rangle d) \otimes \Phi(x),$$
(24)

with a, b, c, d the supernumbers. The conventional scalar product reads [10]

$$\langle \psi_1 | \psi_2 \rangle = \int d^4x \bar{\Phi}_1(x)\Phi_2(x)\left(\bar{a}_1 a_2 + \bar{b}_1 b_2 + \bar{c}_1 c_2 + \bar{d}_1 d_2\right),$$
(25)

and can only formally be regarded as positive definite since involves Grassmann numbers. Note that under this product $\hat{p}_\theta^+ = -i\hat{\theta}, \hat{p}_{\bar{\theta}}^+ = -i\hat{\bar{\theta}}$ which is in a agreement with the definition of the operators like creation and annihilation ones.

The physical subspace in the complete Hilbert space is defined by the standard prescription [27,28]

$$\hat{p}^2|\text{phys}\rangle = 0.$$
(26)

Thus, the Hilbert space of the model contains four essentially different states. To evaluate helicities of the states we reduce the Pauli-Lubanski vector

$$W_a = \frac{1}{2}\epsilon_{abcd}p^b S^{cd},$$
(27)

were $S^{cd} = p_{\theta\gamma}(\sigma^{cd})^\gamma{}_\delta\theta^\delta + p_{\bar{\theta}}^{\dot{\gamma}}(\tilde{\sigma}^{cd})_{\dot{\gamma}}{}^{\dot{\delta}}\bar{\theta}_{\dot{\delta}}$ is the spin part of the Lorentz generators, to the surface of constraints (9) and gauges (20b). Making use of the identities

$$\sigma_{ab} = \frac{i}{2}\epsilon_{abcd}\sigma^{cd}, \qquad \tilde{\sigma}_{ab} = -\frac{i}{2}\epsilon_{abcd}\tilde{\sigma}^{cd},$$
(28)

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[5]More precisely, the F can be endowed with the structure of a supervector space [26].

one gets

$$W_a = -\frac{i}{2} p_a \left(p_\theta \theta - p_{\bar\theta} \bar\theta \right). \tag{29}$$

Passing further to quantum description, it is easy to verify that the relations

$$\begin{aligned}
\hat{W}_a |0\rangle \otimes \Phi(x) &= 0 \hat{p}_a |0\rangle \otimes \Phi(x), \\
\hat{W}_a |p\rangle \otimes \Phi(x) &= \frac{1}{2} \hat{p}_a |p\rangle \otimes \Phi(x), \\
\hat{W}_a |\bar{p}\rangle \otimes \Phi(x) &= -\frac{1}{2} \hat{p}_a |\bar{p}\rangle \otimes \Phi(x), \\
\hat{W}_a |p,\bar{p}\rangle \otimes \Phi(x) &= 0 \hat{p}_a |p,\bar{p}\rangle \otimes \Phi(x), \tag{30}
\end{aligned}$$

hold, provided the pq-ordering procedure for the fermi operators has been adopted. Since $\hat{p}^2 = 0$ on physical states, the equations above determine two particles of helecity 0 and two particles of helicities $\frac{1}{2}, -\frac{1}{2}$ respectively.

In the next section we consider Dirac quantization of the model. It is the analysis above which supports our choice of a wave function to be a *real scalar* superfield.

4 Dirac quantization

Since commutation relations for the variables (x^n, p_m), $(\theta^\alpha, p_{\theta_\alpha})$, $(\bar\theta_{\dot\alpha}, p_{\bar\theta}{}^{\dot\alpha})$ are canonical, we can realize them in the coordinate representation $\hat{x}^n = x^n$, $\hat{p}_m = -i\partial_m$, $\hat{\theta}^\alpha = \theta^\alpha$, $\hat{p}_{\theta_\alpha} = i\partial_\alpha$, $\hat{\bar\theta}_{\dot\alpha} = \bar\theta_{\dot\alpha}$, $\hat{p}_{\bar\theta}{}^{\dot\alpha} = i\bar\partial^{\dot\alpha}$ on a Hilbert space whose elements are chosen to be real scalar superfields

$$\begin{aligned}
V(x,\theta,\bar\theta) &= A(x) + \theta\psi(x) + \bar\theta\bar\psi(x) + \theta^2 F(x) + \bar\theta^2 \bar{F}(x) \\
&\quad + \theta\sigma^n\bar\theta C_n(x) + \bar\theta^2\theta\lambda(x) + \theta^2\bar\theta\bar\lambda(x) + \theta^2\bar\theta^2 D(x). \tag{31}
\end{aligned}$$

In what follows we assume the standard boundary condition

$$V(x,\theta,\bar\theta) \xrightarrow[x\to\pm\infty]{} 0. \tag{32}$$

The physical states in a complete Hilbert space are defined by Dirac's prescription

$$\begin{aligned}
\hat{p}^2 |\text{phys}\rangle &= 0, \\
\tilde\sigma^n \hat{p}_n \hat{p}_\theta |\text{phys}\rangle &= 0, \\
\tilde\sigma^n \hat{p}_n \hat{p}_{\bar\theta} |\text{phys}\rangle &= 0. \tag{33}
\end{aligned}$$

In the representation chosen this yields

$$\begin{aligned}
\tilde\sigma^n \partial_n \psi = 0, \qquad \tilde\sigma^n \partial_n \bar\psi = 0, \\
\Box A = 0; \tag{34a}
\end{aligned}$$

$$\begin{aligned}
(\tilde\sigma^m\sigma^n)^{\dot\alpha}{}_{\dot\beta}\partial_m C_n = 0, \qquad (\sigma^n\tilde\sigma^m)_\alpha{}^\beta \partial_m C_n = 0, \\
\Box C_n = 0, \tag{34b}
\end{aligned}$$

with all other component fields vanishing due to the boundary condition (32). In obtaining Eq. (34) we used the identity

$$\text{Tr}\left(\sigma^n\tilde\sigma^m\right) = -2\eta^{nm}. \tag{35}$$

It is instructive then to simplify Eq. (34b). Taking a trace in the first equation and making use of Eq. (35) one finds

$$\partial^n C_n = 0, \tag{36}$$

which (with the use of the relation $\sigma^n\tilde\sigma^m + \sigma^m\tilde\sigma^n = -2\eta^{nm}$) brings Eq. (34b) to the form

$$\begin{aligned}
(\sigma^{mn})_\alpha{}^\beta \partial_m C_n = 0, \qquad (\tilde\sigma^{mn})^{\dot\alpha}{}_{\dot\beta}\partial_m C_n = 0, \\
\Box C_n = 0 \tag{37}
\end{aligned}$$

Multiplying the first equality in Eq. (37) by $(\sigma^{kl})_\beta{}^\alpha$ and taking into account the identity

$$\text{Tr}\,\sigma^{mn}\sigma^{kl} = -\frac{1}{2}(\eta^{mk}\eta^{nl} - \eta^{ml}\eta^{nk}) - \frac{i}{2}\epsilon^{mnkl}, \tag{38}$$

one gets

$$\partial_k C_l - \partial_l C_k = -i\epsilon_{klmn}\partial^m C^n. \tag{39}$$

Together with its complex conjugate this implies

$$\partial_m C_n - \partial_n C_m = 0, \qquad \epsilon_{klmn}\partial^m C^n = 0. \tag{40}$$

The only solution to Eqs. (36), (40) is

$$C_n = \partial_n B, \tag{41}$$

with B the on-shell massless real scalar field

$$\Box B = 0. \tag{42}$$

Thus, physical states of the first quantized Siegel superparticle look like

$$V_{\text{phys}}(x,\theta,\bar\theta) = A(x) + \theta\psi(x) + \bar\theta\bar\psi(x) + \theta\sigma^n\bar\theta\partial_n B(x), \tag{43}$$

with A, B the on-shell massless real scalar fields (irreps of the Poincaré group of helicity 0) and $\psi, \bar\psi$ the on-shell massless spinor fields (helicities

1/2 and –1/2, respectively). Note that together they fit to form two irreducible representations of the super Poincaré group of superhelicities 0 and –1/2 [26].

It is worth mentioning that Eq. (33) can be rewritten in the manifestly superinvariant form

$$\tilde{\sigma}^{n\dot{\alpha}\alpha}\partial_n D_\alpha V = 0, \qquad \tilde{\sigma}^{n\dot{\alpha}\alpha}\partial_n \bar{D}_{\dot{\alpha}} V = 0, \tag{44}$$

where D_α, $\bar{D}_{\dot{\alpha}}$ are the covariant derivatives, or as a single massless Dirac equation

$$\gamma^n \partial_n \Psi = 0, \tag{45}$$

with $\Psi \equiv \begin{pmatrix} D_\alpha V \\ \bar{D}^{\dot{\alpha}} V \end{pmatrix}$ a (superfield) Majorana spinor.

An effective field theory which reproduces equations (34a), (42) is easy to write

$$S = \int d^4 x \{\frac{1}{2}\partial^m A \partial_m A + \frac{1}{2}\partial^m B \partial_m B + i\psi\sigma^m \partial_m \bar{\psi}\}, \tag{46}$$

which is invariant under global (on-shell) supersymmetry transformations

$$\begin{aligned}
\delta A &= \epsilon\psi + \bar{\epsilon}\bar{\psi}, \qquad \delta B = i\epsilon\psi - i\bar{\epsilon}\bar{\psi}, \\
\delta\psi &= i(\sigma^n\bar{\epsilon})\partial_n A + (\sigma^n\bar{\epsilon})\partial_n B, \\
\delta\bar{\psi} &= -i(\epsilon\sigma^n)\partial_n A + (\epsilon\sigma^n)\partial_n B.
\end{aligned} \tag{47}$$

In Eq. (46) we recognize the massless Wess–Zumino model in the component form [29].

5   Reducibility of the superfield representation and superhelicities

As is known, the superfield formulation of the massless Wess–Zumino model involves chiral and antichiral superfields [30],

$$S = \int d^8 z \, \Phi\bar{\Phi}, \tag{48a}$$

$$\bar{D}_{\dot{\alpha}}\Phi = 0, \tag{48b}$$

$$D_\alpha\bar{\Phi} = 0. \tag{48c}$$

The equations of motion read

$$D^2\Phi = 0, \tag{49a}$$

$$\bar{D}^2\bar{\Phi} = 0. \tag{49b}$$

Let us show that the real scalar superfield (31) satisfying the constraints (44) is the sum of on-shell massless chiral ((48b),(49a)) and antichiral ((48c),(49b)) superfields.

Let us consider Eqs. (48b), (49a). The first of them implies the decomposition [26,30]

$$\Phi(x,\theta,\bar{\theta}) = \alpha(x) + \theta\psi(x) + \theta^2 f(x) + i\theta\sigma^n\bar{\theta}\partial_n\alpha(x) + \frac{1}{2}\theta^2\bar{\theta}\tilde{\sigma}^n\partial_n\psi + \frac{1}{4}\theta^2\bar{\theta}^2\Box\alpha, \tag{50}$$

while the latter, being rewritten in the equivalent form

$$\tilde{\sigma}^{n\dot{\alpha}\alpha}\partial_n D_\alpha\Phi = 0, \tag{51}$$

yields

$$\tilde{\sigma}^n\partial_n\psi = 0, \qquad \Box\alpha = 0, \qquad f = 0. \tag{52}$$

In order to get Eqs. (51), (52) we used the identity

$$[D^2, \bar{D}_{\dot{\alpha}}] = -4i\sigma^n_{\alpha\dot{\alpha}}\partial_n D^\alpha, \tag{53}$$

and assumed the standard boundary condition. Note also that Eqs. (48b), (51) together with the identity $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^n_{\alpha\dot{\alpha}}\partial_n$ imply

$$\Box\Phi = 0, \tag{54}$$

Thus, an on-shell chiral superfield can be written as

$$\begin{aligned}
\Phi(x,\theta,\bar{\theta}) &= \alpha(x) + \theta\psi(x) + i\theta\sigma^n\bar{\theta}\partial_n\alpha(x), \\
\Box\alpha(x) &= 0, \qquad \tilde{\sigma}^n\partial_n\psi(x) = 0.
\end{aligned} \tag{55}$$

Similarly, an on-shell antichiral superfield ((48c),(49b)) has the form

$$\begin{aligned}
\bar{\Phi}(x,\theta,\bar{\theta}) &= \bar{\alpha}(x) + \bar{\theta}\bar{\psi}(x) - i\theta\sigma^n\bar{\theta}\partial_n\bar{\alpha}(x), \\
\Box\bar{\alpha}(x) &= 0, \qquad \tilde{\sigma}^n\partial_n\bar{\psi}(x) = 0.
\end{aligned} \tag{56}$$

Considering now the sum

$$\Phi + \bar{\Phi} = (\alpha + \bar{\alpha}) + \theta\psi + \bar{\theta}\bar{\psi} + \theta\sigma^n\bar{\theta}\partial_n i(\alpha - \bar{\alpha}), \tag{57}$$

and denoting

$$\alpha + \bar{\alpha} \equiv A, \qquad i(\alpha - \bar{\alpha}) \equiv B, \tag{58}$$

one arrives just at Eq. (43).

Thus, the real scalar superfield subject to the constraints (44) is the sum of on-shell chiral and antichiral superfields

$$V_{\text{phys}}(x, \theta, \bar\theta) = \Phi(x, \theta, \bar\theta) + \bar\Phi(x, \theta, \bar\theta). \qquad (59)$$

This is in complete agreement with the results of the previous section. It is worth mentioning that the choice of a wave function to be a *complex scalar* superfield would not reproduce the results (see also the related work [31]).

As is known, on-shell massless scalar chiral superfields form a massless irreducible representation of the super Poincaré group of superhelicity 0 [26]. Analogously, on-shell massless scalar antichiral superfields realize irrep of superhelicity $-1/2$. We finally conclude that quantum states of the first quantized Siegel superparticle form a reducible representation of the super Poincaré group which contains superhelicities 0 and $-1/2$.

Two remarks are relevant here. First, the constraints (9) just coincide with the first-class ones of the CBS superparticle. In Ref. 31 they have been used to covariantly quantize the CBS model within the framework of the Gupta-Bleuler method (see also the related work [32]). We are to stress, however, that the naive omitting of second–class constraints intrinsic in the CBS theory (which generally leads to the Siegel model) in the approach of Ref. 31 would not reproduce the result of the Dirac quantization presented above. It is worth mentioning also Ref. 33, where the technique of quantization with a complex Hamiltonian has been applied to establish a precise relation between on-shell massive chiral superfields and the corresponding particle mechanics. The massless limit of the procedure, however, leads to ghost excitations in the quantum spectrum [33] and, hence, is ill defined. Second, as was mentioned above the C–constraint of the $10d$ $ABC-$, $ABCD-$ superparticles is not necessary in four dimensions. Note in this connection that an alternative possibility $(p_\theta - ip_n\sigma^n\bar\theta)_\alpha (p_{\bar\theta} + i\theta\sigma^n p_n)_{\dot\alpha} = 0$, or $(D_\alpha\bar D_{\dot\alpha} - \bar D_{\dot\alpha}D_\alpha)V = 0$ at the quantum level, leads to the trivial solution $V = 0$ only (see, however, Ref. [34]). By this reason, it is not obvious to us how to extend the model (1) up to a theory equivalent to the $4d$ CBS superparticle along the lines of Ref. 16.

6 Path integral representation of the superfield propagator

Let examine now the possibility to reproduce the Siegel action within the framework of a path integral representation for a (super)field propagator.

For the case concerned the superfield propagator reads [26,30]

$$G(z, z') = \begin{pmatrix} 0 & G_c(z, z') \\ G_a(z, z') & 0 \end{pmatrix}, \qquad (60)$$

$$G_c(z, z') = \frac{1}{4}\frac{\bar D^2}{\Box}\delta_-(z, z'), \qquad \delta_-(z, z') = -\frac{1}{4}D^2\delta^8(z - z'),$$

$$G_a(z, z') = \frac{1}{4}\frac{D^2}{\Box}\delta_+(z, z'), \qquad \delta_+(z, z') = -\frac{1}{4}\bar D^2\delta^8(z - z').$$

Following Ref. 35 we represent it as a matrix element

$$G(z_{out}, z_{in}) = \langle z_{out}|\hat G|z_{in}\rangle, \qquad (61)$$

were $|z\rangle$ are eigenvectors of some coordinate operators $\hat z^M = (\hat x^n, \hat\theta^\alpha, \hat{\bar\theta}_{\dot\alpha})$. Together with the conjugate momenta $\hat p_M = (\hat p_n, \hat p_{\theta\alpha}, \hat p_{\bar\theta}{}^{\dot\alpha})$ they satisfy the relations

$$[\hat x^n, \hat p_m] = i\delta^n{}_m, \qquad \{\hat\theta^\alpha, \hat p_{\theta\beta}\} = i\delta^\alpha{}_\beta, \qquad \{\hat{\bar\theta}_{\dot\alpha}, \hat p_{\bar\theta}{}^{\dot\beta}\} = i\delta_{\dot\alpha}{}^{\dot\beta},$$
$$\hat z^M|z\rangle = z^M|z\rangle, \qquad \langle z|z'\rangle = \delta^8(z - z'), \qquad \int d^8z|z\rangle\langle z| = 1,$$
$$\hat p_M|p\rangle = p_M|p\rangle, \qquad \langle p|p'\rangle = \delta^8(p - p'), \qquad \int d^8p|p\rangle\langle p| = 1,$$
$$-i\partial_m\langle z|\psi\rangle = \langle z|\hat p_m|\psi\rangle, \qquad i\partial_\alpha\langle z|\psi\rangle = \langle z|\hat p_{\theta\alpha}|\psi\rangle,$$
$$i\partial^{\dot\alpha}\langle z|\psi\rangle = \langle z|\hat p_{\bar\theta}{}^{\dot\alpha}|\psi\rangle, \qquad \langle z|\hat z^M|\psi\rangle = z^M\langle z|\psi\rangle,$$
$$\langle z|p\rangle = \frac{1}{\pi^2}e^{ip_nx^n + ip_{\theta\alpha}\theta^\alpha + ip_{\bar\theta}{}^{\dot\alpha}\bar\theta_{\dot\alpha}} \equiv \frac{1}{\pi^2}e^{ipz}, \qquad (62)$$

with $|\psi\rangle$ an arbitrary state. Derivatives with respect to the odd variables entering into Eq. (62) are defined to be left ones. The simplest realization of the relations (62) is given by $|\psi\rangle = \psi(z) = \psi(x, \theta, \bar\theta)$, $|z\rangle_{z_0} = \delta^8(z - z_0)$, $\langle\psi_1|\psi_2\rangle = \int d^8z\bar\psi_1(z)\psi_2(z)$. In the explicit representation (62) Eq. (61) acquires the form

$$G(z_{out}, z_{in}) = \frac{1}{16}\langle z_{out}| \begin{pmatrix} 0 & (\hat p^2)^{-1}\hat{p'}_{\bar\theta}{}^2\hat{p'}_\theta{}^2 \\ (\hat p^2)^{-1}\hat{p'}_\theta{}^2\hat{p'}_{\bar\theta}{}^2 & 0 \end{pmatrix}|z_{in}\rangle, \qquad (63)$$

where we denoted $\hat{p'}_\theta = \hat p_\theta - i\hat p_n\sigma^n\hat{\bar\theta}$, $\hat{p'}_{\bar\theta} = \hat p_{\bar\theta} + i\hat\theta\sigma^n\hat p_n$. Decomposing then the matrix involved into the superpropagator in the sum of $1+1$ γ-matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\{\gamma^n, \gamma^m\} = 2\eta^{nm}, \qquad \eta^{nm} = (+, -);$$
$$\begin{pmatrix} 0 & \hat{p'}_{\bar\theta}{}^2\hat{p'}_\theta{}^2 \\ \hat{p'}_\theta{}^2\hat{p'}_{\bar\theta}{}^2 & 0 \end{pmatrix} = \hat{p'}_{\bar\theta}{}^2\hat{p'}_\theta{}^2\frac{1}{\sqrt 2}\gamma^- + \hat{p'}_\theta{}^2\hat{p'}_{\bar\theta}{}^2\frac{1}{\sqrt 2}\gamma^+,$$

$$(64)$$

where as usual $\gamma^{\pm} = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^1)$, we can conveniently rewrite Eq. (63) in the following equivalent form

$$G(z_{out}, z_{in}) = \frac{1}{16\sqrt{2}} e^{\gamma^+ \frac{\partial}{\partial \mu}} e^{\gamma^- \frac{\partial}{\partial \nu}} \langle z_{out} | (\hat{p}^2)^{-1} \left(\mu \hat{p}_\theta'^2 \hat{p}_{\bar\theta}'^2 + \nu \hat{p}_{\bar\theta}'^2 \hat{p}_\theta'^2 \right) | z_{in} \rangle |_{\mu=\nu=0},$$

(65)

where μ, ν is a pair of auxiliary Grassmann variables, the corresponding derivatives being left ones.

Now we are in a position to represent Eq. (65) as a path integral. Making use of the Schwinger formula for an inverse operator

$$B^{-1} = i \int_0^\infty ds\, e^{-is(B-i\varepsilon)}, \qquad \varepsilon \to 0, \tag{66}$$

and integration over odd variables (below we use Eq. (67) for the odd operator F, hence χ and F will anticommute by definition)

$$F = -i \int d\chi e^{i\chi F}, \tag{67}$$

one gets[6]

$$G(z_{out}, z_{in}) = \frac{1}{32\sqrt{2}} e^{\gamma^+ \frac{\partial}{\partial \mu}} e^{\gamma^- \frac{\partial}{\partial \nu}} \int_0^\infty ds \int d\chi \langle z_{out} | e^{-i\hat{H}(s,\chi)} | z_{in} \rangle |_{\mu=\nu=0}, \quad (68a)$$

$$\hat{H}(s,\chi) = \frac{s}{2} \hat{p}^2 - \chi \mu \hat{p}_\theta'^2 \hat{p}_{\bar\theta}'^2 - \chi \nu \hat{p}_{\bar\theta}'^2 \hat{p}_\theta'^2. \tag{68b}$$

After the standard substitution like $e^{-i\hat{H}} = \left[e^{-i\hat{H}/N} \right]^N$ and the repeated use of the completeness relation $\int d^8 z |z\rangle\langle z| = 1$ this yields

$$G(z_{out}, z_{in}) = \frac{1}{32\sqrt{2}} e^{\gamma^+ \frac{\partial}{\partial \mu}} e^{\gamma^- \frac{\partial}{\partial \nu}} \int_0^\infty ds \int d\chi \int \prod_{k=1}^{N-1} d^8 z_k \langle z_{out} | e^{-i\hat{H}(s,\chi)/N} | z_{N-1} \rangle$$

$$\times \ldots \langle z_1 | e^{-i\hat{H}(s,\chi)/N} | z_{in} \rangle |_{\mu=\nu=0}. \tag{69}$$

It is worth mentioning that the second term entering in Eq. (68b) is a $p_\theta \theta-$ and $\bar\theta p_{\bar\theta}-$ordered operator. Vice versa, the third term is a $\theta p_\theta-$ and $p_{\bar\theta} \bar\theta-$ordered one. This provides the following integral representation (see Ref. 36) for the matrix elements involved into Eq. (69) (note that the integration over odd variable χ in Eq. (69) guarantees that symbol of the exponential can be replaced by the exponential of the symbol)

$$\langle z_k | e^{-i\hat{H}(s,\chi)/N} | z_{k-1} \rangle = \frac{1}{\pi^4} \int d^8 p_k \exp\{ip_k(z_k - z_{k-1}) - \frac{is}{2N} p_k^2\}$$

---

[6]Note that unitarity of the G requires χ to be real Grassmann variable while μ, ν to be imaginary ones. In what follows we omit the infinitesimal quantity ε.

$$+i\frac{\chi\mu}{N}(p_{\theta k} - ip_{nk}\sigma^n\bar{\theta}_k)^2(p_{\bar{\theta}k} + ip_{nk}\theta_{k-1}\sigma^n)^2$$
$$+i\frac{\chi\nu}{N}(p_{\theta k} - ip_{nk}\sigma^n\bar{\theta}_{k-1})^2(p_{\bar{\theta}k} + ip_{nk}\theta_k\sigma^n)^2\} =$$
$$= \frac{1}{\pi^4}\int d^8p_k \exp\{ip_k(z_k - z_{k-1}) - \frac{is}{2N}p_k{}^2 + i\frac{\chi\mu}{N}f_{(\bar{l}r)k} + i\frac{\chi\nu}{N}f_{(l\bar{r})k}\},$$

$$(70)$$

where we formally attached the labels l, r to the classical function $f(\theta, p_\theta; \bar{\theta}, p_{\bar{\theta}}) \equiv (p_\theta - ip_n\sigma^n\bar{\theta})^2(p_{\bar{\theta}} + ip_n\theta\sigma^n)^2$ which specify the calculation prescription for the latter. In particular, $\bar{l}$ means that the argument $\bar{\theta}$ of the function f should be taken in the left point of the interval $[z_k, z_{k-1}]$, while r implies the same for θ in the right point. In considering a path integral below, these labels will determine the descretization prescription.

Now, it only remains to attach indecies to the variables s, χ, μ, ν. For this purpose we can employ the technique developed in Refs. [7,8]. In particular, the ordinary integration over s variable can be transformed into the path integral by introducing N additional integrations over new auxiliary bosonic variables e_k, $k = 1, \ldots, N$ followed by the subsequent use of integral representation for the δ-function

$$G(z_{out}, z_{in}) = \frac{1}{32\sqrt{2}}e^{\gamma^+\frac{\partial}{\partial\mu}}e^{\gamma^-\frac{\partial}{\partial\nu}}\int_0^\infty ds\int d\chi\int\prod_{k=1}^{N-1}d^8z_k\prod_{k=1}^N\frac{d^8p_k}{\pi^4}\prod_{k=1}^N de_k$$
$$\times \exp\{ip_N(z_N - z_{N-1}) - i\frac{e_N}{2N}p_N{}^2 + i\frac{\chi\mu}{N}f_{(\bar{l}r)N} + i\frac{\chi\nu}{N}f_{(l\bar{r})N}\}\ldots$$
$$\times \exp\{ip_1(z_1 - z_0) - i\frac{e_1}{2N}p_1{}^2 + i\frac{\chi\mu}{N}f_{(\bar{l}r)1} + i\frac{\chi\nu}{N}f_{(l\bar{r})1}\}$$
$$\times \delta(e_N - e_{N-1})\delta(e_{N-1} - e_{N-2})\ldots\delta(e_1 - s)|_{\mu=\nu=0} =$$
$$= \frac{1}{32\sqrt{2}}e^{\gamma^+\frac{\partial}{\partial\mu}}e^{\gamma^-\frac{\partial}{\partial\nu}}\int_0^\infty ds\int d\chi\int\prod_{k=1}^{N-1}d^8z_k\prod_{k=1}^N\frac{d^8p_k}{\pi^4}\prod_{k=1}^N de_k\prod_{k=1}^N\frac{d\pi_k}{2\pi}$$
$$\times \exp\{ip_N(z_N - z_{N-1}) - i\frac{e_N}{2N}p_N{}^2 + i\frac{\chi\mu}{N}f_{(\bar{l}r)N} + i\frac{\chi\nu}{N}f_{(l\bar{r})N} + i\pi_N(e_N - e_{N-1})\}$$
$$\times \ldots\exp\{ip_1(z_1 - z_0) - i\frac{e_1}{2N}p_1{}^2 + i\frac{\chi\mu}{N}f_{(\bar{l}r)1} + i\frac{\chi\nu}{N}f_{(l\bar{r})1} + i\pi_1(e_1 - e_0)\}|_{\mu=\nu=0},$$

$$(71)$$

where $z_N \equiv z_{out}, z_0 \equiv z_{in}, e_0 \equiv s$. The odd variables χ, μ, ν can be treated in a similar way and we finally get

$$G(z_{out}, z_{in}) = \frac{1}{32\sqrt{2}}e^{\gamma^+\frac{\partial}{\partial\mu}}e^{\gamma^-\frac{\partial}{\partial\nu}}\int_0^\infty ds\int d\chi\int\prod_{k=1}^{N-1}d^8z_k\prod_{k=1}^N\frac{d^8p_k}{\pi^4}\prod_{k=1}^N de_k$$
$$\times \prod_{k=1}^N\frac{d\pi_k}{2\pi}\prod_{k=1}^N(d\lambda_k d\xi_k)\prod_{k=1}^N(d\sigma_k d\omega_k)\prod_{k=1}^N(d\tilde{\sigma}_k d\tilde{\omega}_k)$$

$$\times \exp i \sum_{k=1}^{N} \{p_k(z_k - z_{k-1})/\Delta t - \frac{e_k}{2}p_k{}^2 + \lambda_k \sigma_k f_{(\bar{l}r)k} + \lambda_k \tilde{\sigma}_k f_{(l\bar{r})k}$$
$$+\pi_k(e_k - e_{k-1})/\Delta t - i\xi_k(\lambda_k - \lambda_{k-1})/\Delta t - i\omega_k(\sigma_k - \sigma_{k-1})/\Delta t$$
$$-i\tilde{\omega}_k(\tilde{\sigma}_k - \tilde{\sigma}_{k-1})/\Delta t\}\Delta t \mid_{\mu=\nu=0}, \qquad (72)$$

where $\Delta t \equiv \frac{1}{N}$ and it is implied that the boundary conditions

$$z_0 = z_{in}, \qquad z_N = z_{out},$$
$$e_0 = s, \qquad \lambda_0 = \chi,$$
$$\sigma_0 = \mu, \qquad \tilde{\sigma}_0 = \nu, \qquad (73)$$

are satisfied.

In the limit of large N one arrives at the path integral

$$G(z_{out}, z_{in}) = \frac{1}{32\sqrt{2}} e^{\gamma^+ \frac{\partial}{\partial\mu}} e^{\gamma^- \frac{\partial}{\partial\nu}} \int_0^\infty ds \int d\chi \int Dz\,Dp\,De\,D\pi\,D\lambda\,D\xi\,D\sigma\,D\omega \times$$
$$\times D\tilde{\sigma}\,D\tilde{\omega}\, e^{i(S_{ssp}+S_{gf})},$$
$$\qquad (74a)$$

$$S_{ssp} = \int_0^1 d\tau (p_n \dot{x}^n + p_{\theta_\alpha}\dot{\theta}^\alpha + p_{\bar{\theta}}{}^{\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}} - \frac{e}{2}p^2$$
$$+\lambda\sigma(p_\theta - ip_n\sigma^n\bar{\theta})^2 (p_{\bar{\theta}} + ip_n\theta\sigma^n)^2_{\bar{l}r} \qquad (74b)$$
$$+\lambda\tilde{\sigma}(p_\theta - ip_n\sigma^n\bar{\theta})^2 (p_{\bar{\theta}} + ip_n\theta\sigma^n)^2_{l\bar{r}}),$$

$$S_{gf} = \int_0^1 d\tau (\pi\dot{e} - i\xi\dot{\lambda} - i\omega\dot{\sigma} - i\tilde{\omega}\dot{\tilde{\sigma}}), \qquad (74c)$$

provided the boundary conditions

$$z(0) = z_{in}, \qquad z(1) = z_{out},$$
$$e(0) = s, \qquad \lambda(0) = \chi,$$
$$\sigma(0) = \mu, \qquad \tilde{\sigma}(0) = \nu, \qquad (74d)$$

hold. Denoting in Eq. (74b) $p_\theta = \rho$, $p_{\bar{\theta}} = \bar{\rho}$ we recover the classical theory (18) in the gauge $\psi = 0$, $\bar{\psi} = 0$. The role of the second term S_{gf} in the path integral (74a) is to fix additional gauge invariance which enters into the problem when adding the variables $\lambda, \sigma, \tilde{\sigma}$ to the original model.

Some comments are in order. First, the gauge fields $\psi, \bar{\psi}$ can easily be restored in the action (74b) by including the gauge fixing conditions $\psi = 0$, $\bar{\psi} = 0$ into Eq. (72) via the δ–function. Second, the labels l, r entering in Eq. (74b), although being inessential in classical theory, play an important role at the quantum level and determine the descretization prescription for the path integral (74a). Moreover, the change of the variables $p'_\theta = p_\theta - ip_n\sigma^n\bar{\theta}$, $p'_{\bar{\theta}} = p_{\bar{\theta}} + ip_n\theta\sigma^n$ in the path integral (74a), which would bring the action to the original Siegel one, is problematic in view of the descretization

prescription. This supports, in particular, the advantage of the geometric formulation [24]. Third, the higher order fermionic constraints, although being not independent in $d = 4$, make a crucial contribution into the path integral.

7 Concluding remarks

Thus, in this paper we have considered operator quantization of the Siegel superparticle in $R^{4|4}$ flat superspace. Quantum states of the model were proven to be the sum of on-shell chiral and antichiral superfields, the corresponding effective field theory being the massless Wess–Zumino model in the component form. Path integral representation for the superfield propagator was constructed and shown to involve the Siegel action in a gauge fixed form. As a further development, it is tempting to compare the result with that of the straightforward BFV quantization combined with the scheme [23]. Another open problem is the generalization of the present analysis to the case of the model coupled to arbitrary external superfields, where the construction of the path integral representation for the superpropagator is known to be much more involved.

Due to the relation to superstring theory, the $10d$ case is of prime interest. The operator quantization presented in this work is rather specific in four dimensions. We hope, however, that BFV path integral quantization will proceed along the same lines both in $4d$ and $10d$. The results on this subject will be presented elsewhere.

References

[1] F.A. Berezin and M.S. Marinov, JETP Lett. **21** (1975) 320; Ann. Phys. (N.Y.) **104** (1977) 336.

[2] L. Brink, S. Deser, B. Zumino, P. Di Vecchia, and P. Howe, Phys. Lett. B **64** (1976) 435;

L. Brink, P. Di Vecchia, and P. Howe, Nucl. Phys. B **118** (1977) 76;
R. Casalbuoni, Nuovo Cimento A **33** (1976) 115;
A. Barducci, R. Casalbuoni, and L. Lusanna, Nuovo Cimento A **35** (1976) 377.

[3] P.P. Srivastava, Nuovo Cimento Lett. **19** (1977) 239;
A.P. Balachandran, P. Salomonson, B. Skagerstam and J. Winnberg, Phys. Rev. **D15**, (1977) 2308;
V.D. Gershun and V.I. Tkach, Pis'ma Zh. Eksper.Teor.Fiz.**29** (1979) 320;
A. Barducci and L. Lusanna, Nuovo Cimento A **77** (1983) 39; J. Phys. A **16** (1983) 1993;
J. Gomis, M. Novell, and K. Rafanelli, Phys. Rev. D **34** (1986) 1072;
P.S. Howe, S. Penati, M. Pernici, and P. Townsend, Phys. Lett. B **215** (1988) 255;
J.W. van Holten, Z. Phys. C. **41** (1988) 497.

[4] D.M. Gitman and I.V. Tyutin, Class. Quantum Grav. **7** (1990) 2131; JETP Lett. **51** (1990) 214;
D.M. Gitman and A.V. Saa, Mod. Phys. A **8** (1993) 463;
D.M. Gitman, A.E. Goncalves, and I.V. Tyutin, Phys. Rev. D **50** (1994) 5439; Int. J. Mod. Phys. A **10** (1995) 701; Phys. of Atom. Nucl. **60** (1997) 748.

[5] D.M. Gitman and I.V. Tyutin, Int. J. Mod. Phys. A **12** (1997) 535; Mod. Phys. Lett. A **11** (1996) 381;
D.M. Gitman and A.E. Goncalves, J. Math. Phys. **38** (1997) 2167; Int. J. Theor. Phys. **35** (1996) 2427;
D.M. Gitman, Nucl. Phys. B (Proc. Suppl.) **57** (1997) 231;
Pseudoclassical theory of relativistic spinning particle, in Topics in Satistical and Theoretical Physics, F.A. Berezin Memorial vol. (Amer. Math. Soc., Providence, PI, 1996).

[6] M.S. Plyushchay, Phys. Lett. B **236** (1990) 291; B **248** (1990) 299; Mod. Phys. Lett. A **8** (1993) 937;
R. Marnelius and U. Märtensson, Nucl. Phys. B **335** (1991) 395; Int. J. Mod. Phys. A **6** (1991) 807;
J.W. van Holten, Int. J. Mod. Phys. A **7** (1992) 7119;
S.M. Kuzenko, S.L. Lyakhovich, and A. Yu. Segal, Int. J. Mod. Phys. A **10** (1995) 1529;

G. Grigorian, R. Grigorian, and I.V. Tyutin, Teor. Mat. Fiz. **111** (1997) 389.

[7] D.M. Gitman, Nucl. Phys. B **488** (1997) 490.

[8] E.S. Fradkin and D.M. Gitman, Phys. Rev. D. **44** (1991) 3230;
D.M. Gitman and A.V. Saa, Class. Quan. Grav. **10** (1993) 1447;
D.M. Gitman and Sh.M. Shvartsman, Phys. Lett. B **318** (1993) 122;
D.M. Gitman and S.I. Zlatev, Phys. Rev. D **55** (1997) 7701.

[9] A.T. Ogielski and J. Sobczuk, J. Math. Phys. **22** (1981) 2060;
M. Henneaux, C. Teitelboim, Ann. Phys. **143** (1982) 127;
N.V. Borisov and P.P. Kulish, Teor. Math. Fiz. **51** (1982) 335;
A.M. Polyakov, *Gauge Fields and Strings*, (Harwood, Chur, Switzerland, 1987);
J.C. Henty, P.S. Howe, and P.K. Townsend, Class. Quan. Grav. **5** (1988) 807;
V.Ya. Fainberg and A.V. Marshakov, JETP Lett. **47** (1988) 565; Phys. Lett. **211B** (1988) 81; Nucl. Phys. **B306** (1988) 659;
T.M. Aliev, V.Ya. Fainberg and N.K. Pak, Nucl. Phys. **B429** (1994) 321;
E. D'Hoker and D.G. Gagné, Nucl. Phys. **B467** (1996) 272; 297.

[10] J.W. van Holten, Nucl. Phys. **B457** (1995) 375;
NIKHEF-H/95-055; Proceedings of 29th Int. Symposium on the Theory of Elementary Particles, Buckow-1995;
Grassmann algebras and spin in quantum dynamics, Lecture notes, AIO School Math. Phys.,(Univ. of Twente, 1992).

[11] L. Brink, M. Henneaux, and C. Teitelboim, Nucl. Phys. B **293** (1987) 505;
C.M. Hull and J.L. Vázquez-Bello, Nucl. Phys. B **416** (1994) 173;
A.A. Deriglazov, A.V. Galajinsky, and S.L. Lyakhovich, Nucl. Phys. B **473** (1996) 245.

[12] R. Casalbuoni, Phys. Lett. B **62** (1976) 49.
L. Brink and J.H. Schwarz, Phys. Lett. B **100** (1981) 310.

[13] Y. Eisenberg and S. Solomon, Nucl. Phys. B **309** (1988) 709;
F. Delduc, A. Galperin, and E. Sokatchev, Nucl. Phys. B **368** (1992) 143;

V.P. Akulov, D.P. Sorokin, and I.A. Bandos, Mod. Phys. Lett. A **3** (1988) 1633.

[14] W. Siegel, Class. Quant. Grav. **2** (1985) L 95.

[15] T.J. Allen, Mod. Phys. Lett. A **2** (1987) 209.

[16] W. Siegel, Nucl. Phys. B **263** (1985) 93; Phys. Lett. B **203** (1988) 79.

[17] A.R. Miković and W. Siegel, Phys. Lett. B **209** (1988) 47.

[18] R. Kallosh, W. Troost, and A. Van Proeyen, Phys. Lett. B **212** (1988) 212.

[19] U. Lindström, M. Rocek, W. Siegel, P. van Nieuwenhuizen, and A.E. van de Ven, J. Math. Phys. **31** (1990) 1761;
A. Miković, M. Rocek, W. Siegel, P. van Nieuwenhuizen, J. Yamron, and A.E. van de Ven, Phys. Lett. B **235** (1990) 106;
F. Ebler, M. Hatsuda, E. Laenen, W. Siegel, J.P. Yamron, T. Kimura, and A. Miković, Nucl. Phys. B **364** (1991) 67.

[20] M.B. Green and C.M. Hull, Nucl. Phys. B **344** (1990) 115.

[21] A. Duncan and M. Moshe, Nucl. Phys. B **268** (1986) 706.

[22] E.S. Fradkin and Sh.M. Shvartsman, Mod. Phys. Lett. A **6** (1991) 1977.

[23] A.A. Deriglazov and A.V. Galajinsky, Phys. Lett. B **381** (1996) 105; Phys. Rev. D **54** (1996) 5195.

[24] A.A. Deriglazov and A.V. Galajinsky, Mod. Phys. Lett. A **9** (1994) 3445.

[25] P. van Nieuwenhuizen, Phys. Rep. **68** (1981) 189.

[26] I.L. Buchbinder and S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*, (Institute of Physics Publishing, Bristol and Philadelphia, 1995).

[27] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Yeshiva University, Belfer Graduate School of Science (Academic Press, New York, 1964).

[28] D.M. Gitman and I.V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, 1990).

[29] J. Wess and B. Zumino, Nucl. Phys. B **70** (1974) 39.

[30] J. Wess and J. Bagger *Supersymmetry and Supergravity*, (Princeton University Press, Princeton, 1983).

[31] S. Aoyama, J. Kowalski–Glikman, J. Lukierski, and J.W. van Holten, Phys. Lett. B **216** (1989) 133; B **201** (1988) 487.

[32] S. Bellucci and A.V. Galajinsky, hep-th/9712247, Phys. Lett. B (to appear).

[33] R. Marnelius and Sh.M. Shvartsman, Nucl. Phys. B **430** (1994) 153.

[34] I. Bengtsson, Phys. Rev. D **39** (1989) 1158.

[35] J. Schwinger, Phys. Rev. 82 (1951) 664.

[36] F.A. Berezin, Uspekhi Fiz. Nauk. **132** (1980) 497;
F.A. Berezin and M.A. Shubin, *Schrödinger Equation* (Moscow State University, Moscow, 1983).