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# Asymptotic behavior of irreducible excitatory networks of analog graded-response neurons


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**Abstract:** In irreducible excitatory networks of analog graded-response neurons, the trajectories of most solutions tend to the equilibria. We derive sufficient conditions for such networks to be globally asymptotically stable. When the network possesses several locally stable equilibria, their location in the phase space is discussed and a description of their attraction basin is given. The results hold even when inter-unit transmission is delayed.

# 1 Introduction

In analog graded-response neural networks, each unit is described by its activation at time $t$, denoted by $a_i(t)$, an output function $\sigma_i(a_i)$, a decay rate $\gamma_i$ and a constant input $K_i$ [1]. $W_{ij}$ represents the connection weight between neurons $j$ and $i$. The behavior of an $n$-neuron network is governed by the following system of ordinary differential equations (ODEs):

$$\tfrac{da_i}{dt}(t) = -\gamma_i a_i(t) + K_i + \sum_{j=1}^{n} W_{ij}\sigma_j(a_j(t)) \quad 1 \leq i \leq n. \tag{1}$$

which can be re-written in vectorial notations:

$$\tfrac{da}{dt}(t) = -\Gamma a(t) + K + W\sigma(a(t)) \tag{2}$$

where $a = (a_1, \cdots, a_n)^T$ is the activation vector, $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_n)$ is the decay matrix, $K = (K_1, \cdots, K_n)^T \in \mathbb{R}^n$ is the input vector, $W = [W_{ij}]$ the connection matrix, and $\sigma(a) = (\sigma_1(a_1), \cdots, \sigma_n(a_n))^T$ is the output vector.

The stationary states (equilibria) of the system (1) are given by the solutions of

$$-\Gamma x + K + W\sigma(x) = 0\ . \tag{3}$$

A network is referred to as recurrent when it includes feedback loops, i.e. there exist $i_1, \cdots,$ $i_p$ in $\{1, \cdots, n\}$ such that $P = W_{i_2 i_1} W_{i_3 i_2} \cdots W_{i_p i_{p-1}} W_{i_1 i_p} \neq 0$, in other words there exists at least a closed directed path connecting a unit to itself. Feedback loops are referred to as positive and negative when $P > 0$ and $P < 0$ respectively. Thus, a positive feedback loop contains zero or an even number of negative (i.e. inhibitory) weights.

Irreducible networks[1] that contain only positive feedback loops compose an important class of recurrent networks. Examples and applications of such networks are presented and discussed in [2, 3, 4]. When all feedback loops are positive, the system can be transformed into an excitatory network satisfying $W_{ij} \geq 0$ for all $i$, $j$ by changing the sign of some of the activations [2]. In order to obtain the appropriate transformation we have to select (arbitrarily) a reference unit (for instance, the unit labeled 1) within the network. The transformed activations are then given by $A_i = \text{sign}(P)a_i$, $P$ being the product of the weights along any directed path connecting unit 1 to unit $i$ (all such paths have necessarily the same sign). So, in the case of a ring network with $2N$ units, all weights being negative, the transformation is $A_{2n-1} = a_{2n-1}$ and $A_{2n} = -a_{2n}$, $n = 1, \ldots, N$. The underlying mathematics and more general methods are given in [5] theorem 5.8 p. 330 and equation (4.1) p. 322. Special transformations suitable for cellular neural networks can be found in [4].

The fact that irreducible networks with positive feedback loops can be transformed into excitatory ones implies that such networks are almost quasi-convergent, that is, almost all trajectories tend to the set of equilibria [2, 3, 4]. This statement shows that such networks do not display stable undamped oscillations which is important for artificial neural network applications relying on convergent dynamics. However, almost quasi-convergence provides

[1] An irreducible network is a network in which there is at least one directed path connecting any given neuron to any other one. Understanding the dynamics of irreducible networks is important because they form the building blocks of recurrent networks and under appropriate conditions, the asymptotic behavior of a network can be derived from that of its constituting irreducible networks in cascade [2].

little information about the organization of the phase portrait of the system. The main goal of the present paper is to describe in some details the dynamics of irreducible excitatory networks depending on the parameters range, and to provide conditions under which such systems are either globally asymptotically stable or display multistability, and in the latter case characterize the attraction basins of the equilibria. It is shown that the results can be generalized for the case in which the inter-unit transmissions are delayed.

In the following we present some general properties of irreducible excitatory networks (section 2), and then analyze their asymptotic dynamics (section 3), and the attraction basins of stable equilibria, and their boundaries (section 4). An example is presented in section (section 5), and the generalization of the results when inter-unit transmissions are delayed is discussed (section 6).

# 2 General properties

Throughout this paper we assume that the two following hypotheses are satisfied.

**Hypothesis 1.** *The neuron output function $\sigma_i$ is sigmoidal,* i.e., *it is a smooth strictly increasing function, bounded between two real numbers $m_i < M_i$, such that there is a unique point $p_i$ such that $\sigma_i''(p_i) = 0$. Without loss of generality, we suppose $p_i = 0$ for all $i$.*

Thus the derivative $\sigma_i'$ has a unique global maximum at zero, *i.e.* $\beta_i = \sigma_i'(0)$ referred to as the neuron gain, and it decreases down to zero for large and low activations *i.e.* $\sigma_i'(a_i) \to 0$ as $|a_i| \to +\infty$.

Examples of sigmoidal functions used in neural networks are $\tanh(\beta_i a_i)$, or $\arctan(\beta_i a_i)$.

**Hypothesis 2.** *The connection matrix $W = [W_{ij}]$ is positive ($W_{ij} \geq 0$ for all $i, j$) and irreducible* i.e. *it does not leave invariant any proper nontrivial subspace generated by a subset of the standard basis vectors for $\mathbb{R}^n$.*

Let $\epsilon = (\epsilon_1, \cdots, \epsilon_n)^T$ with $\epsilon_i = \pm 1$, and $\mathcal{K}_\epsilon$ be the cone defined as $\mathcal{K}_\epsilon = \{x \in \mathbb{R}^n : \epsilon_i x_i > 0 \ \ \forall i \in \{1, \cdots, n\}\}$. For $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ and $y = (y_1, \cdots, y_n)^T \in \mathbb{R}^n$, we define an order relation associated with $\mathcal{K}_\epsilon$ as $x \gg_\epsilon y$ (resp. $x \geq_\epsilon y$) when $x - y$ is in (resp. the closure of) $\mathcal{K}_\epsilon$, that is $\epsilon_i(x_i - y_i) > 0$ (resp. $\epsilon_i(x_i - y_i) \geq 0$), for all $i \in \{1, \cdots, n\}$. Finally, we define $x >_\epsilon y$ when $x \geq_\epsilon y$ and $x \neq y$. For the order associated with the positive cone ($\epsilon = (1, \cdots, 1)^T$), the index $\epsilon$ is not indicated.

Let $s_i = \frac{1}{\gamma_i}(\sum_{j=1}^n W_{ij} m_j + K_i)$ and $S_i = \frac{1}{\gamma_i}(\sum_{j=1}^n W_{ij} M_j + K_i)$, and $s_\epsilon(K) = (s_1^\epsilon, \cdots, s_n^\epsilon)^T$ where

$$s_i^\epsilon = \begin{cases} s_i & \text{if } \epsilon_i = 1 \\ S_i & \text{if } \epsilon_i = -1 \end{cases} \tag{4}$$

**Lemma 0.** *Let $\eta \in \mathbb{R}^n$ such that $\eta \gg_\epsilon 0$. For all $x \in \mathbb{R}^n$ there is $T = T(x) \in \mathbb{R}$ such that $a(t, x) \gg_\epsilon s_\epsilon(K) - \eta$ for all $t > T$.*

**Proof.** *i)* Let $\eta = (\eta_1, \cdots, \eta_n)^T$. For $\eta \gg_\epsilon 0$ we remark that:

$$\epsilon_i(-\gamma_i(s_i^\epsilon - \eta_i) + \sum_{j=1}^n W_{ij}\sigma_j(s_j^\epsilon - \eta_j) + K_i) \geq \epsilon_i \gamma_i \eta_i > 0 \tag{5}$$

Thus if there is $t_0$ such that $\epsilon_i(a_i(t_0, x) - (s_i^\epsilon - \eta_i)) > 0$, then $\epsilon_i(a_i(t, x) - (s_i^\epsilon - \eta_i)) > 0$ for all $t \geq t_0$.

If $\epsilon_i(a_i(t, x) - (s_i^\epsilon - \eta_i)) \leq 0$, for all $t$ in some closed interval $[t_0, t_1]$, then $\epsilon_i \frac{da_i}{dt}(t) > \epsilon_i \gamma_i \eta_i$. So that:

$$\epsilon_i a_i(t_1, x) \geq \epsilon_i a_i(t_0, x) + \epsilon_i \gamma_i \eta_i (t_1 - t_0) \tag{6}$$

The above inequality shows that there is necessarily $T > t_1$ such that $\epsilon_i(a_i(T,x)-(s_i^\epsilon-\eta_i)) > 0$.□

From the lemma we deduce that the solutions are eventually bounded within an $n$-rectangle:

**Theorem: Boundedness.** *For $\eta = (\eta_1, \cdots, \eta_n)^T$ with $\eta_i > 0$ for all $i$, define $\eta_\epsilon = (\epsilon_1\eta_1, \cdots, \epsilon_n\eta_n)^T$ where $\epsilon_i = \pm 1$. For all $x \in \mathbb{R}^n$, there is $T = T(x) \in \mathbb{R}$ such that for all $t \geq T$, $a(t,x)$ is inside the $n$-rectangle $\mathcal{R}(\eta, K)$ defined by the points $s^\epsilon(K) - \eta_\epsilon$.*

A consequence of the uniform asymptotic boundedness of the solutions is that for all $x \in \mathbb{R}^n$, there exists a unique function $a(t,x)$ from $\mathbb{R}$ to $\mathbb{R}^n$, such that $a(0,x) = x$ and $a(t,x)$ satisfies system (1) for all $t \in \mathbb{R}$.

## 3 Asymptotic behavior

All solutions eventually enter the positively invariant compact set $\mathcal{R}(\eta, K)$, so that in order to determine the asymptotic behavior of system (1), we only need to study the behavior of solutions in that set.

We introduce the following notations: minimal decay rate $\gamma = \min_{1\leq i\leq n}(\gamma_i)$, maximal decay rate $\gamma' = \max_{1\leq i\leq n}(\gamma_i)$, the bounds of the output vector $m = (m_1, \cdots, m_n)^T$ and $M = (M_1, \cdots, M_n)^T$ with $m \leq \sigma(x) \leq M$ (with respect to the order associated with the positive cone in $\mathbb{R}^n$) for all $x \in \mathbb{R}^n$.

Let $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, we define the following $n \times n$ matrix: $V(x) = W \times \mathrm{diag}(\sigma_j'(x_j)) = [W_{ij}\sigma_j'(x_j)]$. $V(x)$ is an irreducible positive matrix, thus the positive real number $\lambda(x) = \max\{\mathrm{Re}(\mu) : \mu$ is an eigenvalue of $V(x)\}$ is a simple eigenvalue of $V(x)$ [6].

**Theorem: contraction.** *If $\lambda(x) < \gamma$ for all $x \in \mathcal{R}(\eta, K)$, then system (1) is globally asymptotically stable.*

**Proof.** Consider the map $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by $F(x) = \Gamma^{-1}(W\sigma(x) + K)$. *i)* The set $\mathcal{R}(\eta, K)$ is invariant under $F$ that is $F(\mathcal{R}(\eta, K)) \subset \mathcal{R}(\eta, K)$. *ii)* For $x = (x_1, \cdots, x_n)^T$ and $x' = (x'_1, \ldots, x'_n)^T$ in $\mathcal{R}(\eta, K)$, there is $y = (y_1, \cdots, y_n)^T$ with $y_j \in [x_j, x'_j]$ such that $\sigma(x) - \sigma(x') = [\sigma'_j(y_j)(x_j - x'_j)]$. Thus $||F(x) - F(x')|| = ||\Gamma^{-1}V(y)(x - x')|| \leq \frac{\lambda(y)}{\gamma}||x - x'|| \leq \frac{\lambda}{\gamma}||x - x'||$, where $\lambda = \mathrm{Sup}_{x \in \mathcal{R}(\eta,K)}(\lambda(x))$. Since $\lambda < \gamma$, the map $F$ restricted to $\mathcal{R}(\eta, K)$ is contracting. Hence, there is a unique point $x^* \in \mathcal{R}(\eta, K)$ such that $F(x^*) = x^*$, which is the unique equilibrium point of system (1). *iii)* The map $E : x \to \frac{1}{2}||x - x^*||^2$ is a Lyapunov function for system (1). In fact for $x \neq x^*$ we have:

$$
\begin{aligned}
&\tfrac{dE}{dt}(t) = (\tfrac{da}{dt}(t, x) - \tfrac{da}{dt}(t, x^*))^T(a(t, x) - a(t, x^*))\\
&= (-\Gamma(a(t, x) - a(t, x^*)))^T(a(t, x) - a(t, x^*)) + (W(\sigma(x) - \sigma(x^*)))^T(a(t, x) - a(t, x^*))\\
&= (-\Gamma(a(t, x) - a(t, x^*)))^T(a(t, x) - a(t, x^*)) + (V(y(t))(a(t, x) - a(t, x^*)))^T(a(t, x) - a(t, x^*))\\
&\leq -\gamma||a(t, x) - a(t, x^*)||^2 + \lambda(y(t))||a(t, x) - a(t, x^*)||^2\\
&\leq -\gamma||a(t, x) - a(t, x^*)||^2 + \lambda||a(t, x) - a(t, x^*)||^2 < 0
\end{aligned}
\tag{7}
$$

This is based on the fact that

$$(V(y)x)^T x \leq (V(y)|x|)^T|x| \leq \lambda(y)\,|x|^T|x| = \lambda(y)||x||^2 \tag{8}$$

where for $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, we define $|x| = (|x_1|, \cdots, |x_n|)^T$, and $x^T x'$ denotes the usual scalar product in $\mathbb{R}^n$.

The existence of the Lyapunov function ensures that all solutions tend to $x^*$. □

**Corollary 1.** *If $\lambda(0) < \gamma$ then system (1) is globally asymptotically stable.*

**Proof.** For all $x \in \mathbb{R}^n$, we have $V(x) \leq V(0)$ (componentwise order). As these are irreducible positive matrices we have $\lambda(x) \leq \lambda(0)$ [6]. □

The case of $\lambda(0) > \gamma$ requires extra conditions in order to lead to global asymptotic stability.

**Corollary 2.** *Suppose $\lambda(0) > \gamma$ and let $C_\gamma = \{x \in I\!R^n : \lambda(x) = \gamma\}$, if there is $y \in C_\gamma \bigcap \mathcal{K}_\epsilon$ such that $s^\epsilon(K) \gg_\epsilon y$, then system (1) is globally asymptotically stable.*

**Proof.** $s^\epsilon(K) \gg_\epsilon y$ with $y \in C_\gamma \bigcap \mathcal{K}_\epsilon$ implies that $x \gg_\epsilon y$ for all $x \in \mathcal{R}(\eta, K)$ for $\eta \gg 0$ sufficiently small.

The derivative $\sigma_i'(x_i)$ is a strictly increasing (resp. decreasing) function for $x_i < 0$ (resp. $x_i > 0$). Thus for $x$ and $x'$ in $\mathcal{K}_\epsilon$, if $x \gg_\epsilon x'$, then $V(x) < V(x')$ (componentwise order) and therefore $\lambda(x) < \lambda(x')$. Hence $s^\epsilon(K) \gg_\epsilon y$ implies that $\lambda(x) < \gamma$ for all $x \in \mathcal{R}(\eta, K)$. □

**Remark.** *Let $u \in \mathcal{K}_\epsilon$, the function $c \to \lambda(cu)$ from $I\!R^+$ into $[\lambda(0), 0)$ is strictly decreasing, therefore, when $\lambda(0) > \gamma$, there is a unique $c > 0$ such that $\lambda(cu) = \gamma$. Thus the intersection $C_\gamma \bigcap \mathcal{K}_\epsilon$, is a non empty set. It is unordered with respect to the order associated with $\mathcal{K}_\epsilon$, that is, there are no $x$ and $x'$ in $C_\gamma \bigcap \mathcal{K}_\epsilon$ with $x \neq x'$, such that either $x \geq_\epsilon x'$ or $x' \geq_\epsilon x$.*

**Corollary 3: saturation.** *If $\lambda(0) > \gamma$, and $|K_i|$ is sufficiently large for all $i \in \{1, \cdots, n\}$, then system (1) is globally asymptotically stable.*

**Proof.** Let $\xi = (\xi_1, \cdots, \xi_n)^T \in C_\gamma \bigcap \mathcal{K}_\epsilon$. We have the following equivalence:

$$s^\epsilon(K) \gg_\epsilon \xi \iff \begin{cases} K_i > \gamma_i \xi_i - \sum_{j=1}^n W_{ij} m_j & \text{for } \epsilon_i = +1 \\ K_i < \gamma_i \xi_i - \sum_{j=1}^n W_{ij} M_j & \text{for } \epsilon_i = -1 \end{cases} \tag{9}$$

The inequalities show that when $|K_i|$ is sufficiently large, the hypothesis of the previous corollary is satisfied. □

The above results provide sufficient conditions for system (1) to be globally asymptotically stable. They can also be used to obtain information about the equilibria when system (1) is multistable.

**Proposition.** *If system (1) admits an equilibrium point $x_e$ such that there is $y \in C_\gamma \bigcap \mathcal{K}_\epsilon$ with $x_e \gg_\epsilon y$, then $x_e$ is a hyperbolic locally asymptotically stable equilibrium point and there is no other equilibrium $x$ such that $x \gg_\epsilon x_e$.*

**Proof.** The matrix $A = -\Gamma + V(x_e)$ is irreducible with positive off-diagonal components. Therefore $\mu(A) = \max\{\mathrm{Re}(\mu) : \mu$ is an eigenvalue of $A\}$ is a simple eigenvalue of $A$ and we have $\mu(A) \leq -\gamma + \lambda(x_e) < 0$ [6]. This shows that all eigenvalues of $A$ have strictly negative real parts, which implies that $x_e$ is a hyperbolic locally asymptotically stable equilibrium point. The second statement results from the fact that $\Gamma^{-1}W\sigma(x)$ is contracting in the cone $x \gg_\epsilon x_e$. □

**Remark.** *Unstable equilibria are necessarily located in the region where $\lambda(x) > \gamma$ within the $n$-rectangle $\mathcal{R}(\eta, K)$. This region is a connected set formed by the union of $\{0\}$ and the sets $\{x \in \mathcal{K}_\epsilon \ : \ \exists y \in C_\gamma \bigcap \mathcal{K}_\epsilon$ such that $x \leq_\epsilon y\}$ for all $\epsilon$. Moreover if $\lambda(0) > \gamma' = \max_{1 \leq i \leq n}(\gamma_i)$, then any equilibrium point $x_e$ such that $\lambda(x_e) > \gamma'$ is locally asymptotically unstable.*

For the special case of $\gamma = \gamma' = \gamma_1 = \cdots = \gamma_n$, we see that a hyperbolic equilibrium point $x_e$ is locally asymptotically stable (resp. unstable) if $\lambda(x_e) < \gamma$ (resp. $\lambda(x_e) > \gamma$). So that stable equilibria are in "the corners", and unstable equilibria in the middle region.

In order to obtain a more complete picture of the asymptotic behavior of system (1), we present results concerning global aspects of the dynamics.

For $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ and $y = (y_1, \cdots, y_n)^T \in \mathbb{R}^n$, we say that $x$ is larger (resp. strictly larger) than $y$, denoted $x \geq y$ (resp. $x \gg y$) when $x_i \geq y_i$ (resp. $x_i > y_i$) for all

$1 \leq i \leq n$. Finally $x > y$ indicates that $x \geq y$ and $x \neq y$.

We rewrite system (1) as:

$$\frac{da}{dt}(t) = G(a) \tag{10}$$

where $G(a) = (G_1(a), \cdots, G_n(a))^T$, with $G_i(a) = -\gamma_i a_i(t) + K_i + \sum_{j=1}^{n} W_{ij}\sigma_j(a_j(t))$.

**Cooperative system.** We have $\frac{\partial G_i}{\partial a_j}(a) = W_{ij}\sigma_j'(a_j) \geq 0$ for $i \neq j$. Thus all off-diagonal terms of the $n \times n$ matrix $DG(a) = [\partial G_i/\partial a_j(a)]$ representing the derivative of $G$ at $a$ are positive.

Such a system is referred to as cooperative [6], since increase in any of the activations $a_i$ increments the other activations.

**Irreducible system** Since $W$ is irreducible, $DG(a) = -\Gamma + V(a)$ is also irreducible.

The fact that system (10) is cooperative and irreducible implies that it generates a strongly monotone flow [2, 6]:

$$\text{For } (x, y) \in \mathbb{R}^{2n}, \text{ if } x > y \text{ then } a(t, x) \gg a(t, y) \text{ for all } t > 0. \tag{11}$$

Strong monotonicity added to the boundedness of trajectories (Boundedness theorem) implies that trajectories have a strong tendency to converge to equilibria, as shown by the following results.

**Almost quasi convergence.** *Almost all trajectories of system (1) approach the set of equilibria as $t \to +\infty$ [2].*

**Generic convergence.** *The set of convergent solutions contains an open and dense subset of $\mathbb{R}^n$ [6].*

The above results are general properties of cooperative irreducible systems with bounded trajectories. In the following, we derive properties more specific to system (1).

**Lemma 1.** *There are two equilibrium points $x_m$ and $x_M$ such that:* i) $x_m \leq x_M$, ii) $a(t,x) \to x_m$ *as* $t \to +\infty$ *for all* $x \leq x_m$ *and* iii) $a(t,x) \to x_M$ *as* $t \to +\infty$ *for all* $x \geq x_M$. iv) *If* $a(t,x) \to x_m$ *(resp.* $a(t,x) \to x_M$*), as* $t \to +\infty$*, then* $a(t,x') \to x_m$ *(resp.* $a(t,x') \to x_M$*) as* $t \to +\infty$*, for all* $x' \leq x$ *(resp.* $x' \geq x$*).*

**Proof.** The proof of the statements is divided into two stages. A) The existence of the equilibrium point $x_m$ satisfying (ii) is shown. The existence of the equilibrium point $x_M$ satisfying (iii) follows a similar line. B) it is shown that the arguments used in the construction of these two equilibria imply (i) and (iv).

A) *Existence of* $x_m$ *satisfying (ii).* In order to simplify expressions, we denote $s = s^{(-1,\cdots,-1)}(K)$. Let $\eta \in \mathbb{R}^n$, such that $\eta \gg 0$. The proof is constructed as follows. First we show that the trajectory of *a given* initial condition $x$ satisfying $x \ll s - \eta$ converges to an equilibrium point denoted by $x_m(x)$. Then we remark that the trajectories of *all* initial conditions $x$ with $x \leq s - \eta$ converge to the very same equilibrium, that is $x_m(x) = x_m(s-\eta)$ for all $x \leq s - \eta$. Finally this result is extended to trajectories of all initial conditions $x \leq x_m(s-\eta)$. In other words $x_m(x)$ does not depend on the particular choice of the initial condition $x \leq x_m(s-\eta)$. Thus, by setting $x_m = x_m(s-\eta)$ we have the equilibrium point satisfying (ii).

Let $x \ll s - \eta$, then there is $\theta > 0$ such that $\frac{d}{dt}a(\tau, x) \gg \Gamma\eta \gg 0$ for all $0 < \tau < \theta$. Thus $a(t,x)$ is strictly increasing, and converges to an equilibrium point [6] that we denote by $x_m(x)$, to emphasize the fact that this point may depend on the initial condition.

The asymptotic boundedness of all trajectories (Boundedness Theorem) implies that $x_m(x) \gg$

$s-\eta$. Combined with the fact that $x$ was selected such that $s-\eta \gg x$ and the monotonicity (11) we obtain $x_m(x) = a(t, x_m(x)) \gg a(t, s-\eta) \gg a(t, x)$. Taking the limit $t \to +\infty$, we find that $a(t, s-\eta) \to x_m(x)$. In other words, the trajectories of $x$ and $s-\eta$ tend to the same equilibrium, i.e. $x_m(x) = x_m(s-\eta)$. Since the point $x$ was chosen arbitrarily, it implies that the trajectories of all initial conditions $x \leq s-\eta$ tend to the equilibrium point $x_m(s-\eta)$.

Let $x \leq x_m(s-\eta)$, there exists $y \ll s-\eta$ such that $y \ll x$. Thus from the monotonicity (11) we have: $x_m(s-\eta) = a(t, x_m(s-\eta)) \geq a(t,x) \gg a(t,y)$. The result of the previous paragraph implies that $a(t,y) \to x_m(s-\eta)$ as $t \to \infty$. Thus, taking the limit $t \to \infty$, we obtain that $a(t,x) \to x_m(s-\eta)$ as $t \to \infty$. In other words, we have shown that the trajectories of all $x$ such that $x \leq x_m(s-\eta)$ converge to $x_m(s-\eta)$. Thus, $x_m(s-\eta)$ is an equilibrium point satisfying (ii). This property also implies that the equilibrium point does not depend on the particular choice of $\eta$. Indeed, for $\eta' \gg 0$, there exists $y \ll s-\eta$ such that $y \ll s-\eta'$, so that a similar argument implies that $a(t, s-\eta') \to x_m(s-\eta)$ as $t \to \infty$, i.e. $x_m(s-\eta') = x_m(s-\eta)$, which can thus be denoted $x_m$.

In a similar way, it is possible to define $x_M$ as the limit $x_M(S+\eta)$ of the trajectory of $S+\eta$, where $S = s^{(1,\cdots,1)}(K)$, and show that this equilibrium does not depend on the particular choice of $\eta$, and satisfies (iii).

B) *Statements (i) and (iv).* From $s-\eta \ll S+\eta$ and the monotonicity (11), we obtain $a(t, s-\eta) \ll a(t, S+\eta)$. Taking the limit $t \to \infty$, yields the inequality in (i): $x_m \leq x_M$.

We take $x$ such that $a(t,x) \to x_m$ as $t \to +\infty$, for $x' \leq x_m$, there is $y \in \mathbb{R}^n$ such that $y \leq x_m$ and $y \leq x'$, thus $x'$ is bounded by two points with trajectories tending to $x_m$, so that its trajectory also converges to $x_m$. This argument completes the proof of (iv) for $x_m$, a similar one can be presented for $x_M$. $\square$

**Remark.** i) *If* $x_m = x_M$ *then system (1) is globally asymptotically stable.* ii) *If* $x_m \neq x_M$, *and both are locally asymptotically stable, then there is necessarily an unstable equilibrium point* $x_u$ *with* $x_m \ll x_u \ll x_M$ *[6].*

Sufficient conditions for global asymptotic stability similar to those provided in the previous paragraphs can be formulated. For example if there is $y \in C_\gamma \bigcap \mathcal{K}_{(-1,\cdots,-1)}$ (resp. $y \in C_\gamma \bigcap \mathcal{K}_{(1,\cdots,1)}$) such that $x_M \ll y$ (resp. $x_m \gg y$) then system (1) is globally asymptotically stable.

We define $\kappa = \{K \in \mathrm{IR}^n : \Gamma x - W\sigma(x) = K$ has a unique solution$\}$, and $\mathcal{R}_0 = \mathrm{IR}^n - \kappa$. We have already shown that $\kappa$ is a non-empty set since it contains all vectors $K$ with sufficiently large $|K_i|$. For $K \in \kappa$, system (1) has a unique equilibrium point, so that necessarily $x_m = x_M$ and the system is globally asymptotically stable.

When $K = K_0 = -W\sigma(0)$, the origin, denoted $a = 0$, is an equilibrium point of the system. If $\lambda(0) > \gamma'$, it is a locally asymptotically unstable equilibrium point, and is therefore distinct from both $x_m$ and $x_M$. Thus, system (1) has at least three distinct equilibria $x_m \ll 0 \ll x_M$. We deduce that in this case $\mathcal{R}_0$ is non-empty since it contains $K_0$. In general, applying Sard's theorem shows that [7]:

**Finite equilibria.** *We assume* $\lambda(0) > \gamma'$, *there is a negligible subset* $\mathcal{Q}$ *of* $\mathcal{R}_0$ *such that for all* $K \in \mathcal{R} = \mathcal{R}_0 - \mathcal{Q}$, *system (1) has a finite number* $q > 1$ *of equilibria. All the equilibria are hyperbolic.*

From this point on, and throughout the rest of the paper, we assume $K \in \mathcal{R}$. We denote by $\mathcal{E}$ the set of the equilibria, by $\mathrm{B}(y)$ the basin of attraction of a stable equilibrium point $y$, and by $\partial B(y)$ the boundary of $B(y)$. $B(y)$ is an open set. We have $x_m \neq x_M$ so that there are at least $q = 3$ equilibria, and $x_m \ll y \ll x_M$ for all $y \in \mathcal{E} - \{x_m, x_M\}$.

As the set of equilibria is constituted by isolated hyperbolic points, the almost quasi-convergence and the generic convergence properties can be reformulated as:

**Corollary.** i) *Almost all trajectories of system (1) approach an equilibrium point as* $t \to +\infty$.

*ii) The union of the basins of attraction of the stable equilibria of system (1) is an open and dense subset of* $\mathbb{R}^n$.

# 4 The basins of attraction and their boundaries

We are interested in determining the "shape" of the basins of attraction of stable equilibria. To this end we introduce the following terminology.

A subset $H$ of $\mathbb{R}^n$ is referred to as positively invariant under system (1), if for all $x \in H$, $a(t, x) \in H$ for all $t \geq 0$.

A subset $H$ of $\mathbb{R}^n$ is unordered if there are no $x$ and $x'$ with either $x < x'$ or $x' > x$. For example, in $\mathbb{R}^2$, the diagonal defined as $\{x = (x_1, x_2) \ : x_1 + x_2 = 0\}$ is an unordered set.

For $H$ and $H'$ two subsets of $\mathbb{R}^n$, we say that $H$ is below $H'$, denoted $H \leq H'$, if there are no $x \in H$ and $x' \in H'$ satisfying $x > x'$. When $H$ is below $H'$, the two sets may have a non empty intersection. Furthermore, if the two sets do not contain comparable points, then each one is below the other and vice-versa. This shows that the above definition is not very restrictive, in the sense that it does not define an order. However, in the following we focus on a special family of subsets which can be ordered.

Let $H$ be a hypersurface *i.e.* codimension one manifold in $\mathbb{R}^n$, and assume $H$ is unordered, then for all $x \in (\mathbb{R}^n - H)$, there is $y \in H$, such that either $x < y$ or $x > y$. Thus, an unordered hypersurface divides the space into the two disjoint sets of points above and below it respectively. For example the straight line $\{x = (x_1, x_2) : x_1 + x_2 = 0\}$ divides the plane into regions $\{x = (x_1, x_2) : x_1 + x_2 > 0\}$ and $\{x = (x_1, x_2) : x_1 + x_2 < 0\}$. This allows to see the following. Let $H$ and $H'$ be two unordered hypersurfaces in $\mathbb{R}^n$, such that $H$ is below $H'$, then for all $x \in H - (H \cap H')$, there is $y \in H'$ such that $x < y$. Thus the relation $H$ "is below" $H'$, denoted $H \preceq H'$, defines an order in the set of unordered hypersurfaces.

After these definitions, we describe the basin boundaries. No trajectory in a basin boundary converges to a stable equilibrium point, so that the basin boundaries are necessarily contained in the negligible subset of trajectories that do not converge to any stable equilibrium point.

Conversely, there are finitely many stable equilibria and the union of their basins of attraction is an open and dense subset of $\mathbb{R}^n$, so that any neighborhood of a trajectory that does not converge to a stable equilibrium point intersects necessarily the union of the basins of attraction and is therefore in the boundary of at least one of these basins. Hence, the union of the basin boundaries is exactly the negligible subset of trajectories that do not converge to any stable equilibrium point.

From this characterization of the basin boundaries in terms of unstable trajectories, and the fact that system (1) is strongly monotone with bounded trajectories, and possesses a finite number of equilibria, all of which are hyperbolic, the following result can be derived:

**Basin Boundaries.** *The union of the basin boundaries is formed by a finite number $p \geq 1$ of Lipschitz unordered invariant hypersurfaces, denoted $H_1 \preceq H_2 \preceq \cdots \preceq H_p$ [8].*

**Remark.** *The two open sets defined as $\{x : \text{there is } y \in H_1 \text{ with } y > x\}$ and $\{x : \text{there is } y \in H_p \text{ with } y < x\}$ are invariant. In the same way, for $p \geq 2$, the connected components of the sets $\{x : \text{there are } y \in H_i \text{ and } y' \in H_{i+1} \text{ with } y < x < y'\}$ are open invariant sets.*

The union of the basins of attraction is a dense open subset, so that any open set intersects at least the basin of attraction of one of the equilibria. If a connected open set intersects two attraction basins, then it also intersects the boundary separating them, that is, it intersects the union of the unordered invariant hypersurfaces. Thus we have:

**Basins of attraction.** *Each of the invariant open connected sets defined in the previous remark is exactly the basin of attraction of one of the stable equilibria.*

Thus for the stable equilibria other than $x_m$ and $x_M$, the basin of attraction is caught between two of the hypersurfaces $H_i$ and $H_{i+1}$, and for the two extreme equilibrium points we have:

**Upper and lower boundaries.** *$\partial B(x_m) = H_1$ and $x \in B(x_m)$ if and only if there is $y \in H_1$ such that $x < y$.*

*Similarly $\partial B(x_M) = H_p$ and $x \in B(x_M)$ if and only if there is $y \in H_p$ such that $x > y$.*

**Proof.** Lemma 1 shows that the basins of attraction of $x_m$ and $x_M$ have no lower and upper bound respectively. So that among the open connected sets defined by the hypersurfaces only the one below $H_1$ and above $H_p$ can correspond to these basins. □

Let $y \in \mathcal{E} - \{x_m, x_M\}$, locally asymptotically stable. There are stable equilibria smaller and larger than $y$. We select $x$ and $x'$ in $\mathcal{E}$, such that both are stable, they satisfy $x \ll y \ll x'$ and there are no stable equilibria in the ordered open intervals $[[x, y]] = \{\phi \,:\, x \ll \phi \ll y\}$ and $[[y, x']] = \{\phi \,:\, y \ll \phi \ll x'\}$. These intervals are invariant open subsets of $\mathbb{R}^n$. There is $i \in \{1, \cdots, p-1\}$ such that the basin of attraction of $y$, $\mathrm{B}(y)$ is caught between $H_i$ and $H_{i+1}$, therefore, $H_i$ and $H_{i+1}$ intersect $[[x, y]]$ and $[[y, x']]$ respectively. Each intersection contains necessarily at least one unstable equilibrium point.

Conversely, let $x_u$ be an unstable equilibrium point, then there are $i \in \{1, \cdots, p\}$, and two stable equilibria $x$ and $y$ such that $x_u$ is in the intersection $[[x, y]] \cap H_i$.

These results stem from general properties of strongly monotone systems with bounded trajectories [6] and the fact that unstable equilibria are necessarily in the hypersurfaces $H_i$. They constrain the location of the equilibria. For instance, there cannot be two stable equilibria $x$ and $y$ such that the ordered interval $[[x, y]]$ lies in the saturated area beyond $C_\gamma \cap \mathcal{K}_\epsilon$, *i.e.* for all $z \in [[x, y]]$, there is $z' \in C_\gamma \cap \mathcal{K}_\epsilon$, such that $z \gg_\epsilon z'$.

Trajectories on the boundaries are unordered, that is, if there is $i \in \{1, \cdots, p\}$ such that $x \in H_i$, then for all $t > 0$, $a(t, x) \in H_i$, so that neither $x > a(t, x)$ nor $x < a(t, x)$. This requires the trajectories to be oscillating in the following sense. Let $x_u$ be an unstable equilibrium point in the positively invariant set $[[y, y']] \cap H_i$, where $y$ and $y'$ are stable equilibria. For $x \in [[y, y']] \cap H_i$, $x \neq x_u$, we have neither $a(t, x) \geq x_u$ nor $a(t, x) \leq x_u$ $(t > 0)$. Thus, the components of the vector $a(t, x) - x_u$ are not all of the same sign. We say that $a(t, x)$ oscillates weakly around $x_u$.

## 5 Example

The previous results describe the organization of the phase portrait of irreducible cooperative networks in the general case. In this section, we consider an example such that the concepts introduced in the previous sections can be readily visualized in a simple figure. To this end, we consider the two neuron-network described by the following system of differential equations:

$$\begin{aligned} \tfrac{da_1}{dt} &= -a_1 + 3\sigma(a_1) + \sigma(a_2) \\ \tfrac{da_2}{dt} &= -a_2 + 3\sigma(a_2) + \sigma(a_1) \end{aligned} \tag{12}$$

where $\sigma(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. System (12) satisfies both hypotheses 1 and 2. The dynamics of two-neuron networks have been analyzed previously [9, 10], so that we move directly to the description of the global organization of the phase portrait of (12) in the light of our general results.

**FIGURE 1 HERE**

The phase portrait of system (12) is presented in figure 1. There are two unordered hypersurfaces $H_1$ (thick solid line) and $H_2$ (thick dashed line), with $H_1 \preceq H_2$, that form the basin boundaries of the stable equilibria. These two hypersurfaces intersect at the origin $(0,0)$. They divide the phase space into four regions (I, II, III and IV in Fig. 1). Region I corresponds to points below $H_1$. As expected from the results of the previous section, this region is the basin of attraction of the lowest equilibrium point $x_m$. In the same way, region II represents points above $H_2$ and constitutes the basin of attraction of the highest equilibrium point $x_M$. The two other regions – III and IV – are the basins of attraction of two stable equilibria denoted $x_1$ and $x_2$. Points in each of these basins are above $H_1$ and below $H_2$.

The intersection of the hypersurfaces is an invariant set, which in the case of this example corresponds to an equilibrium point $(0,0)$, this point is an unstable node. The hypersurface $H_1$ contains also two saddle points (large dots in Fig. 1) in the ordered intervals $[[x_m, x_1]]$ and $[[x_m, x_2]]$. Orbits connecting the saddle points to the stable equilibria are schematically represented by the thin dashed lines. As indicated by the arrows on the solid line in Fig. 1, trajectories of all points on $H_1$, except the origin, tend to one of these saddle points. Thus $H_1$ is composed of the stable manifolds of the stable points together with the origin. In a similar way, $H_2$ is composed of the origin and the stable manifolds of two saddle points.

In this example, all trajectories eventually tend to an equilibrium point, so that the system is convergent. However, this is not the case in all irreducible excitatory networks, as for example an excitatory ring network composed of more than five units can display an unstable periodic solution. Such oscillatory solutions are necessarily confined to the boundaries of the basins of attraction, and are not likely to be observed in practical applications.

# 6 Networks with delay

Finite transmission times arising in hardware implementation of analog graded-response neural networks, as well as possible applications of networks with delay have motivated a number of studies on the dynamics of GRN networks with delay [11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Taking delays into account, modifies system (1) into the following system of delay differential equations (DDEs):

$$\frac{da_i}{dt}(t) = -\gamma_i a_i(t) + K_i + \sum_{j=1}^{n} W_{ij}\sigma_j(a_j(t-\tau_{ij})) \quad 1 \leq i \leq n \tag{13}$$

where $\tau_{ij}$ represents the delay between units $j$ and $i$. Let $\tau_j = \max_{1\leq i\leq n}\{\tau_{ij} \ : \ W_{ij} \neq 0\}$, then $S = \mathcal{C}[-\tau_1, 0] \times \cdots \times \mathcal{C}[-\tau_n, 0]$ is the phase space for DDE (13). For any initial condition $\phi$ in $S$, there exists a unique solution of DDE (13) defined for all $t \geq 0$. Under hypotheses 1 and 2, DDE (13) is cooperative irreducible with bounded trajectories. Therefore *i)* the stability of the locally stable equilibria is not affected by the presence of delay, *ii)* most trajectories in the system with delay converge to the stable equilibria and *iii)* the description of the basin boundaries given in section 4 remains valid [6, 8].

# 7 Discussion and conclusion

We have studied in some details the dynamics of irreducible excitatory networks of analog graded-response neural networks. Understanding the dynamics of irreducible networks is important because they constitute the building blocks of other networks, and under appropriate conditions the asymptotic behavior of a network can be derived from that of its constituting irreducible networks [2]. The main purpose of our analysis was to provide a description of the global organization of the phase portrait of a class of neural networks whose connection matrix satisfies a simply verifiable constraint, and that appear in a number of applications [2, 3, 4]. The results are valid irrespective of the size of the network. Besides issues related to the global stability of the dynamics, the general "shape" of the basins of attraction of stable equilibria in irreducible excitatory networks was discussed in some detail.

We have provided several sufficient conditions for the global asymptotic stability of such networks. These results show that in general irreducible excitatory networks are globally asymptotically convergent when either the network is contracting, that is, the linear dissipation is stronger than the nonlinearity, or the inputs are sufficiently large, driving the system to saturation.

It is known that even when irreducible excitatory networks are not globally asymptotically convergent, they are still almost quasi-convergent, that is most trajectories tend to the set of equilibria. We have derived more detailed descriptions of the phase portrait of such networks. We have shown that generically these have a finite number of hyperbolic equilibria, with the stable equilibria usually located in the corners, corresponding to saturated outputs, and each unstable equilibrium point lying "between" two stable equilibria. This result is important for applications which require the stable equilibria to be situated in the saturated zones, and generalizes the studies of networks with high gain [1, 19]. Moreover, our study shows that the attraction basins of the stable equilibria are caught between unordered hypersurfaces. This indicates that attraction basins are not intertwined.

The computation of basin boundaries in specific neural networks is of great importance, providing a numerical algorithm for this task is beyond the scope of the present work which deals mainly with qualitative aspects of the dynamics. Still, our results have been applied to compute numerically the basin boundaries of small networks with delayed interactions, which, despite the low number of units, are, nonetheless, infinite dimensional systems [17, 21]. General algorithms for the computation of basins of attractions of neural networks as well as other systems can be found in [22, 23] and the references therein.

In summary, the contribution of the present work is to improve our understanding of the qualitative dynamics of a class of networks as well as provide some bounds for network parameters that would ensure one form of dynamics, e.g. global asymptotic stability, or another, e.g. multistability. Specific examples that come up in applications can then be better comprehended in the light of the general results.

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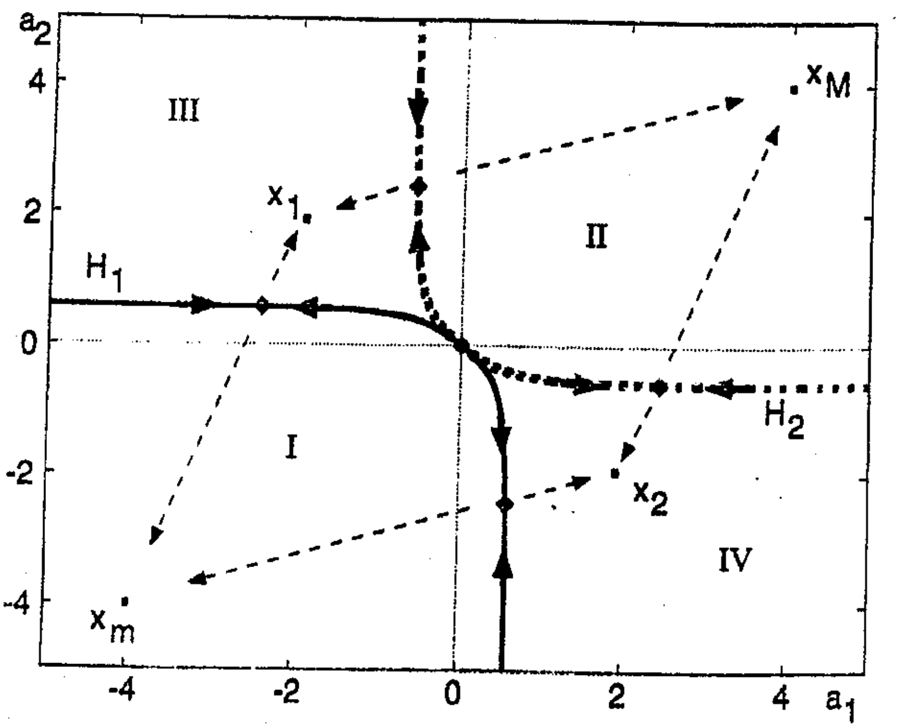
# FIGURE LEGENDS

# and

# FIGURES

Figure 1: *Phase portrait of a two neuron network*

*The solid and dashed thick lines* $H_1$ *and* $H_2$ *represent the boundaries dividing the phase space into four regions I, II, III and IV. Each of these four regions is the basin of attraction of a stable equilibrium point (*$x_m$*,* $x_1$*,* $x_2$ *and* $x_M$*). The boundaries are formed by the origin, which is a source, and the stable manifolds of four saddle points, represented by large dots. The thin dashed lines correspond to the unstable manifolds of these saddle points. The arrows indicate the direction of movement along trajectories. Abscissae activation of neuron 1:* $a_1$*; ordinates activation of neuron 2:* $a_2$*.*


a2
4
2
0
-2
-4
III
II
I
IV
x1
xM
x2
xm
H1
H2
-4
-2
0
2
4
a1