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Effect of symmetry breaking on level curvature distributions*

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An exact general formalism is derived that expresses the eigenvector and the eigenvalue dynamics as a set of coupled equations of motion in terms of the matrix elements dynamics. Combined with an appropriate model Hamiltonian, these equations are used to investigate the effect of the presence of a discrete symmetry in the level curvature distribution. It is shown that this distribution exhibits a nontrivial behavior that explains the recent data regarding frequencies of acoustic vibrations of quartz block.

The usefulness of the study of statistical properties of eigenvalues and eigenvectors of quantum systems has already been demonstrated in many areas of physics. A lot can be learned, specially about symmetries, by just employing the appropriate statistics. It has also become clear that these statistics follow universal patterns that can be modelled by probability distributions extracted from an ensemble of random Hamiltonians of the same class of the underlying symmetry of the system under study [1]. This has been a field of intense investigation over the last two decades [2]. These

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activities have concentrated their effort, firstly, on what we can call the "statics" of the problem, in which stationary Hamiltonians are considered. More recently, however, the interest has also been directed to the dynamical aspects of the same question.

The "dynamics" consists in considering a given Hamiltonian as a function of a parameter (representing "time"). The statistical properties that characterize the evolution is then studied as the parameter is varied. Not any kind of evolution, however, is considered but only those that preserve the symmetry class of the Hamiltonian. Several measures have been introduced to investigate this kind of evolution. One of the most used ones is the probability distribution of the level curvature, which can be thought of as "acceleration" as it is defined in terms of the second derivative with respect to the parameter. These distributions measure correlations among the set of eigenvalues. Another measure that is commonly used is the two-point correlation function between first derivatives ("velocities"). Given some generic level, this two-point correlation function is obtained by calculating the velocity at two different values of the parameter [3]. Measures have also been considered to characterize the evolution of the eigenvectors [4].

These studies started with Wilkinson's pioneering work that investigated the dependence of the eigenvalues of a fully chaotic billiard as a function of its shape [5]. The plot of the trajectories of levels as a function of the parameter that controls the shape, exhibits a typical pattern of avoided crossings. A measure of these is provided by the curvature of the trajectory. There is now an analytical evidence that, in a fully chaotic regime, the curvatures, after an appropriate rescaling, follow an universal sim-

ple distribution. The tail of this distribution has been investigated, and an asymptotic dependence inversely proportional to the third power of the curvature was estabilished, for fully chaotic systems that are time reversal invariant [6]. The expression

$$P(k) = \frac{1}{2(1+k^2)^{\frac{3}{2}}} \tag{1}$$

was then proposed for the entire domain of the curvatures k. Finally, it has been proved that this function gives the exact distribution of curvatures, in the case of random matrix ensembles [7].

Recently, the difficult task of checking experimentally this prediction was undertaken by the experimental group of the Center for Chaos and Turbulence of Niels Bohr Institute [8]. They studied the dependence on the temperature of the spectrum of frequencies of quartz blocks. In previous investigations [9], they have found that the spectra of frequencies of quartz blocks obey statistical models based on random matrix theories. The dynamics of the frequencies, as a function of the temperature, was therefore measured for a quartz block whose statics statistical properties were already previously settled.

The data obtained have shown a deviation from the above expected distribution. This deviation, although slight, is significant and not yet understood. It is the purpose of this letter to show that the random matrix model can also exhibit the same kind of deviation from the universal pattern. Of course, this poses the difficult problem of reconciling the statics and the dynamics statistical aspects. We are going to discuss a solution to this delicate question by showing that this contradiction can be explained

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by the extremely sensitivity of the parametric correlations to relatively weak presence of symmetries which is not detected by the statics statistical measures.

So far, all studies of parametric correlations have been concentrated on the fully chaotic regime when the system statistics are well described by the Gaussian ensembles of Random Matrix Theory (RMT), in particular, the Orthogonal Ensemble (GOE), if there is time-reversal invariance. The partially chaotic situation has been little investigated. We intend here to provide the first systematic discussion of this situation. We start by developping the formalism and the model we are going to use. At the GOE limit, our numerical simulations verify the above universal expression for the level curvature distribution. As some degre of symmetry is introduced, it is found that the distribution starts to exhibit an unexpected nontrivial behavior that is compatible with the data.

We shall now derive a set of equations to describe simultaneously the dynamics of the energy levels and of the eigenvector components of a Hamiltonian H. It is expressed in terms of the equations of motion of the matrix elements of H, whose dependence on the parameter, t, representing the "time", is supposed to be given. Our starting point is the general matrix equation

$$H = U H_D U^\dagger, \qquad\qquad (2)$$

where H is a $N \times N$ real symmetric matrix, H_D is the diagonal matrix constructed with the N eigenvalues, and U is the unitary matrix whose columns are the N eigenvectors. Assuming that H_D and U also depend on the parameter t, differentiating Eq. (1) with

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respect to t we get

$$\dot{H} = U\dot{H}_D U^\dagger + \dot{U} H_D U^\dagger + U H_D \dot{U}^\dagger, \tag{3}$$

where the derivative is indicated by a dot. Multiplying (3) by $U^\dagger$ from the left and by U from the right, and defining the anti-hermitian matrix $S = U^\dagger \dot{U} = -\dot{U}^\dagger U = -S^\dagger$ we obtain the equation of motion

$$\dot{H}_D + [S, H_D] = U^\dagger \dot{H} U. \tag{4}$$

From (4), a system of coupled equations for the evolution of the eigenvalues and the eigenvector components is derived in terms of the matrix elements "velocities", $\dot{H}_{ij}$, that are assumed to be known. The equations for the eigenvalues come from the diagonal part of this matrix equation while its off-diagonal part provides the equations for the eigenvectors components. Explicitly, we find

$$\dot{E}_l = \sum_{i,j=1}^{N} C_i^l \dot{H}_{ij} C_j^l, \tag{5}$$

and

$$\dot{C}_n^m = \sum_{l=1, l \neq m}^{N} \frac{C_n^l}{E_m - E_l} \sum_{i,j=1}^{N} C_i^l \dot{H}_{ij} C_j^m. \tag{6}$$

In deriving (6), use has been made of the anti-hermicity property of S. This set of coupled equations is one of the main results of this paper. All calculations, numerical and analytical, will be based on it. The problem is completly determined once the initial values are given.

Higher order terms can be obtained by taking the derivative of these equations. Thus the equations for the "accelerations" (related to the level curvature), are given by

$$\ddot{E}_l = \sum_{i,j=1}^{N} C_i^l \ddot{H}_{ij} C_j^l + \sum_{i,j=1}^{N} \left(\dot{C}_i^l \dot{H}_{ij} C_j^l + C_i^l \dot{H}_{ij} \dot{C}_j^l \right). \tag{7}$$

By choosing a particular model, i.e., the dependence of the matrix element on the parameter t, the equations derived above can be used in several contexts. For example, they can be used to construct an alternative method of matrix diagonalization. By requiring the matrix elements to satisfy appropriate Langevin equations, these equations lead to Dyson's Brownian motion model [10]. Here, we concentrate on the simple model given by

$$H = H_1 \cos t + H_2 \sin t , \tag{8}$$

where H_1 and H_2 are a couple of fixed, i.e., parameter independent, random matrices taken from the same matrix ensemble, and t is the parameter. If in (8) H_1 and H_2 are taken from the same Gaussian ensemble, the evolution will preserve the probability distribution, so that H will remain in the same ensemble.

We shall work with the Gaussian ensemble that interpolates between GOE and two decoupled GOE's. This ensemble has been already employed with a very satisfactory result in the analysis of data relative to symmetry breaking [11,12] in nuclear [13] and acoustic systems [9]. It can be defined by the following operator equation [14]

$$H = PH^{GOE}P + QH^{GOE}Q + \lambda \left(PH^{GOE}Q + QH^{GOE}P \right), \tag{9}$$

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where $P = \sum_{i=1}^{M} P_i$, $Q = 1 - P$ and $P_i = | i > < i |$, $i = 1, \ldots, N$ are projection operators, $0 \leq \lambda \leq 1$ is the parameter that controls the transition, and H^{GOE} denote a GOE matrix whose elements follow a joint probability distribution given by

$$P\left(H^{GOE}\right) \propto \exp\left[-\alpha \operatorname{tr}\left(H^{GOE}\right)^2\right], \tag{10}$$

with α being an arbitrary scaling parameter. With the above definitions, $\lambda = 1$ corresponds to the GOE case, while $\lambda = 0$ corresponds to block diagonal random matrices, made up of two GOE matrices of sizes $M \times M$ and $(N - M) \times (N - M)$.

In the GOE limit ($\lambda = 1$), the distribution of level curvatures is expected to follow the universal form (1) after a suitable rescaling of variables. This rescaling is obtained in two steps. First, the levels are unfolded which means that a new spectrum is generated by the transformation

$$x_l = \int_{-\infty}^{E_l} dE \overline{\rho}(E) \text{ for } l = 1, \ldots, N , \tag{11}$$

where $\overline{\rho}(E)$ is the averaged level density. Then the parameter t itself is replaced by a new dimensionless parameter τ related to t by [15]

$$\frac{d\tau}{dt} = \sqrt{\langle \dot{x}^2 \rangle} , \tag{12}$$

where the average of the velocity is made over the whole set of eigenvalues or, equivalently, over the ensemble. The level curvature is then defined in terms of these new scaled variables as

$$k = \frac{1}{\pi} \frac{d^2 x}{d\tau^2} = \frac{1}{\pi \langle \dot{x}^2 \rangle} \left(\ddot{E} - \frac{\langle \dot{E} \ddot{E} \rangle}{\langle \dot{x}^2 \rangle} \dot{E} \right) . \tag{13}$$

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The behavior of the distribution, Eq. (1), for large curvatures can be traced to the level spacing distribution. In fact, large curvatures can be considered, approximately, as inversely proportional to the small level spacing s. Thus if we assume $s \propto 1/k$ and use the fact that, in the GOE case, $P(s)$ is linear in s, we obtain

$$P(k) \sim P(s) \left| \frac{ds}{dk} \right| \sim k^{-3} ,$$

(14)

as predicted by (1). As a consequence, as symmetry is introduced by decreasing the parameter λ, one would expect a reduction on the probability of large curvatures with the distribution becoming narrow. We shall see that this will happen only for strong decoupling.

To perform the numerical analysis, it is crucial to have a reliable expression for the average density. As we have considered in the calculations only the symmetric situation in which $N = 2M$, we have an exact expression for the density. It is given by the Wigner's semicircle law [16], with an appropriate scaling in order to give the correct value of the second moment of the eigenvalue

$$\overline{\rho}(E) = \frac{4\alpha}{\pi (1 + \lambda^2)} \sqrt{\frac{N}{2\alpha} (1 + \lambda^2) - E^2} .$$

(15)

In the Fig. 1, we show the nice fit obtained with this expression. This density was used in the calculations of the curvatures.

Our main result is presented in Fig. 2. To avoid density effects, the statistics calculations envolved few eigenvalues, actually 4, right in the middle of the spectra. We see that the curvature distribution shows a nontrivial behavior characterized by initially becoming wider, with an increase of the probability of large curvatures and, then, as

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the chaoticity parameter λ is further reduced, the distribution becomes progressively narrower. The case $\lambda = 0.22$ is particularly important because it has the same shape of the data in Ref. [8]. In Fig. 3, we display, for this special case, $\lambda = 0.22$, the calculated level spacing distribution compared with the Wigner surmise. Here also there is a departure from the theoretical prediction for large separation but it is not so easily seen.

To understand these results, we first remark that there is no reason to expect all statistical measures to respond uniformly to the variation of λ or, in physical terms, to the presence of symmetries. We can argue that the universal distribution is the result of the competition between level repulsion (a short range correlation that acts to increase the curvature), and the crystal lattice [1] nature of Wigner-Dyson spectra (a long range correlation that favours straight line trajectories and therefore small curvatures). As the decoupling of the spectrum into two spectra starts, the long range correlation seems to be relaxed first leading to a wider distribution. By continuing to increase the decoupling, the level repulsion is then reduced, giving rise to smaller curvatures.
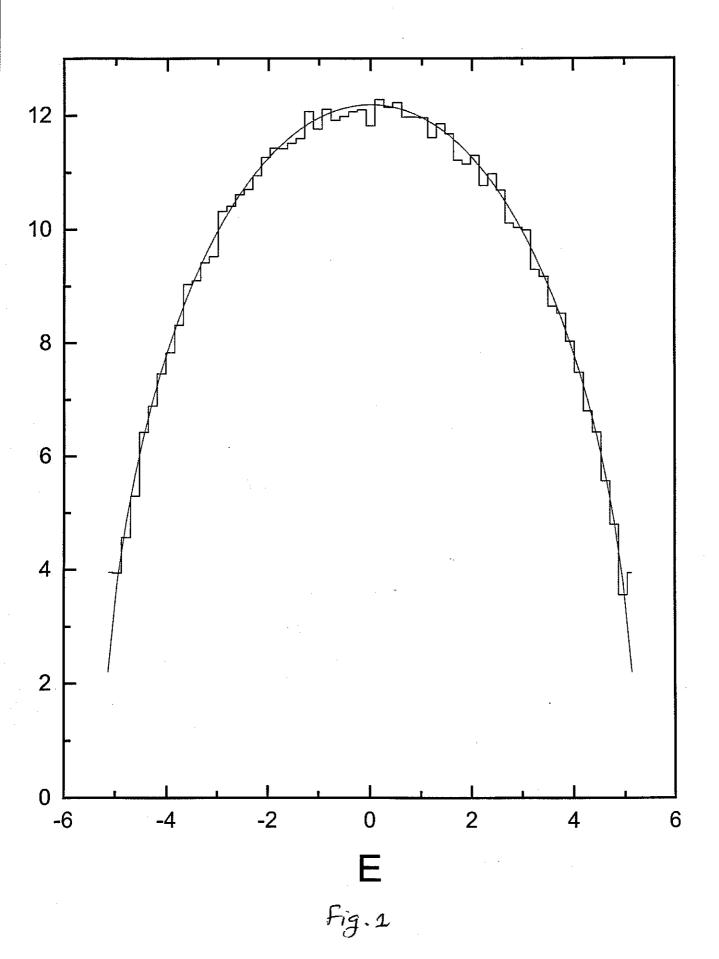
In conclusion, we have performed an analysis of the result of Ref. [8] concerning the full removal of the intrinsic two–GOE symmetry (D_3 symmetry) inherent in the quartz block. Our results indicate that the precision of such symmetry removal can be assessed through the sensitivity exhibitted by the curvature statistics.

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Figure Captions

FIG. 1. Densitiy of levels: comparison of the calculated histogram with the semi-circle law (15)(solid line). The calculation correspond to matrices of dimension $N = 100$, and $\lambda = 0.032$.

FIG. 2. Level curvature distributions: comparison of the calculated histograms with the theoretical prediction (1)(solid line). The calculations correspond to matrices of dimension $N = 100$, and for the values of λ indicated in the figure.

FIG. 3. Level spacing distribution: comparison of the calculated histogram with Wigner surmise (solid line). The calculation correspond to matrices of dimension $N = 100$, and $\lambda = 0.22$.
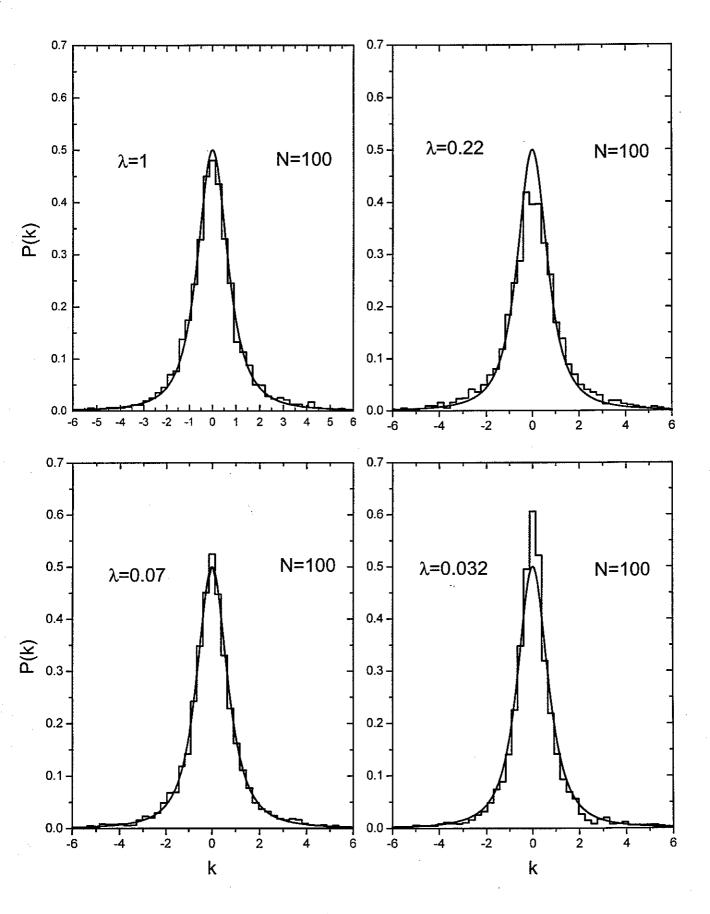
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Fig. 1

Fig. 2

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Fig. 3