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Retarded Greens Functions and Forward Scattering Amplitudes in Thermal Field Theory

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Retarded Green's functions and forward scattering amplitudes in thermal field theory

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In this paper, we give a simple diagrammatic identification of the unique combination of the causal n-point vertex functions in the real time formalism that would coincide with the corresponding functions obtained in the imaginary time formalism. Furthermore, we give a simple calculational method for evaluating the temperature dependent parts of the retarded vertex functions, to one loop, by identifying them with the forward scattering amplitudes of on-shell thermal particles. As an application of the method, we calculate and show that the temperature-dependent parts of all the retarded functions vanish at one loop order for $(1+1)$-dimensional massless QED. We further point out that, in this model, in fact, the temperature-dependent parts of all the retarded vertex functions vanish to all orders in perturbation theory. [S0556-2821(99)06404-8]

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I. INTRODUCTION

In recent years, there has been an increased interest in the study of finite temperature field theory primarily from the point of view of understanding the structure of the early universe as well as the properties of the quark gluon plasma, the latter bearing relevance to experiments planned in the near future [1–5]. These studies, in turn, have led to a better understanding of finite-temperature field theories in general and new structures continue to emerge.

One of the important things, in the context of finite-temperature field theories, is to find simple calculational rules for various quantities of physical importance. Some of the physical phenomena of interest at finite temperature, such as the linear response, involve retarded functions as opposed to the time ordered functions that we are used to in quantum field theories at zero temperature. As is well known [6–8], there are two distinct ways of calculating statistical averages—commonly known as the imaginary time formalism and the real time formalism (furthermore, there are two real time formalisms). In the imaginary time formalism, the calculation of the retarded (or advanced) functions is quite straightforward. We calculate the relevant vertex functions in a Euclidean field theory with appropriate (anti) periodic boundary conditions and then analytically continue the resulting function to the appropriate axis in the complex energy plane. Furthermore, there exists a very simple method for calculating these functions to one loop order, which relates them to forward scattering amplitudes of on-shell thermal particles [9]. In contrast, in the real time formalism, we have a doubling of fields which for the n-point functions leads to (2^n-1) independent causal functions. It is, of course, not clear *a priori* whether there even exists a unique definition of a retarded n-point function in the real time formalism and if so, how it compares with the retarded function calculated in the imaginary time formalism. It is worth pointing out here that a meaningful definition of retarded vertex functions in quantum field theories at zero temperature already exists and a lot of work has been done in recent times showing how this definition, when generalized to finite temperature, coincides with the quantities obtained from the imaginary time formalism [10–13]. In this paper, we would like to describe a simple calculational method which gives in the real time formalism one loop retarded n-point function quite easily.

In Sec. II, we describe briefly a simple approach that works well in the imaginary time formalism, at least to one loop. In Sec. III, we explicitly identify the diagrams that correspond to the imaginary time retarded amplitudes and then drawing from the results of Sec. II, present a simple method for calculating these amplitudes in the real time formalism. In this approach, the temperature-dependent parts of the retarded n-point functions are given a simple diagrammatic representation, which expresses them in terms of Feynman amplitudes with physical (retarded or advanced) propagators and a single statistical factor. As an application of this method, we show in Sec. IV that all the retarded vertex functions for a $(1+1)$-dimensional fermion interacting with an external gauge field at finite temperature vanish to one loop order. We also extract some further properties of this model to show, in Sec. V, that the retarded self-energy, as well as all the other retarded n-point functions actually vanish to all higher orders. We end our paper with a brief conclusion in Sec. VI.

II. IMAGINARY TIME CALCULATIONAL METHOD

The calculation of an amplitude in the imaginary time formalism is exactly similar to that at zero temperature. The only difference is that the energy, instead of taking continuous values, takes discrete values depending on the (anti)periodic boundary condition used (T is the temperature):

$$\omega=\begin{cases}\dfrac{2n\pi}{T} & \text{for bosons,}\\[2ex] \dfrac{(2n+1)\pi}{T} & \text{for fermions.}\end{cases} \tag{1}$$

Consequently, when one has internal loops, the loop energy variable, rather being integrated, is summed over all possible discrete values. The sum, at one loop, gives rise to a single statistical factor for the particle type whose energy is being summed in the temperature-dependent part of the amplitude. This temperature-dependent part of the amplitude can be represented as a forward scattering amplitude where the internal

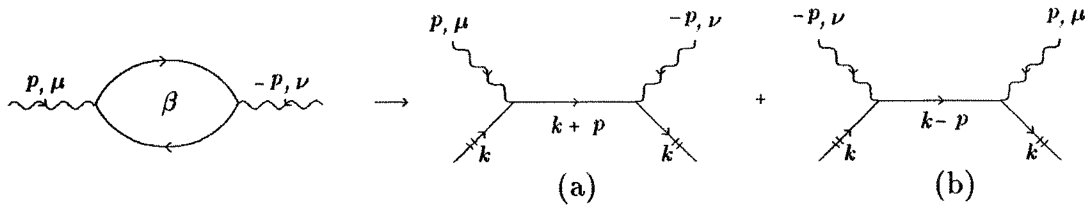


FIG. 1. The forward scattering amplitudes associated with the thermal two-point function. The momentum k of the thermal particles is on-shell, being integrated over with the statistical factor $n_F(|k_0|)$.

line is cut open to be on-shell with the appropriate statistical factor [9] (and the permutations of the external lines which corresponds to different cuttings of the internal lines). Calculationally, this is indeed quite simple. Let us illustrate this with the simple example of the fermionic contribution to the self-energy of an Abelian gauge boson in $3+1$ dimensions. The one loop amplitude can be written in terms of two diagrams as it is indicated in Fig. 1 (the momentum of every internal line that is being cut can always be redefined to be the momentum that is being integrated which leads to the same graph with different permutations of the external lines).

The amplitude itself can now be evaluated trivially to give

$$\Pi^{\mu\nu(\beta)}(p)=-e^2\int\frac{d^4k}{(2\pi)^3}n_F(|k^0|)\,\delta(k^2-m^2)\times\left(\frac{N^{\mu\nu}(k,p)}{[(k+p)^2-m^2]}+\frac{N^{\nu\mu}(k-p,p)}{[(k-p)^2-m^2]}\right). \quad (2)$$

Here β is the inverse temperature in units of the Boltzmann constant

$$n_F(|k^0|)=\frac{1}{e^{\beta|k^0|}+1}$$

and the tensor structure arising from the fermion loop is given by

$$N^{\mu\nu}(k,p)=\mathrm{Tr}[\gamma^\mu(\not k+\not p+m)\gamma^\nu(\not k+m)].$$

The high-temperature limit of the self-energy can now be easily calculated and has the form

$$\Pi^{\mu\nu(\beta)}\simeq\frac{e^2T^2}{3}\int\frac{d\Omega}{4\pi}\left(\frac{p^0\hat{k}^\mu\hat{k}^\nu}{p\cdot\hat{k}}-\eta^{\mu0}\eta^{\nu0}\right), \quad (3)$$

where the angular integration is performed in the 3-dimensional space and $\hat{k}$ is a lightlike four-vector given by $\hat{k}=(1,\hat{\mathbf{k}})$.

There are some aspects to note here. First, in the above expression, $p^0=2ni\pi/T$ even though we have treated it as a real variable. Second, even though this calculation appears to be similar to a real time calculation, there is an essential difference, namely, the propagators in Eq. (2) do not have the $i\epsilon$ term.

This can, in fact, be generalized to the n-point one loop amplitude in a straightforward manner. For example, the temperature-dependent part of the three-point function has the diagrammatic representation shown in Fig. 2.

It is important to note that this method also allows us to calculate the temperature-dependent physical amplitudes, such as the retarded amplitudes, in a simple manner in terms of forward scattering amplitude diagrams with physical propagators. Thus, the temperature-dependent part of the retarded two-point amplitude can be obtained by analytical continuation of the external energy $p^0\to E+i\epsilon$. In general, the retarded n-point amplitude is found by the analytic continuation of the energy $p_n^0\to E_n+(n-1)i\epsilon$ (all other external energies being analytically continued as $p_j^0\to E_j-i\epsilon, j=1,2,\ldots,n-1$), where we have assumed that the vertex with external momentum p_n corresponds to the one with the largest time in the coordinate space. Such amplitudes are, again, straightforward to calculate. They contain physical (advanced or retarded) propagators $G_{A,R}$ in the intermediate states, as illustrated in Fig. 3.

III. REAL TIME METHOD FOR RETARDED FUNCTIONS

The physical two point functions, such as the retarded and the advanced vertex functions, are well known in terms of the causal vertex functions. Thus, for example (Σ_{+-} is defined with a negative sign),

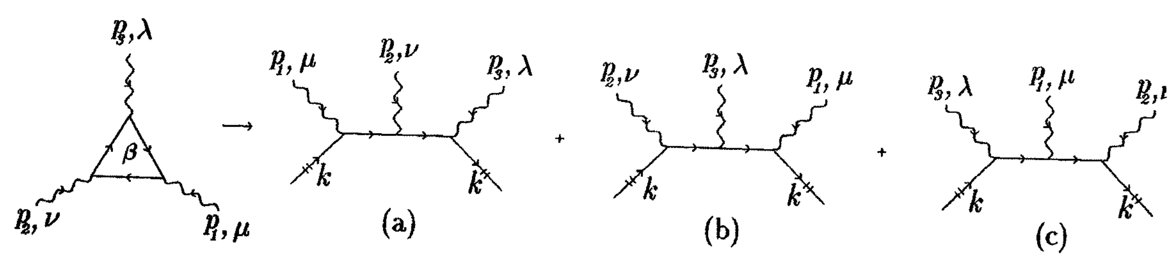


FIG. 2. The forward scattering amplitudes associated with the thermal three-point function. All external momenta p_i are inward with $\Sigma_i p_i=0$.

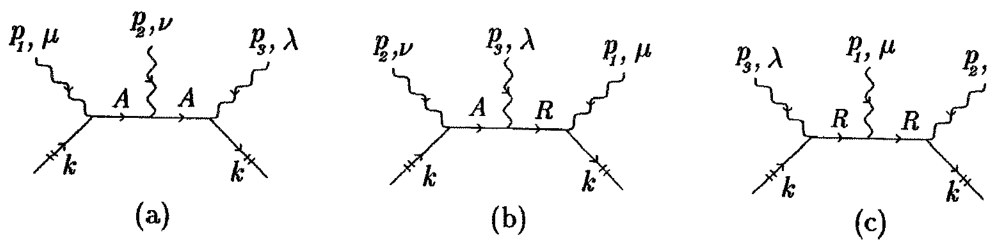


FIG. 3. Examples of forward scattering amplitudes with physical [retarded (R) or advanced (A)] propagators.

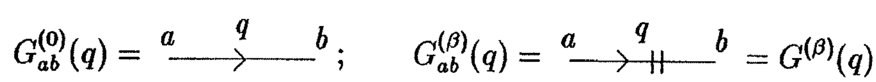
$$\Sigma_R(p) = \Sigma_{++}(p) + \Sigma_{+-}(p) \tag{4}$$

and so on. The problem, however, arises when we go to higher point functions. Because of the doubling of the field degrees of freedom, there are $(2^n - 1)$ independent causal n-point amplitudes. The main difficulty is to determine which combination of these independent amplitudes will correspond to the retarded amplitude that is obtained from the calculations in the imaginary time formalism and how to evaluate them in a simple way. In this connection, it is worth noting from the discussion in the previous section that at the one loop order, the temperature-dependent amplitude in the imaginary time formalism involves only a single statistical factor. In contrast, the temperature-dependent part of every propagator, in the real time formalism has a statistical factor. Consequently, every n-point causal amplitude can, in principle, involve up to a maximum of n statistical factors. Thus, the combination of the causal amplitudes that will correspond to the calculations of the imaginary time formalism has to be such that all the higher order statistical factors except the linear terms cancel out in the combination. This is not hard to determine. In fact, let us start with some explicit lower order results before giving the general result.

Our discussion will be in the closed time path formalism even though everything can be described equally well in the formalism of thermofield dynamics. We note that in the closed time path formalism, the propagator has a 2×2 matrix form given by

$$G_{ab}(q) = G^{(0)}_{ab}(q) + G^{(\beta)}_{ab}(q) \quad a,b = \pm. \tag{5}$$

Here $G^{(0)}_{ab}(q)$ is the zero-temperature propagator whereas $G^{(\beta)}_{ab}(q)$ corresponds to the on-shell thermal correction to the propagator. To keep the discussion completely general, we do not take any particular form for the propagator. Thus, G_{ab} can represent the propagator for a boson or a fermion with the Dirac matrix structure factored. The important thing to note is that no matter what is the propagator being considered, the temperature-dependent part is the same for all values of the indices a, b. For simplicity of discussion, let us introduce a diagrammatic representation for the two parts of the propagator. Thus, a simple line would represent the zero-temperature part of the propagator while a line with a cut would represent the thermal correction



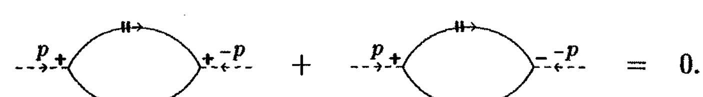
(6)

We also note the following general relations [8]

$$G_R = G_{++} - G_{+-} = G_{-+} - G_{--},$$
$$G_A = G_{++} - G_{-+} = G_{+-} - G_{--}, \tag{7}$$

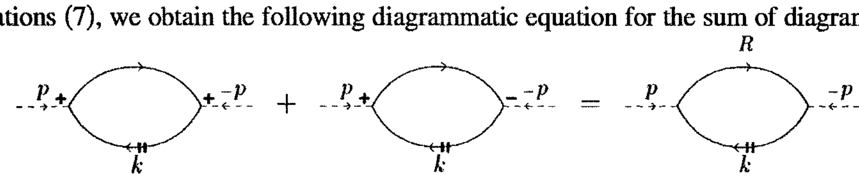
which will be useful in what follows. Furthermore, we remark that in the closed time path formalism, there are two kinds of vertices, the vertices for the $-$ fields having a relative negative sign compared to those for the $+$ fields.

With these observations, let us first look at the retarded two-point function in the real time formalism. We then see that the sum of the graphs with two cut propagators vanishes (to keep the discussion general, we will use dashed external lines and solid internal lines which can stand for any field):



(8)

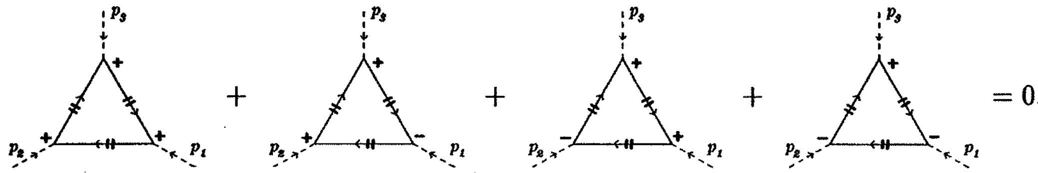
Next, using the relations (7), we obtain the following diagrammatic equation for the sum of diagrams with a single cut line:



(9)

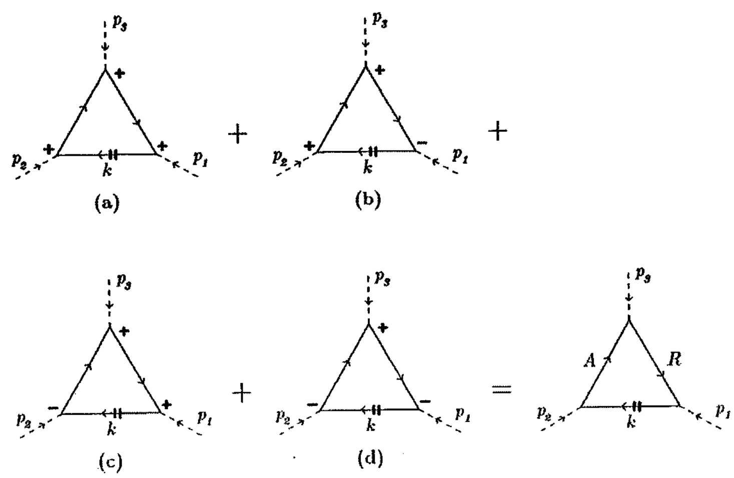
The graph on the right-hand side of Eq. (9) coincides with the forward scattering amplitude in Fig. 1(a) obtained in the imaginary time formalism, by analytic continuation of $p_0 \to E + i\epsilon$. Similarly, the sum of the diagrams where only the other propagator is cut, corresponds to the amplitude shown in Fig. 1(b) (these amplitudes actually give the same contribution).

Let us consider next the sum of the three point functions in the real-time formalism. It is clear that the sum of the following graphs with three cut propagators vanishes:



(10)

The diagrams cancel pairwise and this follows simply from the observation that a cut propagator is the same no matter what are the indices a, b as well as the fact that the $-$ vertices have a relative negative sign compared to the $+$ vertices. For the same reason, the sum of the diagrams with two cut propagators also vanishes. The sum of the diagrams with a single cut propagator, however, do not vanish and, therefore, it is clear that the sum of this set of diagrams is likely to correspond to the temperature dependent part of the retarded three-point amplitude obtained from the imaginary time formalism (because it will be linear in the statistical factor). In fact, let us look at the sum of these four diagrams with a fixed single cut propagator. Remembering that a cut propagator is on-shell and hence can be thought of as a pair of open lines (on-shell and with a statistical factor), we can write



(11)

In order to obtain this result, we make use of the relations (7) and note that the integrands associated with the above diagrams can be combined pairwise as follows:

$$I_{(a)}+I_{(b)} \sim G^{(0)}_{++}(k+p_2)[G^{(0)}_{++}(k-p_1)-G^{(0)}_{+-}(k-p_1)] = G^{(0)}_{++}(k+p_2)G_R(k-p_1), \tag{12}$$

$$I_{(c)}+I_{(d)} \sim -G^{(0)}_{-+}(k+p_2)[G^{(0)}_{++}(k-p_1)-G^{(0)}_{+-}(k-p_1)] = -G^{(0)}_{-+}(k+p_2)G_R(k-p_1). \tag{13}$$

Finally, adding all contributions, we find that

$$I_{(a)}+I_{(b)}+I_{(c)}+I_{(d)} \sim [G^{(0)}_{++}(k+p_2) - G^{(0)}_{-+}(k+p_2)]G_R(k-p_1) = G_A(k+p_2)G_R(k-p_1). \tag{14}$$

The result given in Eq. (11) coincides with that associated with the forward scattering amplitude shown in Fig. 3(b), which was obtained after analytic continuation from the imaginary time formalism.

The sum of the diagrams with distinct single fixed cut propagator would, then, correspond to different permutations of the external lines and it is clear that the sum of all such diagrams would exactly coincide with the diagrams obtained in the earlier section in the imaginary time formalism. Therefore, this set of diagrams uniquely corresponds to the retarded three-point function that would be obtained from the imaginary time formalism

$$\Gamma^{(3)}_R = \Gamma_{+++} + \Gamma_{+-+} + \Gamma_{++-} + \Gamma_{+--}. \tag{15}$$

We parenthetically remark here that this is unique to the extent that the retarded self-energy is unique. In fact, we know that because of the constraints among various causal amplitudes, we can also write

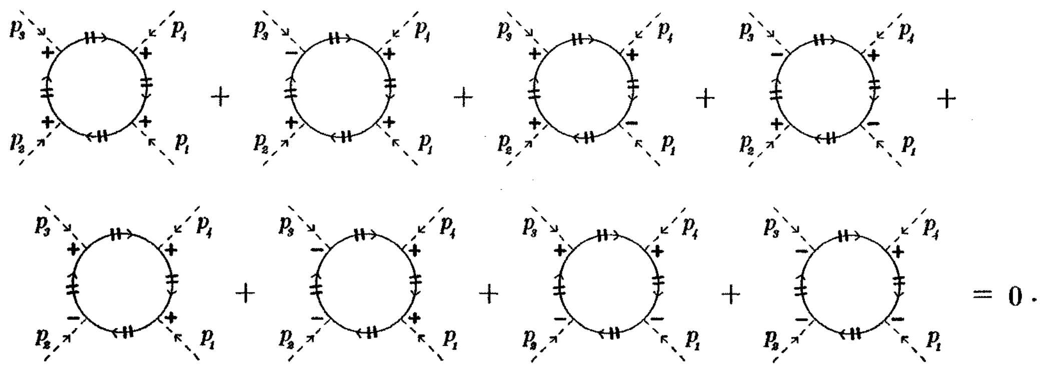
$$\Sigma_R = -\Sigma_{-+} - \Sigma_{--}. \tag{16}$$

Similarly, for the retarded three-point amplitude, we can also write

$$\Gamma_R^{(3)} = -\Gamma_{-++} - \Gamma_{--+} - \Gamma_{-+-} - \Gamma_{---}. \tag{17}$$

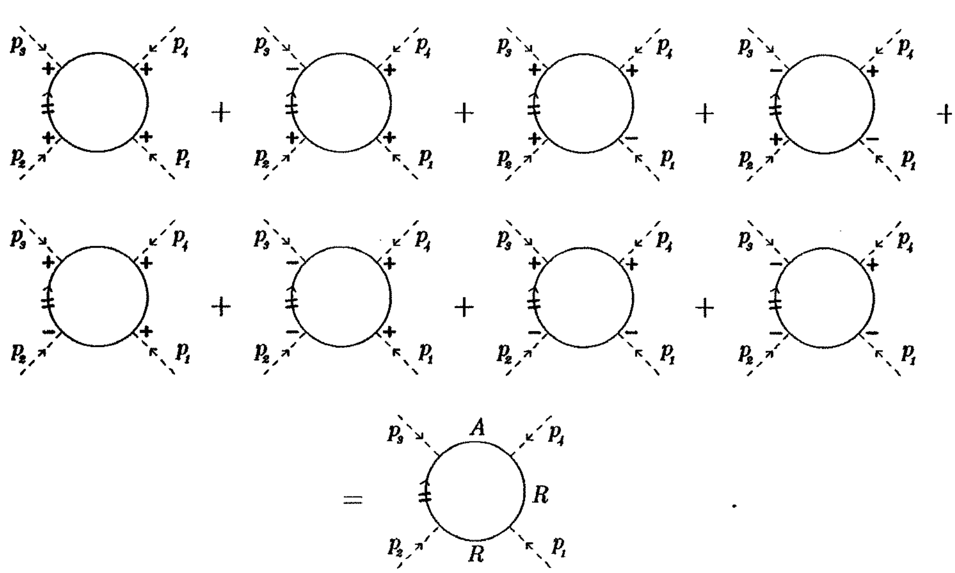
However, the definition that contains the amplitude with all + vertices (namely, the physical amplitude) is uniquely given by Eq. (15). Furthermore, by relating them to the forward scattering amplitude, we, naturally, have a simple way of calculating them.

Let us next look at the retarded four point amplitude. Once again, it is easy to see that the sum of the following graphs with four-cut propagators vanishes:



(18)

Similarly, it can be verified that the sum of three-cut propagators as well as the sum of two-cut propagators also vanish. However, the diagrams with a single-cut propagator do not add up to zero. Thus, it is clear that this set of diagrams is likely to lead to the retarded four-point amplitude that will be obtained from the imaginary time formalism. Once again, it is easy to see that the sum of the diagrams with a fixed single-cut propagator correspond to a given forward scattering amplitude and that distinct fixed single-cut propagators would correspond to cyclic permutations of the external lines. For example, with the help of the relations (7), one finds that



(19)

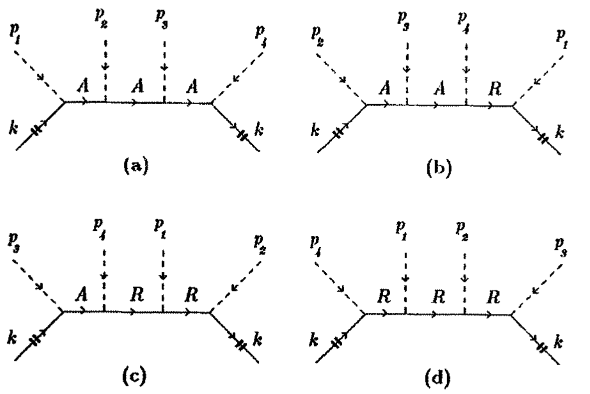


FIG. 4. Forward scattering amplitudes associated with the retarded four-point function. The internal on-shell momentum k is integrated over with an appropriate statistical factor.

The right-hand side of this equation can be represented diagrammatically as the forward scattering amplitude shown in Fig. 4(c).

Thus, this set of diagrams uniquely gives the retarded four point amplitude as

$$\Gamma_R^{(4)}=\Gamma_{++++}+\Gamma_{+++-}+\Gamma_{+-++}+\Gamma_{+-+-}+\Gamma_{++-+}$$
$$+\Gamma_{++--}+\Gamma_{+--+}+\Gamma_{+---}. \tag{20}$$

Furthermore, by relating this sum to the forward scattering amplitudes, we indeed have a simple way of calculating the temperature dependent part of the retarded four-point amplitude.

From these low order examples, it is now clear now how to extend these results to the retarded n-point amplitude.

$$\Gamma_R^{(n)}=\sum_{i_k=\pm}\Gamma_{+i_1\cdots i_{n-1}}. \tag{21}$$

Namely, we should keep the index associated with the largest time vertex fixed to be the physical index (+) and sum over all possible permutations of the thermal indices. It is known [14] that this form of the retarded amplitude is equivalent to the standard one given in terms of multiple nested commutators. There is an even number of diagrams at every order and from the properties already discussed, the sum of all diagrams with more than one cut propagator can easily be seen to vanish (graphs cancel pairwise). Furthermore, the sum of the single-cut propagators can then be given a forward scattering representation which identifies them with the imaginary time result as well as makes the evaluation simple. By a straightforward generalization of the previous results, we see that the rule for calculating the general retarded n-point forward scattering amplitudes is as follows (compare with Fig. 4). Let p_n be the external momenta associated with the latest time vertex. Then, all propagators whose momenta are flowing towards this vertex are advanced, whereas the propagators with momenta flowing outward from the latest time vertex are retarded.

IV. APPLICATION

As an application of this method, let us next calculate the n-point photon amplitudes at one loop for the $(1+1)$-dimensional massless QED. It is well known that in the Schwinger model [15] only the self-energy is nonvanishing. Thus, we are interested mainly in the thermal corrections to these amplitudes. The simplest, of course, is the retarded self-energy which can be obtained from the forward scattering amplitude diagrams shown in Fig. 1:

$$\Gamma_R^{\mu\nu}(p)=-\frac{e^2}{2\pi}\int d^2k\,n_F(|k^0|)\,\delta(k^2)\times\left(\frac{N^{\mu\nu}(k,p)}{[(k+p)^2+i\epsilon(k^0+p^0)]}+\frac{N^{\nu\mu}(k-p,p)}{[(k-p)^2-i\epsilon(k^0-p^0)]}\right). \tag{22}$$

Here, we have

$$N^{\mu\nu}(k,p)=\mathrm{Tr}[\gamma^\mu(\not k+\not p)\gamma^\nu\not k]=k_+^\mu(k+p)_+^\nu+k_-^\mu(k+p)_-^\nu \tag{23}$$

and we have defined

$$k_\pm^\mu=(\eta^{\mu\nu}\pm\epsilon^{\mu\nu})k_\nu, \tag{24}$$

where $\epsilon^{\mu\nu}$ is the antisymmetric tensor with $\epsilon^{01}=1$. It is clear that, under a suitable redefinition of variables, the second term in Eq. (22) becomes identical to the first and the integral can be trivially evaluated to give

$$\Gamma_R^{00}(p)=\Gamma_R^{11}(p)=-\frac{e^2}{2\pi}\left(\frac{1}{(p^0+i\epsilon)-p^1}-\frac{1}{(p^0+i\epsilon)+p^1}\right)\times\int dk^1\,\mathrm{sgn}(k^1)n_F(|k^1|)=0,$$

$$\Gamma_R^{01}(p)=\Gamma_R^{10}(p)=-\frac{e^2}{2\pi}\left(\frac{1}{(p^0+i\epsilon)-p^1}+\frac{1}{(p^0+i\epsilon)+p^1}\right)\times\int dk^1\,\mathrm{sgn}(k^1)n_F(|k^1|)=0. \tag{25}$$

Similarly, the diagrams for the temperature-dependent part of the retarded three-point function are given in Fig. 3. These can also be evaluated in a simple manner. Without going into details, we give the result here, namely, there are only two tensor structures with an even/odd number of spacelike indices. For example,

$$\Gamma_R^{000}(p_1,p_2,p_3)=-\frac{e^2}{2\pi}\left[\frac{1}{p_1^-(p_1^-+p_2^-)}-\frac{1}{p_1^+(p_1^++p_2^+)}+P(p_1,p_2,p_3)\right]\times\int dk^1\,\mathrm{sgn}(k^1)n_F(|k^1|)=0,$$

$$\Gamma_R^{001}(p_1,p_2,p_3)=-\frac{e^2}{2\pi}\left[\frac{1}{p_1^-(p_1^-+p_2^-)}+\frac{1}{p_1^+(p_1^++p_2^+)}+P(p_1,p_2,p_3)\right]\times\int dk^1\,\mathrm{sgn}(k^1)n_F(|k^1|)=0, \tag{26}$$

where $P(p_1,p_2,p_3)$ represents contributions obtained by cyclic permutations of the external momenta, and $p_i^{\mp}=p_i^0\pm p_i^1$. It is to be understood that p_1^0 and p_2^0 are accompanied by a term $-i\epsilon$, whereas p_3^0 is accompanied by a term $2i\epsilon$.

The calculation of the temperature-dependent part of the retarded four-point amplitude is straightforward as well. It corresponds to evaluating the set of diagrams indicated in Fig. 4, where the external lines denote, in this case, photon fields. Once again, there are only two independent tensor structures that arise corresponding to whether there is an even/odd number of spacelike indices with the values

$$\Gamma_R^{0000}(p_1,p_2,p_3,p_4)=-\frac{e^2}{2\pi}\left[\frac{1}{p_1^-(p_1^-+p_2^-)(p_1^-+p_2^-+p_3^-)}-\frac{1}{p_1^+(p_1^++p_2^+)(p_1^++p_2^++p_3^+)}+P(p_1,p_2,p_3,p_4)\right]\times\int dk^1\,\mathrm{sgn}(k^1)n_F(|k^1|)=0,$$

$$\Gamma_R^{0001}(p_1,p_2,p_3,p_4)=-\frac{e^2}{2\pi}\left[\frac{1}{p_1^-(p_1^-+p_2^-)(p_1^-+p_2^-+p_3^-)}+\frac{1}{p_1^+(p_1^++p_2^+)(p_1^++p_2^++p_3^+)}+P(p_1,p_2,p_3,p_4)\right]\times\int dk^1\,\mathrm{sgn}(k^1)n_F(|k^1|)=0, \tag{27}$$

where the appropriate $i\epsilon$ terms are to be understood.

The pattern is clear now. As we calculate the temperature-dependent parts of the retarded higher-point amplitudes, only the external factors change in a predictable manner whereas the coefficient, namely, the integral, does not change and vanishes by antisymmetry. Thus, in this model, the temperature-dependent parts of all the retarded amplitudes, at one loop, vanish. This is indeed consistent with the conclusions of Ref. [16] where it was shown that the temperature-dependent part of the effective action in this theory does not lead to any physical effects.

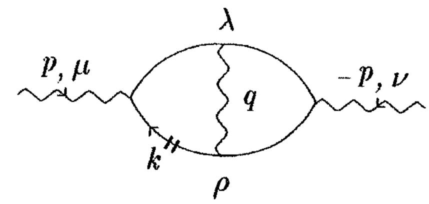


FIG. 5. A typical two-loop self-energy graph with a single fermion line cut.

V. FURTHER PROPERTIES OF 1+1 THERMAL QED

From the results of the previous section, we note that all the finite temperature-dependent parts of the retarded n-point amplitudes in the $(1+1)$-dimensional QED vanish to one loop order. This, however, does not say anything, in general, about the higher loop amplitudes. This is easily seen from the fact that if we were to connect any two external photon lines, the form of the zero-temperature part of the propagator depends on the type of vertices that it connects. And, consequently, the pairwise cancellation that took place between diagrams in the previous section no longer holds. Of course, the temperature-dependent part of the photon propagator, as with any other propagator, does not depend on the type of the vertices it connects (namely, for any pair of thermal indices) and, consequently, leads to a vanishing contribution. Thus, the problematic diagrams at higher loop would appear to be the single cut diagrams where the internal photon lines are not cut as in the example shown in Fig. 5. In this simple theory, however, things are rather special. Let us illustrate this with a general two loop diagram which would correspond to connecting a pair of external photon lines in the four-point vertex functions discussed in Sec. III. First, we note that in the $(1+1)$-dimensional QED, the photon becomes massive and as a result, the photon propagator in the Feynman gauge would have the form

$$D_{\lambda\rho}(q)=\frac{\eta_{\lambda\rho}}{(q^2-m^2+i\epsilon)}, \tag{28}$$

where the photon mass can be identified with $m^2=e^2/\pi$.

The general form for a diagram with all + vertices in the $2n$-point amplitude, at one loop, is easily seen to be (in this theory, only the even point amplitudes are nonzero by charge conjugation invariance)

$$T^{\mu_1\cdots\mu_{2n}}\sim\int\frac{d^2k}{(2\pi)^2}\left[k_+^{\mu_1}(k+p_1)_+^{\mu_2}\cdots(k+\cdots+p_{2n-1})_+^{\mu_{2n}}+k_-^{\mu_1}(k+p_1)_-^{\mu_2}\cdots(k+\cdots+p_{2n-1})_-^{\mu_{2n}}\right]\times\frac{n_F(|k^0|)\,\delta(k^2)}{[(k+p_1)^2+i\epsilon]\cdots[(k+\cdots p_{2n-1})^2+i\epsilon]}. \tag{29}$$

If we now connect any photon line (and consider only the zero-temperature part of the photon propagator because as we have argued before the finite-temperature parts would cancel pairwise among the sum of the diagrams in the combination giving the retarded amplitude), then we trivially obtain

$$\int \frac{d^2q}{(2\pi)^2} T^{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_{2n}} \frac{\eta_{\mu_i\mu_j}}{(q^2-m^2+i\epsilon)}=0. \quad (30)$$

Here we have assumed that the photon line being connected carries a momentum q and the vanishing of the amplitude in Eq. (30) follows from the fact that for any two arbitrary vectors, A^λ and B^ρ,

$$A^\lambda_\pm B^\rho_\pm \eta_{\lambda\rho}=0. \quad (31)$$

A similar argument shows that, in this simple model, each diagram with an internal photon line identically vanishes and, consequently, the retarded amplitude continues to vanish even at higher loops. However, it is to be emphasized that Eq. (31) only leads to the vanishing of the diagram, provided the integral in Eq. (30) is convergent. As is well known, the finite-temperature integrals are ultraviolet finite and hence there is no such problem from the ultraviolet region. For a massive photon, similarly, there is no infrared problem either and hence the diagram, indeed, vanishes. On the other hand, if we did not include the photon mass, these integrals would be infrared divergent and, consequently, we cannot conclude that each of these diagrams vanishes individually. In fact, they do not and only when we sum the amplitude to all orders, all such divergent terms would cancel as Eq. (30) shows.

The discussion, so far, would seem to say that when two external photon lines are contracted, the diagrams individually vanish for a single cut fermion line in the Feynman gauge. This is, however, not true in a general gauge. Without going into details, let us simply summarize here that the $q^\lambda q^\rho$ terms in the photon propagator do contribute graph by graph. However, when their contribution is summed for a fixed single cut fermion line, the effect is to lead to an integral which vanishes by antisymmetry (the number of alternating step functions becomes odd). This could also have been inferred from the gauge invariance of this model. Namely, we know that, in this model, any temperature-dependent correction to any amplitude must be gauge invariant and, consequently, if an amplitude vanishes in the Feynman gauge, it must also vanish in any other gauge. This model, in this sense, is special and immediately leads to the fact that in this $(1+1)$-dimensional QED, the temperature-dependent corrections to all the retarded amplitudes vanish to all loops. This is, indeed, consistent with the conclusions of Ref. [16].

VI. CONCLUSION

In this paper, we have identified, in a simple diagrammatic way, the unique combination of n-point causal amplitudes in the real time formalism that corresponds to the retarded n-point amplitude obtained by analytic continuation from the imaginary time formalism. We have also given a simple method of calculating the temperature-dependent parts of the retarded n-point amplitudes, at least to one loop order, by identifying them with the forward scattering amplitude of on-shell thermal particles. (The extension of this approach to higher orders will be reported elsewhere.) As an application, we have calculated and shown that all the temperature-dependent parts of the retarded n-point amplitudes for $(1+1)$-dimensional massless QED vanish to one loop. For this simple model, it turns out that the temperature-dependent parts of all the retarded n-point amplitudes also vanish to all loops.

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[1] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
[2] H. A. Weldon, Phys. Rev. D **26**, 1394 (1982); **28**, 2007 (1983).
[3] K. Kajantie and J. Kapusta, Ann. Phys. (N.Y.) **160**, 477 (1985).
[4] E. Braaten and R. D. Pisarski, Nucl. Phys. **B337**, 569 (1990); **B339**, 310 (1990); Phys. Rev. D **45**, 1827 (1992).
[5] A. V. Smilga, Phys. Rep. **291**, 1 (1997).
[6] J. I. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989).
[7] M. L. Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
[8] A. Das, *Finite Temperature Field Theory* (World Scientific, New York, 1997).
[9] J. Frenkel and J. C. Taylor, Nucl. Phys. **B374**, 156 (1992); F. T. Brandt and J. Frenkel, Phys. Rev. D **56**, 2453 (1997).
[10] N. P. Landsman and C. G. van Weert, Phys. Rep. **145**, 141 (1987).
[11] T. S. Evans, Nucl. Phys. **B374**, 340 (1992); Phys. Rev. D **47**, 4196 (1993).
[12] R. Baier and A. Niegawa, Phys. Rev. D **49**, 4107 (1994).
[13] M. E. Carrington and U. Heinz, Eur. Phys. J. C **1**, 619 (1998).
[14] Kuang-chao Chou, Zhao-bin Su, Bai-lin Hao, and Lu Yu, Phys. Rep. **118**, 1 (1985).
[15] J. Schwinger, Phys. Rev. **128**, 2425 (1962).
[16] A. Das and A. J. da Silva, hep-th/9808027, 1998.