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Abstract

We use field-theory to calculate the critical exponents $\eta_{\ell 2}$, $\eta_{\ell 4}$, $\nu_{\ell 2}$, and $\nu_{\ell 4}$ for the Lifshitz point (LP) with $m = 2$ and $m = 6$. We were motivated by an old controversy on the order ϵ^2 corrections for the exponents of this multicritical point. In a previous paper, we studied the renormalization of the theory which describes the LP, derived expressions for the exponents and scaling relations that hold to all orders in perturbation. In this work we concentrate on the calculation of the Feynman diagrams that are involved in the determination of the exponents and present our results for the exponents to order ϵ^2. The evaluation of these diagrams, due to a free propagator that contains a quartic term in the momentum and is not rotationally invariant, require special techniques that are presented in detail in appendices.

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I. INTRODUCTION

The Lifshitz point (LP) was introduced by Hornreich, Luban and Shtrikman.[1] At this multicritical point, ordinary paramagnetic and ferromagnetic phases coexist with spatially modulated phases characterized by a wave vector **k**. The LP divides the second-order line, which separates the ferromagnetic and the modulated phases from the paramagnetic phase, into two branches. On one of the branches, the wave vector **k** of the modulated phases goes continuously to zero as one approaches the LP, $|\mathbf{k}| \sim |(p - p_L)/p_L|^{\beta_k}$, where p is an external parameter and β_k is a critical exponent. Magnetic systems, liquid crystals, charge transfer salts and ferroelectric crystals, to mention some examples, may exhibit a LP. An introduction to the LP and a guide in the literature can be found in the good reviews written by Hornreich[2] and Selke.[3]

In this paper we concentrate on the calculation of the critical exponents associated with the LP. More specifically, on the order ϵ^2 corrections obtained using renormalization-group techniques. Our motivation was the existence of conflicting values for the exponents of the LP with $m = 2$ and $m = 6$. Before discussing previous works it is interesting to show why the analytical determination of the exponents is difficult.

All renormalization-group calculations for the LP are based on the Landau-Ginsburg-Wilson Hamiltonian[1]

$$H = \frac{1}{2}\int_q v(q)\,\vec{\Phi}_{-q}\cdot\vec{\Phi}_q + \frac{\lambda}{4!}\int_{q_1}\int_{q_2}\int_{q_3}(\vec{\Phi}_{q_1}\cdot\vec{\Phi}_{q_2})(\vec{\Phi}_{q_3}\cdot\vec{\Phi}_{-q_1-q_2-q_3})\,,$$
$$v(q) = r_0 + q_\beta^2 + c_0 q_\alpha^2 + \sigma_0\left(q_\alpha^2\right)^2\,,$$
$$q_\alpha^2 \equiv \sum_{\mu=1}^{m} q_\mu^2\,; \quad q_\beta^2 \equiv \sum_{\mu=m+1}^{d} q_\mu^2\,, \tag{1}$$

where $\vec{\Phi}_q$ is a n-component order parameter, and the dimensionless parameter $\sigma_0 = 1$ in Ref. 1. The parameters r_0 and c_0 are related with the temperature and with p by $r_0 = T - T_{0L}$ and $c_0 \sim p - p_{0L}$, where T_{0L} and p_{0L} are the mean-field coordinates of the Lifshitz point in the $p - T$ plane. The space is divided into two isotropic subspaces: an α subspace of dimension m, and a β subspace of dimension $d - m$. A large class of models is described by

Hamiltonian (1), each one parametrized by different values of n and m, $1 \leq m \leq 8$.[1] The axial next-nearest-neighbor Ising (ANNNI) model[4,3] corresponds to the $m = n = 1$ case. A LP is associated with a wave vector instability in the m directions of the α subspace. At the Lifshitz point both r_0 and c_0 go to zero and the q_α^4 term has to be kept. Thus, one has to deal with a free propagator of the type $[q_\alpha^4 + q_\beta^2]^{-1}$ which contains a quartic term in the momentum and is not rotationally invariant. Rotational invariance is restricted to each subspace. The difficulty of the problem lies in this unusual propagator.

Using the Wilson-Fisher renormalization-group (WFRG) and an ϵ expansion about the upper critical dimension[1] $d_u(m) = 4 + m/2, \quad m \leq 8$, Hornreich, Luban and Shtrikman[1] calculated, for all m, the exponents $\nu_{\ell 2}$ and $\nu_{\ell 4}$ to order ϵ, and, for $m = 8$, $\nu_{\ell 2}$, $\nu_{\ell 4}$ and $\eta_{\ell 4}$ to order ϵ^2, where the subscript $\ell 4$ ($\ell 2$) refers to the α subspace (β subspace). Mukamel[5] determined $\eta_{\ell 2}$ and $\eta_{\ell 4}$ to order ϵ^2 for all m, and β_k to order ϵ^2 for $m < 6$ (one does not expect helical long-range order for $m \geq 6$). Hornreich and Bruce[6] calculated, for $m = 1$, the exponents $\eta_{\ell 2}$ and $\eta_{\ell 4}$ to order ϵ^2 and the exponent β_k to order ϵ^2 and their result agrees with Mukamel's. However, Sak and Grest[7] performed an independent calculation, for $m = 2$ and $m = 6$, of $\eta_{\ell 2}$, $\eta_{\ell 4}$ and β_k to order ϵ^2, obtaining results which are different from Mukamel's.

All calculations mentioned above use the WFRG and the integrals over momentum shells were evaluated using different approximations. Due to the difficult in deciding which approximations were correct, we used field theory to calculate exactly the dimensional regularization poles of the Feynman diagrams generated by Hamiltonian (1). This information allows us to obtain an ϵ expansion for the exponents in a way analogous to the one for the ϕ^4 theory. The integrals in field theory are simpler because they are over the whole momentum space, instead of over momentum shells as in the WFRG. However, one has to pay a price for this simplification: it is necessary to study the renormalization of the theory described by Hamiltonian (1). This has been done in Ref. 8, where we adapted a technique introduced by Weinberg[9] and applied to critical phenomena by Zinn-Justin.[10] Basically, it consists in expanding the Green functions in the neighborhood of a critical point in terms of the Green functions calculated at the critical point. This method is well explained in Amit's book.[11]

Another crucial ingredient was our showing that there are renormalization prescriptions such that the renormalization constants depend only on the renormalized coupling constant and not on σ—the renormalized σ_0 parameter. These two ingredients simplified the problem enormously, allowing us to determine the renormalization-group equations satisfied by the renormalized Green functions, and by analyzing its solutions we found expressions for the critical exponents and scaling relations that hold to all orders in perturbation. In this paper, we calculate to order ϵ^2, for $m = 2$ and $m = 6$, the exponents $\nu_{\ell 2}$, $\nu_{\ell 4}$, $\eta_{\ell 2}$ and $\eta_{\ell 4}$. The knowledge of these exponents and the scaling relations derived in Ref. 8, enable us to determine the other exponents to order ϵ^2. Our technique does not rely on any approximation and is as exact as the analogous one for the ϕ^4 theory. It is also different from the method used by Nasser and Folk.[14] We have also obtained the critical exponents for the isotropic case ($m = 8$) (Refs. 1, 5, and 7) to order ϵ^2. In this case, since we only have the α subspace, the problem is simpler and the usual techniques of field theory for the ϕ^4 theory can be employed.

The determination of higher order corrections for the critical exponents of the LP is not simply a challenging theoretical problem. It is an essential step for the comparison of theoretical predictions and experimental results. The calculation performed by Nasser, Abdel-Hady, and Folk[12] for the specific-heat amplitude ratio near a LP is not in agreement with experimental measurements in MnP by Bindilatti, Becerra and Oliveira.[13] The problem is that in order to determine universal amplitude ratios to order ϵ^k one has to know the critical exponents and the fixed point to order ϵ^{k+1}. The MnP has a LP with $m = 1$ and for this value of m, except for $\eta_{\ell 2}$, $\eta_{\ell 4}$ and β_k, the exponents are known only to order ϵ, and the amplitudes only to order ϵ^0. We believe that others may improve upon our work and extend the calculations for other values of m. Even using field theory, the calculation of the Feynman diagrams is involved and cannot be done with the existing softwares for symbolic computations. They can only be used in some stages of the calculations. For this reason, we present in detail the techniques we developed to evaluate diagrams with the propagator

$[q_\alpha^4 + q_\beta^2]^{-1}$.

This paper is organized as follows: in Sec. II we present a brief review of the results derived in Ref. 8 that will be used in this paper; in Sec. III we derive expressions for exponents in terms of Feynman diagrams; and in Sec. IV we discuss our results and conclude. Finally, a detailed description of the calculation of the diagrams is given in appendices.

II. RENORMALIZATION OF THE FIELD THEORY FOR THE LP

In perturbative field theory one calculates the one-particle irreducible (1PI) Green functions $\Gamma^{(N,L)}(k_1, ..., k_N, p_1, ..., p_L; \sigma_0, c_0, r, \lambda, \overline{\phi}, \Lambda)$, which contain N external legs, L insertions of $\phi^2(p_i)$ operators, as an infinite sum of diagrams obtained using the Feynman rules[11] extracted from Hamiltonian (1). The magnetization $\overline{\phi}$ is zero in the paramagnetic phase and not null in the ferromagnetic phase in zero magnetic field. The field theory that describes the LP is renormalizable.[8] In other words, it is possible to define renormalized 1PI Green functions that are finite in the infinite cutoff limit ($\Lambda \to \infty$) for $d \leq d_u(m)$, where $d_u(m)$ is the upper critical dimension.

All physical information is contained in the 1PI Green functions. For example, the inverse of the zero field susceptibility χ is proportional to $\Gamma^{(2,0)}$ calculated at zero external momenta:[11]

$$\chi^{-1} = \beta^{-1}\Gamma^{(2,0)}(0, 0, \sigma_0, c_0, r_0, \lambda, \bar{\phi} = 0, \Lambda) \,. \tag{2}$$

At criticality χ diverges and the equation which determines the critical line $T_c(p)$ is given by

$$\Gamma^{(2,0)}\left(0, 0, \sigma_0, c_0, r_0, \lambda, 0, \Lambda\right)\Big|_{T=T_c} = 0 \,. \tag{3}$$

At the Lifshitz point the coefficient of k_α^2 is zero and, in addition to Eq. (3) with $T = T_L$, the coordinates (T_L, p_L) of the Lifshitz point also satisfy

$$\frac{\partial}{\partial k_\alpha^2}\Gamma^{(2,0)}\left(k, -k, \sigma_0, c_0, r_0, \lambda, 0, \Lambda\right)\Big|_{\substack{T=T_L,\, p=p_L,\\ k_\alpha^2=0}} = 0 \,. \tag{4}$$

Recall that $r_0 = T - T_{0L}$ and $c_0 \sim p - p_{0L}$, and, to lowest order in perturbation theory (mean-field approximation), $r_L = T_L - T_{0L} = 0$ and $c_L \sim p_L - p_{0L} = 0$. As we take fluctuations into account T_L and p_L move away from their mean-field values. The corrections are determined by expanding r_L and c_L in the coupling constant λ, inserting these expansions in Eqs. (3) and (4) and solving them perturbatively. When we expand the propagators in the Feynman diagrams about $r_L = c_L = 0$ we obtain integrals without any dimensional parameters. These integrals in the dimensional regularization scheme vanish and all corrections to the mean-field coordinates of the Lifshitz point are exactly zero. Thus, Green's functions at the LP are calculated with the massless propagator $(\sigma_0 q_\alpha^4 + q_\beta^2)^{-1}$. From now on we shall use dimensional regularization, calculating integrals below the upper critical dimension $d_u(m) = 4 - m/2$, and taking the limit $\Lambda \to \infty$.

In Ref. 8 there is a misprint in the equation that gives the degree of divergence, δ, of $\Gamma^{(N,L)}$. The correct expression reads

$$\delta = \left[d - \left(4 + \frac{d_\alpha}{2}\right)\right] I + \left(2 - L - \frac{N}{2}\right)\left(\frac{d_\alpha}{2} + d_\beta\right) . \tag{5}$$

Thus, at $d = d_u(m)$ the term proportional to I—the number of internal lines— cancels out and the only 1PI functions with primitive divergences ($\delta \geq 0$) are $\Gamma^{(2,0)}$, $\Gamma^{(4,0)}$, $\Gamma^{(2,1)}$, and $\Gamma^{(0,2)}$. These are the same as in the ϕ^4 theory and here we can also neglect $\Gamma^{(0,1)}$ which gives an infinite constant.

As in the ϕ^4 theory, all 1PI Green functions $\Gamma^{(N,L)}$, except for $\Gamma^{(0,2)}$ that requires additive renormalization, are renormalized multiplicatively,[8]

$$\Gamma_R^{(N,L)}(k_i, p_i, \sigma, g, \kappa) = Z_\phi^{N/2} Z_{\phi^2}^L \Big[\Gamma^{(N,L)}(k_i, p_i, \sigma_0, \lambda) - \delta_{N,0}\delta_{L,2}\Gamma^{(0,2)}(p, -p, \sigma_0, \lambda)\Big|_{\substack{\sigma p_\alpha^4 = \kappa^2 \\ p_\beta^2 = 0}}\Big] , \tag{6}$$

where g is the renormalized coupling constant, $\sigma = Z_\sigma \sigma_0$ is the renormalized σ_0 parameter, Z_σ, Z_ϕ, Z_{ϕ^2} are renormalization constants, and κ is an arbitrary momentum scale.

However, there are differences. The divergent part of $\Gamma^{(2,0)}$ has the structure

$$\Gamma^{(2,0)} = \frac{I_{11}^{\alpha}}{\epsilon} k_{\alpha}^4 + \frac{I_{11}^{\beta}}{\epsilon} k_{\beta}^2 + \mathcal{O}(\epsilon^0) \ , \tag{7}$$

with $I_{11}^{\alpha} \neq I_{11}^{\beta}$. Besides field renormalization we need the renormalization of the σ_0 parameter to eliminate the poles of $\Gamma^{(2,0)}$. Therefore, the renormalization conditions[8] that define the renormalized parameters are

$$\frac{\partial}{\partial k_{\alpha}^4} \Gamma_R^{(2,0)}(k, -k, \sigma, g, \kappa)\Big|_{\substack{\sigma k_{\alpha}^4 = \kappa^2 \\ k_{\beta}^2 = 0}} = \sigma \ , \tag{8}$$

$$\frac{\partial}{\partial k_{\beta}^2} \Gamma_R^{(2,0)}(k, -k, \sigma, g, \kappa)\Big|_{\substack{\sigma k_{\alpha}^4 = 0 \\ k_{\beta}^2 = \kappa^2}} = 1 \ , \tag{9}$$

$$\Gamma_R^{(4,0)}(k_1, \ldots, k_4, \sigma, g, \kappa)\Big|_{sp_{\alpha}} = g \ , \tag{10}$$

$$\Gamma_R^{(2,1)}(k_1, k_2, p, \sigma, g, \kappa)\Big|_{\overline{sp}_{\alpha}} = 1 \ , \tag{11}$$

$$\Gamma_R^{(0,2)}(p, -p, \sigma, g, \kappa)\Big|_{\substack{\sigma p_{\alpha}^4 = \kappa^2 \\ p_{\beta}^2 = 0}} = 0 \ , \tag{12}$$

where the renormalization points are defined as follows: sp_{α} means $\sigma^{1/2} k_{i\alpha} \cdot k_{j\alpha} = \kappa(4\delta_{ij} - 1)/4$; $\overline{sp}_{\alpha}$ means $\sigma^{1/2} k_{i\alpha}^2 = 3\kappa/4$, $\sigma^{1/2} k_{1\alpha} \cdot k_{2\alpha} = -\kappa/4$, $\sigma^{1/2}(k_1 + k_2)_{\alpha}^2 = \sigma^{1/2} p_{\alpha}^2 = \kappa$, and, except in Eq. (9), the external momenta at which the values of the Green functions are evaluated have no components in the β subspace. This choice of renormalization points has a remarkable property: it makes the renormalization constants σ-independent.[8] Our choosing σ_0 dimensionless implies that k_{α} has dimension of square root of mass, this explains why the scalar products of the α components of the momenta above are proportional to κ and not to κ^2 as usual. Parameters and $\Gamma_R^{(N,L)}$ have the following dimensions: $[r_0] = [\kappa^2]$, $[c_0] = [\kappa^0]$, $[\sigma_0] = [\kappa^0]$, $[k_{\beta}] = [\kappa]$, $[x_{\beta}] = [\kappa^{-1}]$, $[k_{\alpha}] = [\kappa^{1/2}]$, $[x_{\alpha}] = [\kappa^{-1/2}]$, $[\lambda] = [\kappa^{4+d_{\alpha}/2-d}]$, $[\phi(x)] = [\kappa^{-1+d_{\alpha}/4+d_{\beta}/2}]$, and $[\Gamma^{(N,L)}(k_i, p_i, \ldots)] = [\kappa^{(d_{\alpha}/2+d_{\beta})(1-N/2)+N-2L}]$.

Before proceeding, it is convenient to define dimensionless coupling constants u_0 and u such that

$$u\kappa^{4-D} = g\sigma^{-d_{\alpha}/4} \ ,$$

$$u_0 \kappa^{4-D} = \lambda \ ,$$

$$D \equiv d_{\alpha}/2 + d_{\beta} \ . \tag{13}$$

Next, we write $u_0\sigma^{-d_\alpha/4}$ and the renormalization constants as power series in u:

$$u_0\sigma^{-d_\alpha/4} = u(1 + a_1 u + a_2 u^2) , \tag{14}$$

$$Z_\phi = 1 + bu^2 , \tag{15}$$

$$Z_{\phi^2} = 1 + c_1 u + c_2 u^2 , \tag{16}$$

$$Z_\sigma = 1 + du^2 . \tag{17}$$

Inserting Eqs. (13)–(17) into Eqs. (6) and (8)–(12), all dependence on σ cancels out.[8] Thus, u_0 has a trivial dependence on σ—a multiplicative factor $\sigma^{d_\alpha/4}$—, and the renormalization constants Z_ϕ, Z_{ϕ^2} and Z_σ depend only on u.[8]

All results presented so far apply only at the Lifshitz point. This suffices to determine the exponent η. However, to determine the other exponents it is necessary to know the Green functions also in the neighborhood of the LP. We have adapted[8] to the LP a technique that allows one to expand the Green function around an ordinary critical point in terms of the Green functions calculated at the critical point.[11] This expansion reads

$$\begin{aligned}
&\Gamma_R^{(N,L)}(k_i, p_i, \sigma, t, u, M) \\
&= \sum_{I,J} \frac{t^I M^J}{I!J!} \Gamma_R^{(N+J,L+I)}(k_i, l_i = 0, p_i, q_i = 0, \sigma, u, \kappa) \\
&= Z_\phi^{\frac{N}{2}} Z_{\phi^2}^{L} \Big[\Gamma^{(N,L)}(k_i, p_i, \sigma_0, \delta r, u_0\kappa^{4-D}, \bar{\phi}) \\
&- \delta_{N,0}\delta_{L,2}\Gamma^{(0,2)}(p, -p, \sigma_0, 0, u_0\kappa^{4-D}, 0)\Big|_{\substack{\sigma p_\alpha^4=\kappa^2 \\ p_\beta^2=0}} \Big] ,
\end{aligned} \tag{18}$$

where $t \equiv Z_{\phi^2}^{-1}\delta r$ and $M \equiv Z_\phi^{-\frac{1}{2}}\bar{\phi}$ are finite, and the renormalization constants are equal to the ones calculated at the LP. Above T_L, in the paramagnetic phase, one obtains an analogous expression without M.

The next step consists in deriving the renormalization-group equations for $\Gamma_R^{(N,L)}$. The solutions of these equations and the determination of the dependence of $\Gamma_R^{(N,L)}$ on σ enabled us to identify the critical exponents. We found[8]

$$\eta_{\ell 2} = \gamma_1^* , \tag{19}$$

$$\eta_{\ell 4} = 4\left(\frac{\gamma_1^* - \gamma_\sigma^*}{2-\gamma_\sigma^*}\right) , \tag{20}$$

$$\nu_{\ell 4} = \frac{1}{4}\left(\frac{2-\gamma_\sigma^*}{2-\gamma_2^*}\right) , \tag{21}$$

$$\nu_{\ell 2} = \frac{1}{2-\gamma_2^*} , \tag{22}$$

where $\gamma_1^* \equiv \gamma_\phi(u^*)$, $\gamma_2^* \equiv \gamma_{\phi^2}(u^*)$, $\gamma_\sigma^* \equiv \gamma_\sigma(u^*)$, u^* is the fixed point, and

$$\gamma_\sigma(u) = \left(\kappa\frac{\partial ln Z_\sigma}{\partial\kappa}\right)_{\lambda,\sigma_0} , \tag{23}$$

$$\gamma_\phi(u) = \left(\kappa\frac{\partial ln Z_\phi}{\partial\kappa}\right)_{\lambda,\sigma_0} , \tag{24}$$

$$\gamma_{\phi^2}(u) = -\left(\kappa\frac{\partial ln Z_{\phi^2}}{\partial\kappa}\right)_{\lambda,\sigma_0} . \tag{25}$$

The fixed point u^* satisfies $\beta(u^*) = 0$, where

$$\beta(u) = \left(\kappa\frac{\partial u}{\partial\kappa}\right)_{\lambda,\sigma_0} . \tag{26}$$

After determining $\eta_{\ell 2}$, $\eta_{\ell 4}$, $\nu_{\ell 2}$, and $\nu_{\ell 4}$, one may calculate the other exponents using the scaling relations[1,8]

$$\gamma_\ell = \nu_{\ell 4}(4-\eta_{\ell 4}) = \nu_{\ell 2}(2-\eta_{\ell 2}) , \tag{27}$$

$$\gamma_\ell = \beta_\ell(\delta_\ell - 1) , \tag{28}$$

$$\alpha_\ell + 2\beta_\ell + \gamma_\ell = 2 , \tag{29}$$

and

$$2 - \alpha_\ell = d_\beta \nu_{\ell 2} + d_a \nu_{\ell 4} . \tag{30}$$

All these relations hold to all orders in perturbation.

III. CALCULATION OF THE CRITICAL EXPONENTS

First we determine the coefficients in the expansions of $u_0\sigma^{-d_\alpha/4}$, Z_ϕ, Z^2_ϕ, and Z_σ, Eqs. (14)–(17), using Eq. (6) and the renormalization conditions for $\Gamma_R^{(2,0)}$, $\Gamma_R^{(4,0)}$, and $\Gamma_R^{(2,1)}$, given in Eqs. (8)–(11).

The diagrams that renormalize the mass can be neglected.[11,8] Actually, in the dimensional regularization scheme they vanish exactly. Thus, only the diagram D_1 in Fig. 1 contributes to $\Gamma_R^{(2,0)}$ to the order of two loops. The diagrams D_2, D_3, and D_4 of $\Gamma_R^{(4,0)}$; and D_5, D_6, and D_7 of $\Gamma_R^{(2,1)}$ are shown in Figs. 2 and 3, respectively. Due to the choice of the renormalization points (see Eqs. (8)–(12) and the text below them), several of these diagrams give the same contributions. If we define I'_i as the contribution of the diagram D_i after extracting the symmetry factor and all powers of $-\lambda$, associated with each ϕ^4 vertex of the diagram, then the following relations hold

$$I'_5|_{\overline{sp}_\alpha} = I'_2|_{sp_\alpha} \, , \tag{31}$$

$$I'_6|_{\overline{sp}_\alpha} = I'_3|_{sp_\alpha} \, , \tag{32}$$

$$I'_7|_{\overline{sp}_\alpha} = I'_4|_{sp_\alpha} = \left(I'_2|_{sp_\alpha}\right)^2 \, . \tag{33}$$

For example, consider the diagrams D_2 and D_5. The integral I'_5 is obtained by replacing p_α for $(k_1+k_2)_\alpha$ in I'_2 (see Figs. 2, 3, App. A and recall that the renormalization points $\overline{sp}_\alpha$ and sp_α have no components in the β subspace). The α components of the momenta are multiplied by $\sigma_0^{1/4}$ or equivalently by $\sigma^{1/4}$, since $\sigma_0 = Z_\sigma^{-1}\sigma = \sigma + \mathcal{O}(u^2)$. Moreover, rotational invariance in the α subspace implies that I'_5 and I'_2 depend only on the scalar products $\sigma^{1/2}p_\alpha^2$ and $\sigma^{1/2}(k_1+k_2)^2_\alpha$, respectively. The renormalization point sp_α is chosen in such a way that these two products are identical and Eq. (31) is satisfied. Equations (32) and (33) can be verified analogously. From now on we drop the subscripts $\overline{sp}_\alpha$ and sp_α.

In this way, we only have to calculate three integrals, namely I'_1, I'_2, and I'_3, associated with the diagrams D_1, D_2, and D_3. Following Amit,[11] we also extract a power of κ from the integrals I'_i to render them dimensionless. We define I_i as follows

$$I_1' \equiv \kappa^{2D-14} I_1 \,, \quad I_2' \equiv \kappa^{D-4} I_2 \,, \quad I_3' \equiv \kappa^{2D-16} I_3 \,, \tag{34}$$

and $D \equiv d_\alpha/2 + d_\beta$ (see Eq. (13)). The renormalization points at which these dimensionless integrals are evaluated are given in the text below Eq. (12) with κ replaced by 1.

We perform our calculations at the LP with the massless free propagator $[\sigma_0 q_\alpha^4 + q_\beta^2]^{-1}$. Since $\sigma_0 = Z_\sigma^{-1}\sigma$ and $Z_\sigma = 1 + \mathcal{O}(u^2)$, if we choose our renormalized σ parameter equal to 1, then we can put the propagators equal to $[q_\alpha^4 + q_\beta^2]^{-1}$ in all diagrams that have at least one-loop.

It is important to mention how the dimensional regularization scheme was implemented. We calculate integrals below the upper critical dimension $d_u = 4 + m/2$ in the following way: for $m = 2$, $d_\alpha = 2 - \epsilon$ and $d_\beta = 3$; for $m = 3$, $d_\alpha = 6 - \epsilon$ and $d_\beta = 1$. Note that the ϵ expansion is performed only in the α subspace. It is convenient to extract for each integration loop a geometric angular factor $S_{d_\alpha} S_{d_\beta}$, where

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \,, \tag{35}$$

and absorb it in a redefinition of the coupling constant.[11]

We obtained the following expressions for $\beta(u)$, $\gamma_\phi(u)$, $\gamma_{\phi^2}(u)$, and $\gamma_\sigma(u)$ in terms of the dimensionless integrals I_i

$$\beta(u) = -\frac{\epsilon}{2} u \left[1 - a_1 u + 2\left(a_1^2 - a_2 + \frac{d_\alpha}{4} d\right) u^2\right] + \mathcal{O}\left(u^4\right) \,, \tag{36}$$

$$\gamma_\phi(u) = -\epsilon b u^2 + \mathcal{O}\left(u^3\right) \,, \tag{37}$$

$$\gamma_\sigma(u) = -\epsilon d u^2 + \mathcal{O}\left(u^3\right) \,, \tag{38}$$

$$\gamma_{\phi^2}(u) = \frac{\epsilon}{2} c_1 u + \frac{\epsilon}{2}\left(2c_2 - c_1^2 - c_1 a_1\right) u^2 + \mathcal{O}\left(u^3\right) \,, \tag{39}$$

where a_1, a_2, b, c_1, c_2, and d are given by

$$a_1 = \left(\frac{n+8}{6}\right) I_2 + \mathcal{O}(\epsilon) \,, \tag{40}$$

$$a_2 = \frac{n^2 + 26n + 108}{36} (I_2)^2 - \frac{5n+22}{9} I_3 - \frac{n+2}{9} \frac{I_{11}^\beta}{\epsilon} + \mathcal{O}\left(\epsilon^0\right) \,, \tag{41}$$

$$b = \left(\frac{n+2}{18}\right) \frac{I_{11}^\beta}{\epsilon} + \mathcal{O}\left(\epsilon^0\right) \,, \tag{42}$$

$$c_1 = \left(\frac{n+2}{6}\right) I_2 + \mathcal{O}(\epsilon) \ , \tag{43}$$

$$c_2 = \left(\frac{n^2+10n+16}{36}\right)(I_2)^2 - \left(\frac{n+2}{6}\right) I_3 - \left(\frac{n+2}{18}\right)\frac{I_{11}^{\beta}}{\epsilon} + \mathcal{O}\left(\epsilon^0\right) \ , \tag{44}$$

$$d = \frac{n+2}{18}\left(\frac{I_{11}^{\beta} - I_{11}^{\alpha}}{\epsilon}\right) + \mathcal{O}\left(\epsilon^0\right) \ . \tag{45}$$

Recall that n is the number of components of the order parameter $\vec{\Phi}$; I_{11}^{α} and I_{11}^{β}—the coefficients of the poles of $\Gamma^{(2,0)}$—are defined in Eq. (7). Note that we only need I_2 to order ϵ^0 and I_3 to order ϵ^{-1}.

Defining I_{ij} as the term of I_i proportional to ϵ^{-j} we can write

$$I_2 \equiv \frac{I_{21}}{\epsilon} + I_{20} + \mathcal{O}(\epsilon) \ , \tag{46}$$

$$I_3 \equiv \frac{I_{32}}{\epsilon^2} + \frac{I_{31}}{\epsilon} + \mathcal{O}\left(\epsilon^0\right) \ . \tag{47}$$

We also define the quantities

$$\Delta \equiv I_{31} - I_{21} I_{20} \ , \tag{48}$$

$$\Pi \equiv I_{11}^{\beta} + \frac{m}{8}\left(I_{11}^{\beta} - I_{11}^{\alpha}\right) \ , \tag{49}$$

$$S(n,m) \equiv (7n+20)(I_{31} - I_{21}I_{20}) + (n-4) I_{11}^{\beta} + (n+2)\frac{m}{4}\left(I_{11}^{\beta} - I_{11}^{\alpha}\right) \ , \tag{50}$$

to write the expressions for the fixed point and for the exponents in a more compact way.

Keeping these definitions in mind, the fixed point, u^*, is given by

$$u^* = \frac{6}{(n+8) I_{21}}\epsilon + \frac{48\left[(5n+22)\Delta + (n+2)\Pi\right] - 6(n+8)^2 I_{20} I_{21}}{(n+8)^3 I_{21}^3}\epsilon^2 \ . \tag{51}$$

Inserting Eqs. (37)–(45) and Eq. (51) into Eqs. (19)–(22) we obtain the critical exponents

$$\eta_{\ell\alpha} = -4\frac{n+2}{(n+8)^2}\frac{I_{11}^{\alpha}}{I_{21}^2}\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \ , \tag{52}$$

$$\eta_{\ell\beta} = -2\frac{n+2}{(n+8)^2}\frac{I_{11}^{\beta}}{I_{21}^2}\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \ , \tag{53}$$

$$\nu_{\ell\alpha} = \frac{1}{4} + \frac{n+2}{16(n+8)}\epsilon + \frac{n+2}{4(n+8)^2 I_{21}^2}\left(I_{11}^{\beta} - I_{11}^{\alpha} + \frac{n+2}{16} I_{21}^2 + \frac{S(n,m)}{n+8}\right)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \ , \tag{54}$$

$$\nu_{\ell\beta} = \frac{1}{2} + \frac{n+2}{8(n+8)}\,\epsilon + \frac{n+2}{2(n+8)^2 I_{21}^2}\left(\frac{n+2}{16}I_{21}^2 + \frac{S(n,m)}{n+8}\right)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) . \tag{55}$$

Note that the results derived so far hold for all values of n and for $1 \leq m \leq 7$. For $m = 8$ the propagator at the LP is simply $(q_\alpha^4)^{-1}$ and the usual techniques of field theory can be used. Finally, specializing for the the cases $m = 2$ and $m = 6$, and using the values of the integrals calculated in the appendices we obtain, for m=2,

$$\eta_{\ell\alpha} = -\frac{2}{27}\frac{n+2}{(n+8)^2}\epsilon^2 , \tag{56}$$

$$\eta_{\ell\beta} = \frac{1}{9}\frac{n+2}{(n+8)^2}\epsilon^2 , \tag{57}$$

$$\nu_{\ell\alpha} = \frac{1}{4} + \frac{n+2}{16(n+8)}\epsilon + \frac{n+2}{4(n+8)^3}\left(\frac{n^2}{16} + \frac{11n}{24} + \frac{5}{9} - \frac{7n+20}{4}\ln 3\right)\epsilon^2 , \tag{58}$$

$$\nu_{\ell\beta} = \frac{1}{2} + \frac{n+2}{8(n+8)}\epsilon + \frac{n+2}{2(n+8)^3}\left(\frac{n^2}{16} + \frac{115n}{216} + \frac{31}{27} - \frac{7n+20}{4}\ln 3\right)\epsilon^2 ; \tag{59}$$

and, for $m = 6$,

$$\eta_{\ell\alpha} = -\frac{7}{54}\frac{n+2}{(n+8)^2}\epsilon^2 , \tag{60}$$

$$\eta_{\ell\beta} = \frac{2}{3}\left(1 + 3\ln\frac{3}{4}\right)\frac{n+2}{(n+8)^2}\epsilon^2 , \tag{61}$$

$$\nu_{\ell\alpha} = \frac{1}{4} + \frac{n+2}{16(n+8)}\epsilon + \frac{n+2}{4(n+8)^3}\left[\frac{n^2}{16} + \frac{361n}{432} + \frac{535}{216} + \left(\frac{7n}{4} + 8\right)\ln\frac{3}{4}\right]\epsilon^2 , \tag{62}$$

$$\nu_{\ell\beta} = \frac{1}{2} + \frac{n+2}{8(n+8)}\epsilon + \frac{n+2}{2(n+8)^3}\left[\frac{n^2}{16} + \frac{173n}{144} + \frac{389}{72} + \left(\frac{11n}{4} + 16\right)\ln\frac{3}{4}\right]\epsilon^2 . \tag{63}$$

The other exponents are obtained using the scaling relations (Eqs. (27)–(30)).

IV. DISCUSSION OF THE RESULTS AND CONCLUSIONS

Our results for the critical exponents associated with the LP agree with the ones determined by Sak and Grest[7] after taking into account the different definitions of the ϵ parameter. Sak and Grest perform an ϵ expansion around the upper critical dimension, $d_u(m) = 4 - m/2$, and, as in Ref. 1, define

$$\epsilon_\ell = 4 + \frac{m}{2} - d = \frac{\epsilon_\alpha}{2} + \epsilon_\beta \ , \qquad (64)$$

where $d \equiv d_\alpha + d_\beta$. We also perturb around $d_u(m)$ but we take $d_\alpha = m - \epsilon$ and $d_\beta = 4 - m/2$, in other words we perform the ϵ expansion only in the α subspace. In our case $\epsilon_\alpha = \epsilon$, $\epsilon_\beta = 0$ and $\epsilon_\ell = \epsilon/2$. This last relation has to kept in mind when comparing the results.

In this paper we have shown how to calculate critical exponents for the LP using field theory. We have derived expressions for the exponents that hold for all n, m, $0 \le m \le 8$ in terms of diagrams. In particular, to order ϵ^2, we have reduced the problem to the evaluation of the leading contributions of three diagrams, namely D_1, given in Fig. 1, and D_2 and D_3, in Fig. 2. The technique of expanding around the LP has simplified the problem considerably in that we only have to deal with a massless free propagator of the type $[q_\beta^2 + \sigma_0 (q_\alpha^2)^2]^{-1}$ instead of the complete propagator, $[r_0 + q_\beta^2 + c_0 q_\alpha^2 + \sigma_0 (q_\alpha^2)^2]^{-1}$. We have demonstrated the feasibility of our method by calculating the exponents for the $m = 2$ and $m = 6$ cases for which there were conflicting results in the literature.

Finally, it is important to emphasize that the determination of critical exponents for the LP is not complete. As we discussed in Sec. I, $\nu_{\ell\alpha}$ and $\nu_{\ell\beta}$ are known only to order ϵ for the $m = 1$ case, the most interesting from the experimental point of view. Thus, more than twenty years after the seminal paper by Hornreich, Luban and Shtrikman,[1] one does not know the critical exponents of the LP for all m to order ϵ^2. We believe that the field theory approach to the LP is an interesting alternative to the existing methods and that further work along these lines may yield the ϵ^2 corrections for other values of m.

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Defining $P \equiv k_1 + k_2$, the integral I_2, associated with the diagram D_2 in Fig. 2, is given by

$$I_2 = \int \frac{d^dq}{(2\pi)^d} \frac{1}{q_\alpha^4 + q_\beta^2} \frac{1}{(q_\alpha + P_\alpha)^4 + (q_\beta + P_\beta)^2} \,, \tag{A1}$$

where $P = (P_\alpha, P_\beta)$, $\alpha = 1, \ldots, d_\alpha$ (α subspace), $\beta = 1, \ldots, d_\beta$ (β subspace) and $d_\alpha + d_\beta = d$.

1. Case m=2

For $m = 2$, $d_\alpha = 2 - \epsilon$ and $d_\beta = 3$. We begin the calculation as in Ref. 7 rewriting I_2 as

$$I_2 = \int \frac{d^dq_1}{(2\pi)^d} \frac{d^dq_2}{(2\pi)^d} (2\pi)^d \delta(q_1 + q_2 + P) \frac{1}{q_{1\alpha}^4 + q_{1\beta}^2} \frac{1}{q_{2\alpha}^4 + q_{2\beta}^2} \,. \tag{A2}$$

Using the integral representation of the δ function, $\delta(q) = \int d^dr \exp^{iq\cdot r} / (2\pi)^d$, we obtain

$$I_2 = \int d^dr (J(r))^2 e^{ir\cdot P} \,, \tag{A3}$$

where

$$J(r) \equiv \int \frac{d^dq}{(2\pi)^d} \frac{e^{ir\cdot q}}{q_\alpha^4 + q_\beta^2} \,, \tag{A4}$$

and $q \cdot r \equiv q_\alpha \cdot r_\alpha + q_\beta \cdot r_\beta$. $J(r)$ can be calculated using spherical coordinates. In the α subspace first we perform the integration for an arbitrary integer dimension, d_α, and then use the resulting expression to define the analytic continuation for $d_\alpha = 2 - \epsilon$. Recalling that $d_\beta = 3$ we can write

$$J = \frac{1}{(2\pi)^d} \int_0^\infty dk_1 k_1^{d_\alpha - 1} \int_0^{2\pi} d\theta_1 \left(\prod_{i=2}^{d_\alpha - 1} \int_0^\pi d\theta_i sen^{i-1}\theta_i \right) e^{ir_1 k_1 cos\theta_{d_\alpha - 1}}$$
$$\times \int_0^\infty dk_2 k_2^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin(\theta) e^{ir_2 k_2 cos\theta} \frac{1}{k_1^4 + k_2^2} \,, \tag{A5}$$

where $k_1 \equiv\mid q_\alpha \mid$, $k_2 \equiv\mid q_\beta \mid$, $r_1 \equiv\mid r_\alpha \mid$ and $r_2 \equiv\mid r_\beta \mid$. Using formulas 3.621-1 and 3.915-5, given in the table of integrals by Gradshteyn and Ryzhik (GrR) (Ref. 15), we can perform the angular integrals. Formulas 3.723-3 and 6.631-4 in GrR enable us to integrate over k_2 and k_1, respectively. The final result reads

$$J(r) = \frac{1}{(4\pi r_2)^{1+d_\alpha/2}} e^{-r_1^2/(4r_2)} \; . \tag{A6}$$

Next, we insert Eq. (A6) into Eq. (A3), write $r \cdot P = r_\alpha \cdot P_\alpha + r_\beta \cdot P_\beta$, and use spherical coordinates, as in Eq. (A5). The angular part of the resulting integral can be calculated using formulas 3.621-1 and 3.915-5 in GrR; and the integrals over $|r_\beta|$ and $|r_\alpha|$ with formulas 3.944-5 and 6.631-4. After this, we arrive at the exact result

$$I_2 = \frac{1}{2^{2-\epsilon}(2\pi)^{2-\epsilon/2}\epsilon} \frac{\Gamma(1+\epsilon/2)}{1-\epsilon/2} \frac{1}{|P_\beta|\left(P_\alpha^4/4 + P_\beta^2\right)^{\epsilon/4-1/2}} \sin\left[\left(1-\frac{\epsilon}{2}\right)\arctan\left(\frac{2|P_\beta|}{P_\alpha^2}\right)\right] \; . \tag{A7}$$

Extracting the angular factor $8\pi^{2-\epsilon/2}/[\Gamma(1-\epsilon/2)(2\pi)^{5-\epsilon}]$, which will be absorbed in the coupling constant, as explained in Sec. III, and expanding the result to order ϵ^0 we obtain

$$I_2 = \frac{\pi}{4}\left[\frac{1}{\epsilon} + \frac{1}{2} + \frac{1}{2}\ln 2 - \frac{1}{4}\ln\left(\frac{P_\alpha^4}{4} + P_\beta^2\right) - \frac{P_\alpha^2}{4|P_\beta|}\arctan\left(\frac{2|P_\beta|}{P_\alpha^2}\right)\right] \; . \tag{A8}$$

At the renormalization point sp_α, $P_\beta = 0$, and $P_\alpha^2 \equiv (k_{1\alpha} + k_{2\alpha})^2 = 1$ (we chose $\sigma = 1$). Thus,

$$I_2|_{sp_\alpha} = \frac{\pi}{4}\left(\frac{1}{\epsilon} + \ln 2\right) \; . \tag{A9}$$

2. Case m=6

We use Eq. (A1) again. However, for $m = 6$, $d_\alpha = 6 - \epsilon$ and $d_\beta = 1$. Since the β subspace is unidimensional we can use complex integration to calculate the integral over q_β. After taking the two poles into account, we are left with two integrals over q_α. It is convenient to add two other integrals, obtained from the first ones by making the change of variables $q_\alpha \to -q_\alpha - P_\alpha$. In order not to change the result we divide each integral by 2. Adding the four integrals, we arrive at the result

$$I_2 = \frac{1}{2}\int \frac{d^{d_\alpha} q_\alpha}{(2\pi)^{d_\alpha}} \frac{q_\alpha^2 + (q_\alpha + P_\alpha)^2}{q_\alpha^2\, (q_\alpha + P_\alpha)^2\, \{P_\beta^2 + [q_\alpha^2 + (q_\alpha + P_\alpha)^2]^2\}} \,. \tag{A10}$$

The two terms in the numerator of Eq. (10) give the same contribution as can be seen by separating each contribution and making the change of variables $q_\alpha \to -q_\alpha - P_\alpha$ in one of the integrals. Thus,

$$I_2 = \int \frac{d^{d_\alpha} q_\alpha}{(2\pi)^{d_\alpha}} \frac{1}{q_\alpha^2\, \{P_\beta^2 + (q_\alpha^2 + (q_\alpha + P_\alpha)^2)^2\}} \,. \tag{A11}$$

The identity

$$\frac{1}{P_\beta^2 + A^2} = \frac{1}{2iP_\beta}\left[\frac{1}{A - iP_\beta} - \frac{1}{A + iP_\beta}\right] , \tag{A12}$$

with $A = q_\alpha^2 + (q_\alpha + P_\alpha)^2$, enables us to rewrite I_2 as the sum of two integrals whose denominators contain the product of two terms quadratic in q_α. The standard Feynman parameter method[11] allows us to put these two terms together and integrate over q_α. After extracting the angular factor $4\pi^{d_\alpha/2}[\Gamma(d_\alpha/2)(2\pi)^d]^{-1}$, we obtain

$$I_2 = -\frac{\pi}{2P_\beta}\Gamma(3 - \epsilon/2)\Gamma(\epsilon/2 - 1) \int_1^2 dx (x-1)^{1-\epsilon/2} x^{\epsilon-4} (P_\alpha^4 + x^2 P_\beta^2)^{1/2-\epsilon/4}$$
$$\times \sin\left[(1 - \epsilon/2)\arctan\left(xP_\beta/P_\alpha^2\right)\right] \,. \tag{A13}$$

The expansion of Eq. (A13) to order ϵ^0 reads

$$I_2 = \frac{\pi}{4}\left\{\frac{1}{\epsilon} + \frac{3}{4} - \ln 2 - \frac{1}{4}\ln p_\alpha^4 + \frac{1}{12\Delta}\left[-8\arctan\Delta + 24\Delta^2\arctan\Delta\right.\right.$$
$$+ 4\arctan(2\Delta) - 24\Delta^2\arctan(2\Delta) + 8\Delta^3\ln 2 - 12\Delta\ln(1+\Delta^2)$$
$$\left.\left. + 4\Delta^3\ln(1+\Delta^2) + 9\Delta\ln(1+4\Delta^2) - 4\Delta^3\ln(1+4\Delta^2)\right]\right\} , \tag{A14}$$

where $\Delta \equiv P_\beta/P_\alpha^2$.

At the renormalization point sp_α, $P_\beta^2 = 0$, $P_\alpha^2 = 1$ and we finally obtain

$$I_2|_{sp_\alpha} = \frac{\pi}{4}\left(\frac{1}{\epsilon} + \frac{3}{4} - \ln 2\right) \,. \tag{A15}$$

APPENDIX B: DIAGRAM D_1

The integral I_1, associated with diagram D_1 in Fig. 1, is given by

$$I_1 = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_{1\alpha}^4+q_{1\beta}^2}\frac{1}{q_{2\alpha}^4+q_{2\beta}^2}\frac{1}{(q_{1\alpha}+q_{2\alpha}+P_\alpha)^4+(q_{1\beta}+q_{2\beta}+P_\beta)^2}\,. \tag{B1}$$

1. Case m=2

For $m = 2$, $d_\alpha = 2-\epsilon$ and $d_\beta = 3$. The integral representation of the δ function (see App. A 1), enables us to write Eq. (B1) as

$$I_1 = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\int d^dr e^{ir\cdot(q_1+q_2+q_3+P)}\frac{1}{q_{1\alpha}^4+q_{1\beta}^2}\frac{1}{q_{2\alpha}^4+q_{2\beta}^2}\frac{1}{q_{3\alpha}^4+q_{3\beta}^2}\,. \tag{B2}$$

Using the function $J(r)$ defined in Eq. (A4), I_1 can be rewritten as

$$I_2 = \int d^dr (J(r))^3 e^{ir\cdot P}\,. \tag{B3}$$

Inserting Eq. (A6) that gives the value of $J(r)$ into Eq. (B3), integrating over r as in App. A 1, and extracting the angular factor $[\Gamma(1-\epsilon/2)(2\pi)^{5-\epsilon}/8\pi^{2-\epsilon/2}]^{-2}$ we obtain

$$I_1 = \frac{\pi^2}{288}\left(\frac{1}{\epsilon}+\frac{11}{6}+\frac{1}{2}\ln 3\right)\frac{1}{P_\beta}\left(\frac{P_\alpha^4}{9}+P_\beta^2\right)^{3/2-\epsilon/2}\sin\left[(3-\epsilon)\arctan\left(\frac{3P_\beta}{P_\alpha^2}\right)\right]\,. \tag{B4}$$

Expanding to order ϵ^0 we arrive at

$$I_1 = \frac{\pi^2}{288}(\frac{P_\alpha^4}{3}-P_\beta^2)\left\{\frac{1}{\epsilon}+\frac{11}{6}+\frac{1}{2}\ln 3-\frac{1}{2}\ln\left(\frac{P_\alpha^4}{9}+P_\beta^2\right)\right.$$
$$\left.-\frac{P_\alpha^2\arctan\left[(3P_\beta/P_\alpha^2)\left(1-27P_\beta^2/P_\alpha^4\right)\right]}{9P_\beta\left(1-3P_\beta^2/P_\alpha^4\right)}\right\}\,. \tag{B5}$$

From Eq. (B5) one can easily determine I_{11}^α and I_{11}^β—the coefficients of P_α^4/ϵ and P_β^2/ϵ, respectively.

2. Case m=6

For $m = 6$, $d_\alpha = 6 - \epsilon$ and $d_\beta = 1$. We can perform the integration over $q_{2\beta}$ in Eq. (B1) by noting that it is analogous to the integration over q_β in Eq. (A1). Thus, we can use Eq. (A11) with $q_{1\alpha} + P_\alpha$ replaced for P_α, and $q_{1\beta} + P_\beta$ for P_β.

$$I_1 = \int \frac{d^d q_1}{(2\pi)^d} \frac{d^{d_\alpha} q_{2\alpha}}{(2\pi)^{d_\alpha}} \frac{1}{q_{1\alpha}^4 + q_{1\beta}^2} \frac{1}{q_{2\alpha}^2} \frac{1}{(q_{2\alpha}^2 + (q_{1\alpha} + q_{2\alpha} + P_\alpha)^2)^2 + (q_{1\beta} + P_\beta)^2} . \tag{B6}$$

Next, we integrate over $q_{1\beta}$ in the complex plane. After a very long algebra we obtain

$$\begin{aligned} I_1 = &\frac{1}{2} \int\limits_{q_1} \int\limits_{q_2} \frac{1}{q_1^2} \frac{1}{q_2^2} \frac{1}{P_\beta^2 + (q_1^2 + q_2^2 + (q_1 + q_2 + P)^2)^2} + \\ &+ \frac{1}{2} \int\limits_{q_1} \int\limits_{q_2} \frac{1}{q_2^2} \frac{1}{q_2^2 + (q_1 + q_2 + P)^2} \frac{1}{P_\beta^2 + (q_1^2 + q_2^2 + (q_1 + q_2 + P)^2)^2} , \end{aligned} \tag{B7}$$

where we defined $q_1 \equiv q_{1\alpha}$, $q_2 \equiv q_{2\alpha}$, $P \equiv P_\alpha$, and

$$\int\limits_q \equiv \int \frac{d^{d_\alpha} q}{(2\pi)^{d_\alpha}} . \tag{B8}$$

Let us denote I_{1A} and I_{1B} the first and second integrals in Eq. (B7), respectively. We calculate I_{1A} by first using Eq. (A12) to rewrite the term that contains P_β. We obtain two integrals, one the complex conjugate of the other. After this, we use a single Feynman parameter to put together the two denominators that contain q_1 and integrate over this variable. The resulting expression is combined with the remaining q_2^2 term in the denominator using another Feynman parameter. Integration over q_2 yields

$$\begin{aligned} I_{1A} = &\frac{\pi^2}{i32P_\beta} (\Gamma(3 - \epsilon/2))^2 \, \Gamma(\epsilon - 3) \int\limits_0^1 dx \left[x \left(1 - \frac{x}{4} \right) \right]^{1-\epsilon/2} \int\limits_0^1 dy y^{1-\epsilon/2} \\ &\times \left\{ \left[\left(\frac{1/2 - x/4}{1 - x/4} \right) \left(1 - \frac{1/2 - x/4}{1 - x/4} y \right) P_\alpha^2 - \frac{iP_\beta}{2 - x/2} \right]^{3-\epsilon} - c.c. \right\} . \end{aligned} \tag{B9}$$

Extracting the angular factor, expanding the result to order ϵ^{-1} and integrating over x and y we obtain

$$I_{1A} = -\frac{\pi^2}{48\epsilon} \left[\left(\frac{2}{3} + 2 \ln \frac{3}{4} \right) P_\beta^2 - \frac{7}{108} P_\alpha^4 \right] + \mathcal{O}\left(\epsilon^0\right) . \tag{B10}$$

The integral I_{1B} is calculated following the same steps as in I_{1A}. The result is given by

$$I_{1B} = \frac{\pi^2}{i32P_\beta}(\Gamma(3-\epsilon/2))^2\,\Gamma(\epsilon-3)\int_1^2 dx\left[x\left(1-\frac{x}{4}\right)\right]^{1-\epsilon/2}\int_0^1 dy y^{1-\epsilon/2}$$
$$\times\left\{\left[\left(\frac{1/2-x/4}{1-x/4}\right)\left(1-\frac{1/2-x/4}{1-x/4}y\right)P_\alpha^2 - \frac{iP_\beta(2-x)}{2x(1-x/4)}\right]^{3-\epsilon} - c.c.\right\}. \qquad \text{(B11)}$$

After expanding Eq. (B11) to order ϵ^{-1}, we can integrate over x and y:

$$I_{1B} = -\frac{\pi^2}{48\epsilon}\left[\left(\frac{1}{3}+\ln\frac{3}{4}\right)P_\beta^2 - \frac{7}{216}P_\alpha^4\right] + \mathcal{O}\left(\epsilon^0\right)\,. \qquad \text{(B12)}$$

Putting Eqs. (B10) and (B12) together we obtain the final result

$$I_1 = \frac{\pi^2}{48\epsilon}\left[\frac{7}{72}P_\alpha^4 - \left(1+3\ln\frac{3}{4}\right)P_\beta^2\right] + \mathcal{O}\left(\epsilon^0\right), \qquad \text{(B13)}$$

from which we can easily read I_{11}^α and I_{11}^β—the coefficients of P_α^4/ϵ and P_β^2/ϵ.

APPENDIX C: DIAGRAM D_3

Defining $P \equiv k_1 + k_2$ we can write the integral I_3 associated with diagram D_3, given in Fig. 2, as

$$I_3^\alpha = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_{1\alpha}^4+q_{1\beta}^2}\frac{1}{q_{2\alpha}^4+q_{2\beta}^2}\frac{1}{(q_{1\alpha}+P_\alpha)^4+q_{1\beta}^2}$$
$$\times\frac{1}{(q_{1\alpha}+q_{2\alpha}-k_{3\alpha})^4+(q_{1\beta}+q_{2\beta})^2}\,. \qquad \text{(C1)}$$

Note that we are restricting the calculation of I_3 to the case where the external momenta have components only in the α subspace. Due to the choice of renormalization points that we made, this is sufficient for our purposes.

1. Case m=2

For $m = 2$, $d_\alpha = 2-\epsilon$ and $d_\beta = 3$. Using the integral representation of the δ function, given in App. A 1, we can write Eq. C1 as

$$I_3 = \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{d^d q_3}{(2\pi)^d} \frac{1}{q_{1\alpha}^4 + q_{1\beta}^2} \frac{1}{q_{2\alpha}^4 + q_{2\beta}^2} \frac{1}{(q_{1\alpha}+P_\alpha)^4 + q_{1\beta}^2}$$
$$\times \int d^d r \frac{e^{ir\cdot(q_1+q_2+q_3-k_{3\alpha})}}{q_{3\alpha}^4 + q_{3\beta}^2} . \tag{C2}$$

Using the function $J(r)$, defined in Eq. (A4) and evaluated in Eq. (A6), we obtain

$$I_3 = \int \frac{d^d q_1}{(2\pi)^d} \int d^d r \frac{1}{q_{1\alpha}^4 + q_{1\beta}^2} \frac{e^{ir\cdot(q_1-k_3)}}{(q_{1\alpha}+P_\alpha)^4 + q_{1\beta}^2} \frac{e^{-r_1^2/2r_2}}{(4\pi r_2)^{2+d_\alpha}} , \tag{C3}$$

where $r_1 \equiv |r_\alpha|$, $r_2 \equiv | \, |r_\beta|$ and $k_3 \equiv (k_{3\alpha}, 0)$.

The integration over $q_{1\beta}$ can be performed using spherical coordinates ($d_\beta = 3$). After calculating the angular integrals we integrate over $|q_{1\beta}|$ using Eq. 3.728-2 in GrR.[15] We arrive at the expression

$$I_3 = \frac{2\pi^2}{(4\pi)^{2+d_\alpha}(2\pi)^d} \int d^{d_\alpha} q_\alpha \frac{1}{(q_\alpha+P_\alpha)^4 - q_\alpha^4}$$
$$\times \int d^d r e^{-r_1^2/2r_2 + ir_\alpha\cdot(q_\alpha - k_{3\alpha})} \left(\frac{e^{-r_2 q_\alpha^2} - e^{-r_2(q_\alpha+P_\alpha)^2}}{r_2^{3+d_\alpha}} \right) . \tag{C4}$$

The integral over r in Eq. (C4) can be written as

$$4\pi \int_0^\infty dr_2 \frac{e^{-r_2 q_\alpha^2} - e^{-r_2(q_\alpha+P_\alpha)^2}}{r_2^{1+d_\alpha}} \int d^{d_\alpha} r_\alpha e^{-r_1^2/2r_2 + ir_\alpha\cdot(q_\alpha-k_{3\alpha})}$$
$$= 2(2\pi)^{2-\epsilon/2}\Gamma(\frac{\epsilon}{2}-1) \left\{ \left[q_\alpha^2 + \frac{1}{2}(q_\alpha - k_{3\alpha})^2 \right]^{1-\epsilon/2} \right.$$
$$\left. - \left[(q_\alpha+P_\alpha)^2 + \frac{1}{2}(q_\alpha - k_{3\alpha})^2 \right]^{1-\epsilon/2} \right\} , \tag{C5}$$

where $d_\alpha = 2-\epsilon$, $d_\beta = 3$, and the integration over r_α was made using Eqs. 3.915-5 and 6.631-4 in GrR; and the integration over r_2 with Eq. 3.381-4.

Inserting Eq. (C5) into Eq. (C4) and using the identity

$$\frac{1}{\Delta}\left[\frac{1}{a^\nu} - \frac{1}{(a+\Delta)^\nu}\right] = \nu \int_0^1 dx \frac{1}{(a+x\Delta)^{\nu+1}} , \quad (\nu \neq 0) , \tag{C6}$$

we obtain

$$I_3 = \frac{\Gamma(\epsilon/2)}{3^{\epsilon/2} 2^{5-\epsilon/2}(2\pi)^{5-3\epsilon/2}} \int_0^1 dx \int d^{d_\alpha} q_\alpha \frac{1}{q_\alpha^2 + q_\alpha P_\alpha + P_\alpha^2/2}$$
$$\times \frac{1}{[q_\alpha^2 + 2q_\alpha(4P_\alpha x/3 - k_{3\alpha}/3) + k_{3\alpha}^2/3 + 2xP_\alpha^2/3]^{\epsilon/2}} . \tag{C7}$$

We can use Feynman parameters to put the two denominators together and integrate over q_α (Ref. 11). Thus,

$$I_3 = \frac{\Gamma(\epsilon)}{3^{\epsilon/2}2^{6-\epsilon}(2\pi)^{4-\epsilon}} \int_0^1 dx \int_0^1 dy y^{\epsilon/2-1}$$
$$\times \left[k_{3\alpha}^2 \frac{y}{3} + P_\alpha^2 \left(\frac{2xy}{3} + \frac{1-y}{2} \right) - \left(P_\alpha \left(\frac{1-y}{2} + \frac{4xy}{3} \right) - k_{3\alpha} \frac{y}{3} \right)^2 \right]^{-\epsilon} . \qquad \text{(C8)}$$

At this stage we can integrate over the remaining variables using a very elegant procedure described in App. 9-3 of Amit's book.[11] We obtain

$$I_3 = \frac{\Gamma(1+\epsilon)}{3^{\epsilon/2}2^{5-3\epsilon}(2\pi)^{4-\epsilon}\epsilon^2} \frac{1}{(P_\alpha^4)^{\epsilon/2}} . \qquad \text{(C9)}$$

Finally, extracting the angular factor and expanding to order ϵ^{-1} the result takes the form

$$I_3 = \frac{\pi^2}{32} \left[\frac{1}{\epsilon^2} + \frac{1}{2\epsilon} \left(\ln \frac{16}{3} - \ln P_\alpha^4 \right) \right] . \qquad \text{(C10)}$$

2. Case m=6

For $m = 6$, $d_\alpha = 6 - \epsilon$ and $d_\beta = 1$. Since the β subspace is one-dimensional, we can use the complex plane to integrate over $q_{1\beta}$ and $q_{2\beta}$. First, with the help of Eqs. (A1) and (A11), we integrate over $q_{2\beta}$ in Eq. (C1). Next, we integrate over $q_{1\beta}$ arriving after a very long algebra at the result

$$I_3 = \frac{1}{2} \int \frac{d^{d_\alpha} q_1}{(2\pi)^{d_\alpha}} \frac{d^{d_\alpha} q_2}{(2\pi)^{d_\alpha}} \frac{1}{q_1^4} \frac{1}{q_2^2} \frac{1}{(q_1+P)^2 + q_2^2 + (q_1+q_2-k_3)^2}$$
$$\times \left[\frac{1}{(q_1+P)^2} \frac{1}{q_2^2 + (q_1+q_2-k_3)^2} - \frac{1}{q_1^2 + q_2^2 + (q_1+q_2-k_3)^2} \frac{1}{q_1^2 + (q_1+P)^2} \right] . \qquad \text{(C11)}$$

We shall denote I_{3A} and I_{3B} the integrals associated with the first and second terms in the square brackets in Eq. (C11). The identity

$$\frac{1}{2abc} = \int_0^1 dx_1 \int_0^{x_1} dx_2 \frac{1}{[a + (b-a)x_1 + (c-b)x_2]^3} , \qquad \text{(C12)}$$

with $a = q_2^2$, $b = q_2^2 + (q_1 + q_2 - k_3)^2$ and $c = q_2^2 + (q_1+P)^2 + (q_1+q_2-k_3)^2$, enables us to put together the three last denominators in I_{3A}. In this way

$$I_{3A} = \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_{q_1}\int_{q_2} \frac{1}{q_1^4} \frac{1}{(q_1+P)^2} \frac{1}{\left(q_2^2 + (q_1+q_2-k_3)^2 x_1 + (q_1+P)^2 x_2\right)^3}, \tag{C13}$$

where $\int_q$ was defined in Eq. (B8).

The integral over q_2 is performed first with the standard formulas of dimensional regularization. As a result one obtains a term in denominator raised to the power ϵ. To integrate over q_1 we first use a Feynman parameter, y, to combine the denominators q_1^4 and $(q_1+P)^2$. The resulting term is then combined using another Feynman parameter, z, with the term that comes from the integration over q_2. After this, it is possible to perform the integration over q_1 and extracting the angular factor we obtain

$$I_{3A} = \frac{\pi^2 \Gamma(\epsilon)(\Gamma(3-\epsilon/2))^2}{8} \int_0^1 dx_1 \int_0^{x_1} dx_2 \frac{(1+x_1)^{\epsilon-3}}{(x_1+x_2(1+x_1))^{\epsilon/2}}$$
$$\times \int_0^1 dy(1-y) \int_0^1 dz(1-z)^2 z^{\epsilon/2-1} \left\{ P^2 y(1-z) + \frac{P^2 x_2(1+x_1) + k_3^2 x_1}{x_1+x_2(1+x_1)} z \right.$$
$$\left. - \left[Py(1-z) + \frac{P x_2(1+x_1) - k_3 x_1}{x_1+x_2(1+x_1)} z \right]^2 \right\}^{-\epsilon}. \tag{C14}$$

At this stage it is possible to use the procedure described in App. 9-3 of Amit's book[11] and calculate the integrals over y and z. The result takes the form

$$I_{3A} = \frac{\pi^2}{4} \frac{\Gamma(\epsilon)\Gamma(\epsilon/2)}{P^{2\epsilon}} \frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)(\Gamma(3-\epsilon/2))^2}{\Gamma(3-2\epsilon)\Gamma(3+\epsilon/2)}$$
$$\times \int_0^1 dx_1 \int_0^{x_1} dx_2 \frac{(1+x_1)^{\epsilon-3}}{(x_1+x_2(1+x_1))^{\epsilon/2}}. \tag{C15}$$

Expanding Eq. (C15) to order ϵ^{-1} we can perform the integrations over x_1 and x_2 and we obtain the expression

$$I_{3A} = \frac{\pi^2}{16} \left[\frac{1}{\epsilon^2} + \frac{1}{2\epsilon} \left(\frac{19}{6} + 9 \ln \frac{3}{4} - \ln P_\alpha^4 \right) \right]. \tag{C16}$$

Integral I_{3B} that corresponds to the second term in Eq. (C11), is calculated following the same steps as in I_{3A}, discussed above. Using again the results in App. 9-3 of Amit's book[11] we arrive at

$$I_{3B} = -\frac{\pi^2\Gamma(\epsilon)\Gamma(\epsilon/2)}{4P^{2\epsilon}\Gamma(3+\epsilon/2)} \int_0^1 dx_1 \frac{x_1(1+x_1)^{\epsilon-3}}{(x_1(2+x_1))^{\epsilon/2}} \int_0^1 dz \frac{(1-z)z^{-\epsilon}}{(1+z)^{3-2\epsilon}} \,. \tag{C17}$$

After expanding the above result to order ϵ^{-1} we can calculate the integrals over x_1 and x_2 and Eq. (C18) becomes

$$I_{3B} = -\frac{\pi^2}{32}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{5}{4} + \frac{15}{2}\ln\frac{3}{4} + 2\ln 2 - \frac{1}{2}\ln P_\alpha^4\right)\right] \,. \tag{C18}$$

Finally, adding I_{3A} and I_{3B} given in Eqs. (C16) and (C18), respectively, we finish the calculation of I_3,

$$I_3 = \frac{\pi^2}{32}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{23}{12} + \frac{3}{2}\ln 3 - 5\ln 2 - \frac{1}{2}\ln P_\alpha^4\right)\right] \,. \tag{C19}$$

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FIGURES

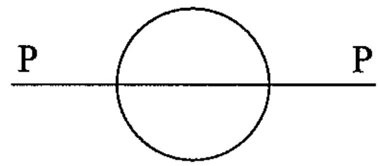

FIG. 1. Diagram D_1 of $\Gamma^{(2,0)}$.

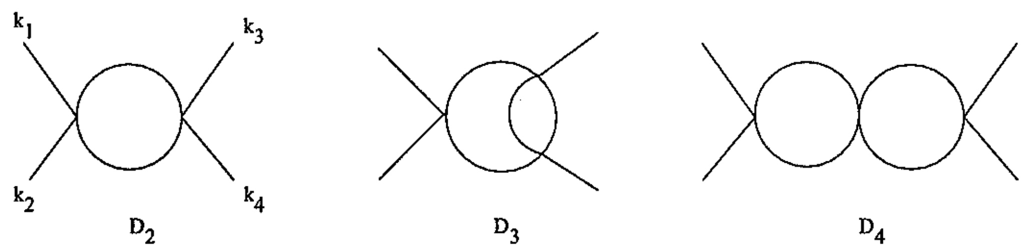

FIG. 2. Diagrams that contribute to $\Gamma^{(4,0)}$.

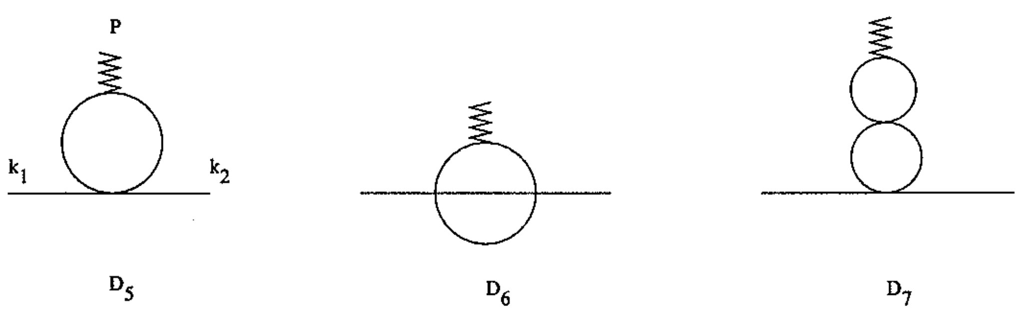

FIG. 3. Diagrams that contribute to $\Gamma^{(2,1)}$. The wiggly line indicates the point where the operator ϕ^2 was inserted.