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TIME-DEPENDENT VARIATIONAL PRINCIPLE FOR ϕ^4 FIELD THEORY: RPA APPROXIMATION AND RENORMALIZATION (II)

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ABSTRACT

The Gaussian-time-dependent variational equations are used to explored the physics of $(\phi^4)_{3+1}$ field theory. We have investigated the static solutions and discussed the conditions of renormalization. Using these results and stability analysis we show that there are two viable non-trivial versions of $(\phi^4)_{3+1}$. In the continuum limit the bare coupling constant can assume $b \to 0^+$ and $b \to 0^-$, which yield well defined asymmetric and symmetric solutions respectively. We have also considered small oscillations in the broken phase and shown that they give one and two meson modes of the theory. The resulting equation has a closed solution leading to a "zero mode" and vanished scattering amplitude in the limit of infinite cutoff.

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I. Introduction

In a recent paper [1](hereafter referred to as I) we have obtained the RPA equations for ϕ^4 field theory by linearizing the time-dependent variational equations. The method was implemented for the case of symmetric vacuum, $\langle\phi\rangle = 0$, which allows us to investigate two-meson physics. We have shown that it is a simple nonperturbative method to study scattering processes. Using this framework the problem of stability of vacuum can be explored from the RPA modes. In continuation of I we will consider here the stability of the theory for other critical points of the Gaussian parameter space[2]. In particular, we discuss the solutions and renormalization conditions for the asymmetric vacuum. In this case, the excitations of the vacuum are identified with the one and two-particle wavefunctions and the system of equations can also be solved analytically. As result, a stable zero mode is found for certain range of renormalized coupling constant and as well as a complete form of the scattering amplitude.

For completeness, let us first repeate here the key equations of I. The bare parameter hamiltonian for the ϕ^4 theory is [We use the notation: $\int_{\mathbf{x}} = \int d^3x$]

$$\hat{H} = \int_{\mathbf{x}} \Big(\frac{1}{2}\hat{\pi}^2(\mathbf{x}) + \frac{1}{2}(\nabla\hat{\phi}(\mathbf{x}))^2 + \frac{a}{2}\hat{\phi}^2(\mathbf{x}) + \frac{b}{24}\hat{\phi}^4(\mathbf{x}) \Big) . \tag{1.1}$$

As an approximation, we take a Gaussian trial wave functional

$$\psi(\phi,t) = N exp\Big\{ - \int_{\mathbf{x},\mathbf{y}} \big[\phi(\mathbf{x}) - \phi_0(\mathbf{x},t)\big]\big[\frac{G^{-1}(\mathbf{x},\mathbf{y},t)}{4} - i\Sigma(\mathbf{x},\mathbf{y},t)\big]\big[\phi(\mathbf{y}) - \phi_0(\mathbf{y},t)\big] + i\int_{\mathbf{x}} \pi_0(\mathbf{x},t)\big[\phi(\mathbf{x}) - \phi_0(\mathbf{x},t)\big] \Big\} , \tag{1.2}$$

where N is a normalization and $\phi_0(\mathbf{x},t), \pi_0(\mathbf{x},t), G(\mathbf{x},\mathbf{y},t)$ and $\Sigma(\mathbf{x},\mathbf{y},t)$ are our variational parameters. These quantities are related to the following mean-values:

$$\langle\psi,t\mid\hat{\phi}(\mathbf{x})\mid\psi,t\rangle=\phi_0(\mathbf{x},t)\,, \tag{1.3}$$

$$\langle\psi,t\mid\hat{\pi}(\mathbf{x})\mid\psi,t\rangle=\pi_0(\mathbf{x},t)\,, \tag{1.4}$$

$$\langle\psi,t\mid\hat{\phi}(\mathbf{x})\hat{\phi}(\mathbf{y})\mid\psi,t\rangle=G(\mathbf{x},\mathbf{y},t)+\phi_0(\mathbf{x},t)\phi_0(\mathbf{y},t)\,, \tag{1.5}$$

$$\langle\psi,t\mid i\frac{\delta}{\delta t}\mid\psi,t\rangle=\int_{\mathbf{x},\mathbf{y}}\Sigma(\mathbf{x},\mathbf{y},t)\dot{G}(\mathbf{x},\mathbf{y},t)+\int_{\mathbf{x}}\pi_0(\mathbf{x},t)\dot{\phi}_0(\mathbf{x},t)\,. \tag{1.6}$$

From (1.1) and (1.2) one can compute the effective hamiltonian,

$$\mathcal{H}=\langle\psi,t\mid\hat{H}\mid\psi,t\rangle, \tag{1.7}$$

which is the energy of the system.

$$\begin{aligned}\mathcal{H} &= \int_{\mathbf{x}}\Big[\frac{1}{2}\pi_0^2(\mathbf{x},t)+\frac{1}{2}(\nabla\phi_0(\mathbf{x},t))^2+\frac{a}{2}\phi_0^2(\mathbf{x},t)+\frac{b}{24}\phi_0^4(\mathbf{x},t)\Big]\\ &+ \frac{1}{8}\int_{\mathbf{x}}G^{-1}(\mathbf{x},\mathbf{x},t)+2\int_{\mathbf{x},\mathbf{y},\mathbf{z}}\Sigma(\mathbf{z},\mathbf{x},t)G(\mathbf{x},\mathbf{y},t)\Sigma(\mathbf{y},\mathbf{z},t)\\ &+ \frac{1}{2}\int_{\mathbf{x}}[-\nabla_{\mathbf{x}}^2+a+\frac{b}{2}\phi_0^2(\mathbf{x},t)]G(\mathbf{x},\mathbf{y},t)|_{\mathbf{x}=\mathbf{y}}+\frac{b}{8}\int_{\mathbf{x}}G(\mathbf{x},\mathbf{x},t)G(\mathbf{x},\mathbf{x},t)\,.\end{aligned} \tag{1.8}$$

The variational equations of motion read

$$\dot{\phi}_0(\mathbf{x},t) = \pi_0(\mathbf{x},t)\,, \tag{1.9}$$

$$\dot{\pi}_0(\mathbf{x},t) = \nabla^2\phi_0(\mathbf{x},t)-a\phi_0(\mathbf{x},t)-\frac{b}{6}\phi_0^3(\mathbf{x},t)-\frac{b}{2}\phi_0(\mathbf{x},t)G(\mathbf{x},\mathbf{x},t)\,, \tag{1.10}$$

$$\dot{G}(\mathbf{x},\mathbf{y},t) = 2\int_{\mathbf{z}}\Big[G(\mathbf{x},\mathbf{z},t)\Sigma(\mathbf{z},\mathbf{y},t)+\Sigma(\mathbf{x},\mathbf{z},t)G(\mathbf{z},\mathbf{y},t)\Big]\,, \tag{1.11}$$

$$\begin{aligned}\dot{\Sigma}(\mathbf{x},\mathbf{y},t) &= -2\int_{\mathbf{z}}\Sigma(\mathbf{y},\mathbf{z},t)\Sigma(\mathbf{z},\mathbf{x},t)+\frac{1}{8}G^{-2}(\mathbf{x},\mathbf{y},t)\\ &+ \frac{1}{2}\Big[\nabla_{\mathbf{x}}^2-a-\frac{b}{2}\phi_0^2(\mathbf{x},t)-\frac{b}{2}G(\mathbf{x},\mathbf{x},t)\Big]\delta(\mathbf{x}-\mathbf{y})\,.\end{aligned} \tag{1.12}$$

These are nonlinear time-dependent field equations. Therefore, a closed solution is not easily constructed. Here, we will consider the equations in the equilibrium situation and the small oscillation regime. In these cases, an explicit solution can be obtained allowing us to examine diverse properties of the theory. The structure of this paper is as follows. Section II discusses the time independent Gaussian equations. Particular attention is paid to the question of the renormalization and we study solutions for the resulting gap equation. We also discuss the so called Gaussian effective potential for the case when the bare coupling constant $b\to 0^+$. Section III discusses stability of the stationary points found in the section II by investigating the properties of variational space. In Section IV we shall derive the RPA equation by considering near equilibrium dynamics about the critical points. Since the excitations of vacuum are quantum particles, Section V will solve the RPA equation with boundary conditions of scattering processes. In Sec VI we will use the properties of separable potential to get solutions for the $\mathbf{T}$ matrix. The spectrum of RPA modes is discussed using this result and conditions of stability will be analysed within this context.

II. Time-Independent Gaussian Variational Equations

This section will discuss the variational equations, Eqs.(1.8)-(1.12), in the equilibrium situation. We investigate the solutions of these equations and study the renormalization conditions. We will consider ϕ_0 uniform, $\phi_0(\mathbf{x})\equiv\varphi$, and parametrize the kernel G as $\frac{1}{4}G^{-2}(\mathbf{x},\mathbf{y})\equiv\langle\mathbf{x}\mid\hat{p}^2+m^2\mid\mathbf{y}\rangle$. Therefore, the parameters φ and m are taken to be independent of $\mathbf{x}$. With these assumptions the expectation value of H, (1.8), with kinetic term set to zero, reads as

$$U(\varphi,m) = \int_{\mathbf{x}} u(\varphi,m)\,,$$
$$u(\varphi,m) = \frac{1}{4}G^{-1}(m) - \frac{m^2}{2}G(m) + \frac{1}{2b}\Big[a + \frac{b}{2}G(m)\Big]^2 + \frac{1}{2}\Big[a + \frac{b}{2}G(m)\Big]\varphi^2 + \frac{b}{24}\varphi^4\,. \tag{2.1}$$

In this equation $G(m)$ and $G^{-1}(m)$ are defined as $\langle \mathbf{x} \mid G \mid \mathbf{x}\rangle$ and $\langle \mathbf{x} \mid G^{-1} \mid \mathbf{x}\rangle$ respectively. By virtue of parametrization of G these matrix elements are given by

$$G(m) = \langle \mathbf{x} \mid \frac{1}{2\sqrt{\hat{p}^2 + m^2}} \mid \mathbf{x}\rangle = \frac{1}{8\pi^2}\left[\Lambda_{\mathbf{p}}^2 - m^2 \log(\frac{2\Lambda_{\mathbf{p}}}{\sqrt{e}m})\right]\,, \tag{2.2}$$

$$G^{-1}(m) = \langle \mathbf{x} \mid 2\sqrt{\hat{p}^2 + m^2} \mid \mathbf{x}\rangle = \frac{1}{8\pi^2}\left[2\Lambda_{\mathbf{p}}^4 + 2m^2\Lambda_{\mathbf{p}}^2 - m^2\log(\frac{2\Lambda_{\mathbf{p}}}{\sqrt{e}m}) - \frac{m^4}{4}\right]\,, \tag{2.3}$$

where $\Lambda_{\mathbf{p}}$ is a cutoff in the integrals. Notice that the energy density u is a function of two variables φ and m. The next step is to obtain the critical points of this functions and find the renormalization conditions.

II–a. Gap Equation and Renormalization

Minimization of u with respect to φ and m yields

$$\frac{\partial u}{\partial \varphi} = \varphi\Big(a + \frac{b}{6}\varphi^2 + \frac{b}{2}G(m)\Big) = 0\,, \tag{2.4}$$

$$\frac{\partial u}{\partial m} = mG'(m)\Big[a + \frac{b}{2}\varphi^2 + \frac{b}{2}G(m) - m^2\Big] = 0\,, \tag{2.5}$$

where $G'(m) = dG(m)/dm^2$. Notice that one may have several sets of solutions for the system of equations (2.4-2.5).

By analysing the stability matrix (see section III) one can show that the solution $m = 0$ of (2.5) is always a saddle point of $u(\varphi, m)$ because the determinant is negative in this case. Thus, the possible stable points of (2.1) are determined through the gap equation, defined by

$$m^2 = a + \frac{b}{2}\varphi^2 + \frac{b}{2}G(m)\,, \tag{2.6}$$

and (2.4). From its solutions, $\varphi = 0$ and

$$a + \frac{b}{6}\varphi^2 + \frac{b}{2}G(m) = 0, \tag{2.7}$$

one can eliminate φ in favor of m and arrive at the following key equations to be solved,

$$m^2 = \alpha\Big(a + \frac{b}{2}G(m)\Big)\,. \tag{2.8}$$

In this expression, we have parametrized the two solutions of (2.4) as follows: $\alpha = 1$ when $\varphi = 0$ and $\alpha = -2$ for $b\varphi^2 = -6[a + b/2G(m)]$. On the other hand, by combining (2.6) and (2.7) it is easy to show that

$$b\varphi^2 = 3m^2 \tag{2.9}$$

which we define as the broken phase. Notice that this phase might degenerate with the massless symmetric phase if $m = 0$ is a solution of (2.8). We will come back to this point later.

Let us now examine the renormalization conditions for the gap equation (2.8). First, we use (2.2) to rewrite (2.8) as

$$\frac{m^2}{\alpha} = a + \frac{b\Lambda_{\mathbf{p}}^2}{18\pi^2} - \frac{bm^2}{16\pi^2}\log\frac{2\Lambda_{\mathbf{p}}}{\sqrt{e}m}\,. \tag{2.10}$$

Next we introduce an arbitrary mass scale μ by writing $\log(\Lambda_P/m) = \log(\Lambda_P/\mu) + \log(\mu/m)$, the equation (2.10) becomes

$$m^2 = \frac{a + \frac{b\Lambda_P^2}{16\pi^2}}{\frac{1}{\alpha} + \frac{b}{16\pi^2}\log\frac{2\Lambda_P}{\sqrt{e}\mu}} + \frac{\frac{b}{32\pi^2}}{\frac{1}{\alpha} + \frac{b}{16\pi^2}\log\frac{2\Lambda_P}{\sqrt{e}\mu}} m^2 \log\frac{m^2}{\mu^2}. \tag{2.11}$$

This equation renders finite result if we choose the following self-consistency condition

$$\epsilon\mu^2 = \frac{a + \frac{b\Lambda_P^2}{16\pi^2}}{\frac{1}{\alpha} + \frac{b}{16\pi^2}\log\frac{2\Lambda_P}{\sqrt{e}\mu}}, \qquad \epsilon = 0, \pm 1, \tag{2.12}$$

$$\frac{1}{g_\mu} = \frac{2}{\alpha b} + L_\mu, \qquad L_\mu = \frac{1}{8\pi^2}\log\frac{2\Lambda_P}{e\mu}. \tag{2.13}$$

Thus, the renormalized version of Eq.(2.8) reads

$$m^2 = \epsilon\mu^2 + \frac{\bar{g}_\mu}{1+\bar{g}_\mu} m^2 \log\frac{m^2}{\mu^2}, \tag{2.14}$$

where $\bar{g}_\mu = g_\mu/16\pi^2$ and ϵ indicates different renormalization conditions. The above result can be interpreted as follows. We start with two bare parameters a and b. With the help of the transformations rules (2.12)-(2.13) we arrive at a new set of paramteters μ and g. Being μ a renormalized mass, which also defines a mass scale, and g a dimensionless renormalized coupling constant, in a such way that the resulting gap equation involves finite quantities only. Notice also from (2.12)-(2.14) that two different theories of $(\phi^4)_{3+1}$ are involved. One is the version with $b \propto 1/L_\mu$ $(\alpha = 1)$ and the other is $b \propto -1/L_\mu$ $(\alpha = -2)$ which belong to two distinct field theories [3-4]. In the limit of infinite cutoff the renormalized coupling constant g_μ can assume any value.

II–b. Massive and Massless Solution

In order to discuss the possible solutions of (2.8) and investigate the roles played by ϵ we compute $u(\varphi(m), m)$, being $\varphi(m)$ the solutions of (2.4),

$$\begin{aligned} u(m) &= u(\varphi(m), m) \\ &= \frac{1}{4}G^{-1}(m) - \frac{m^2}{2}G(m) - \frac{\alpha}{2b}\left[a + \frac{b}{2}G(m)\right]^2 \qquad (2.15) \\ &= \frac{1}{128\pi^2}\left(m^4\log\frac{m^4}{\mu^4} - m^4 + \mu^4\right) - \left(\frac{1}{\bar{g}_\mu} + 1\right)\frac{(m^2 - \epsilon\mu^2)^2}{64\pi^2}, \qquad (2.16) \end{aligned}$$

where we have used (2.3) and the counterterms introduced in (2.12)-(2.13)[1,2]. The different behavior of $u(m)$ with $\bar{g}_\mu < 0$ and $\epsilon = 1$ are shown in the fig.1a of I. In this case $u(m = \mu)$ and $u(m = 0)$ are local or true minima depeding on the values of $\bar{g}_\mu$. However $m = 0$ shown there actually corresponds to the massless solution of (2.5), which is a saddle point in the (φ, m) space. In addition, $u(m)$ calculated with $\bar{g}_\mu > 0$ reduce to those in fig.1a if one uses the scale $\bar{m}$ defined as minimum of $u(m^2)$ (see appendix B of I) . A similar consideration can be made also for the case $\epsilon = -1$ as follows.

Recalling (2.13) we can write a relation of renormalized coupling constants at mass scales μ and $\bar{m}$ readly as

$$\frac{1}{\bar{g}_m} = \frac{1}{\bar{g}_\mu} - \log\frac{\bar{m}^2}{\mu^2}, \tag{2.17}$$

where $\bar{m}$ is defined as the minimum of $u(m)$. Combining now (2.14) with (2.17) one can get the following simple relation

$$\bar{m}^2\left(1 + \frac{1}{\bar{g}_m}\right) = -\mu^2\left(1 + \frac{1}{\bar{g}_\mu}\right). \tag{2.18}$$

Using this result we can rewrite (2.16), at scale of $\bar{m}$, as

$$\frac{64\pi^2}{\bar{m}^4}u(m) = \frac{m^4}{\bar{m}^4}(\log\frac{m^2}{\bar{m}^2} - \frac{1}{2}) - (\frac{1}{\bar{g}_m} + 1)(\frac{m^2}{\bar{m}^2} - 1)^2. \tag{2.19}$$

Disregarding some unimportant additive constants this equation is equal to (2.16) in unit of $\bar{m}$. Therefore, this renormalization scheme does not add any new physics.

The case when $\epsilon = 0$ corresponds to the renormalization prescription $a + b\Lambda^2/16\pi^2 = 0$ [2,5]. Thus, the gap equation (2.8) becomes

$$m^2\Big(\frac{1}{\alpha} + \frac{b}{16\pi^2}\log\frac{2\Lambda}{\sqrt{e}m}\Big) = 0 \tag{2.20}$$

which allows a massless solution. It is important to notice that there is only one free parameter involved in this case. Thus, we may fix it using the mass scale $\bar{m}$ defined by the second solution of (2.20), namely

$$\frac{2}{\alpha b} = -\frac{1}{8\pi^2}\log\frac{2\Lambda}{\sqrt{e}\bar{m}}. \tag{2.21}$$

Comparing this to (2.13) and using (2.17) one has $\bar{g}_m = -1$. With these ingredients we find the following the renormalized version of u for this case:

$$u(m) = \frac{m^4}{128\pi^2}(\log\frac{m^4}{\bar{m}^4} - 1). \tag{2.22}$$

On the other hand, one can obtain the above result rewriting directly (2.16) in unit of $\bar{m}$ defined by (2.20). In other words, the case of $\epsilon = 0$ corresponds to the result given by (2.19) with $\bar{g}_m = -1$. Therefore, we can conclude after this analysis that all possible physics situations in $u(\varphi(m), m)$ are contained in the case of $g_m < 0$ and $\epsilon = 1$ when $\bar{m}$ is utilized as the mass scale. It has minima at $m = \bar{m}$, and massless solution $m = 0$, for the particular

case of $\bar{g}_m = -1$. Thus, the time-independent variational equations (2.4)-(2.5) allow simple analytic solution and energy density $u(\varphi, m)$ has equilibrium points at $(\varphi = 0, \bar{m})$ and $(b\varphi^2 = 3\bar{m}^2, \bar{m})$.

II–c. Gaussian Effective Potential

In the previous subsections we have studied the energy density $u(\varphi, m)$ when it is minimum with respect to φ, i.e., $u(\varphi(m), m)$, being $\varphi(m)$ given by (2.4). Now we want to calculate $u(\varphi, m)$ when it is minimum with respect to m. In particular, when the resulting expression is written as a function of φ, it is known as Gaussian Effective Potential (GEP) [3-4]. However, an explicit expression of $m(\varphi)$ from (2.6) is not straighforward. Instead, we write φ in terms of m,

$$\frac{b\varphi^2}{2} = m^2 - a - \frac{b}{2}G(m). \tag{2.23}$$

Substituting this into (2.1) yields

$$u(m) = \frac{1}{4}G^{-1}(m) - \frac{m^2}{2}G(m) - \frac{1}{b}\Big[a + \frac{b}{2}G(m)\Big]^2 + D(m), \tag{2.24}$$

where

$$D(m) = \frac{m^4}{6b} + \frac{2m^2}{3b}\Big[a + \frac{b}{2}G(m)\Big] + \frac{2}{3b}\Big[a + \frac{b}{2}G(m)\Big]^2. \tag{2.25}$$

Notice that $u(m) - D(m)$ is exactly equal to (2.15) for the of case $\alpha = -2$. Using now the renormalization procedure given by (2.12)-(2.13) it is straightforward to show that $D \to \mathcal{O}\big(\frac{1}{L_\mu}\big)$. Therefore, the energy density at the curves defined by (2.6) as well as by (2.7) in the (φ, m) plane has the same result. The coincidence is not casual and let us look at (2.1) more carefully. Recalling the relation (2.9) we note that since m^2 is a finite physical quantity, therefore it suggests that φ requires a scaling factor. Thus, we define a finite mean-field value as

$$b\varphi^2 = \Phi^2 \, . \tag{2.26}$$

Using now (2.12)-(2.13) one can rewrite (2.6) and (2.7) respectively as

$$\Phi^2 = 3m^2 + \frac{1}{L_\mu}\Big[\frac{m^2-\mu^2}{16\pi^2}\Big(\frac{1}{\bar{g}_\mu}+1\Big) - \frac{m^2}{8\pi^2}\log\frac{m}{\mu}\Big] , \tag{2.27}$$

$$\Phi^2 = 3m^2 + \frac{3}{L_\mu}\Big[\frac{m^2-\mu^2}{16\pi^2}\Big(\frac{1}{\bar{g}_\mu}+1\Big) - \frac{m^2}{8\pi^2}\log\frac{m}{\mu}\Big] . \tag{2.28}$$

This result shows that in the limit of infinite cutoff, the two curves converge to $b\varphi^2 = 3m^2$. Using this relation one can write GEP from (2.16) immediately as

$$V_{GEP}(\Phi) = \frac{\Phi^4}{576\pi^2}\Big(\log\frac{\Phi^2}{3\mu^2} - \frac{1}{2}\Big) - \Big(\frac{1}{\bar{g}_\mu}+1\Big)\frac{\Phi^2}{192\pi^2}\Big(\frac{\Phi^2}{3} - 2\mu^2\Big) . \tag{2.29}$$

In particular, (2.29) recovers the GEP obtained by the authors of Ref.[3] after some changes of variables (cf. their Eq.(17)).

III. Stability Analysis

In this section we shall analyse the stability conditions for the solutions obtained in the previous discussion. Since the energy density $u(\varphi, m)$, (2.1), is a function of two variables, we can define its stability matrix as

$$\begin{aligned} A &= \begin{pmatrix} \frac{\partial^2 u}{\partial\varphi\partial\varphi} & \frac{\partial^2 u}{\partial\varphi\partial m} \\ \frac{\partial^2 u}{\partial m\partial\varphi} & \frac{\partial^2 u}{\partial m\partial m} \end{pmatrix} \\ &= \begin{pmatrix} a + \frac{b}{2}G(m) + \frac{b}{2}\varphi^2 & b\varphi m G'(m) \\ b\varphi m G'(m) & [G'(m) + 2m^2 G''(m)]\Gamma + 2m^2 G'(m)[\frac{b}{2}G'(m) - 1] \end{pmatrix} , \end{aligned} \tag{3.1}$$

where

$$\Gamma = a + \frac{b}{2}\varphi^2 + \frac{b}{2}G(m) - m^2 . \tag{3.2}$$

Now the stability analysis reduce to a discussion about the signs of the eigenvalues of A at $u(\bar{\varphi}, \bar{m})$, being $\bar{\varphi}$ and $\bar{m}$ the solutions of (2.4)-(2.5).

Section II has showed that when $\Gamma = 0$, the system might have two solutions at $\varphi = 0$ and $b\bar{\varphi}^2 = 3\bar{m}^2$. For the former case A is diagonal,

$$A = \begin{pmatrix} \bar{m}^2 & 0 \\ 0 & 2\bar{m}^2 G'(\bar{m})[\frac{b}{2}G'(\bar{m}) - 1] \end{pmatrix} . \tag{3.3}$$

Hence φ and m are the eigenvectors of A. In addition, $\bar{m}^2$ is the oscillation mode in the φ direction. Recalling now the renormalization condition (2.12)-(2.13) with $\alpha = 1$ we get the following expression for the eigenvalue in the m direction,

$$2\bar{m}\Big(-\frac{1}{g_\mu} + \frac{1}{16\pi^2}\log\frac{\bar{m}^2}{\mu^2}\Big) . \tag{3.4}$$

The system is stable if

$$\frac{1}{g_\mu} < \frac{1}{16\pi^2}\log\frac{\bar{m}^2}{\mu^2} . \tag{3.5}$$

This is the result we have obtained in I from the RPA analysis.

For the broken phase the stability matrix is no longer diagonal,

$$A = \begin{pmatrix} \bar{m}^2 & b\varphi\bar{m}G'(\bar{m}) \\ b\varphi\bar{m}G'(\bar{m}) & 2\bar{m}^2 G'(\bar{m})[\frac{b}{2}G'(\bar{m}) - 1] \end{pmatrix} . \tag{3.6}$$

Its determinant is equal to

$$\det A = -2\bar{m}^4 G'(\bar{m})\big(bG'(\bar{m}) + 1\big) = 2\bar{m}^4\Big(-\frac{1}{g_\mu} + \frac{1}{16\pi^2}\log\frac{\bar{m}^2}{\mu^2}\Big), \tag{3.7}$$

where we have used the relation $b\bar{\varphi}^2 = 3\bar{m}$ and the renormalization condition (2.12)-(2.13) with $\alpha = -2$ in the second equality. The criterion of minimum requires the eigenvalues to be positive. A straighforward calculation yields

$$\lambda^+ = \bar{m}^2 L_\mu\Big\{3 + \mathcal{O}\big(\frac{1}{L_\mu}\big)\Big\}, \tag{3.8}$$

$$\lambda^- = \frac{2\bar{m}^2}{3L_\mu}\big(-\frac{1}{g_\mu} + \frac{1}{16\pi^2}\log\frac{\bar{m}^2}{\mu^2}\big) + \mathcal{O}\big(\frac{1}{L_\mu^2}\big). \tag{3.9}$$

The eigenvalue λ^+ is always positive while $\lambda^- > 0$ if the Eq.(3.5) is satisfied. The result shows that $u(\varphi, m)$ is quite singular at this critical point, because λ^- is a zero mode for the limit of infinite cutoff and the curvature correspondent to the eigenvalue λ^+ is infinitely sharp.

IV. RPA Equations

Section II discussed in detail the possible minima of $\mathcal{H}$ as a function of the variational parameters. In this and next sections we will investigate near equilibrium dynamics around the stationary points. In I we have obtained the RPA equations by linearizing the time-dependent variational equations. Here we will procede differently: we first expand the hamiltonian, $\mathcal{H}$, around the stationary points and then the RPA equations are obtained from Hamilton's equation with the new hamiltonian [6]. Of course these two approachs are equivalent, but the method discussed here has advantage of eliminating the step of introducing a new auxiliary variable δv (see Section IV of I).

Let us first consider fluctuations around $\bar{\varphi}$ and $\bar{m}$, which are generic solutions obtained

in the previous section for the uniform system,

$$\delta\phi(\mathbf{x},t) \equiv \phi(\mathbf{x},t) - \bar{\varphi}, \tag{4.1}$$

$$\delta G(\mathbf{x},\mathbf{y},t) \equiv G(\mathbf{x},\mathbf{y},t) - \bar{G}(\mathbf{x},\mathbf{y}). \tag{4.2}$$

They define the one- and two-particle wavefunctions respectively. These quantities and their canonical conjugate momenta, π_0 and Σ, are assumed to be small in our approximation. Next, we expand the hamiltonian, Eq.(1.8), up to second order in $\delta\phi$, π_0, δG and Σ,

$$\begin{aligned}
\mathcal{H}_{RPA} &= \int_{\mathbf{x}}\Big(\frac{1}{2}[a + \frac{b}{2}\bar{G}(\mathbf{x},\mathbf{x})]\bar{\varphi}^2 + \langle\mathbf{x}\mid \frac{1}{8}\hat{\bar{G}}^{-1} + \frac{1}{2}\hat{p}^2\hat{\bar{G}}\mid\mathbf{x}\rangle + \frac{1}{2b}[a + \frac{b}{2}\bar{G}(\mathbf{x},\mathbf{x})]^2 + \frac{b}{24}\bar{\varphi}^4\Big) \\
&+ \int_{\mathbf{x}}\Big(\big[a + \frac{b}{6}\bar{\varphi}^2 + \frac{b}{2}\bar{G}(\mathbf{x},\mathbf{x})\big]\bar{\varphi}\Big)\,\delta\phi(\mathbf{x},\mathbf{x},t) \\
&+ \int_{\mathbf{x},\mathbf{y}}\Big(\frac{1}{2}\big[-\nabla_{\mathbf{x}}^2 + a + \frac{b}{2}\bar{\varphi}^2 + \frac{b}{2}\bar{G}(\mathbf{x},\mathbf{x})\big]\delta(\mathbf{x}-\mathbf{y}) - \frac{1}{4}\bar{G}^{-2}(\mathbf{x},\mathbf{y})\Big)\,\delta G(\mathbf{x},\mathbf{y},t) \\
&+ \frac{1}{2}\int_{\mathbf{x}}\pi_0^2(\mathbf{x},t) + 2\int_{\mathbf{x},\mathbf{y},\mathbf{z}}\Sigma(\mathbf{z},\mathbf{x},t)\bar{G}(\mathbf{x},\mathbf{y})\Sigma(\mathbf{y},\mathbf{z},t) \\
&+ \int_{\mathbf{x}}\frac{1}{2}\big[-\nabla_{\mathbf{x}}^2 + a + \frac{b}{2}\bar{\varphi}^2 + \frac{b}{2}\bar{G}(\mathbf{x},\mathbf{x})\big]\delta\phi^2(\mathbf{x},t) + \frac{b\bar{\varphi}}{2}\int_{\mathbf{x}}\delta\phi(\mathbf{x},t)\delta G(\mathbf{x},\mathbf{x},t) \\
&+ \frac{b}{8}\int_{\mathbf{x}}\delta G(\mathbf{x},\mathbf{x},t)\delta G(\mathbf{x},\mathbf{x},t) + \frac{1}{8}\int_{\mathbf{x}}\langle\mathbf{x}\mid\hat{\bar{G}}^{-1}\delta\hat{G}\hat{\bar{G}}^{-1}\delta\hat{G}\hat{\bar{G}}^{-1}\mid\mathbf{x}\rangle.
\end{aligned} \tag{4.3}$$

The first term of this expression is the time-independent part of the hamiltonian. The second and third terms vanish because of the equilibrium conditions, (2.4) and (2.5). The Eq. (2.6) can also be used to simplify the sixth term. The matrix element of the last term can be written explicitly in momentum space as

$$\begin{aligned}
&\frac{1}{8}\int_{\mathbf{x}}\langle\mathbf{x}\mid\hat{\bar{G}}^{-1}\delta\hat{G}\hat{\bar{G}}^{-1}\delta\hat{G}\hat{\bar{G}}^{-1}\mid\mathbf{x}\rangle \\
&= \frac{1}{2}\int_{\mathbf{p}_1,\mathbf{p}_2}\omega_{\mathbf{p}_1}\omega_{\mathbf{p}_2}(\omega_{\mathbf{p}_1} + \omega_{\mathbf{p}_2})\delta G(\mathbf{p}_1,\mathbf{p}_2,t)\delta G(-\mathbf{p}_1,-\mathbf{p}_2,t)
\end{aligned} \tag{4.4}$$

where $\omega_{\mathbf{p}}$ is a matrix element of the operator in momentum space,

$$\langle \mathbf{p}_1 \mid \hat{\omega} \mid -\mathbf{p}_2 \rangle \equiv \langle \mathbf{p}_1 \mid \sqrt{\hat{p}^2 + \bar{m}^2} \mid -\mathbf{p}_2 \rangle = \delta(\mathbf{p}_1 + \mathbf{p}_2)\omega_{\mathbf{p}}.$$

It is useful to introduce the relative and total momentum coordinates given respectively by

$$\mathbf{p} \equiv (\mathbf{p}_1 - \mathbf{p}_2)/2, \tag{4.5}$$

$$\mathbf{P} \equiv (\mathbf{p}_1 + \mathbf{p}_2). \tag{4.6}$$

Thus, from these remarks the Eq.(4.3) can be rewritten, in terms of $\mathbf{p}$, $\mathbf{P}$ coordinates, as

$$\begin{aligned} \mathcal{H}_{RPA} &= \mathcal{H}_{EST} + \frac{1}{2}\int_{\mathbf{P}} \pi_0(\mathbf{P},t)\pi_0(-\mathbf{P},t) + \int_{\mathbf{p},\mathbf{P}} \Sigma(\mathbf{p},\mathbf{P},t)\Big(\frac{1}{2\omega_+} + \frac{1}{2\omega_-}\Big)\Sigma(-\mathbf{p},-\mathbf{P},t) \\ &+ \int_{\mathbf{P}} \frac{1}{2}\Big[\mathbf{P}^2 + \bar{m}^2\Big]\delta\phi(\mathbf{P},t)\delta\phi(-\mathbf{P},t) \\ &+ \frac{b\bar{\varphi}}{4}\int_{\mathbf{p},\mathbf{P}}\Big[\delta\phi(\mathbf{P},t)\delta G(\mathbf{p},-\mathbf{P},t) + \delta\phi(-\mathbf{P},t)\delta G(\mathbf{p},\mathbf{P},t)\Big] \\ &+ \frac{b}{8}\int_{\mathbf{p},\mathbf{p}',\mathbf{P}} \delta G(\mathbf{p},\mathbf{P},t)\delta G(\mathbf{p}',-\mathbf{P},t) \\ &+ \frac{1}{2}\int_{\mathbf{p},\mathbf{P}} \omega_+\omega_-(\omega_+ + \omega_-)\delta G(\mathbf{p},\mathbf{P},t)\delta G(-\mathbf{p},-\mathbf{P},t), \end{aligned} \tag{4.7}$$

where $\omega_\pm = \sqrt{(\mathbf{p}\pm\mathbf{P})^2 + \bar{m}^2}$. It is convenient to make the following changes of variables for the further use,

$$\Sigma(\mathbf{p},\mathbf{P},t)h(\mathbf{p},\mathbf{P},t) \equiv \Pi(\mathbf{p},\mathbf{P},t), \tag{4.8}$$

$$\delta G(\mathbf{p},\mathbf{P},t)h^{-1}(\mathbf{p},\mathbf{P},t) \equiv \rho(\mathbf{p},\mathbf{P},t), \tag{4.9}$$

being

$$h(\mathbf{p},\mathbf{P}) = \sqrt{\frac{\omega_+ + \omega_-}{2\omega_+\omega_-}}. \tag{4.10}$$

Thus,

$$\begin{aligned} \mathcal{H}_{RPA} &= H_{EST} + \frac{1}{2}\int_{\mathbf{P}} \pi_0(\mathbf{P},t)\pi_0(-\mathbf{P},t) + \frac{1}{2}\int_{\mathbf{p},\mathbf{P}} \Pi(\mathbf{p},\mathbf{P},t)\Pi(-\mathbf{p},-\mathbf{P},t) \\ &+ \int_{\mathbf{P}} \frac{1}{2}\Big[\mathbf{P}^2 + \bar{m}^2\Big]\delta\phi(\mathbf{P},t)\delta\phi(-\mathbf{P},t) \\ &+ \frac{b\bar{\varphi}}{4}\int_{\mathbf{p},\mathbf{P}} h(\mathbf{p},\mathbf{P})\Big[\delta\phi(\mathbf{P},t)\delta\rho(\mathbf{p},-\mathbf{P},t) + \delta\phi(-\mathbf{P},t)\delta\rho(\mathbf{p},\mathbf{P},t)\Big] \\ &+ \frac{b}{8}\int_{\mathbf{p},\mathbf{p}',\mathbf{P}} \Big(h(\mathbf{p},\mathbf{P})\delta\rho(\mathbf{p},\mathbf{P},t)\Big)\Big(h(\mathbf{p}',\mathbf{P})\delta\rho(\mathbf{p}',-\mathbf{P},t)\Big) \\ &+ \frac{1}{2}\int_{\mathbf{p},\mathbf{P}} (\omega_+ + \omega_-)^2\delta\rho(\mathbf{p},\mathbf{P},t)\delta\rho(-\mathbf{p},-\mathbf{P},t). \end{aligned} \tag{4.11}$$

From this one can get the linearized equations of motion by following directly Hamiltion's equations [1]:

$$\delta\dot{\phi}(\mathbf{P},t) = \pi_0(-\mathbf{P},t), \tag{4.12}$$

$$\dot{\pi}_0(\mathbf{P},t) = -\Big[\mathbf{P}^2 + \bar{m}^2\Big]\delta\phi(-\mathbf{P},t) - \frac{b\bar{\varphi}}{2}\int_{\mathbf{p}} h(\mathbf{p},\mathbf{P})\delta\rho(\mathbf{p},-\mathbf{P},t), \tag{4.13}$$

$$\delta\dot{\rho}(\mathbf{p},\mathbf{P},t) = \Pi(-\mathbf{p},-\mathbf{P},t), \tag{4.14}$$

$$\begin{aligned} \dot{\Pi}(\mathbf{p},\mathbf{P},t) &= -(\omega_+ + \omega_-)^2\delta\rho(-\mathbf{p},-\mathbf{P},t) - \frac{b}{4}h(\mathbf{p},\mathbf{P})\int_{\mathbf{p}} h(\mathbf{p}',\mathbf{P})\delta\rho(\mathbf{p}',-\mathbf{P},t) \\ &- \frac{b\bar{\varphi}}{2}\int_{\mathbf{p}} h(\mathbf{p},\mathbf{P})\delta\phi(-\mathbf{P},t). \end{aligned} \tag{4.15}$$

Eliminating the canonical momenta π_0 and Π we arrive finally at

$$\delta\ddot{\phi}(\mathbf{P},t) + \omega_{\mathbf{P}}^2\delta\phi(\mathbf{P},t) + \frac{b\bar{\varphi}}{2}\int_{\mathbf{p}'} h(\mathbf{p}',\mathbf{P})\delta\rho(\mathbf{p}',\mathbf{P},t) = 0, \tag{4.16}$$

$$\delta\ddot{\rho} + (\omega_+ + \omega_-)^2\delta\rho(\mathbf{p},\mathbf{P},t) + \frac{b}{4}h(\mathbf{p},\mathbf{P})\int_{\mathbf{p}} h(\mathbf{p}',\mathbf{P})\delta\rho(\mathbf{p}',\mathbf{P},t) + \frac{b\bar{\varphi}}{2}h(\mathbf{p},\mathbf{P})\delta\phi(\mathbf{P},t) = 0. \tag{4.17}$$

These are linear oscillator equations as usual in RPA treatment. The solutions for this problem involves determining the modes of oscillation and their eigenfrequences.

We have seen in Sec.II that $\bar{\varphi}$ may have solutions at $\varphi = 0$ and $\varphi^2 = 3\bar{m}^2/b$. Note that in the symmetric phase the two equations decouple. The two-particle equation reduces to Eq.(4.20) of I while the one-particle equation becomes simple oscillator equations, one for each $\mathbf{P}$ with frequence $\omega_{\mathbf{P}}$,

$$\delta\ddot{\phi}(\mathbf{P},t) + \omega_{\mathbf{P}}^2\delta\phi(\mathbf{P},t) = 0\,. \tag{4.18}$$

For the solution $\bar{\varphi}^2 = 3\bar{m}^2/b$, however, the coupling between the one-particle and two-particle wavefunction is nontrivial, we shall discuss the solution of this problem in the next section.

V. Small Oscillation Equation and Scattering Problem

Let us first remove the time dependence of (4.16)-(4.17) by writting

$$\delta\phi(\mathbf{P},t) = \delta\phi^{(0)}(\mathbf{P})\cos[\Omega(\mathbf{P})t]\,, \tag{5.1}$$

$$\delta\rho(\mathbf{p},\mathbf{P},t) = \delta\rho^{(0)}(\mathbf{p},\mathbf{P})\cos[\Omega(\mathbf{P})t]\,, \tag{5.2}$$

where $\delta\phi^{(0)}$ and $\delta\rho^{(0)}$ are the amplitude of oscillation for the one- and two-particle modes respectively and $\Omega(\mathbf{P})$ are the eigenfrequences (see section IV-b of I for interpretation of the wavefunctions). Substituting these into (4.16)-(4.17) we have the following eigenvalue problem:

$$\omega_{\mathbf{P}}^2\delta\phi^{(0)}(\mathbf{P}) + \frac{b\bar{\varphi}}{2}\int_{\mathbf{p}'} h(\mathbf{p}',\mathbf{P})\delta\rho^{(0)}(\mathbf{p}',\mathbf{P}) = \Omega^2\delta\phi^{(0)}(\mathbf{P})\,, \tag{5.3}$$

$$\begin{aligned}(\omega_+ + \omega_-)^2\delta\rho^{(0)}(\mathbf{p},\mathbf{P}) &+ \frac{b}{4}h(\mathbf{p},\mathbf{P})\int_{\mathbf{p}} h(\mathbf{p}',\mathbf{P})\delta\rho^{(0)}(\mathbf{p}',\mathbf{P}) \\ &+ \frac{b\bar{\varphi}}{2}h(\mathbf{p},\mathbf{P})\delta\phi^{(0)}(\mathbf{P}) = \Omega^2\delta\rho^{(0)}(\mathbf{p},\mathbf{P}).\end{aligned} \tag{5.4}$$

Since the amplitudes of oscillations are wavefunctions of quantum particles it is interesting to treat this system as a coupled channel scattering problem with appropriate boundary conditions. Henceforth, we shall use α and β to denote the one and two-particle channel respectively. In the following discussion we will consider the cases separately.

a. $\alpha \longrightarrow \alpha$

In this process one has a incident wave of $\delta\phi^{(0)}$; it couples to $\delta\rho^{(0)}$ through the term $b\bar{\varphi}\int h\delta\rho^{(0)}$ and reemits $\delta\phi^{(0)}$ at the exit channel. Thus, we can formally solve (5.4) as follows:

$$\delta\rho^{(0)}(\mathbf{p},\mathbf{P}) = \frac{1}{\Omega^2 - (\omega_+ + \omega_-)^2 + i\epsilon}\Big[\frac{b}{4}h(\mathbf{p},\mathbf{P})\int_{\mathbf{p}} h(\mathbf{p}',\mathbf{P})\delta\rho(\mathbf{p}',\mathbf{P},t) + \frac{b\bar{\varphi}}{2}h(\mathbf{p},\mathbf{P})\delta\phi(\mathbf{P},t)\Big]\,. \tag{5.5}$$

This expression includes the boundary condition that there is no indent wave of $\delta\rho^{(0)}$. The term $\int_{\mathbf{p}} h(\mathbf{p},\mathbf{P})\delta\rho^{(0)}(\mathbf{p},\mathbf{P})$ couples $\delta\rho^{(0)}$ of different relative momenta. We can solve for this term by multiplying (5.5) by $h(\mathbf{p},\mathbf{P})$ and integrates it with respect to $\mathbf{p}$,

$$\int_{\mathbf{p}'} h(\mathbf{p}',\mathbf{P})\delta\rho^{(0)}(\mathbf{p},\mathbf{P}) = \frac{2\bar{\varphi}I_{\mathbf{P}}(\Omega)}{\frac{2}{b} - I_{\mathbf{P}}(\Omega)}\delta\phi^{(0)}(\mathbf{P})\,, \tag{5.6}$$

where

$$I_{\mathbf{P}}(\Omega) = \int_{\mathbf{p}} \frac{h^2(\mathbf{p},\mathbf{P})}{\Omega^2 - (\omega_+ + \omega_-)^2 + i\epsilon}\,. \tag{5.7}$$

The details of the computation of this integral can be found in the appendix B of I. For the present analysis it is sufficient to say

$$I_{\mathbf{P}}(\Omega) = -L_\mu + F_{\mathbf{P}}(\Omega)\,, \tag{5.8}$$

where L_μ is the logarithmic divergent term defined in (2.12) and $F_{\mathbf{P}}(\Omega)$ the finite part of $I_{\mathbf{P}}(\Omega)$ [See (6.3)-(6.6) for $I_{\mathbf{P}}(\Omega)$]. Substituing now (5.6) into (5.3) we have

$$\left[\Omega^2 - \omega_{\mathbf{P}}^2 - \frac{b\bar{\varphi}^2 I_{\mathbf{P}}(\Omega)}{2/b - I_{\mathbf{P}}(\Omega)}\right]\delta\phi^{(0)}(\mathbf{P}) = 0 . \tag{5.9}$$

For the asymmetric solution, $b\bar{\varphi}^2 = 3\bar{m}^2$, and (5.9) becomes

$$\left[\Omega^2 - \mathbf{P}^2 - \bar{m}^2\left(1 + \frac{3I_{\mathbf{P}}(\Omega)}{2/b - I_{\mathbf{P}}(\Omega)}\right)\right]\delta\phi^{(0)}(\mathbf{P}) = 0 . \tag{5.10}$$

Using the renormalization condition (2.13) and (5.8) one arrives at

$$\left[\Omega^2 - \mathbf{P}^2 - \bar{m}^2\mathcal{O}\left(\frac{1}{L_\mu}\right)\right]\delta\phi^{(0)}(\mathbf{P}) = 0 . \tag{5.11}$$

Notice that the effect of the coupling $\delta\phi^{(0)}$ to the two-particle modes switch the mass of the particle. In the limit of infinity cutoff its effective mass goes to zero.

b. $\beta \longrightarrow \beta$

It is also an elastic scattering process, where the entrance channel as well as the exit channel exhibit two-particle wave $\delta\rho^{(0)}$. Analogously to the previous discussion we first solve (5.3) as

$$\delta\phi^{(0)}(\mathbf{P}) = \frac{b\bar{\varphi}/2}{\Omega^2 - \omega_{\mathbf{P}}^2 + i\epsilon}\int_{\mathbf{p}'} h(\mathbf{p}', \mathbf{P})\delta\rho(\mathbf{p}', \mathbf{P}) . \tag{5.12}$$

Using this and (5.4) we get the following integral equation for $\delta\rho^{(0)}$:

$$\delta\rho^{(0)}(\mathbf{k}, \mathbf{p}, \mathbf{P}; \Omega) = \gamma_\beta\delta(\mathbf{k}-\mathbf{p}) + \frac{b}{4}\left(\frac{1 + \dfrac{b\bar{\varphi}^2}{\Omega^2 - \omega_{\mathbf{P}}^2 + i\epsilon}}{\Omega^2 - (\omega_+ + \omega_-)^2 + i\epsilon}\right)h(\mathbf{p}, \mathbf{P})\int_{\mathbf{p}'} h(\mathbf{p}', \mathbf{P})\delta\rho^{(0)}(\mathbf{k}, \mathbf{p}', \mathbf{P}; \Omega) . \tag{5.13}$$

In this expression $\mathbf{k}$ is the relative momentum of the two incident meson; γ_β indicates a phase factor to keep $\delta\rho^{(0)}$ real and the subindex β denotes the source of incident wave. From (5.13) one finds immediately

$$\int_{\mathbf{p}} h(\mathbf{p}, \mathbf{P})\delta\rho^{(0)}(\mathbf{p}, \mathbf{P}) = \gamma_\beta\frac{b}{2}\frac{1}{\Delta_{\mathbf{P}}^{+}(\Omega)}h(\mathbf{p}, \mathbf{P}) ,$$

where

$$\Delta_{\mathbf{P}}^{+}(\Omega) = \frac{2}{b} - \left(1 + \frac{b\bar{\varphi}^2}{\Omega^2 - \omega_{\mathbf{P}}^2 + i\epsilon}\right)I_{\mathbf{P}}(\Omega) . \tag{5.14}$$

Using this result in (5.13) we find

$$\delta\rho^{(0)}(\mathbf{k}, \mathbf{p}, \mathbf{P}; \Omega) = \gamma_\beta\delta(\mathbf{k} - \mathbf{p}) + \frac{1}{\Omega^2 - (\omega_+ + \omega_-)^2 + i\epsilon}\gamma_\beta h(\mathbf{k}, \mathbf{P})\Xi(\mathbf{P}, \Omega)h(\mathbf{p}, \mathbf{P}) \tag{5.15}$$

with

$$\Xi(\mathbf{P}, \Omega) = \left(1 + \frac{b\bar{\varphi}^2}{\Omega^2 - \omega_{\mathbf{P}}^2 + i\epsilon}\right)\frac{1}{\Delta_{\mathbf{P}}^{+}(\Omega)} . \tag{5.16}$$

Of course, this result reduce to Eq.(5.5) of I in the case of $\bar{\varphi} = 0$. For the broken solution, we can rewrite (5.16) in terms of renormalized parameters as

$$\Xi(\mathbf{P}, \Omega) = \frac{\Omega^2 - \mathbf{P}^2 + 2\bar{m}^2}{3L_\mu(\Omega^2 - \mathbf{P}^2) - \dfrac{2}{g_\mu}(\Omega^2 - \mathbf{P}^2 - \bar{m}^2) - F(\mathbf{P}, \Omega)[\Omega^2 - \mathbf{P}^2 + 2\bar{m}^2]} . \tag{5.17}$$

Notice that $\Xi \to L_\mu^{-1}$ for large $\Lambda_{\mathbf{p}}$, which means that the scattering waves is infinitesimaly small. Therefore, **effectively** there is no interaction between the particles coming from this broken vacuum.

c. $\alpha \longrightarrow \beta$

In this case we want to obtain $\delta\rho^{(0)}$ with a source of one particle, which is an inelastic processes. To do so, (5.3) is solved as

$$\delta\phi^{(0)}(\mathbf{P},\mathbf{K},,\Omega) = \gamma_\alpha\delta(\mathbf{K}-\mathbf{P}) + \frac{1}{\Omega^2-\omega_{\mathbf{P}}^2+i\epsilon}\frac{b\bar{\varphi}}{2}\int_{\mathbf{p}'} h(\mathbf{p}',\mathbf{P})\delta\rho^{(0)}(\mathbf{p}';\mathbf{P},\Omega). \qquad (5.18)$$

Here we have included an incident wave of $\delta\phi^{(0)}$ with moment $\mathbf{K}$. Using this solution in (5.4) one gets an integral equation for $\delta\rho^{(0)}$ with a source in α,

$$\begin{aligned}[-\Omega^2+(\omega_+ + \omega_-)^2]\delta\rho^{(0)}(\mathbf{p},\mathbf{P}) &= \gamma_\alpha\delta(\mathbf{K}-\mathbf{P})\frac{b\bar{\varphi}}{2}h(\mathbf{p}',\mathbf{P}) \\ &+\frac{b}{4}\Big(1 + \frac{b\bar{\varphi}^2}{\Omega^2-\omega_{\mathbf{P}}^2+i\epsilon}\Big)h(\mathbf{p},\mathbf{P})\int_{\mathbf{p}'} h(\mathbf{p}',\mathbf{P})\delta\rho^{(0)}(\mathbf{p}',\mathbf{P}) \quad (5.19)\end{aligned}$$

This equation can be solved as usual given

$$\begin{aligned}\delta\rho^{(0)}(\mathbf{K},\mathbf{p},\mathbf{P};\Omega) &= \gamma_\alpha\delta(\mathbf{K}-\mathbf{P})\frac{b\bar{\varphi}}{2}h(\mathbf{p}',\mathbf{P})\big[1+\Xi(\mathbf{P},\Omega)\big] \\ &= \gamma_\alpha\delta(\mathbf{K}-\mathbf{P})\frac{b\bar{\varphi}}{2}h(\mathbf{p}',\mathbf{P})\Big[1+\mathcal{O}\Big(\frac{1}{L_\mu}\Big)\Big]. \qquad (5.20)\end{aligned}$$

In the second line we have used (5.17). Recalling the discussion in the section II, we have learned that in the broken solution, the mean-field φ requires a scaling factor. As consequence, we have essencially $b\bar{\varphi} \propto 1/\sqrt{L_\mu}$, which vanishes for large cutoff. Therefore, in the continuum limit $\delta\rho$ cannot be observed in this reaction.

d. $\beta \longrightarrow \alpha$

For the last case, the Equation (5.4) is solved with the source in β,

$$\begin{aligned}\delta\rho^{(0)}(\mathbf{k},\mathbf{p},\mathbf{P};\Omega) &= \gamma_\beta\delta(\mathbf{k}-\mathbf{p}) + \frac{1}{\Omega^2-(\omega_+ + \omega_-)^2+i\epsilon}\times \\ &\Big[\frac{b}{4}h(\mathbf{p},\mathbf{P})\int_{\mathbf{p}'} h(\mathbf{p}',\mathbf{P})\delta\rho^{(0)}(\mathbf{k},\mathbf{p}',\mathbf{P};\Omega) + \frac{b\bar{\varphi}}{2}h(\mathbf{p},\mathbf{P})\Big]. \qquad (5.21)\end{aligned}$$

From this, one gets immediately

$$\int_{\mathbf{p}} h(\mathbf{p},\mathbf{P})\delta\rho^{(0)}(\mathbf{p},\mathbf{P}) = \frac{1}{1-bI_{\mathbf{P}}(\Omega)/2}\big[\gamma_\beta h(\mathbf{k}) + b\bar{\varphi}I_{\mathbf{P}}(\Omega)\delta\phi\rho^{(0)}(\mathbf{p})\big]. \qquad (5.22)$$

Combining this with (5.3) results

$$\Big[\Omega^2-\omega_{\mathbf{P}}^2-\frac{b\bar{\varphi}^2 I_{\mathbf{P}}(\Omega)}{2/b-I_{\mathbf{P}}(\Omega)}\Big]\delta\phi^{(0)}(\mathbf{P}) = \gamma_\beta b\bar{\varphi}h(\mathbf{k})\frac{1}{1-bI_{\mathbf{P}}(\Omega)/2}. \qquad (5.23)$$

Above equation differs (5.9) from the boundary condition for this scattering process. Here we have incoming waves of $\delta\rho$ characterized by γ_β in its rhs. It is proportional to $b\bar{\varphi}$, or $\delta\phi^{(0)} \propto 1/\sqrt{L_\mu}$. Therefore, we conclude that this physical process actually cannot happen when the cutoff is taken to infinite.

In summary, this section has discussed the solution for the RPA equations in the context of $b \to 0^+$. The excitations are interpreted as quantum particles and the RPA equations are treated as a coupled channel scattering problem. The system of equations allows analytical solutions and we have obtained the amplitudes of oscillation which are wavefunction of one and two-particles. In particular, we have shown that at one-particle channel the effect of coupling is to reduce the effective mass of the particles and two-particle channel leads to infinitesimal scattering waves for large cutoff. Furthermore, the inelastic processes, cases c. and d., cannot occur when the renormalized field theory is considered.

VI. Discussion

From the previous sections and I we have seen that the Gaussian variational approximation indicates two distinct nontrivial version of $(\phi^4)_{3+1}$ theories. They are characterized by the bare coupling to be infinitesimal of the form $b \to \pm L_\mu^{-1}$ in the continuum limit. Although classical intuition and perturbation calculation indicate that the theory with $b \to 0^-$ is unstable, but preceding analysis and I reveal that it has a well defined symmetric phase. In particular the renormalized version of the theory allows a stable vacuum and finite scattering amplitude[1][4]. The other one, $b \to 0^+$, shows a renormalized broken phase solution, $b\bar{\varphi}^2 = 3\bar{m}^2$, which is degenerate with the symmetric phase. In section III we have discussed the conditions of stability for these vacuums by studying the eigenvalues of the stability matrix. An alternative approach to investigate this problem is through analyses of the positions of the RPA modes. The method is more general because it includes inhomogeneous degree of freedom. In the following discussion we will consider the cases of symmetric and broken solution separately.

a. Symmetrical Phase

a-1. Massive Theory

In this case $\delta\phi$ decouples from $\delta\rho$ and (5.3) becomes

$$(s - \bar{m}^2)\delta\phi^{(0)} = 0\,, \tag{6.1}$$

where we have introduced the covariant variable $s \equiv \Omega^2 - \mathbf{P}^2$ [7]. Thus, this channel yields a simple free-particle spectrum. The Eq.(5.4), on the other hand, describes the nontrivial sector of this vacuum where one has excitations of two particles.

This case was explored in I and here we will summarize the main results. In order to get

the spectrum we write the scattering matrix for this process as

$$\mathbf{T}(\mathbf{p},\mathbf{p}',\mathbf{P};\Omega) = h(\mathbf{p},\mathbf{P})\frac{1}{\frac{2}{b} - I_{\mathbf{P}}(\Omega)}h(\mathbf{p}',\mathbf{P})\,. \tag{6.2}$$

In appendix B of I we have performed the calculation of $I(\Omega)$. In terms of covariant variable s the result is

$$I(s) = -L_m - \frac{1}{16\pi^2}f(s) - \theta(s - 4\bar{m}^2)\frac{i}{16\pi}\sqrt{\frac{s - 4\bar{m}^2}{s}}\,, \tag{6.3}$$

where, $L_m = 1/8\pi^2 \log(2\Lambda_p/e\bar{m})$ and

$$f(s) = 2 + \sqrt{\frac{s-4\bar{m}^2}{s}}\log\frac{\sqrt{s} - \sqrt{s-4\bar{m}^2}}{\sqrt{s}+\sqrt{s-4\bar{m}^2}} \qquad s > 4\bar{m}^2, \tag{6.4}$$

$$f(s) = 2 - 2\sqrt{\frac{-4\bar{m}^2}{s}}\tan^{-1}\sqrt{\frac{s}{-4\bar{m}^2}} \qquad 0 < s < 4\bar{m}^2, \tag{6.5}$$

$$f(s) = 2 + \sqrt{\frac{s-4\bar{m}^2}{s}}\log\frac{\sqrt{s-4\bar{m}^2} - \sqrt{s}}{\sqrt{s-4\bar{m}^2}+\sqrt{s}} \qquad s < 0 \tag{6.6}$$

(cf. fig.3 of I). Depending on the value of s the system can have different dynamical behavior. When $s < 0$ the system is unstable and for $s > 4\bar{m}^2$ it has a continuum spectrum. In the interval of $0 < s < 4\bar{m}^2$ the system may present stable bound states if one finds s_B such as

$$\Delta^+(s_B) = \frac{2}{b} + L_m + \frac{f(s_B)}{16\pi^2} = 0\,. \tag{6.7}$$

In this interval, $0 \le f(s) \le 2$, this equation has solution if, only if $b \to -L_m^{-1}$. In particular one can choose the renormalization condition used for the static equations, i.e., (2.12)-(2.13). In this way, we can show that this theory will result a stable bound state when the renormalized coupling constant $g_m < -1/8\pi^2$ (cf. section V of I).

a-2. Massless Theory

In section II we have shown that with the renormalization condition $a + b\Lambda/16\pi^2 = 0$ the system has a massless solution. In particular the broken phase is degenerate with the symmetric phase. In this case it is still convenient to use $\bar{m}$ as the mass scale defined by the relation (2.21). Using these ingredients we get

$$\Delta^+(s<0) = \frac{2}{b} - I(s) = -\frac{\alpha}{8\pi^2}\log\frac{2\Lambda_p}{\sqrt{e}\bar{m}} + \frac{1}{16\pi^2}\log\frac{\Lambda_s}{|s|}. \tag{6.8}$$

In this expression α labels the two theories as we have discussed before.

Case 1: $\alpha = 1$ ($b \to 0^-$)

The equation (6.8) becomes

$$\Delta^+(s<0) = -\frac{1}{8\pi^2}\log\frac{2\Lambda_p}{e\bar{m}} + \frac{1}{16\pi^2}\log\frac{\Lambda_s}{|s|} = \frac{1}{16\pi^2}\log\frac{e^2\bar{m}^2}{|s|}, \tag{6.9}$$

where we have used the relation $\Lambda_s = (2\Lambda_p)^2$. The system is stable if $\Delta^+(s<0) \neq 0$ for any s. However, from (6.8) one can always find s_i such as $\Delta^+(s_i) = 0$. Therefore the vacuum is unstable when $b \to 0^-$.

Case 2: $\alpha = -2$ ($b \to 0^+$)

In this case the broken phase collapses to the symmetric vacuum. The stability analysis is analogous to the previous one:

$$\Delta^+(s<0) = \frac{2}{8\pi^2}\log\frac{2\Lambda_p}{e\bar{m}} + \frac{1}{16\pi^2}\log\frac{\Lambda_s}{|s|} = \frac{1}{16\pi^2}\log\frac{\Lambda_s^3}{e^4\bar{m}^4|s|}. \tag{6.10}$$

The solution for $\Delta^+(s) = 0$ is

$$|s| = \frac{\Lambda_s^3}{e^4\bar{m}^4}. \tag{6.11}$$

This solution is unphysical and has to be eliminated. Therefore, the system is stable in this case.

Thus, the Gaussian variational approximation indicates that $(\phi^4)_{3+1}$ with $b \to 0^-$ might be a nontrivial theory. It has a well defined potential for the sector of $(\varphi = 0, m)$ of the variational space. For certain range of renormalized coupling constant the theory allows stable vacuum. Its excitations yield one (free) and two-particle modes. The two-meson equation leads to a single bound state and the scattering amplitude in the continuum. We also find a massless solution with the renormalization condition $a + b\Lambda^2/16\pi^2 = 0$. The RPA analyses indicate that the solution is: i)unstable if $b \to 0^-$, which suggest that the true vacuum is not homogeneous; ii)stable for theory with $b \to 0^+$.

b. Broken Phase

In this case we have the one- and two-particle wavefunction coupled through of $b\bar{\varphi}$. In order to discuss the RPA equations (5.10) is rewritten in terms of covariant variable s as

$$\Big[s - \bar{m}^2\Big(1 + \frac{3I^+(s)}{2/b - I^+(s)}\Big)\Big]\delta\phi^{(0)} = 0. \tag{6.12}$$

In the interval of $0 < s \leq 4\bar{m}^2$, $I^+(s)$ is real and the system may present a stable state if one finds a solution $s_B > 0$ such as

$$\begin{aligned} s_B &= \bar{m}^2\Big[1 + \frac{3I^+(s_B)}{2/b - I^+(s_B)}\Big] \\ &= \bar{m}^2\Big[1 - \frac{3}{1 + \frac{2}{b(L_m + f(s_B)/16\pi^2)}}\Big], \end{aligned} \tag{6.13}$$

where we have used (6.3). Using now (2.12) with $\alpha = -2$, we arrive at the follwing equations for s_B:

$$s_B = \frac{\bar{m}^2}{24\pi^2 L_\mu}\Big[-\frac{1}{\bar{g}_\mu} + \log\frac{\bar{m}^2}{\mu^2} + f(s_B)\Big]. \tag{6.14}$$

Thus, one may have a solution for $s_B \to L_\mu^{-1}$ and it is stable if the bracket is positive. From fig.3 of I one can see that $f(s)$ goes to zero for small s. Therefore, we might neglect the last term as first approximation to yield

$$s_B \approx \frac{\bar{m}^2}{24\pi^2 L_\mu}\Big[-\frac{1}{\bar{g}_\mu} + \log\frac{\bar{m}^2}{\mu^2}\Big]. \tag{6.15}$$

This is precisely the eigenvalue found in Sec. III (cf.Eq.(3.9)).

When $s > 4\bar{m}^2$ $I^+(s)$ has an imaginary part and (6.15) becomes, after the renormalization,

$$\Big[s - \frac{\bar{m}^2}{24\pi^2 L_\mu}\Big(-\frac{1}{\bar{g}_\mu} + \log\frac{\bar{m}^2}{\mu^2} + f(s) + i\pi\sqrt{\frac{s-4\bar{m}^2}{s}}\Big)\Big]\delta\phi^{(0)} = 0. \tag{6.16}$$

Hence, we have continuum spectrum in this case.

It is still illustrative to see the above results from the elastic channel, i.e., $\beta \to \beta$, where physics is described by the integral equation (5.19). Note that the potential term is also separable and the scattering **T** matrix can be obtained as usual [8]:

$$\mathbf{T}(\mathbf{p},\mathbf{p}',\mathbf{P};\Omega) = h(\mathbf{p},\mathbf{P})\Xi(\mathbf{P};\Omega)h(\mathbf{p}',\mathbf{P}), \tag{6.17}$$

where Ξ is given by (5.22). In terms of s it can be rewritten as

$$\Xi(s) = \Big(1 + \frac{3\bar{m}^2}{s-\bar{m}^2}\Big)\frac{1}{\Delta^+(s)}, \qquad \Delta^+(s) = \frac{2}{b} - \Big(1 + \frac{3\bar{m}^2}{s-\bar{m}^2}\Big)I^+(s). \tag{6.18}$$

From this viewpoint several dynamical behavior is described by denominator of the scattering matrix **T**, i.e., $\Delta^+(s) = 0$. In particular, the solution found in the equation (6.15) can be

seen as a bound state of two massless particles. It is stable for $g_m < 0$. Another intereting point is to obsever $\Xi(s)$ when $s > 4\bar{m}^2$:

$$\Xi(s) = \frac{s + 2\bar{m}^2}{3s(L_\mu - \frac{1}{g_\mu}) + (s+2\bar{m}^2)\Big\{\frac{1}{g_m} + \frac{1}{16\pi^2}\Big[f(s) + i\pi\sqrt{\frac{s-\bar{m}^2}{\bar{m}^2}}\Big]\Big\}} \tag{6.19}$$

In this range of energy the spectrum is continuum, but the scattering amplitude vanishes as $1/L_\mu$. In other words, the particles **effectively** do not interact, but in a very non-trivial way.

In summary, we have obtained RPA equations for ϕ^4 field theory by considering near equilibrium dynamics about the critical points of the gaussian parameter space. A simple analytical solution can be obtained and it is a nonperturbative method to investigate one and two-meson physics. Using this framework we have investigate the stability of the theory and shown that in the continuum limit two distinct version of $(\phi^4)_{3+1}$ are viable. One has $b \to 0^-$ and its renormalized theory is stable at the symmetric phase. The other, $b \to 0^+$, indicates spontaneous symmetric breaking. However, the excitation fields **effectively** do not interact leading to infinitesimal scattering amplitude [9]. Finally, we comment that the techniques developed here are general and can be readily extented to other relativistic field models [10] and low energy many-body problems such as Boson condensation [11].

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