Instituto de Física
Universidade de São Paulo

Fields on the Poincaré group

I. Spin and relativistic wave equations

Gitman, D. M.; Shelepin, A.L.
Instituto de Física , Universidade de São Paulo, SP, Brasil

Publicação IF - 1372/99

We construct the quantum fields on the Poincaré group instead of Minkowski space (which is a coset space of Poincaré group) in 2,3,4 dimensions. We study the scalar field $f(x,z)$ on the Poincaré group, where x are coordinates on Minkowski space and z, parametrizing elements of the Lorentz group, correspond to spin degrees of freedom. It is shown that usual spin fields arize in the framework of a decomposition of this unique field. Doing a classification of the functions $f(x,z)$, we naturally obtain relativistic wave equations, describing a particle with fixed spin and mass. One and the same spin may be described both in terms of a finite-dimensional nonunitary representation and in terms of a infinite-dimensional unitary representation of the Lorentz group. The consideration of the scalar field on the Poincaré group allows one to give an uniform description for an arbitrary spin. In particular, the present approach is a convenient tool for the theory of higher spins, where usual multicomponent matrix approach is too cumbersome.

I. INTRODUCTION

The problem of constructing of the relativistic wave equations for particles with arbitrary spin is far from its completion and continue attract attention till now. In fact, the question is to construct physically sensible description of particles with arbitrary spin and mass in different dimensions. This problem is closely connected with the representation theory of the space-time symmetry groups, in particular with representation theory of Poincaré group in arbitrary dimensions.

In the present paper we develop one of the possible approaches for the obtaining relativistic wave equations, basing on the analysis of generalized regular representation (GRR) of Poincaré group. The consideration of GRR (i.e. representation in the space of functions on the group) suppose use of the method of harmonic analysis as a basic method [1–4]. This general group-theoretical method is in some sense alternative with respect to the method of induced representations, started from [5] and usually used for semi-direct products [4,6,7].

Studying GRR, we encounter with the problem of classification of scalar functions of some form on the Poincaré group. The specific character of this functions connected with the fact, that they depend not only on space-time coordinates x, but also on some complex coordinates z, which describe spin degrees of freedom.

In the framework of such an approach the construction of relativistic wave equations looks as a separation of invariant subspaces in the space of such functions on the group. That means the uniform approach to obtaining of distinct types of relativistic wave equations, and also the possibility of regular construction of new equations in different dimensions by uniform way.

Besides, one may consider this investigation as an alternative approach to construct a detail theory of Poincaré group.

In section 2 we develop an uniform scheme for the analysis of the field on the group, starting from the construction of left $T_L(g)$ and right $T_R(g)$ GRR of Poincaré group. On this base we give the definition and consider general properties of the scalar field on the Poincaré group,

$$f'(x',z') = f(x,z),$$

where $f'(x,z) = T_L(g)f(x,z)$, x are coordinates on Minkowski space and z are coordinates on the Lorentz group. Spin operators are differential operators on z. We show, that this field contain arbitrary spin, and then establish the connection with usual spin description by means of tensor fields in the Minkowski space. The action of discrete transformations C, P, T are written in terms of scalar field on the group.

Then on this base we consider in detail the description of spin and obtain relativistic wave equations in 2,3,4 dimensions by a regular method.

In section 3 we study scalar fields on two-dimensional Poincaré and Euclidean groups. We consider the parity transformation, construct the relativistic wave equations for massive particles and find its solutions.

In section 4 we study scalar fields on three-dimensional Poincaré and Euclidean groups. Apart from finite-component equations we also consider positive energy wave equations, connected with unitary infinite-dimensional irreducible representations (IR) of 2+1 Lorentz group, describing, in particular, particles with fractional spin (anyons).

In section 5 we study scalar fields on four-dimensional Poincaré group. It is shown, that consistent consideration, taking into account symmetry with respect to parity transformation, is possible on the base of representation theory of the de Sitter $SO(3,2)$ group, whose IR combine the components with different chirality. The connection of the present approach with traditional approaches in the theory of relativistic wave equations is considered in detail. In particular, we pay significant attention to equations with subsidiary conditions (the Dirac-Fiertz-Pauli equations, the Rarita-Schwinger equations and the Bargmann-Wigner equations), which also arise in present approach, but as equations for systems with composite spin. General Gel'fand-Yaglom equations and, in particular, Bhabha equations connect a few scalar functions on the group.

Under classification of the scalar functions on the group we obtain equations with uniform basic properties in 2,3,4 dimensions, which describe a particle with fixed mass and spin. General properties of these equations we consider in section 6.

II. FIELDS ON THE POINCARÉ GROUP AND SPIN DESCRIPTION

A. Parametrization of the Poincaré group

To begin with we remember basic definitions and introduce some notations.

Consider linear non-homogeneous real transformations

$$x' = gx, \qquad x'^{\nu} = \Lambda^{\nu}{}_{\mu} x^{\mu} + a^{\nu}, \tag{2.1}$$

of the coordinates $x = (x^{\mu},\ \mu = 0, \ldots, D)$ in d-dimensional $(d = D + 1)$ pseudo-Euclidean space, which leave the interval square invariant,

$$ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu}, \tag{2.2}$$

where $\eta_{\mu\nu} = \mathrm{diag}\{1, -1, \ldots, -1\}$ is Minkowski metric tensor. In (2.1) vectors a define translations and the matrices Λ define rotations, that means that the latter belong to the vector representation of $O(D,1)$ group. We are also going to consider D-dimensional Euclidean case, when $ds^2 = \eta_{ik} dx^i dx^k$ and $\eta_{ik} = \mathrm{diag}\{1, 1, \ldots, 1\}$, $i, k = 1, \ldots, D$. In this case the matrices Λ belong to the vector representation of $O(D)$ group.

The transformations (2.1), which may be connected continuously to the unit transformation, form a real Lie group which is called Poincaré group (or proper Poincaré group) $M_0(D,1)$. In the homogeneous case, when $a=0$, this is the Lorentz group (or proper Lorentz group) $SO_0(D,1)$. In Euclidean case those are $M_0(D)$ and $SO_0(D)$ respectively. The composition low and the inverse element of the group have the form

$$(a_2,\Lambda_2)(a_1,\Lambda_1)=(a_2+\Lambda_2 a_1,\Lambda_2\Lambda_1),\quad g^{-1}=(-\Lambda^{-1}a,\Lambda^{-1}). \tag{2.3}$$

Thus, the groups $M_0(D,1)$ and $M_0(D)$ are semidirect products

$$M_0(D,1)=T(d)\times)SO_0(D,1),\qquad M_0(D)=T(D)\times)SO_0(D),$$

where T(d) is d-dimensional translations group.

In pseudo-Euclidean spaces of 2,3 and 4 dimensions, there exist one-to-one correspondence between the vectors x and 2×2 Hermitian matrices X *,

$$x\leftrightarrow X,\quad X=x^\mu\sigma_\mu. \tag{2.4}$$

Namely:

$$d=3+1:\quad X=\begin{pmatrix} x^0+x^3 & x^1-ix^2\\ x^1+ix^2 & x^0-x^3\end{pmatrix}, \tag{2.5}$$

$$d=2+1:\quad X=\begin{pmatrix} x^0 & x^1-ix^2\\ x^1+ix^2 & x^0\end{pmatrix}, \tag{2.6}$$

$$d=1+1:\quad X=\begin{pmatrix} x^0 & x^1\\ x^1 & x^0\end{pmatrix}, \tag{2.7}$$

In all the above cases

$$\det X=\eta_{\mu\nu}x^\mu x^\mu,\quad x^\mu=\frac{1}{2}\mathrm{Tr}(X\sigma^\mu). \tag{2.8}$$

In Euclidean spaces of 2 and 3 dimensions similar correspondence has the form

$$D=3:\qquad X=\begin{pmatrix} x^3 & x^1-ix^2\\ x^1+ix^2 & -x^3\end{pmatrix}, \tag{2.9}$$

$$D=2:\qquad X=\begin{pmatrix} x^2 & x^1\\ x^1 & -x^2\end{pmatrix}. \tag{2.10}$$

If x is subjected to the transformation (2.1) then X transforms as follows (see, for example, [2,4,8]):

---

*We will use two sets of 2×2 matrices $\sigma_\mu=(\sigma_0,\sigma_k)$ and $\bar\sigma_\mu=(\sigma_0,-\sigma_k)=\sigma^\mu$,

$$\sigma_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

$$X' = gX = UXU^\dagger + A, \tag{2.11}$$

where $A = a^\mu \sigma_\mu$, and U are some matrices obeying the conditions

$$\sigma_\nu \Lambda^\nu{}_\mu = U \sigma_\mu U^\dagger. \tag{2.12}$$

It follows from (2.12) that $\det U = e^{i\phi}$. Clear, that it is enough to consider only unimodular matrices $\det U = 1$. One can also see that U and $-U$ generate the same transformations of X. Thus, U belong to double covering group Spin$(D, 1)$ of the Lorentz group $SO_0(D, 1)$ or in Euclidean case to double covering Spin(D) of the group $SO_0(D)$). In the dimensions under consideration those groups are isomorphic to following once:

$$d = 3+1: \quad U \in SL(2, C), \quad U = \begin{pmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{pmatrix}, \quad u_1^1 u_2^2 - u_1^2 u_2^1 = 1, \tag{2.13}$$

$$d = 2+1: \quad U \in SU(1, 1), \quad U = \begin{pmatrix} u_1 & u_2 \\ \overset{*}{u}_2 & \overset{*}{u}_1 \end{pmatrix}, \quad |u_1|^2 - |u_2|^2 = 1, \tag{2.14}$$

$$D = 3: \quad U \in SU(2), \quad U = \begin{pmatrix} u_1 & u_2 \\ -\overset{*}{u}_2 & \overset{*}{u}_1 \end{pmatrix}, \quad |u_1|^2 + |u_2|^2 = 1, \tag{2.15}$$

$$d = 1+1: \quad U \in SO(1, 1), \quad U = \begin{pmatrix} \cosh\frac{\beta}{2} & \sinh\frac{\beta}{2} \\ \sinh\frac{\beta}{2} & \cosh\frac{\beta}{2} \end{pmatrix}, \tag{2.16}$$

$$D = 2: \quad U \in SO(2), \quad U = \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}. \tag{2.17}$$

We retain both elements U and $-U$ in the consideration. That means, in fact, a transition to the groups

$$M(D, 1) = T(d) \times) \mathrm{Spin}(D, 1), \qquad M(D) = T(D) \times) \mathrm{Spin}(D).$$

As it is known, that allows one to avoid double valued representations for half integer spin description. There exist one-to-one correspondence between the element g of the group $M(D, 1)$ or $M(D)$ and two 2×2 matrices, $g \leftrightarrow (A, U)$, where A correspond to translations and U correspond to rotations. The formula (2.11) describes the action of $M(D, 1)$ on the Minkowski space, the latter is coset space $M(D, 1)/\mathrm{Spin}(D, 1)$.

As a consequence of (2.11) one can obtain the composition low and the inverse element,

$$(A_2, U_2)(A_1, U_1) = (U_2 A_1 U_2^\dagger + A_2,\ U_2 U_1)\ , \quad g^{-1} = (-U^{-1} A (U^{-1})^\dagger,\ U^{-1})\ . \tag{2.18}$$

The matrices U in the dimensions under consideration satisfy the following identities:

$$U \in SL(2, C): \quad \sigma_2 U \sigma_2 = (U^T)^{-1}; \tag{2.19}$$

$$U \in SU(1, 1): \quad \sigma_1 U \sigma_1 = \overset{*}{U}, \quad \sigma_2 U \sigma_2 = (U^T)^{-1}, \quad \sigma_3 U \sigma_3 = (U^\dagger)^{-1}, \tag{2.20}$$

$$U \in SU(2): \quad \sigma_2 U \sigma_2 = (U^T)^{-1} = \overset{*}{U}. \tag{2.21}$$

An equivalent description can be done in terms of the matrices $\overline{X} = x^\mu \bar{\sigma}_\mu$. Using the relation $\overline{X} = \sigma_2 X^T \sigma_2$, the transformation low of X (2.11) and the identity (2.19), one can find that

$$\overline{X}' = (U^\dagger)^{-1}\overline{X}U^{-1} + \overline{A}. \tag{2.22}$$

Thus, $\overline{X}$ is transformed by means of the element $(\overline{A}, (U^\dagger)^{-1})$. The relation $(A, U) \to (\overline{A}, (U^\dagger)^{-1})$ define an automorphism of the Poincaré group $M(D,1)$. In Euclidean case the matrices U are unitary and the latter relation is reduced to $(A,U) \to (-A,U)$.

The representation of the transformations in the form (2.11) is closely connected to a representation of finite rotations in $\mathbb{R}^d$ in terms of the Clifford algebra. In higher dimensions the transformation low has the same form, where A is a vector element and U correspond to an invertible element (spinor element) of the of Clifford algebra [9]. Besides, the representation of finite transformations in the form (2.11) can be useful for spin description by means of Grassmannian variables ξ, since ξ and $\partial\xi$ give a realization of the Clifford algebra [10].

B. Regular representation and scalar functions on the group

Remember that a generalized regular representation (GRR, left $T_L(g)$ or right $T_R(g)$) is defined in the space of functions on a group G,

$$T_L(g)f(g_0) = f'(g_0) = f(g^{-1}g_0), \tag{2.23}$$
$$T_R(g)f(g_0) = f'(g_0) = f(g_0 g), \tag{2.24}$$

where $g \in G$ and $g_0 \in G$ is a fixed element. It is well known that any IR of a group is equivalent to one of sub-representation of the left (right) GRR [1–3]. Thus, the study of GRR is an effective method of the analysis of IR of the group. It is easy to see that the transformation law (2.23), (2.24) under the action of GRR is reduced to the transformation law of scalar functions on the group

$$f'(g_0') = f(g_0), \tag{2.25}$$

if the transformation law of arguments is defined as

$$g_0' = g g_0 \qquad \text{for left GRR or} \tag{2.26}$$
$$g_0' = g_0 g^{-1} \quad \text{for right GRR.} \tag{2.27}$$

Thus, the consideration of all IR of the group may be based on the classification of the scalar functions on the group with the transformation law (2.26) or (2.27).

Let us consider the transformation law for the groups $M(D,1)$ in the parametrization of group elements by means of 2×2 matrices, described belove. Let $g_0 \leftrightarrow (A_0, U_0)$ and $g \leftrightarrow (A, U)$. Then, according to (2.18),

$$g_0' = g g_0 \leftrightarrow (A_0', U_0') = (UA_0U^\dagger + A, UU_0) \tag{2.28}$$
$$g_0' = g_0 g^{-1} \leftrightarrow (A_0', U_0') = (A_0 + U_0^{-1}A(U_0^{-1})^\dagger, U_0U^{-1}). \tag{2.29}$$

Matrices A_0 and U_0 correspond to the element g_0. The former matrix A_0 is transformed similar matrix X, i.e. as an element of coset space $M(D,1)/\mathrm{Spin}(D,1)$, see (2.11). The latter U_0, belongs to $\mathrm{Spin}(D,1)$ and transforms under rotations only. Therefore, we identify

A_0 with the coordinate dependence and U_0 with the spin dependence of the scalar functions on the group.

According to what has been said there will be convenient for us to use the notations $X = A_0$, $Z = U_0$, and denote variables, which parametrize this matrices, by x and z.

In these notations one can rewrite (2.28) in the form

$$g_0' = gg_0 \leftrightarrow (X', Z') = (UXU^\dagger + A, UZ), \tag{2.30}$$

from where follows $x'^\nu = \Lambda^\nu{}_\mu x^\mu + a^\nu$. The latter transformation law coincides with (2.1). It allows us to interpret the arguments of the scalar functions on the group with the transformation law (2.30) as the space and spin coordinates. In this case (2.30) may be considered as the transformation law of the coordinates under the transformation of the frame of references.

The right transformations act on $g_0 \leftrightarrow (X, Z)$ according to formula

$$g_0' = g_0 g^{-1} \leftrightarrow (X', Z') = (X + Z^{-1}A(Z^{-1})^\dagger, ZU^{-1}), \tag{2.31}$$

from where follows $x'^\nu = x^\nu + (\Lambda^{-1}(z))^\nu{}_\mu a^\mu$, $\bar\sigma_\nu \Lambda^\nu{}_\mu = Z\bar\sigma_\mu Z^\dagger$. These transformations differ from the coordinates transformation (2.1). The inconsistency of the latter transformation law with the interpretation of arguments x as coordinates is stressed by the fact, that for $a \neq 0$ the interval square (2.2) does not conserve under the transformations (2.31).

Therefore below we will consider scalar functions on the group $f(x, z)$ with the transformation law (2.30), or what is the same, only the left GRR, with the aim to keep the interpretation of x as the coordinates in Minkowski space. Using the parametrization, described above, we obtain for the groups $M(D)$ and $M(D,1)$

$$T_L(g)f(x, z) = f(g^{-1}x,\ g^{-1}z),\ \ g^{-1}x \leftrightarrow U^{-1}(X - A)(U^{-1})^\dagger,\ \ g^{-1}z \leftrightarrow U^{-1}Z, \tag{2.32}$$

According to (2.32), X is transformed with respect to the adjoint (vector) representation and Z with respect to the spinor representation of corresponding Lorentz (rotation) group. One can also see that Z is invariant under the translations. If one restricts itself by Z-independent functions, then (2.32) reduces to the left quasi-regular representation,† which corresponds to the scalar field case $f'(x') = f(x)$. If one restricts itself by X-independent functions, then (2.32) reduces to the left GRR of Lorentz (or rotation) group.

Generators, which correspond to translations and rotations will be written as

$$\hat p_\mu = -i\partial/\partial x^\mu, \quad \hat J_{\mu\nu} = \hat L_{\mu\nu} + \hat S_{\mu\nu}, \tag{2.33}$$

where $\hat L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ are angular momentum operators, and $\hat S_{\mu\nu}$ are spin operators.

The algebra of the generators (2.33) has the form

$$\begin{aligned} &[\hat p_\mu, \hat p_\nu] = 0, \quad [\hat J_{\mu\nu}, \hat p_\rho] = i(\eta_{\nu\rho}\hat p_\mu - \eta_{\mu\rho}\hat p_\nu), \\ &[\hat J_{\mu\nu}, \hat J_{\rho\sigma}] = i\eta_{\nu\rho}\hat J_{\mu\sigma} - i\eta_{\mu\rho}\hat J_{\nu\sigma} - i\eta_{\nu\sigma}\hat J_{\mu\rho} + i\eta_{\mu\sigma}\hat J_{\nu\rho}\,. \end{aligned} \tag{2.34}$$

---

†Let H be a subgroup of G. The equality $T_L^q(g)f(g_0H) = f(g^{-1}g_0H)$ gives a representation of G on the function space on G/H. This representation is called left quasi-regular representation corresponding to the subgroup H [3].

Making Fourier transformation

$$\varphi(p) = (2\pi)^{-d/2} \int f(x) e^{ipx} dx \tag{2.35}$$

in the variables x, i.e. considering representations in the space of functions $\varphi(p,z)$, one can get an analog of the formulas (2.32) in this representation,

$$T_L(g)\varphi(p,z) = e^{iap'}\varphi(p', g^{-1}z), \quad p' = g^{-1}p \leftrightarrow P' = U^{-1}P(U^{-1})^\dagger, \quad P = p_\mu \sigma^\mu \tag{2.36}$$

It is seen that the combination $\det Z$ and $\det P = p^2$ are conserved under the transformations (2.36).[‡] and p^2 is an eigenvalue of Casimir operator $\hat{p}^2$.

For the groups $M(D)$ one can consider two types of representations depending on the eigenvalues p^2 of Casimir operator $\hat{p}^2$: (1) $p^2 \neq 0$ (moving particle); (2) $p^2 = 0$; then all $p_i = 0$ (rest particle) and IR differ by eigenvalues of Casimir operators of Rotation subgroup.

For the groups $M(D,1)$ one can consider four types of representations depending on the eigenvalues m^2 of Casimir operator $\hat{p}^2$: (1) $m^2 > 0$ (particle of mass m); (2) $m^2 < 0$ (tachyon); (3) $m^2 = 0$, $p_0 \neq 0$ (massless particle); (4) $m^2 = p_0 = 0$, IR differ by eigenvalues of Casimir operators of Lorentz subgroup and corresponding functions do not depend on x.

For the decomposition of the left GRR we are going to construct a full set of a commuting operators in the space of functions on the group. Together with Casimir operators a functions of right generators[§] are a part of this set. Therefore it is necessary to know the explicit form of right generators. As a consequence of the formulas

$$T_R(g)f(x,z) = f(xg,\ zg), \quad xg \leftrightarrow X + ZAZ^\dagger, \quad zg \leftrightarrow ZU, \tag{2.37}$$

$$T_R(g)\varphi(p,z) = e^{-ia'p}\varphi(p,zg), \quad a' \leftrightarrow A' = ZAZ^\dagger, \tag{2.38}$$

one can obtain

$$\hat{p}^R_\mu = -(\Lambda^{-1}(z))_\mu{}^\nu p_\nu, \quad \hat{J}^R_{\mu\nu} = \hat{S}^R_{\mu\nu}. \tag{2.39}$$

Operators of right translations may be written also in the form $\hat{P}^R = -Z^{-1}\hat{P}(Z^{-1})^\dagger$; operators $\hat{S}_{\mu\nu}$ and $\hat{S}^R_{\mu\nu}$ are the left and right generators of $\mathrm{Spin}(D,1)$ (or $\mathrm{Spin}(D)$) subgroup and depend on z only. All right generators (2.39) commute with all left generators (2.33) and obey the same commutation relations (2.34).

In accordance with the theory of harmonic analysis on the Lie groups [1,4] a complete set of commuting operators consist of a Casimir operators, commuting with all (left and right) generators, the certain number of left generators and the same number of right generators. The number of commuting operators is equal to the number of parameters of the group.

[‡]We denoted the square of the vector p_μ by p^2, $p^2 = \eta^{\mu\nu}p_\mu p_\nu$. Since we will not use p with upper indices this will not lead to a misunderstanding.

[§]The physical meaning for right generators is usually not so transparent as left generators, however, the right generators of three-dimensional rotations in the rotator theory are interpreted as operators of the angular momentum components in a rotating system of coordinates [11].

The nonequivalent representations in the decomposition of the left GRR are differed by eigenvalues of the Casimir operators, equivalent representations are differed by eigenvalues of right generators, and the states inside IR are differed by eigenvalues of left generators.

In particular, Casimir operators of spin Lorentz subgroup are the functions of $\hat{S}^R_{\mu\nu}$ (or $\hat{S}_{\mu\nu}$) and commute with all left generators (and with left translations and rotations correspondingly), but do not commute with the generators of right translations. These operators differ equivalent representations in the decomposition of the left GRR.

On every finite-parameter Lie group it is possible to define an invariant measure $d\mu(g)$. If GRR acts in the space of all functions on the group G, then regular representation acts in the space of functions $L^2(G,\mu)$, such that the norm

$$\int \overset{*}{f}(g)f(g)d\mu(g) \tag{2.40}$$

is finite [3,1]. The regular representation is unitary, as it follows from (2.40) and the invariance of the measure $d\mu(g)$.** However we will use also nonunitary representations (in particular, finite-dimensional representations of the Lorentz group). Therefore we consider the GRR as a more useful concept.

C. Field on the Poincaré group

Tensor fields in the Minkowski space describe particles with different spins. These fields are defined by means of the transformation low of multicomponent functions (i.e. the functions, depend on both x and some discrete parameter) under the coordinate transformation.

The relations $f'(g_0') = f(g_0)$, $g_0' = gg_0$, connected with the left GRR (2.23), also define the transformation low under transformation of the frame of references, but in a extended space. Besides the coordinates x scalar functions on the Poincaré group $f(g_0)$, $g_0 = (x,z)$ depend on the set of variables z,

$$f'(x',z') = f(x,z), \tag{2.41}$$

$$x' = gx = \Lambda x + a \leftrightarrow UXU^\dagger + A, \quad z' = gz \leftrightarrow UZ. \tag{2.42}$$

In contrast to tensor fields on Minkowski space this field is reducible with respect to both mass and spin.

Let us consider more detail the transformation low of x and z for distinct dimensions.

In two dimensional case matrix Z depend on only one parameter (angle or hyperbolic angle, see (2.16),(2.17)). The functions on the group depend on $x = (x^\mu)$ and $z = e^\alpha$ (or $x = (x^k)$ and $z = e^{i\alpha}$ in Euclidean case); it is appropriate to consider these functions as functions of real parameter α directly.

**There exist unitary representations which are not found in the decomposition of regular representation. These representations of so-called supplementary series are characterized by nonlocal scalar product $\int \overset{*}{f}(\tilde{g}_1)f(\tilde{g}_2)I(\tilde{g}_1,\tilde{g}_2)d\mu(\tilde{g}_1)d\mu(\tilde{g}_2)$, where kernel function $I(\tilde{g}_1,\tilde{g}_2)$ has to be invariant under the group transformations, $\tilde{g}_k \in G/H$, $H \subset G$.

In three dimensional case according to (2.14),(2.15)

$$D=3: \quad Z=\begin{pmatrix} z_1 & -\overset{*}{z}_2 \\ z_2 & \overset{*}{z}_1 \end{pmatrix}, \qquad d=2+1: \quad Z=\begin{pmatrix} z_1 & \overset{*}{z}_2 \\ z_2 & \overset{*}{z}_1 \end{pmatrix}, \qquad \det Z=1. \tag{2.43}$$

Functions $f(x,z)$ depend on $x=(x^\mu)$ (in Euclidean case $x=(x^k)$) and $z=(z_\alpha,\overset{*}{z}_{\dot\alpha})$, where z_α are the elements of the first column of matrix (2.43). Let us write the relation (2.42) for $d=2+1$ in component-wise form,

$$x'^\nu\sigma_{\nu\alpha\dot\alpha}=U_\alpha{}^\beta\sigma_{\mu\beta\dot\beta}x^\mu\overset{*}{U}{}^{\dot\beta}{}_{\dot\alpha}, \tag{2.44}$$

$$z'_\alpha=U_\alpha{}^\beta z_\beta, \quad \overset{*}{z}{}'_{\dot\alpha}=\overset{*}{U}{}_{\dot\alpha}{}^{\dot\beta}\overset{*}{z}_{\dot\beta}, \quad z'^\alpha=(U^{-1})^\alpha{}_\beta z^\beta, \quad \overset{*}{z}{}'^{\dot\alpha}=(\overset{*}{U}{}^{-1})^{\dot\alpha}{}_{\dot\beta}\overset{*}{z}{}^{\dot\beta}. \tag{2.45}$$

Undotted and dotted indices correspond to spinors which transformed by means of matrix U and complex conjugated matrix $\overset{*}{U}$. Invariant tensor $\sigma_{\nu\alpha\dot\alpha}$ has one vector index and two spinor indices of distinct types.

For the group $M(3,1)$ matrices Z has the form

$$d=3+1: \quad Z=\begin{pmatrix} z_1 & \tilde z_1 \\ z_2 & \tilde z_2 \end{pmatrix}, \qquad \det Z=z_1\tilde z_2-z_2\tilde z_1=1. \tag{2.46}$$

The functions $f(x,z)$ depend on $x=(x^\mu)$ and $z=(z_\alpha,\overset{*}{z}_{\dot\alpha},\tilde z_\alpha,\tilde z^*_{\dot\alpha})$. There are some reasons to consider functions $f(x,z)$ as functions of z and $\overset{*}{z}$ rather then real parameters $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$. Variables z and $\overset{*}{z}$ are subjected to simple transformation rule. Besides, use the spaces of analytical and antianalytical functions is suitable for the problem of decomposition of GRR. Under action of group $M(3,1)$ the elements z^α and $\tilde z^\alpha$ of first and second columns of matrix (2.46) are subjected to the same transformation low.

In accordance with (2.42) and (2.22) one may write the transformation low of x^μ, z_α, $\overset{*}{z}_{\dot\alpha}$ in component-wise form,

$$x'^\nu\sigma_{\nu\alpha\dot\alpha}=U_\alpha{}^\beta\sigma_{\mu\beta\dot\beta}x^\mu\overset{*}{U}{}^{\dot\beta}{}_{\dot\alpha}, \qquad x'^\nu\bar\sigma_\nu{}^{\dot\alpha\alpha}=(\overset{*}{U}{}^{-1})^{\dot\alpha}{}_{\dot\beta}\bar\sigma_\mu{}^{\dot\beta\beta}x^\mu(U^{-1})_\beta{}^\alpha, \tag{2.47}$$

$$z'_\alpha=U_\alpha{}^\beta z_\beta, \quad \overset{*}{z}{}'_{\dot\alpha}=\overset{*}{U}{}_{\dot\alpha}{}^{\dot\beta}\overset{*}{z}_{\dot\beta}, \quad z'^\alpha=(U^{-1})^\alpha{}_\beta z^\beta, \quad \overset{*}{z}{}'^{\dot\alpha}=(\overset{*}{U}{}^{-1})^{\dot\alpha}{}_{\dot\beta}\overset{*}{z}{}^{\dot\beta}. \tag{2.48}$$

Corresponding to transformation low (2.47), the tensors

$$\sigma_{\mu\alpha\dot\alpha}=(\sigma_\mu)_{\alpha\dot\alpha}, \quad \bar\sigma_\mu{}^{\dot\alpha\alpha}=(\bar\sigma_\mu)^{\dot\alpha\alpha}, \tag{2.49}$$

are invariant. These tensors are usually used to convert vector indices into spinor ones and vice versa or to construct vector from two spinors of different types,

$$x^\mu=\frac{1}{2}\bar\sigma^{\mu\dot\alpha\alpha}x_{\dot\alpha\alpha}, \quad x_{\alpha\dot\alpha}=\sigma_{\mu\alpha\dot\alpha}x^\mu, \quad q^\mu=\frac{1}{2}\bar\sigma^{\mu\dot\alpha\alpha}z_\alpha\overset{*}{z}_{\dot\alpha}. \tag{2.50}$$

In consequence of the unimodularity of 2×2 matrices U there exist invariant antisymmetric tensors $\varepsilon^{\alpha\beta}=-\varepsilon^{\beta\alpha}$, $\varepsilon^{\dot\alpha\dot\beta}=-\varepsilon^{\dot\beta\dot\alpha}$, $\varepsilon^{12}=\varepsilon^{\dot1\dot2}=1$, $\varepsilon_{12}=\varepsilon_{\dot1\dot2}=-1$. Now spinor indices are lowered and raised according to the rules

$$z_\alpha = \varepsilon_{\alpha\beta} z^\beta, \quad z^\alpha = \varepsilon^{\alpha\beta} z_\beta. \tag{2.51}$$

Below we will also use the notations $\partial_\alpha = \partial/\partial z^\alpha$, $\partial^{\dot\alpha} = \partial/\partial \overset{*}{z}_{\dot\alpha}$.

In the framework of theory of the scalar functions on the Poincaré group a standard spin description in terms of multicomponent functions arises under **separation of space and spin variables.**

Since z does not transformed under translations, then functions, which depend on z only, are transformed under a representation of the Lorentz group. Let a function $f(x,z)$ allows the representation

$$f(x,z) = \phi^n(z)\psi_n(x), \tag{2.52}$$

where $\phi^n(z)$ form the basis in the representation space of the Lorentz group. The latter means that one may decompose the functions $\phi^n(z')$ of transformed argument $z' = gz$, on the functions $\phi^n(z)$,

$$\phi_n(z') = \phi^l(z) L_l{}^n(U). \tag{2.53}$$

The action of Poincaré group on a line composed of $\phi^n(z)$ reduced to multiplication by matrix $L(U)$, where $U \in \mathrm{Spin}(D,1)$, $\phi(z') = \phi(z)L(U)$.

One may compare decompositions of the function $f'(x',z') = f(x,z)$ over the changed basis $\phi(z')$ and over the initial basis $\phi(z)$,

$$f'(x',z') = \phi(z')\psi'(x') = \phi(z)L(U)\psi'(x') = \phi(z)\psi(x),$$

where $\psi(x)$ is a column with components $\psi_n(x)$, and obtain the transformation low of tensor field on Minkowski space

$$\psi'(x') = L(U^{-1})\psi(x). \tag{2.54}$$

This transformation low is connected with the representation of the Poincaré group acting in a linear space of tensor fields as follows $T(g)\psi(x) = L(U^{-1})\psi(\Lambda^{-1}(x-a))$. According to (2.53), (2.54) functions $\phi(z)$ and $\psi(x)$ are transformed under contragradient representations of the Lorentz group.

For example, let us consider scalar functions on the Poincaré group $f_1(x,z) = \psi_\alpha(x)z^\alpha$ and $f_2(x,z) = \bar\psi_{\dot\alpha}(x)\overset{*}{z}^{\dot\alpha}$, which correspond to spinor representations of Lorentz group. According to (2.52) and (2.54)

$$\psi'_\alpha(x') = U_\alpha{}^\beta \psi_\beta(x), \quad \bar\psi'_{\dot\alpha}(x') = \overset{*}{U}_{\dot\alpha}{}^{\dot\beta} \bar\psi_{\dot\beta}(x). \tag{2.55}$$

The product $\psi_\alpha(x)\bar\psi^{*\alpha}(x)$ is Poincaré invariant.

Since, tensor fields of all spins are contained in the decomposition of the field (2.41) on the Poincaré group and the problems of their classification and construction of explicit realizations are reduced to problem of the decomposition of the left GRR.

Note, that above we reject the phase transformations, which correspond to $U = e^{i\phi}$. This transformations of $U(1)$ group do not change space-time coordinates x, but change the phase of z. According to (2.53) and (2.54), that lead to the transformation of phase of tensor field components $\psi_n(x)$. The consideration of this transformations means the transition to the functions on the group $T(d)\times)\mathrm{Spin}(D,1) \times U(1)$.

D. Discrete transformations: C,P,T

Three automorphisms of $M(D,1)$,

$$(A,U) \to (\overline{A},(U^\dagger)^{-1}), \tag{2.56}$$

$$(A,U) \to (\overset{*}{A},\overset{*}{U}), \tag{2.57}$$

$$(A,U) \to (-A,U), \tag{2.58}$$

generate finite group consist of six elements. The discrete transformations of x and z are connected with these automorphisms.

The space reflection (or parity transformation) P is defined by the relations $x^0 \to x^0$, $x^k \to -x^k$, or $X \to \overline{X}$. If X is transformed by means of the group element (A,U), then $\overline{X}$ is transformed by means of the group element $(\overline{A},(U^\dagger)^{-1})$, see (2.22). Therefore the space reflection represent a realization of the automorphism (2.56) of the Poincaré group,

$$(X,Z) \overset{P}{\to} (\overline{X},(Z^\dagger)^{-1}). \tag{2.59}$$

Thus, under the space reflection x and z in all the constructions have to be changed according to (2.59). In particular, $P \to \overline{P}$, where $\overline{P} = p_\mu\bar{\sigma}^\mu$. The generators of the rotations are not changed and the generators of the boosts change their signs only.

The time reversal T is defined by the relation $x^\mu \to (-1)^{\delta_{0\mu}}x^\mu$, or $X \to -\overline{X}$ (the time reflection transformation T'), with the supplementary condition of energy sign conservation, that means $P \to \overline{P}$. For the time reflection one can obtain

$$(X,Z) \overset{T'}{\to} (-\overline{X},(Z^\dagger)^{-1}). \tag{2.60}$$

Besides, as it follows from the conditions $X \to -\overline{X}$, $P \to \overline{P}$, the conditions $\hat{p}_\mu \to -(-1)^{\delta_{0\mu}}\hat{p}_\mu$, $\hat{L}_{\mu\nu} \to -(-1)^{\delta_{0\mu}+\delta_{0\nu}}\hat{L}_{\mu\nu}$, $\hat{S}_{\mu\nu} \to -(-1)^{\delta_{0\mu}+\delta_{0\nu}}\hat{S}_{\mu\nu}$ take place. It is sufficient for that in addition to (2.60) to use the substitution $z \to \overset{*}{z}$, $i \to -i$.

The charge conjugation changes the signs of charges, of energy and of chirality. In the frame of the characteristics connected with the Poincaré group the charge conjugation changes the signs of energy and chirality (spin orientation). One may show, that the charge conjugation corresponds to the complex conjugation of the functions $f(x,z)$,

$$f(x,z) \overset{C}{\to} \overset{*}{f}(x,z). \tag{2.61}$$

Let us consider a term in the decomposition (2.52) of $f(x,z)$, which corresponds to a plain wave. Such term has the form $e^{ipx}\phi_n(z)$. Complex conjugation changes the sign of energy and for $d = 2,3,4$ we will see below (using the explicit form of spin projection operators), that transformation $\phi_n(z) \overset{C}{\to} \overset{*}{\phi}_n(z)$ changes the sign of chirality. It is easy to see, that the charge conjugation is connected with automorphism (2.57).

The improper Poincaré group is defined as a group, which includes continuous transformations of the proper Poincaré group $g \in M(D,1)$, the space reflection P and the time reflection T'.

In the Euclidean case the space reflection is reduced to the substitution $(X,Z) \overset{P}{\to} (-X,Z)$. The charge conjugation inverts the momentum and spin orientation.

E. Equivalent representations

In the decomposition of scalar field (2.41) on the Poincaré group (or, that is the same, of the left GRR) there are equivalent representations, differed by the right generators.

Remember, that representations $T_1(g)$ and $T_2(g)$, acting in linear spaces L_1 and L_2 correspondingly, are equivalent, if there exist a invertible operator $A : L_1 \to L_2$, that

$$AT_1(g) = T_2(g)A. \tag{2.62}$$

In particular, the left and the right GRR of a Lee group G are equivalent. The operator $(Af)(g) = f(g^{-1})$ realizes the equivalence [1,2].

Let us consider the transformation of functions $f(x, z)$, that correspond to the transition between two equivalent representations in the decomposition of the left GRR. If the representations $T_1(g)$ and $T_2(g)$ of the group $M(D, 1)$ (or $M(D)$), acting in the different subspaces L_1 and L_2 of the space of functions on the group, are equivalent, then

$$AT_1(g)f_1(x, z) = T_2(g)Af_1(x, z), \quad f_2(x, z) = Af_1(x, z),$$

where $f_1(x, z) \in L_1$ and $f_2(x, z) \in L_2$. In particular, if operator $A : L_1 \to L_2$ is a function of the right translations generators $\hat{p}^R_\mu$, then one can't map the function $f_1(x, z)$ to the function $f_2(x, z)$ by the group transformation, which leave the interval square invariant. Therefore, the physical equivalence of the states, that correspond to equivalent IR in the decomposition of the left GRR at least is not evident.

Below we will consider a number of examples in different dimensions. In particular, in the framework of the representation theory of three-dimensional Euclidean group $M(3)$ IR, characterized by different spins (but the same spin projection on the direction of propagation), are equivalent. There are no contradictions in the fact, that in this case different particles are described by equivalent IR, since one can't map corresponding wave functions one to another by the rotations or translations of the frame of references.

In some cases more general consideration may be based on the representation theory of an extanded group. In framework of the latter there are two possibilities: either IR, labelled by different eigenvalues of right generators of initial group, are nonequivalent, or some equivalent IR of initial group are combined into one IR. For example, in nonrelativistic theory spin becomes the characteristic of nonequivalent IR after the transition from $M(3)$ to Galilei group. In 3+1 dimensions the representations of proper Poincaré group, characterized by different chiralities, are equivalent. At the transition from the Lorentz group to the de Sitter group all states with different chiralities λ, characterized by spin s, $\lambda = -s, -s+1, \ldots, s$ combined into one IR.

The space of functions $f(x, z)$ contain functions, transformed under equivalent representations of the Poincaré group, and is sufficiently wide to define transformations of improper Poincaré group, which include space and time reflections, and charge conjugation. These discrete transformations, connected with automorphisms of the group, also combine equivalent IR of proper Poincaré group into one IR of improper group. For example, in 3+1 dimensions space reflection combine two equivalent IR of proper group labelled by λ and $-\lambda$ into one IR of improper group.

Besides, as we will see below, the different types of relativistic wave equations (finite-component and infinite-component equations) also connected with equivalent representations in the decomposition of the left GRR.

Thus, initially it is appropriate to consider all representations in the decomposition of the left GRR, including equivalent ones.

F. Quasiregular representations and spin description

The consideration of GRR of Poincaré group ensures the possibility of consistent description of particles with arbitrary spin in the space of scalar functions on $\mathbb{R}^d \times \mathrm{Spin}(D,1)$. At the same time, there are a number of papers, (see, for example, [12–19]), where spin described by means of operators, acting on the functions on some spaces (one or two-sheeted hyperboloid, cone, complex disk, projective space and so on).

The use of the spaces $\mathbb{R}^d \times M$, where the action of the Poincaré group are defined on $\mathbb{R}^d$ and action of the Lorentz subgroup are defined on M, are common for these papers. (In momentum representation $\mathbb{R}^d$ are replaced on the surface, defined by the equation $p_\mu p^\mu = m^2$.) In some papers such spaces are treated as phase spaces of some classic mechanics, and the latter are treated as models of spinning relativistic particles. These models can be naturally obtained in frameworks of the next group-theoretical scheme.

Let us consider left quasiregular representation of Poincaré group

$$T(g)f(g_0H) = f(g^{-1}g_0H), \quad H \subset \mathrm{Spin}(D,1). \tag{2.63}$$

H is a subgroup of $\mathrm{Spin}(D,1)$, and since x is invariant under right rotations (see (2.37)),

$$g_0 \leftrightarrow (X,Z), \quad g_0H \leftrightarrow (X,ZH).$$

Therefore, the relation (2.63) define the representation of Poincaré group in the space of functions $f(x,zH)$ on

$$\mathbb{R}^d \times (\mathrm{Spin}(D,1)/H). \tag{2.64}$$

In the decomposition of the representation in the space of functions on $\mathrm{Spin}(D,1)/H$ (or $\mathbb{R}^d \times (\mathrm{Spin}(D,1)/H)$) there are, generally speaking, only part of IR of the Lorentz (or Poincaré) group. In particular, the case $H \sim \mathrm{Spin}(D,1)$ correspond to scalar field.

Thus, the consideration of quasiregular representations allow one to construct in any dimension a number of spin models, classified by subgroups $H \subset \mathrm{Spin}(D,1)$. But the existence of nontrivial subgroup H lead to rejection of the part of equivalent (with different characteristics with respect to the Lorentz subgroup) or, possibly, nonequivalent IR of the Poincaré group.

G. Relativistic wave equations

Let us consider a problem of the classification of the scalar functions on the Poincaré group. The classification can be based on the use of the operators $\hat{C}_k$, commuting with $T_L(g)$ (and correspondingly with all left generators). For these operators as a consequence of a relation $\hat{C}f(x,z) = cf(x,z)$ one can obtain $\hat{C}f'(x,z) = cf'(x,z)$, where $f'(x,z) = T_L(g)f(x,z)$. Therefore, different eigenvalues c correspond to subspaces, which are invariant with respect to action of $T_L(g)$. The invariant subspaces correspond to subrepresentations of the left GRR.

For the classification in addition to the Casimir operators one may use the right generators, since all right generators commute with all left generators. The right generators, as was mentioned, differ equivalent representations in the decomposition of the left GRR.

There is some freedom to choose the commuting operators, which are functions of the right generators of the Poincaré group. We will use only functions of the generators of the right rotations (2.39), in particular, for the coordination with standard formulation of theory.

Following the general scheme of harmonic analysis, for $M(D,1)$ one may consider the system, consisting of d equations

$$\hat{C}_k f(x,z) = c_k f(x,z), \tag{2.65}$$

where $\hat{C}_k$ are Casimir operators of the Poincaré group and of the spin Lorentz subgroup. Just the system we will use below for $d = 2+1$.

On the other hand, for relativistic equations there exist two general requirements, connected with physical interpretation:
1. Invariance under space reflection.
2. Presence of the first order equation on $\partial/\partial t$.[††]

A Casimir operator linear on $\hat{p}_\mu$ exist in odd dimensions. As we will see below, the system (2.65) in 2+1 dimensions is invariant under space reflections.

Casimir operators of the Poincaré group are constructed by generators $\hat{p}_\mu$ and $\hat{J}_{\mu\nu}$. In even dimensions the invariant tensor $\varepsilon^{\mu\ldots\nu}$ also have even number of indices and therefore linear on $\hat{p}_\mu$ Casimir operator does not exist. Besides, in even dimensions IR of proper Poincaré group under space reflection converts to equivalent IR, labelled by another eigenvalues of Casimir operators of spin Lorentz subgroup. These two representations form IR of improper Poincaré group.

Nevertheless, in odd dimensions there exist operator $\hat{C}' = \hat{p}_\mu\hat{\Gamma}^\mu$, $\hat{\Gamma}^\mu = \hat{\Gamma}^\mu(z,\partial/\partial z)$, commuting with all left generators and connecting the states, which mapping one to another under space reflections. In contrast to Casimir operators this operator is not a function of generators of Poincaré group and does not commute with some right generators. A first order equation

$$\hat{p}_\mu\hat{\Gamma}^\mu f(x,z) = \varkappa f(x,z) \tag{2.66}$$

connect at least two IRs of the group $M(D,1)$, differed by eigenvalues of Casimir operator of spin Lorentz subgroup. The equations (2.65) and (2.65) have the same form; namely, invariant operator acts on the scalar function $f(x,z)$ on the group $M(D,1)$. Note, that the addition of the operators $\hat{\Gamma}^\mu$ in fact means the transition from the Lorentz group to

---

[††]As a consequence of relativistic invariance, a linear on $\partial/\partial t$ equation can be either first order or infinite order on space derivatives (square-root Klein-Gordon equation [20–23]). The latter type of equations are naturally obtained in the theory of Markov processes for probability amplitudes [24].

more wide group (in particular, in four dimensions to the de Sitter group $SO(3,2)$). The equation (2.66) replaces the equations of the system (2.65), which are not invariant under space reflection.

In approach under consideration equations for all spins have the same form. The separation of the components with fixed spin and mass is realized by fixing eigenvalues of the Casimir operators of the Poincaré group (or operator $\hat{p}_\mu\hat{\Gamma}^\mu$). Fixing the representation of the Lorentz group, under which transforming $\phi(z)$ in the decomposition

$$f(x,z) = \phi^n(z)\psi_n(x)$$

one can obtain relativistic wave equations in standard multicomponent form. This fixation is realized by the Casimir operator of spin Lorentz subgroup.

There are two types of equations to describe one and the same spin, one on functions $f(x,z)$, where $\phi^n(z)$ transforms under finite-dimensional nonunitary IR of the Lorentz group and another on functions $f(x,z)$, where $\phi^n(z)$ transforms under infinite-dimensional unitary IR of the Lorentz group. In matrix representation these equations are written in the form of finite-component or infinite-component equations correspondingly. The latter type of equations (for example, Majorana equations [25–27]) is interesting because it give the possibility to combine the relativistic invariance and probability amplitudes. Desirability of this combination was emphasized in [28].

Let us briefly consider the possibility of existence of particles with fractional spin. The restrictions on the spin value in the representation theory of $M(D)$ and $M(D,1)$ arise, if one restrict the consideration by (1) unitary, (2) finite-dimensional (on spin), or (3) single-valued representations. (The latter means that the representation acts in the space of single-valued functions.) The restriction by single-valued functions (although often supposed in mathematical papers connected with classification of IR) is omitted in some physical problems, that allow to consider particles with fractional spin (anyons). Thus, we will consider also multi-valued representations of $M(D)$ and $M(D,1)$ in the space of the functions $f(x,z)$ on the group.

III. TWO DIMENSIONAL CASE

A. Field on the group $M(2)$

In two dimensional case the general formulas are simplified. Matrices U (2.17) of $SO(2)$ subgroup depend on only one parameter, namely an angle β, $0 \leq \beta \leq 4\pi$. Using the correspondence $g_0 \leftrightarrow (X, U(\alpha/2))$, $g \leftrightarrow (A, U(\beta/2))$, one may write the action of GRR,

$$T_L(g)f(x,\alpha/2) = f(x', \alpha/2 - \beta/2), \tag{3.1}$$

$$x'_1 = (x_1 - a_1)\cos\beta + (x_2 - a_2)\sin\beta, \quad x'_2 = (x_2 - a_2)\cos\beta - (x_1 - a_1)\sin\beta,$$

$$T_R(g)f(x,\alpha/2) = f(x'', \alpha/2 + \beta/2), \tag{3.2}$$

$$x''_1 = x_1 + a_1\cos\alpha - a_2\sin\alpha, \quad x''_2 = x_2 + a_2\cos\alpha + a_1\sin\alpha.$$

Left and right generators, which correspond to parameters α and β, is given by

$$\hat{p}_k = -i\partial_k, \quad \hat{J} = \hat{L} + \hat{S}, \tag{3.3}$$
$$\hat{p}_k^R = i\Lambda_k{}^i\partial_i, \quad \hat{J}^R = -\hat{S}, \tag{3.4}$$

where

$$\hat{L} = i(x_1\partial_2 - x_2\partial_1) = -i\frac{\partial}{\partial\varphi}, \quad \hat{S} = -i\frac{\partial}{\partial\alpha}, \quad \Lambda = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

The functions on the group are the functions on $\mathbb{R}^2 \times S^1$ and invariant measure on the group may be written as

$$d\mu(x,\alpha) = (4\pi)^{-1}dx_1dx_2d\alpha, \quad -\infty < x < +\infty, \quad 0 \leq \alpha < 4\pi.$$

We will consider two full sets of commuting operators, $\hat{p}_1$, $\hat{p}_2$, $\hat{S}$ and $\hat{p}^2$, $\hat{J}$, $\hat{S}$. The eigenfunctions of these operators are

$$\langle x_1x_2\alpha|p_1\,p_2\,s\rangle = (2\pi)^{-1}\exp(ip_1x_1 + ip_2x_2 + is\alpha), \tag{3.5}$$
$$\langle r\varphi\alpha|p\,j\,s\rangle = (2\pi)^{-1/2}i^l J_l(pr)\exp(il\varphi)\exp(is\alpha), \tag{3.6}$$

where $l = j - s$ is orbital momentum, $J_l(pr)$ is the Bessel function. IR are labelled by eigenvalues p^2 of the Casimir operator $\hat{p}^2$. At $p \neq 0$ the representation is irreducible, at $p = 0$ split into one-dimensional IR of spin subgroup $U(1)$, which are labelled by eigenvalues s of the spin projection operator (or, simply speaking, spin operator) $\hat{S}$.

At $p \neq 0$ the representations characterized by the spin s and $s' = s + n$, where n is integer number, are equivalent. Really, operator $\hat{S}$ commute with all left generators, but do not commute with the generators of right translations, which mix spin and space coordinates. Operators $\hat{p}_+^R = p_1^R - ip_2^R$ and $\hat{p}_-^R = p_1^R + ip_2^R$ are raising and lowering operators with respect to spin s,

$$\hat{p}_\pm^R|p_1\,p_2\,s\rangle = (ip_1 \pm p_2)|p_1\,p_2\,s \pm 1\rangle. \tag{3.7}$$

Right translations do not conserve both interval (distance) and spin s.

The functions (3.6) satisfy the relations of orthogonality and completeness

$$\int \langle p\,j\,s|r\varphi\alpha\rangle\langle r\varphi\alpha|p\,j\,s\rangle r dr d\varphi d\alpha = \frac{\delta(p-p')}{p}\delta_{jj'}\delta_{ss'}, \tag{3.8}$$
$$\int \sum_{l,s}\langle r\varphi\alpha|p\,j\,s\rangle\langle p\,j\,s|r\varphi\alpha\rangle dp = \frac{\delta(r-r')}{r}\delta(\varphi-\varphi')\delta(\alpha-\alpha'). \tag{3.9}$$

Therefore, we obtained the decomposition of left regular representation. Spin operator $\hat{S}$ differs equivalent IR (except the case $p = 0$, when IR are labelled by its eigenvalues). The decomposition of the functions of α on the eigenfunctions of $\hat{S}$ correspond to the Fourier series expansion of functions on a circle.

Thus, the representations of $M(2)$ are single-valued at integer and half-integer s. The fractional values of s correspond to multi-valued representations. IR are equivalent, if are labelled by the same $p \neq 0$ and the difference $s - s' = n$ is an integer number. At fixed $p \neq 0$ there are only two nonequivalent single-valued representations, which correspond to integer and half-integer spin. Nonequivalent multi-valued representations at fixed $p \neq 0$ are labelled by $\tilde{s} \in [0,1)$, $\tilde{s} = s - [s]$.

B. Field on the group $M(1,1)$

Matrices U (2.16) of $SO(1,1)$ subgroup, which is isomorphic to an additive group of real numbers, depend on a hyperbolic angle β. Using the correspondence $g_0 \leftrightarrow (X, U(\alpha/2))$, $g \leftrightarrow (A, U(\beta/2))$, one may write the action of GRR,

$$T_L(g)f(x,\alpha/2) = f(x', \alpha/2 - \beta/2), \tag{3.10}$$
$$x'_0 = (x_0 - a_0)\cosh\beta + (x_1 - a_1)\sinh\beta, \quad x'_1 = (x_1 - a_0)\cosh\beta + (x_0 - a_0)\sinh\beta,$$
$$T_R(g)f(x,\alpha/2) = f(x'', \alpha/2 + \beta/2), \tag{3.11}$$
$$x''_0 = x_0 + a_0\cosh\alpha - a_1\sinh\alpha, \quad x''_1 = x_1 + a_1\cosh\alpha - a_0\sinh\alpha.$$

The functions on the group are functions on $\mathbb{R}^2 \times \mathbb{R}$ and invariant measure on the group may be written as

$$d\mu(x,\alpha) = dx_1 dx_2 d\alpha, \quad -\infty < x, \alpha < +\infty,$$

As above, we will consider two full sets of commuting operators, $\hat{p}_1$, $\hat{p}_2$, $\hat{S}$ and $\hat{p}^2$, $\hat{J}$, $\hat{S}$. The eigenfunctions of the first set are

$$\langle x_1 x_2 \alpha | p_1\, p_2\, \lambda\rangle = (2\pi)^{-3/2}\exp(ip_\mu x^\mu + i\lambda\alpha), \tag{3.12}$$

where λ is an eigenvalue of the spin projection (chirality) operator $\hat{S}$. The form of eigenfunctions of the second set depend on the type of IR. There are four types of unitary IR, labelled by eigenvalue m^2 of operator $\hat{p}^2$ [29].

1. $m^2 > 0$. Representations correspond to the particles of nonzero mass, the eigenfunctions of operators $\hat{p}^2$, $\hat{J}$, $\hat{S}$ are

$$\langle r\varphi\alpha | m\, j\, \lambda\rangle = (4\pi)^{-1} i \exp(\pi l/2) H^{(2)}_{il}(mr)\exp(il\varphi)\exp(i\lambda\alpha), \tag{3.13}$$

where $H^{(2)}_{il}(mr)$ is Hankel function, $r^2 = (x^0)^2 - (x^1)^2$.

2. $m^2 < 0$. Representations correspond to tachyons, which in $d = 1+1$ are more similar to usual particles because of symmetry of space and time variables. The form of $\langle r\varphi\alpha | m\, j\, \lambda\rangle$ coincide with (3.13), but m is imaginary.

3. $m = 0$, $p_1 = \pm p_0$. Representations correspond to the massless particles. According to (2.36), one may obtain for the action of finite transformations $T_0(g)$ on the functions $f(p, \pm p, \alpha/2)$

$$T_0(g)f(p, \pm p, \alpha/2) = e^{iap'} f(p', \pm p', \alpha/2 - \beta/2), \quad p' = e^{\mp\beta}p.$$

Therefore the representation $T_0(g)$ is reducible and split into four IR, which correspond to sign of energy ($p_0 > 0$ or $p_0 < 0$) at $p_1 = \pm p_0$, and reducible representation, which corresponds to $m = p_0 = 0$.

4. $m = p_0 = 0$. This representation split into one-dimensional IR of group $SO(1,1)$, which are labelled by eigenvalues of $\hat{S}$.

There are no integer value restrictions for the spectrum of $\hat{S}$ and chirality can be fractional, $-\infty < \lambda < +\infty$. The decomposition of the functions of α on the eigenfunctions of $\hat{S}$ correspond to the Fourier integral expansion of functions on a line. The equivalence of the representations characterized by different λ is connected with the fact, that as in Euclidean case, operator $\hat{S}$ does not commute with right translations.

C. Relativistic wave equations in 1+1 dimensions

A IR of the group $M(1,1)$ may be extract from GRR by means of equations

$$\hat{p}^2 f(x,\alpha) = m^2 f(x,\alpha), \tag{3.14}$$
$$\hat{S} f(x,\alpha) = \lambda f(x,\alpha), \tag{3.15}$$

where chirality λ differs equivalent IR, labelled by identical eigenvalues m^2 of the Casimir operator $\hat{p}^2$. Solutions of this system have the form $f(x,\alpha) = \psi(x)e^{i\lambda\alpha}$, where $\hat{p}^2\psi(x) = m^2\psi(x)$.

According to (2.59), parity transformation converts $e^{i\lambda\alpha}$ to $e^{-i\lambda\alpha}$ and therefore combines two equivalent representations of the proper Poincaré group $M(1,1)$, characterized by chiralities λ and $-\lambda$, into one IR of the improper Poincaré group, characterized by spin $s = |\lambda|$. Thus, if the symmetry with respect to parity transformation takes place, it is necessary to transit to the equations, which, in contrast to (3.15), combines states with chiralities $\pm\lambda$.

The general form of the linear on $\hat{p}^\mu$ equations is

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,\alpha) = \varkappa f(x,\alpha), \tag{3.16}$$

where $\hat{\Gamma}^\mu = \hat{\Gamma}^\mu(\alpha, \partial/\partial\alpha)$. It is necessary for invariance of (3.16) under the parity transformation and the hyperbolic rotations respectively

$$\hat{\Gamma}^\mu \xrightarrow{P} (-1)^{\delta_{1\mu}}\hat{\Gamma}^\mu,$$
$$[\hat{\Gamma}^0, \hat{S}] = \hat{\Gamma}^1, \qquad [\hat{\Gamma}^1, \hat{S}] = \hat{\Gamma}^0. \tag{3.17}$$

The operators

$$\hat{\Gamma}^0 = s\cos\alpha - \sin\alpha\,\frac{\partial}{\partial\alpha}, \quad \hat{\Gamma}^1 = is\sin\alpha + i\cos\alpha\,\frac{\partial}{\partial\alpha}, \quad [\hat{\Gamma}^0, \hat{\Gamma}^1] = -\hat{S}, \tag{3.18}$$

obey these relations. One may construct the operators, which raise and lower chirality λ by 1,

$$\hat{\Gamma}_+ = \hat{\Gamma}^0 + \hat{\Gamma}^1 = e^{i\alpha}(s + i\frac{\partial}{\partial\alpha}), \quad \hat{\Gamma}_- = \hat{\Gamma}^0 - \hat{\Gamma}^1 = e^{-i\alpha}(s - i\frac{\partial}{\partial\alpha}). \tag{3.19}$$

Operators $\hat{\Gamma}^0$, $\hat{\Gamma}^1$ and $\hat{\Gamma}^2 = -i\hat{S} = -\partial/\partial\alpha$ obey the commutation relations of the generators of the $SO(2,1) \sim SU(1,1)$ group,

$$[\hat{\Gamma}^a, \hat{\Gamma}^b] = -i\epsilon^{abc}\hat{\Gamma}_c, \quad \hat{\Gamma}_a = \eta_{ab}\hat{\Gamma}^b, \quad \eta_{00} = \eta_{22} = -\eta_{11} = 1, \quad \hat{\Gamma}_a\hat{\Gamma}^a = s(s+1).$$

Thus, in the presence of symmetry with respect to the parity transformation the condition of mass irreducibility (3.14) may be supplement by the equation (3.16) instead of (3.15). Let us consider the system

$$\hat{p}^2 f(x,\alpha) = m^2 f(x,\alpha), \tag{3.20}$$
$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,\alpha) = ms f(x,\alpha). \tag{3.21}$$

The operator $\hat{S}$ does not commute with $\hat{p}_\mu \hat{\Gamma}^\mu$, and the particle with nonzero mass, described by equation (3.21), can't be characterized by certain chirality. In the rest frame a general solution of (3.20)-(3.21) is

$$f(x, \alpha) = C_1 e^{imx^0} [\cos(\alpha/2)]^{2s} + C_2 e^{-imx^0} [\sin(\alpha/2)]^{2s}. \tag{3.22}$$

Therefore, for a fixed spin s there are only two independent components with positive and negative frequency. Plane wave solutions, which correspond to moving particle, may be obtain from (3.22) by a hyperbolic rotation at the angle 2β,

$$C_1(e^{i\alpha+\beta} + e^{-i\alpha-\beta})^{2s} e^{ik_0x^0+ik_1x^1} + C_2(e^{i\alpha+\beta} - e^{-i\alpha-\beta})^{2s} e^{-ik_0x^0-ik_1x^1}$$

where $k_0 = m\cosh 2\beta$, $k_1 = m\sinh 2\beta$. In the ultrarelativistic limit $\beta \to \pm\infty$ we have two states with chirality $\lambda = \pm s$ respectively. Thus, if in the rest frame one may differ two components with positive and negative frequency, then in massless limit one may differ two components with positive and negative chirality.

Matrix form of the system (3.20)-(3.21) can be obtained by the decomposition of $f(x, \alpha)$ on the basis $e^{\lambda\alpha/2}$, $\lambda = -s, -s+1, \ldots, s$. There are $2s+1$ components $\psi(x)$ in this form, but only two of them are independent.

At $s = 1/2$, substituting the function $f(x, \alpha) = \psi_1(x)e^{i\alpha/2} + \psi_2(x)e^{-i\alpha/2}$ into equation (3.21), we obtain two-dimensional Dirac equation for $\Psi(x) = (\psi_1(x)\ \psi_2(x))^T$,

$$\hat{p}_\mu \gamma^\mu \Psi(x) = m\Psi(x), \qquad \gamma^0 = \sigma_1, \quad \gamma^1 = i\sigma_2, \quad 2\hat{S} = \gamma^3 = \sigma_3. \tag{3.23}$$

Matrix γ^3 corresponds to chirality operator and satisfies the conditions $\gamma^3 = -i\gamma^0\gamma^1$, $\{\gamma^3, \gamma^\mu\}_+ = 0$.

At $s = 1$, substituting the function $f(x, \alpha) = \psi_{11}(x)e^{i\alpha} + \psi_{12}(x) + \psi_{22}(x)e^{-i\alpha}$ into equation (3.21), we obtain

$$(\hat{p}_\mu \Gamma^\mu - m)\Psi(x) = 0,$$
$$\Gamma^0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Gamma^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \hat{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.24}$$

where $\Psi(x) = (\psi_{11}(x)\ \psi_{12}(x)/\sqrt{2}\ \psi_{22}(x))^T$. If at $s = 1/2$ and at $s = 1$ the first equation of the system (3.20)-(3.21) is the consequence of the second equation, then at $s > 1$ there are the solutions of the equation (3.21) with mass spectrum, $m_i|s_i| = ms$, $s_i = s, s-1, \ldots, -s$. For the extraction of IR of improper Poincaré group, characterized by certain mass m and spin s, it is necessary consider both equations of the system.

Note, that the chirality λ of a particle, described by (3.14)-(3.15), can be fractional, but the spin s of a particle, described by (3.20)-(3.21), at $m \neq 0$ and finite number of components $\psi(x)$ can be only integer or half-integer.

Really, if $2s$ is not integer, then acting by the raising operator on the state with label $\lambda = -s$, we not get into the state with label $\lambda = s$, which connected with initial state by the parity transformation; moreover, the spectrum of λ is not bounded above.

On the other hand it is possible to develop an alternative approach (in particular, for the particles with fractional spin), based on the using of infinite-dimensional unitary IR of $SO(2,1)$. That approach we will consider below in 2+1-dimensional case.

IV. THREE DIMENSIONAL CASE

A. Field on the group $M(3)$

The case of $M(3)$ group is characterized by many-dimensional spin space. On the other hand, the constructions allow the simple physical interpretation.

Using the operators $\hat{J}^i = \hat{L}^i + \hat{S}^i = (1/2)\epsilon^{ijk}\hat{J}_{jk}$, it is possible to rewrite the commutation relations (2.34) in the more compact form,

$$[\hat{p}_i, \hat{p}_k] = 0, \quad [\hat{p}^i, \hat{J}^j] = -i\epsilon^{ijk}\hat{p}_k, \quad [\hat{J}^i, \hat{J}^j] = -i\epsilon^{ijk}\hat{J}_k \,. \tag{4.1}$$

The invariant measure on the group is given by the formulas

$$d\mu(x,z) = Cd^3x\delta(|z_1|^2 + |z_2|^2 - 1)d^2z_1 d^2z_2 = \frac{1}{16\pi^2}d^3x \sin\theta d\theta d\phi d\psi. \tag{4.2}$$
$$-\infty < x < +\infty, \quad 0 < \theta < \pi, \quad 0 < \phi < 2\pi, \quad -2\pi < \psi < 2\pi,$$

where $z_1 = \cos\frac{\theta}{2}e^{i\phi/2+i\psi/2}$, $z_2 = i\sin\frac{\theta}{2}e^{-i\phi/2+i\psi/2}$ are the elements of the first column of matrix (2.43), $z^2 = -z_1$, $z^1 = z_2$, and θ, ϕ, ψ are the Euler angles. The spin projection operators, acting in the space of the functions on the group $f(x,z)$, have the form

$$\hat{S}_k = \frac{1}{2}(z\sigma_k\partial_z - \overset{*}{z}\overset{*}{\sigma}_k\partial_{\overset{*}{z}}), \quad z = (z_1\ z_2),\ \partial_z = (\partial/\partial z_1\ \partial/\partial z_2)^T,$$
$$\hat{S}^R_k = -\frac{1}{2}(\chi\overset{*}{\sigma}_k\partial_\chi - \overset{*}{\chi}\sigma_k\partial_{\overset{*}{\chi}}), \quad \chi = (z_1\ -\overset{*}{z}_2),\ \partial_\chi = (\partial/\partial z_1\ -\partial/\partial\overset{*}{z}_2)^T. \tag{4.3}$$

In the parametrization by the Euler angles one may obtain

$$\hat{S}_3 = -i\partial/\partial\phi, \quad \hat{S}^R_3 = i\partial/\partial\psi. \tag{4.4}$$

The operator $\hat{\mathbf{p}}^2$ and the operator of the spin projection on the direction of propagation $\hat{W} = \hat{\mathbf{p}}\hat{\mathbf{J}} = \hat{\mathbf{p}}\hat{\mathbf{S}}$ are the Casimir operators. The eigenvalues $S(S+1)$ of the Casimir operator of rotation subgroup in z-space $\hat{\mathbf{S}}^2 = \hat{\mathbf{S}}^2_R$ define spin S. Full sets of the commuting operators $\{\hat{p}_k, \hat{W}, \hat{\mathbf{S}}^2, \hat{S}^3_R\}$, $\{\hat{\mathbf{p}}^2, \hat{W}, \hat{\mathbf{J}}^2, \hat{S}_3, \hat{\mathbf{S}}^2, \hat{S}^R_3\}$ consist of six operators (two Casimir operators, two left generators and two right generators). The Casimir operator $\hat{W}$ does not commutes with $\hat{L}_k$ and $\hat{S}_k$ separately, but only with the generators $\hat{J}_k = \hat{L}_k + \hat{S}_k$, therefore there are sets, which do not include $\hat{W}$, for example, $\{\hat{\mathbf{p}}^2, \hat{p}_3, \hat{L}_3, \hat{S}_3, \hat{\mathbf{S}}^2, \hat{S}^R_3\}$ and $\{\hat{p}_\mu, \hat{S}_3, \hat{\mathbf{S}}^2, \hat{S}^R_3\}$.

We will consider the first set, since in this case eigenfunctions have the most simple form. This set includes two Casimir operators, the operator of spin square $\hat{\mathbf{S}}^2$ and the generator $\hat{S}^R_3$. The latter two generators commute with all left generators, but do not commute with right generators and label equivalent representations in the decomposition of the left GRR.

According to (4.4), the eigenfunctions of $\hat{S}^R_3$, $\hat{S}^R_3|\ldots n\rangle = n|\ldots n\rangle$, has the form $|\ldots n\rangle = F(x,\theta,\phi)\exp(-in\psi)$ and are differed only by a phase factor. As a consequence of the commutation relations of generators $\hat{S}^R_k$ the operators $\hat{S}^R_\pm = \hat{S}^R_1 \pm i\hat{S}^R_2$ are the raising and lowering operators for the eigenfunctions of $\hat{S}^R_3$,

$$\hat{S}^R_\pm|\ldots n\rangle = C(S,n)|\ldots n \pm 1\rangle. \tag{4.5}$$

The intertwining operators $\hat{S}^R_\pm$ consist of the generators of right rotations, which conserve the interval square, according to (2.31). Moreover, the right rotations do not act on x.

In spite of the fact that there is no transformations (rotations and translations) of the frame of references, which connect the representations with different n, this give some reason to consider only one of the equivalent representations, labelled by $\hat{S}^R_3$.

The operator $\hat{\mathbf{S}}^2$ also labels equivalent representations of $M(3)$ group. This operator commute with all generators except right translations and therefore an intertwining operator is a function of the latter generators. Right translations change both the interval and spin. Therefore it is naturally to characterize free particle in three dimensional Euclidean space not only by momentum and spin projection on the direction of propagation, but also by spin S.

There are two standard realizations of the representation spaces, which correspond to eigenvalues $n = \pm 2S$ and $n = 0$ of the operator $\hat{S}^R_3$.

The first realization is the space of analytical ($n = -2S$) or antianalytical ($n = 2S$) functions $f(x, z)$ of two complex variables z_1, z_2, $|z_1|^2 + |z_2|^2 = 1$, i.e. the space of functions of two-component spinors. In particular, according to (4.3), for the space of analytical functions $\hat{S}^R_3 = -(z_1\partial/\partial z_1 + z_2\partial/\partial z_2)$,

$$\hat{S}_k = \frac{1}{2} z\sigma_k\partial_z, \tag{4.6}$$

and $\hat{\mathbf{S}}^2 = \hat{S}^R_3(\hat{S}^R_3 - 1)$. The eigenfunctions of the operator of spin square are polynomials of the power $2S$ in z_1, z_2. The charge conjugation transformation connects equivalent IR labeled by $n = \pm 2S$ and the spaces of analytical and antianalytical function. This transformation reverses the direction of momentum and spin.

The second realization is the space of functions, which do not depend on the angle ψ, and corresponds to $n = 0$. That is the space of functions of five real variables on the manifold

$$\mathbb{R}^3 \times S^2, \qquad d\mu = (4\pi)^{-1} d^3x \sin\theta d\theta d\phi.$$

The point in the spin space (i.e. on the sphere $S^2 \sim \mathbb{C}P^1 \sim SU(2)/U(1)$) can be define by the spherical coordinates θ, ϕ, or by two complex variables $z_1 = \cos\frac{\theta}{2} e^{i\phi/2}$, $z_2 = \sin\frac{\theta}{2} e^{-i\phi/2}$ (in this case one may use (4.6) for the spin projection operators) or by one complex number $z = z_1/z_2$ (this case corresponds to the realization in terms of projective space $\mathbb{C}P^1$). In terms of variables θ, ϕ the eigenfunctions of operators $\hat{S}$, $\hat{S}_3$ are $P^s_S(\cos\theta)e^{-is\phi}$, where $P^s_S(\cos\theta)$ are associated Legendre functions [2].

Let us consider eigenfunctions of the set of the operators $\{\hat{p}_\mu, \hat{W}, \hat{\mathbf{S}}^2\}$ in the space of analytical functions of z_1, z_2,

$$\hat{p}_\mu f(x, z) = p_\mu f(x, z), \quad \hat{\mathbf{S}}^2 f(x, z) = S(S+1) f(x, z), \quad \hat{\mathbf{p}}\hat{\mathbf{S}} f(x, z) = ps f(x, z). \tag{4.7}$$

The eigenfunctions of $\hat{\mathbf{S}}^2$ are polynomials of the power $2S$ on z (the unitary IR of $SU(2)$ are finite-dimensional, therefore spin S and spin projection on the direction of propagation s are integer or half-integer). Let $p_\mu = (0, 0, p)$, then the normalized solutions of the system (4.7) are

$$|0\,0\,p\,S\;s\rangle = (2\pi)^{-3/2} \left(\frac{(2S)!}{(S+s)!(S-s)!} \right)^{1/2} (z_1)^{S+s} (z_2)^{S-s} e^{ix_3 p}.$$

The states with arbitrary direction of vector p may be obtain by the rotation $P = UP_0U^\dagger$, $Z = UZ_0$, $P_0 = p\sigma^3$, $Z_0 = (z_1\ z_2)^T$,

$$|p_1\, p_2\, p_3\, S\, s\rangle = (2\pi)^{-3/2}\left(\frac{(2S)!}{(S+s)!(S-s)!}\right)^{1/2}(z_1u^1 + z_2u^2)^{S+s}(-z_1\overset{*}{u}{}^2 + z_2\overset{*}{u}{}^1)^{S-s}e^{ipx}, \quad (4.8)$$

where u^1, u^2 are the elements of the first line of matrix U. Note, that for the parametrization of matrix U it is sufficient to use only two angles, since the initial state has a stationary subgroup $U(1)$.

For the rest particle $\hat{\mathbf{p}}^2 = \hat{\mathbf{p}}\hat{\mathbf{S}} = 0$ and only in this case IR of $M(3)$, labelled by different S, are nonequivalent.

In general case the function, corresponding to the particle of spin S, has the form

$$f_S(x,z) = \sum_{n=0}^{2S}\phi_n(z)\psi^n(x), \quad \phi_n(z) = (C_{2S}^n)^{1/2}(z_1)^{S-n}(z_2)^n, \quad (4.9)$$

$$\int \overset{*}{f}_S(x,z)f'_{S'}(x,z)d\mu(x,z) = \delta_{SS'}\int\sum_{n=0}^{2S}\overset{*}{\psi}{}^n(x)\psi'^n(x)d^3x. \quad (4.10)$$

where C_n^{2S} is the binomial coefficient and $d\mu(x,z)$ is the invariant measure (4.2). The relation (4.9) gives the connection between the description by the functions $f(x,z)$ and the standard description by the multicomponent functions $\psi^n(x)$. It is easy to see, that the action of the operators $\hat{S}_k = \frac{1}{2}z\sigma_k\partial_z$ on the function (4.9) reduces to the multiplication of the column $\psi(x)$ by $(2S+1)\times(2S+1)$ matrices S_k of $SU(2)$ generators in the representation T_S, $\hat{S}_kf(x,z) = \phi(z)S_k\psi(x)$. Matrices S_k obey the commutation relations of spin projection operators, $[S^i, S^j] = \epsilon^{ijk}S_k$.

In particular, the linear function of z_1,z_2 correspond to spin $S = 1/2$, and the action of the operators $\hat{S}_k$ on $\psi(x)$ is reduced to the multiplication by σ-matrices.

The operator $\hat{\mathbf{S}}^2$, as was mentioned above, is not a Casimir operator of $M(3)$, and labels equivalent representations of the group. This operator is the direct analog of the Lorentz spin operator in pseudoeuclidean case and we will consider its properties in detail.

1. Operator $\hat{\mathbf{S}}^2$ is composed of right generators and commute with all left generators, therefore is not changed under the coordinate transformation (left transformations of the Euclidean group). The right transformations do not change the spin projection s on the direction of propagation, but change both spin S and interval (distance).

2. Operator $\hat{\mathbf{S}}^2$ does not depend on x and commutes with operators $x_k, \hat{p}_k, \hat{S}_k$, therefore in the presence of interactions is conserved for any Hamiltonian $\hat{H} = \hat{H}(x_k, \hat{p}_k, \hat{S}_k)$.

3. The eigenvalues of $\hat{\mathbf{S}}^2$ label IR of the rotation subgroup in the spin space and define the possible values of the spin projection s, which can arise under the interactions.

Note, that in the representation theory of Galilei group (symmetry group of nonrelativistic mechanics, which includes $M(3)$ and ensures more general description) IR, labeled by different eigenvalues of $\hat{\mathbf{S}}^2$, are not equivalent. The classification of IR of Galilei group can be based on the use of two invariant equations. The Schrödinger equation fixes the mass m, and the second equation fixes the eigenvalue of spin operator $\hat{\mathbf{S}}^2$ [30,31].

B. Field on the group $M(2,1)$ and fractional spin

Using the operators $\hat{J}^\rho = \hat{L}^\rho + \hat{S}^\rho = (1/2)\epsilon^{\rho\mu\nu}\hat{J}_{\mu\nu}$, it is possible to rewrite the commutation relations (2.34) in the next form:

$$[\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{p}^\mu, \hat{J}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{p}_\eta, \quad [\hat{J}^\mu, \hat{J}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{J}_\eta \,. \tag{4.11}$$

The invariant measure on the group is given by the formulas [2]

$$d\mu(x,z) = d\mu(z)d^3x = Cd^3x\delta(|z_1|^2 - |z_2|^2 - 1)d^2z_1d^2z_2 = \frac{1}{8\pi^2}d^3x \sinh\theta d\theta d\phi d\psi. \tag{4.12}$$
$$-\infty < x < +\infty, \quad 0 < \theta < \infty, \quad 0 < \phi < 2\pi, \quad -2\pi < \psi < 2\pi,$$

where $z_1 = \cosh\frac{\theta}{2}e^{i\phi/2+i\psi/2}$, $z_2 = \sinh\frac{\theta}{2}e^{-i\phi/2+i\psi/2}$ are the elements of the first column of matrix Z (2.43), and θ, ϕ, ψ are the analogs of Euler angles. The spin projection operators, acting in the space of the functions on the group $f(x,z)$, have the form

$$\hat{S}^\mu = \frac{1}{2}(z\gamma^\mu\partial_z - \overset{*}{z}\overset{*}{\gamma}{}^\mu\partial_{\overset{*}{z}}), \quad z = (z_1\ z_2),\ \partial_z = (\partial/\partial z_1\ \partial/\partial z_2)^T,$$
$$\hat{S}^\mu_R = -\frac{1}{2}(\chi\overset{*}{\gamma}_\mu\partial_\chi - \overset{*}{\chi}\gamma_\mu\partial_{\overset{*}{\chi}}), \quad \chi = (z_1\ \overset{*}{z}_{\dot{2}}),\ \partial_\chi = (\partial/\partial z_1\ \partial/\partial\overset{*}{z}_{\dot{2}})^T. \tag{4.13}$$

where γ^μ are three-dimensional γ-matrices,

$$\gamma^\mu = (\sigma_3, i\sigma_2, -i\sigma_1), \qquad \gamma^\mu\gamma^\nu = \eta^{\mu\nu} - i\varepsilon^{\mu\nu\rho}\gamma_\rho. \tag{4.14}$$

In the parametrization by the Euler angles one may obtain $\hat{S}^0 = -i\partial/\partial\phi$, $\hat{S}^0_R = i\partial/\partial\psi$. The sets of commuting operators are the same as in Euclidean case.

One may show, that in consequence of the identity $\sigma_1\overset{*}{U}\sigma_1 = U$ matrix σ_1 is the invariant symmetrical tensor, which converting dotted and undotted indices, $\overset{*}{z}_\alpha = (\sigma_1)_\alpha{}^{\dot{\alpha}}\overset{*}{z}_{\dot{\alpha}}$.

According to (2.44), the invariant tensor $\sigma_{\mu\alpha\dot{\alpha}}$ connect vector index and two spinor indices of different types. On the other hand, using the identity, mentioned above, one may rewrite (2.44) in the form $x'^\nu(\sigma_\mu\sigma_1) = x^\mu U(\sigma_\mu\sigma_1)U^T$. Thus, the invariant tensor, which we denote as

$$\check{\sigma}_{\mu\alpha\beta} = (\sigma_\mu\sigma_1)_{\alpha\beta}, \tag{4.15}$$

connect vector index and two spinor indices of one type. Raising first or second index of $\check{\sigma}_{\mu\alpha\beta}$, one may obtain two sets of three-dimensional γ-matrices, differed only by the sign of γ^0 and γ^2.

Similarly to the Euclidean case, there are two standard realizations of the representation spaces, which correspond to eigenvalues $n = \pm 2S$ and $n = 0$ of the operator $\hat{S}^R_3$.

The first realization is the space of analytical ($n = -2S$) or antianalytical ($n = 2S$) functions $f(x,z)$ of two complex variables z_1, z_2, $z^2 = -z_1$, $z^1 = z_2$, $|z_1|^2 - |z_2|^2 = 1$, i.e. the space of functions of two-component spinors. The eigenfunctions of $\hat{\mathbf{S}}^2$ are homogeneous functions of degree $2S$ in z. According to (4.3), for the space of analytical functions $\hat{S}^0_R = -(z_1\partial/\partial z_1 + z_2\partial/\partial z_2)$, and for the space of antianalytical functions $\hat{S}^0_R = \overset{*}{z}_{\dot{1}}\partial/\partial\overset{*}{z}_{\dot{1}} + z_{\dot{2}}\partial/\partial z_{\dot{2}}$.

The eigenfunctions of $\hat{\mathbf{S}}^2$ in these spaces are also eigenfunctions of $\hat{S}^0_R$ with eigenvalues $n = \mp 2S$ correspondingly.

The second realization is the space of functions, which are not depend on the angle ψ, and corresponds to the eigenfunctions of $\hat{S}^0_R$ with zero eigenvalue. That is the space of functions of five real parameters on the manifold

$$\mathbb{R}^3 \times \mathbb{C}D^1, \qquad d\mu = (2\pi)^{-1} d^3x \sinh\theta d\theta d\phi,$$

where $\mathbb{C}D^1 \sim SU(1,1)/U(1)$ is a complex disk.

Remember some facts from the representation theory of $SU(1,1)$. Spin projection s (the eigenvalue of $\hat{S}^0$) for finite-dimensional nonunitary IR T^0_S of 2+1 Lorentz group $SU(1,1) \sim SO(2,1)$ can be only integer or half-integer, $s = -S, \ldots, S$, $S \geq 0$.

However, 2+1 dimensional Lorentz group has not compact non-Abelian subgroup. Therefore there are infinite-dimensional unitary representations, corresponding to fractional S. These representations are multi-valued representations of $SU(1,1)$. For single-valued representations of $SU(1,1)$ ($SO(2,1)$) the spin projection s can be only integer or half-integer (only integer).

The representations of discrete seria correspond to $S < -1/2$. IR of the positive discrete series T^+_S bounded by lowest weight $s = -S$, IR of the negative discrete series T^-_S bounded by highest weight $s = S$, IR of the principal series correspond to $S = -1/2 + i\lambda$, and can be bounded by highest (lowest) weight only for $S = -1/2$. For other IR of principal series the spectrum of s is not bounded. Supplementary series correspond to $-1/2 < S < 0$ and are characterized by nonlocal scalar product.

A visual picture for weight diagrams of all seria on the plain S, s one can find in [8,32].

Thus, there are only two possibilities for description of a particle with fractional spin by means of unitary IR of $SU(1,1)$ with local scalar product. The first correspond to IR of discrete or principal seria of the Lorentz group, bounded by lowest (highest) weight, $|s| \geq |S| \geq 1/2$. The second correspond to IR of principal series, which is not bounded.

Unitary IR of discrete series are used for the description of anyons [8,18,19].

C. Relativistic wave equations in 2+1 dimensions

Let us consider the system on the eigenvalues of the Casimir operators of Poincare group and spin Lorentz subgroup,

$$\hat{p}^2 f(x,z) = m^2 f(x,z), \tag{4.16}$$

$$\hat{p}\hat{S} f(x,z) = K f(x,z), \tag{4.17}$$

$$\hat{\mathbf{S}}^2 f(x,z) = S(S+1) f(x,z). \tag{4.18}$$

The operator $\hat{\mathbf{S}}^2$ we will call below as operator of the Lorentz spin square.

The equations (4.16),(4.17) define some sub-representation of the left GRR of $M(2,1)$, which is characterized by mass m, Lorentz spin S, and by the eigenvalue K of Lubanski-Pauli operator. At $m = 0$ we suppose $K = 0$, that is true for IR with finite number of spinning degrees of freedom. The general cases $m = 0$ and m imaginary was discussed in [33,8].

Possible values of K can be easily described in the massive case. Here we can use a rest frame, where $\hat{p}_\mu \hat{S}^\mu = \hat{S}^0 m \,\mathrm{sign}\, p_0$. Thus, for particles $K = sm = s^0 m$ and for antiparticles $K = sm = -s^0 m$, where s^0 is the eigenvalue of $\hat{S}^0$. The latter spectrum depends on the representation of the Lorentz group.

Variable s labels IR of the group $M(2,1)$ and can take both positive and negative values. It is the analogy with massless particles in 3+1 dimensions, characterized by helicity. In both cases $SO(2)$ is the little group, and single-valued IR of $SO(2)$ are labelled by integer number $2s$. (It is a particular case of the connection between the massive fields in d dimensions and massless fields in $d+1$ dimensions, see [34,35]). Therefore we will call s as helicity (correspondingly $s \,\mathrm{sign}\, p_0$ as chirality) and $|s|$ as spin.

Corresponding to (2.59) space reflection convert Z to $(Z^\dagger)^{-1} = \sigma^3 Z \sigma^3$, or $z_1 \to z_1$, $z_2 \to -z_2$, and operators $\hat{p}^0$, $\hat{S}^0$ do not changed. Thus, distinct from 3+1-dimensional case, chirality also does not changed under space reflection.

Fixing of S in (4.18) ensures the transition to the space of homogeneous functions of degree $2S$ in z_1, z_2. According to the sign of S, below we consider two possible choices of IR of $SU(1,1)$, bounded on two or one side.

Finite-dimensional nonunitary IR T^0_S of $SU(1,1)$ are labelled by positive integer or half-integer S. The basis in the representation space is formed by the polynomials of power $2S$ in z, see (8.2). Corresponding representations of $M(2,1)$ we denote by $T^0_{m,s}$.

Infinite-dimensional unitary IR T^-_S (T^+_S) of $SU(1,1)$ are labelled by negative $S < -1/2$ and are bounded by highest (lowest) weight. The basis in the representation space is formed by the quasipolynomials of power $2S$ in z, see (8.3). Corresponding representations of $M(2,1)$ we denote by $T^-_{m,s}$ ($T^+_{m,s}$).

One may present a function $f(x,z)$ in the form

$$f(x,z) = \phi(z)\psi(x), \tag{4.19}$$

where $\phi(z)$ is a line, composed of the elements $\phi_n(z)$ of the basis of corresponding IR $SU(1,1)$, and $\psi(x)$ is a column composed of the coefficients in the decomposition on this basis. The action of the differential operators $\hat{S}^\mu$ on a function $f(x,z)$ may be presented in terms of matrices

$$\hat{S}^\mu f(x,z) = \phi_n(z)(S^\mu)^n{}_{n'}\psi^{n'}(x), \tag{4.20}$$

where S^μ are generators of $SU(1,1)$ in the representation T_S, described in Appendix (see also [8]). They obey the commutation relations of the $SU(1,1)$ group,

$$[S^\mu, S^\nu] = -i\epsilon^{\mu\nu\eta} S_\eta.$$

At S fixed and in the matrix representation the equations (4.16),(4.17) have the form

$$(\hat{p}^2 - m^2)\psi(x) = 0, \tag{4.21}$$

$$(\hat{p}_\mu S^\mu - sm)\psi(x) = 0, \tag{4.22}$$

According to (4.22),

$$\psi^\dagger(x)(iS^{\dagger\mu}\overleftarrow{\partial_\mu} + sm) = 0.$$

It follows from the explicit expressions (8.4), that for $T^0_{m,s}$ the relation $S^{\dagger\mu} = \Gamma S^\mu \Gamma$, where $(\Gamma)_{nn'} = (-1)^n \delta_{nn'}$, take place. For $T^+_{m,s}$ and $T^-_{m,s}$ matrices S^μ are Hermitian, $S^{\dagger\mu} = S^\mu$, according to (8.5). Let us introduce the notation

$$\overline{\psi} = \psi^\dagger \Gamma \quad \text{for} \quad T^0_{m,s},$$
$$\overline{\psi} = \psi^\dagger \quad \text{for} \quad T^+_{m,s}, T^-_{m,s}.$$

The function $\overline{\psi}(x)$ obeys the equation

$$\overline{\psi}(x)(iS^\mu \overleftarrow{\partial_\mu} + sm) = 0. \tag{4.23}$$

Therefore $\overline{\psi}(x)\psi(x)$ is a scalar density, and one may define a scalar product

$$(\psi'(x), \psi(x)) = \overline{\psi}(x)\psi(x). \tag{4.24}$$

The scalar density $\overline{\psi}(x)\psi(x)$ is positive definite for $T^+_{m,s}$ and $T^-_{m,s}$ in contrast to the $T^0_{m,s}$.

As a consequence of (4.22) and (4.23), the continuity equation holds

$$\partial_\mu j^\mu = 0, \quad j^\mu = \overline{\psi} S^\mu \psi. \tag{4.25}$$

By analogy with four-dimensional case [26], along with the current vector j^μ, one can connect with the linear equation (4.22) the energy-momentum tensor $T^{\mu\nu}$ and the energy density $W = -T^{00}$,

$$T^{\mu\nu} = \text{Im}\left(S^\mu \frac{\partial\psi}{\partial x^\nu}, \psi\right), \qquad W = -T^{00} = -\text{Im}\left(S^0 \frac{\partial\psi}{\partial x^0}, \psi\right). \tag{4.26}$$

If matrix S^0 is diagonable, then the positiveness of $W(x)$ is equivalent to the requirement that

$$(S^0\psi, S^0\psi) \geq 0. \tag{4.27}$$

for all vectors ψ [26]. In particular, for $T^+_{m,s}$ and $T^-_{m,s}$ the relation $(S^0\psi, S^0\psi) = \psi^\dagger S^0 S^0 \psi \geq 0$ takes place and energy density is positive definite.

There are two cases when the equations (4.21) and (4.22) are dependent. Indeed, multiplying the equation (4.17) by $\hat{p}_\mu S^\mu + ms$ one gets

$$(\hat{p}_\mu S^\mu + ms)(\hat{p}_\mu S^\mu - ms)\psi(x) = \left(\hat{p}_\mu \hat{p}_\nu \{S^\mu,\ S^\nu\} - m^2 s^2\right)\psi(x) = 0. \tag{4.28}$$

In the particular case $S = 1/2$ we have $s = \pm 1/2$, $S^\mu = \gamma^\mu/2$ and (4.28) is merely the Klein-Gordon equation (4.21). In general case the matrices S^μ are not γ-matrices in higher dimensions and the squared equation (4.28) do not coincide with the Klein-Gordon equation (4.21). Using the rest frame, one may show that the equation (4.21) follows from (4.22) also in the case of vector representation $S = 1$, $s = \pm 1$. In the other cases for the isolation of the IR of $M(2,1)$ it is necessary to use both equations of the system (4.21)-(4.22).

It is naturally to connect spin value with the highest (lowest) weight of IR of Lorentz group, $s = \pm S$. That means, that up to a sign (+ for particles, − for antiparticles) s is equal to maximal or minimal eigenvalue of the operator $\hat{S}^0$ in the representation T_S of the Lorentz group. According to (4.16)-(4.18), in this case functions $f(x, z)$ obey the equations

$$\hat{p}^2 f(x,z) = m^2 f(x,z), \tag{4.29}$$

$$\hat{p}\hat{S} f(x,z) = ms f(x,z), \quad s = \pm S, \tag{4.30}$$

$$\hat{\mathbf{S}}^2 f(x,z) = S(S+1) f(x,z). \tag{4.31}$$

In the frame of system (4.29)-(4.31) **there are two possibilities to describe the same spin**:

1. Equations for $f(x, z) = \phi(z)\psi(x)$, where $\phi(z)$ is transformed under finite-dimensional nonunitary IR of the Lorentz group.

2. Equations for $f(x, z) = \phi(z)\psi(x)$, where $\phi(z)$ is transformed under infinite-dimensional unitary IR of the Lorentz group. These equations allow also to describe particles with fractional spin (anyons).

(1) Consider first representations $T^0_{m,s}$ of the Poincaré group, connected with **finite-dimensional non-unitary IR** of $SU(1,1)$. In this case S has to be positive, integer or half-integer. In the rest frame the solutions of the system (4.29)-(4.31) in the space of analytical functions (polynomials of power $2S$ in z^1, z^2) are

$$s = S > 0: \quad f(x,z) = C_1 (z_1)^S e^{imx^0} + C_2 (z_2)^S e^{-imx^0}, \tag{4.32}$$

$$s = -S < 0: \quad f(x,z) = C_3 (z_1)^S e^{-imx^0} + C_4 (z_2)^S e^{imx^0}. \tag{4.33}$$

At certain mass and spin there exist 4 independent components, differed by the signs of p_0 and s. The separation by the sign of helicity s has absolute character since these states are solutions of different equations. But the states with different sign of p_0 are solutions of one and the same equation. Correspondingly, the energy spectrum of solutions are not bounded below or above.

In the space of antianalytical functions (polynomials of power $2S$ on $\overset{*}{z}_1$, $\overset{*}{z}_2$), connected with previous case by charge conjugation, the solutions of the system (4.29)-(4.31) are

$$s = S > 0: \quad f(x,z) = C_1 (\overset{*}{z}_1)^S e^{-imx^0} + C_2 (\overset{*}{z}_2)^S e^{imx^0},$$

$$s = -S < 0: \quad f(x,z) = C_3 (\overset{*}{z}_1)^S e^{imx^0} + C_4 (\overset{*}{z}_2)^S e^{-imx^0}.$$

These solutions have the same characteristics as the solutions (4.32),(4.33). Thus, in this case the descriptions by means of analytical and antianalytical functions are equivalent.

The wave function (4.32) in the rest frame, corresponding to the helicity $s = S$, has the form $(z_2)^{2S} e^{ip_0 x^0}$, $p_0 = m$. Acting on it by finite transformations, we get a solution in the form of the plane wave, which is characterized by the momentum p,

$$P = U P_0 U^\dagger, \quad P_0 = mI, \quad Z = U Z_0, \quad Z_0 = (z_1 \, z_2)^T,$$

$$f(x,z) = (2\pi)^{-3/2} \left(z_1 u^1 + z_2 u^1 \right)^{2S} e^{ipx}. \tag{4.34}$$

The state with $P_0 = mI$ has the stationary subgroup $U(1)$, and we can take elements $u^1 = \cosh\theta/2$ and $u^2 = \sinh\theta/2\, e^{i\omega}$ of the first line of matrix U, that depend on two parameters only. Thus $p_0 = E = m\cosh\theta$, $\; -p_1 + ip_2 = m\sinh\theta e^{i\omega}$ and one can express the parameters u_1 and u_2 via the momentum p,

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E+m \\ -p_1+ip_2 \end{pmatrix}. \tag{4.35}$$

$2S+1$ components $\psi_n(x)$ are the coefficients in the decomposition of the function (4.34) on the basis $\phi_n(z)$,

$$f(x,z) = \phi_n(z)\psi^n(x), \quad n = 0,1,\ldots,2S, \tag{4.36}$$

$$\psi_n(x) = (2\pi)^{-3/2}\left(C^n_{2S}\right)^{1/2}(u^1)^{2S}(u^2)^n e^{ipx} = (2\pi)^{-3/2}\left(C^n_{2S}\right)^{1/2}\frac{(E+m)^{2S-n}(-p_1+ip_2)^S}{(2m(E+m))^S}e^{ipx}.$$

In the particular case $S = 1/2$ we get

$$\psi(x) = \frac{1}{\sqrt{2m(E-m)}}\begin{pmatrix} E+m \\ -p_2+ip_1 \end{pmatrix} e^{ipx}.$$

Considering the system (4.30)-(4.31) without the condition of the mass irreducibility (4.29), it is easy to see, that the charge density $j^0 = \psi^\dagger\Gamma S^0\psi$ is positive definite only at $S = 1/2$, and the energy density $-T^{00}$ is positive definite only at $S = 1$. The scalar density $\overline{\psi}\psi = \psi^\dagger\Gamma\psi$ is not positive definite.

Let us show that for the particles with half-integer spin, described by the system (4.29)-(4.31), the charge density j^0 (4.25) is positive definite. In the rest frame solutions of the system (4.29)-(4.31) have only two components (labelled by $s_0 = \pm S$), which we denote as $\psi_S(x)$ and $\psi_{-S}(x)$. For half-integer spin $j^0 = \psi^\dagger\Gamma S^0\psi = S(|\psi_S|^2 + |\psi_{-S}|^2) > 0$ holds. At $S \geq 3/2$ from the explicit form of matrices S^1 and S^2 (8.4) one can obtain, that in the rest frame $j^1 = j^2 = 0$, therefore the square of the current vector $(j^0)^2 - (j^1)^2 - (j^2)^2$ is positive. Correspondingly, $j^0 > 0$ for any plane wave.

Thus, for particles with half-integer spin, described by representations $T^0_{m,s}$ of $M(2,1)$, the charge density j^0 is positive definite. The scalar density and the energy density in the rest frame are proportional to $\psi^\dagger\Gamma\psi = |\psi_S|^2 - |\psi_{-S}|^2$ and therefore are indefinite.

Let us consider now particles with integer spin. In the rest frame solutions of the system also have only two components, $\psi_S(x)$ and $\psi_{-S}(x)$, $(S^0\psi, S^0\psi) = \psi^\dagger\Gamma S^0 S^0\psi = S^2(|\psi_S|^2 + |\psi_{-S}|^2) > 0$. Thus, for particles with half-integer spin, described by representations $T^0_{m,s}$ of $M(2,1)$, the energy density j^0 is positive definite. The charge density in the rest frame are proportional to $|\psi_S|^2 - |\psi_{-S}|^2$ and therefore is indefinite.

Consider two particular cases explicitly. At $S = 1/2$ the decomposition (4.19) has the following form,

$$f(x,z) = z_1\psi^1(x) + z_2\psi^2(x), \quad \psi'(x') = U^{-1}\psi(x), \quad \psi(x) = (\psi^1(x)\ \psi^2(x))^T. \tag{4.37}$$

Taking into account the relation $U^{-1} = \sigma^3 U^\dagger\sigma^3$, which is valued for the $SU(1,1)$ matrices, we get the transformation low for the line $\bar{\psi} = \psi^\dagger\sigma^3$, $\bar{\psi}'(x') = \bar{\psi}(x)U$. The product $\bar{\psi}(x)\psi(x) = |\psi_1(x)|^2 - |\psi_2(x)|^2$ is the scalar density.

Thus, in the case under consideration, we have two equivalent descriptions. One in terms of functions (4.37) and another one in terms of lines $\bar{\psi}(x)$ or columns $\psi(x)$. One can find the action of the operators $\hat{S}^\mu$ in the latter representation and the equation (4.22) can be rewritten in the form of $2+1$ Dirac equation,

$$\hat{S}^{\mu}\psi(x) = \frac{1}{2}\gamma^{\mu}\psi(x), \quad (\hat{p}_{\mu}\gamma^{\mu} \mp m)\psi(x) = 0, \tag{4.38}$$

where minus correspond to $s = 1/2$, plus to $s = -1/2$, and γ^{μ} are 2×2 γ-matrices (4.14) in $2+1$ dimensions. The functions $\psi = (\psi^1\ 0)^T$ and $\psi = (0\ \psi^2)^T$ are eigenvectors for the operator $\hat{S}^0$ with the eigenvalues $(\pm 1/2)$.

Sometimes two equations for $s = \pm 1/2$ are written as one equation on the four-component reducible representation [36], $(\hat{p}_{\mu}\Gamma^{\mu} - m)\Psi(x) = 0$, where $\Gamma^{\mu} = \mathrm{diag}(\gamma^{\mu}, -\gamma^{\mu})$, that correspond to the simultaneous consideration of particles with opposite helicities.

At $S = 1$ the decomposition (4.19) has the following form

$$f(x,z) = \psi^{11}(x)(z_1)^2 + \psi^{12}(x)z_1z_2 + \psi^{22}(x)(z_2)^2, \tag{4.39}$$

where $\psi(x) = (\psi^{22}(x)\ \psi^{12}(x)/\sqrt{2}\ \psi^{11}(x))^T$ is subjected to the equation (4.22) with the matrices

$$S^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S^1 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^2 = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.40}$$

If instead of the cyclic components $\psi^{\alpha\beta}(x)$ one introduces the new (Cartesian) components $\mathcal{F}_{\mu} = \breve{\sigma}_{\mu\alpha\beta}\psi^{\alpha\beta}(x)$, where $\breve{\sigma}_{\mu\alpha\beta}$ is defined in (4.15), then the equation (4.22) takes the form [8]

$$\partial_{\mu}\varepsilon^{\mu\nu\eta}\mathcal{F}_{\eta} + sm\mathcal{F}^{\nu} = 0. \tag{4.41}$$

A transversality condition follows from (4.41), $\partial_{\mu}\mathcal{F}^{\mu} = 0$. One can see now that the equations (4.41) are in fact field equations of the so called "self-dual" free massive field theory [37]. As remarked in [38] this theory is equivalent to the topologically massive gauge theory with the Chern-Simons term. Indeed, the transversality condition allows introducing gauge potentials A_{μ}, namely a transverse vector can be written as a curl:

$$\mathcal{F}^{\mu} = \varepsilon^{\mu\nu\lambda}\partial_{\nu}A_{\lambda} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda},$$

where $F_{\nu\lambda} = \partial_{\nu}A_{\lambda} - \partial_{\lambda}A_{\nu}$ is the field strength. Thus, $\mathcal{F}^{\mu}$ appears to be dual field strength, which is a tree-component vector in 2+1 dimensions. Then (4.41) implies the following equations for $F_{\mu\nu}$

$$\partial_{\mu}F^{\mu\nu} + \frac{sm}{2}\varepsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0, \tag{4.42}$$

which are the field equations of the topologically massive gauge theory.

(2)Consider now representations $T^{+}_{m,s}$ and $T^{-}_{m,s}$ of the Poincaré group, connected with **unitary infinite-dimensional IR** of $SU(1,1)$ with highest (lowest) weight. In this case S can be non-integer, $S < -1/2$ (discrete series) or $S = -1/2$ (principal series). For discrete positive series s_0 can take on only positive values, $s_0 = -S + n$, and for negative one only negative $s_0 = S - n$, $n = 0, 1, 2, \ldots$.

Let us consider the energy spectrum of the system (4.29)-(4.31) at $m \neq 0$. According to the first equation $p_0 = \pm m$. The second equation ensures the connection between the spectra of operators $\hat{p}_0$ and $\hat{s}^0$,

$$p_0 s^0 = ms. \tag{4.43}$$

For representations $T^0_{m,s}$, which correspond to finite-dimensional IR T^0_S of the Lorentz group, the value of s^0 can be both positive and negative. Therefore, at any s there are both positive-frequency and negative-frequency solutions.

For unitary IR with highest (lowest) weight the spectrum of s^0 has definite sign. For T^+_S, which act in the space of antianalytical functions, the spectrum of operator $\hat{s}^0$ is positive, and for T^-_S, which act in the space of analytical functions, is negative. Therefore, for T^+_S the sign of energy p_0 coincides with the sign of s, and for T^-_S the signs of p_0 and s are opposite.

As well as for representations $T^0_{m,s}$, at fixed mass and spin there are four states, differed by the signs of p_0 and s. In the rest frame there are two solutions in the space of antianalytical functions,

$$p_0 > 0, s > 0: \quad f(x,z) = (2\pi)^{-3/2}(\overset{*}{z}_1)^S e^{imx^0} \tag{4.44}$$

$$p_0 < 0, s < 0: \quad f(x,z) = (2\pi)^{-3/2}(\overset{*}{z}_1)^S e^{-imx^0}. \tag{4.45}$$

The solutions correspond to positive chirality and connected by CPT-transformation. There are also two solutions in the space of analytical functions, but for negative chirality,

$$p_0 > 0, s < 0: \quad f(x,z) = (2\pi)^{-3/2}(z_1)^S e^{imx^0} \tag{4.46}$$

$$p_0 < 0, s > 0: \quad f(x,z) = (2\pi)^{-3/2}(z_1)^S e^{-imx^0} \tag{4.47}$$

Thus, there exist four equations, defined by the sign of s and by used functional space (IR T^+_S or T^-_S of the Lorentz group), and any equation has the solutions only with definite sign of p_0.

Thus, in contrast to the case of $T^0_{m,s}$, where the energy spectrum p_0 is not bounded both above and below, the energy spectrum has definite sign. In any inertial frame the spectrum is bounded below by $p_0 = m$ for the solutions (4.44), (4.46) and above by $p_0 = -m$ for the solutions (4.45), (4.45).

For the unitary IR of $M(2,1)$ under consideration, which correspond to IR of the discrete seria of the Lorentz group, the integration of the functions (8.3) in the invariant measure (4.12) gives

$$\int \overset{*}{f}_{S_1}(x,z) f'_{S_2}(x,z) d\mu(x,z) = \delta_{S_1 S_2} \int \sum_{n=0}^{\infty} \psi^n(x) \psi'^n(x) d^3x, \tag{4.48}$$

$$\int \overset{*}{f}_{S_1}(x,z) f'_{S_2}(x,z) d\mu(z) = \delta_{S_1 S_2} \psi^\dagger(x) \psi'(x),$$

At the same time, for the representations $T^0_{m,s}$, which correspond to finite-dimensional IR of the Lorentz group, the integral on the spin space diverges. In particular, the states (4.44)-(4.45) are normalized according to $\delta_{SS'}\delta(p-p')$. For the principal series $j = -1/2 + i\lambda$, and $\delta_{j_1 j_2}$ in (4.48) is changed by $\delta(\lambda_1 - \lambda_2)$.

Arbitrary plain wave solution may be obtained by analogy with the case of $T^0_{m,s}$, considered above. For example, for the states (4.44) one can get the formula (4.36), where C^n_{2S} are now the coefficients from (8.3) and $n = 0, 1, 2, \ldots$. The power $2S$ is negative and the decomposition $f(x,z) = \phi_n(z)\psi^n(x)$ contain infinite number of terms.

Let us summarize some properties of unitary IR under consideration. IR $T^+_{m,s}$ and $T^-_{m,s}$ of Poincaré group describe particles with positive and negative chirality correspondingly. Charge density $j^0 = \psi^\dagger S^0\psi$ is positive (negative) definite for the particle with positive (negative) chirality. The energy density is positive definite in both cases, since $(S^0\psi, S^0\psi) = \psi^\dagger S^0 S^0\psi > 0$. Besides, for unitary IR the scalar density $\psi^\dagger\psi$ is also positive definite in contrast to the finite-dimensional case. The existence of positive definite scalar density ensures the possibility of interpretation of $\psi(x)$ as an probability amplitudes.

Thus, in 2+1 dimensions the problem of the construction of **positive-energy relativistic wave equations** is solved by the system (4.29)-(4.31), using of the infinite-dimensional unitary IR T^+_S and T^-_S of the Lorentz group with lowest (highest) weight. The wave functions are transformed under unitary IR $T^+_{m,s}$ (signs of p_0 and s are the same) or $T^-_{m,s}$ (signs of p_0 and s are opposite) of Poincaré group $M(2,1)$, characterized by mass m and helicity s. These IR of Poincaré group are connected by charge conjugation, which changes the signs of p_0 and chirality, but conserves the sign of helicity s.

The interesting problem is to find an explicit form of the intertwining operator A for the unitary IR $T^+_{m,s}$, $T^-_{m,s}$ and the representation $T^0_{m,s}$, labelled by the same mass m and spin s, but connected with finite-dimensional nonunitary IR of the Lorentz group, $AT^0_{m,s} = T^\pm_{m,s}A$. The intertwining operator is nonunitary and must be a function of the generators of right translations, since other generators commute with operator $\hat{\mathbf{S}}^2$ and can't change the representation of spin Lorentz subgroup.

Note, that the 2+1 Dirac equation arises also in the case of unitary infinite-dimensional IR T^+_S and T^-_S of the Lorentz group, not as an equation on a true wave function, but as an equation for spin coherent states evolution. In this case the equation includes effective mass $m_s = |\frac{s}{S}|m$, $s = -S, -S+1, \ldots$ [8].

Among the above considered relativistic wave equations are ones which describe particles with fractional real spin. These equations are connected with unitary multi-valued IR of the Lorentz group and can be used to describe anyons.

In spite of the fact that the number of independent polarization states for massive 2+1 particle is one, the vectors of the corresponding representation space of IR $T^+_{m,s}$, $T^-_{m,s}$ have infinite number of components in matrix representation. Thus, z-representation is more convenient in this case.

V. FOUR DIMENSIONAL CASE

A. Field on the group $M(3,1)$

The action of the left GRR on the functions $f(x,z)$ and the generators are given by formulas (2.32)-(2.34). For spin projection operators it is convenient to use three-dimensional vector notations $\hat{S}_k = \frac{1}{2}\epsilon_{ijk}\hat{S}^{ij}$, $\hat{B}_k = \hat{S}_{0k}$. The explicit calculation gives

$$
\begin{aligned}
\hat{S}_k &= \frac{1}{2}(z\sigma^k\partial_z - \overset{*}{z}\overset{*}{\sigma}{}^k\partial_{\overset{*}{z}}) + \ldots , \\
\hat{B}_k &= \frac{i}{2}(z\sigma^k\partial_z + \overset{*}{z}\overset{*}{\sigma}{}^k\partial_{\overset{*}{z}}) + \ldots , \quad z = (z_1\ z_2), \quad \partial_z = (\partial_{z_1}\ \partial_{z_2})^T; \qquad (5.1)\\
\hat{S}^R_k &= -\frac{1}{2}(\chi\bar{\sigma}^{*k}\partial_\chi - \overset{*}{\chi}\bar{\sigma}^k\partial_{\overset{*}{\chi}}) + \ldots ,
\end{aligned}
$$

$$\hat{B}^R_k = -\frac{i}{2}(\chi\bar{\sigma}^{*k}\partial_\chi + \overset{*}{\chi}\bar{\sigma}^k\partial_{\overset{*}{\chi}}) + \dots, \quad \chi = (z_1\ \tilde{z}_1), \quad \partial_\chi = (\partial_{z_1}\ \partial_{z_3})^T; \tag{5.2}$$

Dots in the formulas replace analogous expressions, obtaining by the substitutions $z \to z' = (\tilde{z}_1\ \tilde{z}_2)$, $\chi \to \chi' = (z_2\ \tilde{z}_2)$.

Since $\det Z = 1$, then any of z can be expressed in terms of other three, for example $\tilde{z}_2 = (1 - z_2\tilde{z}_1)/z_1$. Invariant measure on $\mathbb{R}^4 \times SL(2,C)$ has the form [39]

$$d\mu(x,z) = (i/2)^3 d^4x d^2z_1 d^2z_2 d^2\tilde{z}_1 |z_1|^{-2}. \tag{5.3}$$

The functions on Poincaré group depend on 10 parameters, and correspondingly there are 10 commuting operators (two Casimir operators, four left and four right generators).

Both the Poincaré group and the spin Lorentz subgroup have two Casimir operators,

$$\hat{p}^2 = \hat{p}_\mu\hat{p}^\mu, \quad \hat{W}^2 = \hat{W}_\mu\hat{W}^\mu = \frac{1}{2}\hat{\mathbf{p}}^2\hat{S}_{\mu\nu}\hat{S}^{\mu\nu} - \hat{p}^\mu\hat{p}_\nu\hat{S}_{\mu\rho}\hat{S}^{\nu\rho},$$

$$\text{where} \quad \hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{p}_\nu\hat{J}_{\rho\sigma} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\hat{\partial}_\nu\hat{S}_{\rho\sigma}, \tag{5.4}$$

$$\frac{1}{2}\hat{S}_{\mu\nu}\hat{S}^{\mu\nu} = \frac{1}{2}\hat{S}^R_{\mu\nu}\hat{S}^{\mu\nu}_R = \hat{\mathbf{S}}^2 - \hat{\mathbf{B}}^2, \quad \frac{1}{16}\epsilon^{\mu\nu\rho\sigma}\hat{S}_{\mu\nu}\hat{S}_{\rho\sigma} = \frac{1}{16}\epsilon^{\mu\nu\rho\sigma}\hat{S}^R_{\mu\nu}\hat{S}^R_{\rho\sigma} = \hat{\mathbf{S}}\hat{\mathbf{B}}. \tag{5.5}$$

Let us consider a set of commuting operators

$$\hat{p}_\mu, \ \hat{W}^2, \ \hat{\mathbf{p}}\hat{\mathbf{S}}, \ \hat{\mathbf{S}}^2 - \hat{\mathbf{B}}^2, \ \hat{\mathbf{S}}\hat{\mathbf{B}}, \ \hat{S}^R_3, \ \hat{B}^R_3. \tag{5.6}$$

This set consist of operators of momenta, Lubanski-Pauli operator $\hat{W}^2$, the proportional to helicity operator $\hat{\mathbf{p}}\hat{\mathbf{J}} = \hat{\mathbf{p}}\hat{\mathbf{S}}$ and four operators, which are the functions of the right generators. This four operators commute with the left rotations and translations and allow one to differ equivalent IR in the decomposition of GRR.

Functions $f(x,z)$ on the group $M(3,1)$ are the functions of four real variables x^μ and four complex variables z_α, $\tilde{z}_\alpha$, $z_1\tilde{z}_2 - z_2\tilde{z}_1 = 0$.

The space of functions on the Poincaré group contain the subspace of analytical functions $f(x^\mu, z^\alpha, \bar{z}_{\dot\alpha})$ of the elements of the Dirac z-spinor

$$Z_D = (z^\alpha, \bar{z}_{\dot\alpha}), \quad \bar{z}_{\dot\alpha} = \{\bar{z}_{\dot 1}, \bar{z}_{\dot 2}\} = \{\tilde{z}^*_{\dot 1}, \tilde{z}^*_{\dot 2}\}. \tag{5.7}$$

Charge conjugation means the transition to subspace of antianalytical functions of elements of Dirac z-spinor (i.e. analytical functions $\tilde{z}^\alpha, \overset{*}{z}_{\dot\alpha}$).

According to (2.59), for the space inversion we have $Z \xrightarrow{P} (Z^{-1})^\dagger$, or

$$\begin{pmatrix} z^1 & \tilde{z}^{\dot 1} \\ z^2 & \tilde{z}^{\dot 2} \end{pmatrix} \xrightarrow{P} \begin{pmatrix} -\bar{z}_1 & \overset{*}{z}_{\dot 1} \\ -\bar{z}_2 & \overset{*}{z}_{\dot 2} \end{pmatrix},$$

This transformation reverses the sign of the boost operators $\hat{B}_k$. It is easy to see that, in contrast to charge conjugation, space inversion conserve the analyticity (or antianalyticity) of functions of Z_D.

Similarly to three-dimensional case (see (4.5)), eigenfunctions of $\hat{S}^3_R$ and $\hat{B}^3_R$ differ only by a phase factor. Fixing eigenvalues of operators $\hat{S}^3_R$ and $\hat{B}^3_R$, one may pass to the space of functions of 8 real independent variables on the manifold

$$\mathbb{R}^4 \times \mathbb{C}^2, \qquad d\mu = d^4x d^2 z_1 d^2 z_2. \tag{5.8}$$

of x^μ and elements of Majorana z-spinor

$$Z_M = (z^\alpha, \overset{*}{z}_{\dot\alpha}).$$

Thus, in this space we have 8 commuting operators (2 Casimir operators, 4 operators differ states inside IR, 2 operators differ equivalent IR). Notice, that physical argumentation of necessity to use 8 variables for describing spinning particles contained in [40].

The space reflection maps the functions of Z_M to the functions of $\tilde{Z}_M = (\tilde{z}^\alpha, \bar{z}_{\dot\alpha})$. As was mentioned above, the elements $\tilde{z}_1, \tilde{z}_2$ and z_1, z_2 have the same transformation rule. The charge conjugation leave the space of functions of Z_M invariant. Therefore, one may use the functions $f^*(x, z) = f(x, z)$ to describe particles, which coincide with their antiparticles.

Below we will consider the massive case, characterized by the symmetry with respect to space reflection and, therefore, the space of the analytical functions of Dirac z-spinor Z_D. The action of $M(3,1)$ on this space is given by formula

$$T(g)f(x^\mu,\ z^\alpha,\ \bar{z}_{\dot\alpha}) = f((\Lambda^{-1})^\nu{}_\mu x^\mu,\ U^\beta{}_\alpha z^\alpha,\ (\overset{*}{U}{}^{-1})_{\dot\beta}{}^{\dot\alpha}\bar{z}_{\dot\alpha}). \tag{5.9}$$

Spin projection operators have the form

$$\hat{S}_k = \frac{1}{2}(z\sigma_k\partial_z - \bar{z}\overset{*}{\sigma}_k\partial_{\bar{z}}), \quad \hat{B}_k = \frac{i}{2}(z\sigma_k\partial_z + \bar{z}\overset{*}{\sigma}_k\partial_{\bar{z}}). \tag{5.10}$$

It is known, that one can compose the combinations $\hat{M}_k$, $\hat{N}_k$,

$$\begin{aligned} \hat{M}_k &= \frac{1}{2}(\hat{S}_k - i\hat{B}_k) = z\sigma_k\partial_z, \quad \hat{M}_+ = z_1\partial/\partial z_2, \quad \hat{M}_- = z_2\partial/\partial z_1, \\ \hat{N}_k &= -\frac{1}{2}(\hat{S}_k + i\hat{B}_k) = \bar{z}\sigma_k\partial_{\bar{z}}, \quad \hat{N}_+ = \bar{z}_{\dot 1}\partial/\partial\bar{z}_{\dot 2}, \quad \hat{N}_- = \bar{z}_{\dot 2}\partial/\partial\bar{z}_{\dot 1}, \end{aligned} \tag{5.11}$$

that $[\hat{M}_i, \hat{N}_k] = 0$. According to $\hat{S}_k^\dagger = \hat{S}_k$, $\hat{B}_k^\dagger = \hat{B}_k$, for unitary representations of the Lorentz group these operators obey the relation $\hat{M}_k^\dagger = \hat{N}_k$ (for finite-dimensional nonunitary IR $\hat{S}_k^\dagger = \hat{S}_k$, $\hat{B}_k^\dagger = -\hat{B}_k$ and $\hat{M}_k^\dagger = -\hat{N}_k$ correspondingly). Casimir operators of the Lorentz subgroup one can rewrite in the form

$$\begin{aligned} &\hat{\mathbf{S}}^2 - \hat{\mathbf{B}}^2 = 2(\hat{\mathbf{M}}^2 + \hat{\mathbf{N}}^2) = 2j_1(j_1+1) + 2j_2(j_2+1) = -\frac{1}{2}(k^2 - \rho^2 - 4), \\ &\hat{\mathbf{S}}\hat{\mathbf{B}} = -i(\hat{\mathbf{M}}^2 - \hat{\mathbf{N}}^2) = -i\left(j_1(j_1+1) - j_2(j_2+1)\right) = k\rho, \\ &\text{where} \quad \rho = -i(j_1 + j_2 + 1), \quad k = j_1 - j_2. \end{aligned} \tag{5.12}$$

Thus, IR of the Lorentz group $SL(2,C)$ are labelled by the pair (j_1, j_2). It is convenient label unitary IR by $[k, \rho]$, where IR $[k, \rho]$ and $[-k, -\rho]$ are equivalent [4,39].

Notice, that the formulas (5.9)-(5.12) are also valid, if we consider the functions of elements of Majorana z-spinor Z_M instead of Z_D, using correspondingly substitution $\bar{z}_{\dot\alpha} \to \overset{*}{z}_{\dot\alpha}$.

The formulas of reduction on the compact $SU(2)$-subgroup have the form

$$T_{(j_1,j_2)} = \sum_{j=|j_1-j_2|}^{j_1+j_2} T_j, \qquad T_{[k,\rho]} = \sum_{j=k}^{\infty} T_j, \tag{5.13}$$

for finite-dimensional nonunitary IR and infinite-dimensional unitary IR correspondingly [39].

Consider monomial basis in the space of functions $\phi(z, \bar{z})$,

$$(z^1)^a (z^2)^b (\bar{z}_1)^c (\bar{z}_2)^d.$$

The values $j_1 = (a+b)/2$ and $j_2 = (c+d)/2$ are conserved under the action of generators (5.11). Therefore, the space of IR (j_1, j_2) is the space of homogeneous analytical functions of two pairs of complex variables of power $(2j_1, 2j_2)$, which we denote as $\phi_{j_1 j_2}(z, \bar{z})$. For finite-dimensional nonunitary IR of $SL(2,C)$ a, b, c, d are integer nonnegative, therefore j_1, j_2 are integer or half-integer nonnegative numbers.

B. Relativistic wave equations as the conditions of irreducibility of $M(3,1)$ representations

Fixing eigenvalues of Casimir operators of the Poincaré group and the Lorentz subgroup

$$\hat{p}^2 f(x,z) = m^2 f(x,z), \tag{5.14}$$

$$\hat{W}^2 f(x,z) = -s(s+1)m^2 f(x,z), \tag{5.15}$$

$$\hat{\mathbf{M}}^2 f(x,z) = j_1(j_1+1) f(x,z), \tag{5.16}$$

$$\hat{\mathbf{N}}^2 f(x,z) = j_2(j_2+1) f(x,z), \tag{5.17}$$

allows one to decompose the representation, acting in the space of scalar functions $f(x, z)$ on the Poincaré group. Thus, in general case we have second order equations on $\partial/\partial x$ and $\partial/\partial z$.

The exception is the equations for massless particles (or, more precisely, massless particles with discrete spin). The latter case corresponds to zero eigenvalues of Casimir operators $\hat{p}^2$ and $\hat{W}^2$ and IR are labeled by chirality λ,

$$(\hat{p}_k \hat{S}^k - \lambda \hat{p}^0) f(x,z) = 0, \quad \lambda = j_1 - j_2. \tag{5.18}$$

The proof of the relation $\lambda = j_1 - j_2$ is contained in [4]. The explicit form of chirality operator in the space of analytical functions $f(x, z)$ is given by the formula

$$\hat{\lambda} = \frac{1}{2}\left(z^\alpha \frac{\partial}{\partial z^\alpha} - \bar{z}_{\dot{\alpha}} \frac{\partial}{\partial \bar{z}_{\dot{\alpha}}} \right). \tag{5.19}$$

At $m \neq 0$ IR of proper Poincaré group, labelled by different chirality, are equivalent. Correspondingly, the system (5.14)-(5.17) for fixed mass m and spin $s = j_1 + j_2$ has $2s+1$ solutions, differed by $\lambda = j_1 - j_2$.

Analogously with 2+1 case, there are two types of representations of the Poincaré group, describing the same spin s:

1. $s = j_{\max} = j_1 + j_2$ for nonunitary finite-dimensional IR (j_1, j_2),

2. $s = j_{\min} = j_0 = |j_1 - j_2|$ for unitary infinite-dimensional IR $[j_0, \rho]$,
where $j_{\max}$ and $j_{\min}$ are the maximal and minimal j in the decomposition (5.13) of IR of the Lorentz group over IR T_j of compact $SU(2)$ subgroup. Below we will consider the first case.

The space inversion connects two equivalent IR of proper Poincaré group, labelled by Lorentz indices (j_1, j_2) and (j_2, j_1) (by chiralities $\pm\lambda$) into IR of improper Poincaré group. Therefore, the system (5.14)-(5.17) is not symmetric with respect to space inversion and some correction is necessary to consider massive particles.

The simplest possibility is the transition to a system

$$\hat{p}^2 f(x,z) = m^2 f(x,z), \tag{5.20}$$

$$\hat{W}^2 f(x,z) = -s(s+1)m^2 f(x,z), \tag{5.21}$$

$$s = j_1 + j_2. \tag{5.22}$$

The last equation fix the power $2s$ of the function $f(x,z)$ as polynomial in z. The first two equations are the conditions of mass and spin irreducibility. Thus, the system describe fixed mass and spin, but the representation of Poincaré group, defined by this system, are reducible. This representation split on $2s+1$ equivalent IR, differed by chirality $\lambda = -s, \ldots, s$.

The equations of the system (5.20)-(5.22) do not contain operators, which not commute with chirality operator (5.18), and therefore, do not describe the transitions between the states with different chiralities. Moreover, in the rest frame it is easy to see, that solutions of the system contain $2(2s+1)^2$ independent components instead of $2(2s+1)$. (Each IR of the Lorentz group (j_1, j_2), $j_1 + j_2 = s$, contain spin s IR of $SU(2)$ subgroup, see (5.13)).

Thus, if the symmetry with respect to the space reflection take place, it is necessary to consider equations, which combine equivalent IR of the proper Poincaré group, labelled by different chiralities $\lambda = j_1 - j_2$. In the other words, it is necessary to consider supplementary operators, which define some of extension the Lorentz group.

C. Relativistic wave equations, invariant under improper Poincare group

The general form of the invariant equations linear on $\hat{p}$ is

$$\hat{p}_\mu \hat{V}^\mu f(x,z) = \varkappa f(x,z), \tag{5.23}$$

where $\hat{V}^\mu$ is a function of z and ∂/z, transforming as four-vector.

Using invariant tensor $\sigma^\mu_{\dot{\alpha}\alpha}$ and spinors z^α, $\bar{z}_{\dot{\alpha}}$, $\partial_\alpha = \partial/\partial z^\alpha$, $\tilde{\partial}^{\dot{\alpha}} = \partial/\partial \bar{z}_{\dot{\alpha}}$, it is possible to construct just four vectors:

$$\hat{V}^\mu_{12} = \frac{1}{2}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\bar{z}_{\dot{\alpha}}\partial_\alpha, \quad \hat{V}^\mu_{21} = \frac{1}{2}\sigma^\mu{}_{\alpha\dot{\alpha}} z^\alpha \tilde{\partial}^{\dot{\alpha}}, \tag{5.24}$$

$$\hat{V}^\mu_{11} = \frac{1}{2}\bar{\sigma}^{\mu\dot{\alpha}}{}_\alpha z^\alpha \bar{z}_{\dot{\alpha}}, \quad \hat{V}^\mu_{22} = \frac{1}{2}\sigma^{\mu\alpha}{}_{\dot{\alpha}}\partial_\alpha \tilde{\partial}^{\dot{\alpha}}. \tag{5.25}$$

These operators are not functions of generators of $M(3,1)$ and define transitions between IR with different (j_1, j_2). Operators $\hat{V}^\mu_{12}$, $\hat{V}^\mu_{21}$ conserve $j_1 + j_2$, and operators $\hat{V}^\mu_{11}$, $\hat{V}^\mu_{22}$ conserve $j_1 - j_2$. Any of four relations, connecting two scalar functions,

$$\hat{p}_\mu \hat{V}^\mu_{12} f_{j_1,j_2}(x,z) = \varkappa_{12} f_{j_1-\frac{1}{2},j_2+\frac{1}{2}}(x,z), \quad \hat{p}_\mu \hat{V}^\mu_{21} f_{j_1,j_2}(x,z) = \varkappa_{21} f_{j_1+\frac{1}{2},j_2-\frac{1}{2}}(x,z), \tag{5.26}$$

$$\hat{p}_\mu \hat{V}^\mu_{11} f_{j_1,j_2}(x,z) = \varkappa_{11} f_{j_1+\frac{1}{2},j_2+\frac{1}{2}}(x,z), \quad \hat{p}_\mu \hat{V}^\mu_{22} f_{j_1,j_2}(x,z) = \varkappa_{22} f_{j_1-\frac{1}{2},j_2-\frac{1}{2}}(x,z). \tag{5.27}$$

one may consider as a relativistic wave equation. Thus, the operator $\hat{V}^\mu$ in (5.23) is a linear combination of $\hat{V}^\mu_{ik}$.

Let us consider finite-component (on spin) equations invariant with respect to space reflection. That means:

1. The operator $\hat{p}_\mu \hat{V}^\mu$ is invariant under space reflection.

2. The equation has solutions $f(x,z)$, which transform under representation, containing finite number of IR (j_1, j_2).

It is easy to see, that at $\varkappa \neq 0$ operator $\hat{V}^\mu_{22}$ can't be contained in $\hat{V}^\mu$. In this case one can separate from the system of equations on functions $f_{j_1,j_2}(x,z)$, $f(x,z) = \sum f_{j_1,j_2}(x,z)$ the independent equation on the function $f_j(x,z) = \sum_{j_1+j_2=j} f_{j_1,j_2}(x,z)$, characterized by maximal $j_1 + j_2$, which does not contain $\hat{V}^\mu_{11}$. (Besides, it is not necessary to use operators $\hat{V}^\mu_{11}$ and $\hat{V}^\mu_{22}$, since these operators leave $j_1 - j_2$ invariable and can't connect IR with different λ.)

Relating to operators $\hat{V}^\mu_{12}$ and $\hat{V}^\mu_{21}$, one can see, that only the combination $\hat{p}_\mu \hat{\Gamma}^\mu$,

$$\hat{\Gamma}^\mu = \hat{V}^\mu_{12} + \hat{V}^\mu_{21} = \frac{1}{2}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha} \bar{z}_{\dot{\alpha}} \partial_\alpha + \sigma^\mu{}_{\alpha\dot{\alpha}} z^\alpha \bar{\partial}^{\dot{\alpha}}\right), \tag{5.28}$$

is invariant under space reflections. Operators $\hat{\Gamma}^\mu$ connect representation (j_1, j_2) with $(j_1 + 1, j_2 - 1)$ and $(j_1 - 1, j_2 + 1)$ and conserve $j_1 + j_2$. These operators obey the commutation relations

$$[\hat{S}^{\lambda\mu}, \hat{\Gamma}^\nu] = i(\eta^{\mu\nu}\hat{\Gamma}^\lambda - \eta^{\lambda\nu}\hat{\Gamma}^\mu), \tag{5.29}$$

$$i[\hat{\Gamma}^\mu, \hat{\Gamma}^\nu] = \hat{S}^{\mu\nu}, \tag{5.30}$$

which coincide with the commutation relations of matrices $\gamma^\mu/2$. The explicit calculation shows, that $\hat{\Gamma}_\mu \hat{\Gamma}^\mu$ depend on IR of the Lorentz subgroup,

$$\hat{\Gamma}_\mu \hat{\Gamma}^\mu = 2j_1 + 2j_2 + 4j_1 j_2. \tag{5.31}$$

Supplementing generators of the Lorentz group by four operators

$$\hat{S}^{4\mu} = \hat{\Gamma}^\mu, \quad \hat{S}^{ab} = -\hat{S}^{ba}, \tag{5.32}$$

we obtain

$$[\hat{S}^{ab}, \hat{S}^{cd}] = i(\eta_{bc}\hat{S}^{ad} - \eta_{ac}\hat{S}^{bd} - \eta_{bd}\hat{S}^{ac} + \eta_{ad}\hat{S}^{bc}), \quad \eta_{44} = \eta_{00} = 1. \tag{5.33}$$

Thus, operators $\hat{S}^{ab}$, $a, b = 0,1,2,3,4$, obey the commutation relations of the generators of de Sitter group $SO(3,2) \sim Sp(4,R)$. This group has two fundamental IR, four-dimensional spinor IR $T_{[10]}$ (by matrices $Sp(4,R)$) and five-dimensional vector IR $T_{[01]}$ (by matrices $SO(3,2)$).

Using (5.5), (5.12) and (5.31), we obtain for second order Casimir operator of the group $Sp(4,R)$

$$\hat{S}_{ab}\hat{S}^{ab} f(x,z) = 4j(j+2) f(x,z).$$

Thus, we are deal with symmetric representations of $Sp(4,R)$, denoted as $T_{[2s0]}$ (see Appendix). These IR can be obtained as symmetric term in the decomposition of the direct product $(\otimes T_{[10]})^{2s}$. IR $T_{[2s0]}$ of the group $Sp(4,R)$, characterized by dimensionality $(2s+3)!/(6(2s)!)$, combine all finite-dimensional IR of the Lorentz group with $j_1+j_2=s$.

However, it is obvious, that the equation

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,z) = \varkappa f(x,z) \tag{5.34}$$

by itself does not fix spin and mass, defined by (5.14) and (5.15), or the power j_1+j_2 of the $f(x,z)$ in z. In the rest frame it is easy to see, that even at $s=j_1+j_2$ this equation fix only composition $ms=\varkappa$.

Let us consider the system

$$\hat{\mathrm{p}}^2 f(x,z) = m^2 f(x,z), \tag{5.35}$$
$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,z) = ms f(x,z), \tag{5.36}$$
$$\hat{S}_{ab}\hat{S}^{ab} f(x,z) = 4j(j+2) f(x,z). \tag{5.37}$$

Below we will show, that the equations of this system fix m, s and $j=j_1+j_2$, in contrast to four equations of the system (5.14)-(5.17), which fix j_1 and j_2 separately.

Really, the last equation of the system fixes IR $T_{[2j0]}$ of de Sitter group and the power $2j=2j_1+2j_2$ of polynomial in $z,\bar{z}$. On these polynomials can be realized IR of Poincaré group, characterized by spin $s\leq j$. Below we restrict our consideration by the condition $s=j$. The condition $s=j$ allow to describe spin s by means of the IR of the de Sitter group with minimal possible dimensionality.

In the rest frame

$$\hat{p}_0^2 f(x,z) = m^2 f(x,z),$$
$$\hat{p}_0\hat{\Gamma}^0 f(x,z) = ms f(x,z), \quad \hat{\Gamma}^0 = \frac{1}{2}(\sigma^{0\dot{\alpha}\alpha}\bar{z}_{\dot{\alpha}}\partial_\alpha + \sigma^0{}_{\alpha\dot{\alpha}} z^\alpha \bar{\partial}^{\dot{\alpha}}). \tag{5.38}$$

According to the first equation $p_0=\pm m$. At $p_0=m$ any function, characterized by $n_1-n_2=2s$ is the solution of the equation (5.38), where n_1 is the power of homogeneity on the variables $(z^1+\bar{z}_1)$, $(z^2+\bar{z}_2)$, and n_2 is the power of homogeneity on the variables $(z^1-\bar{z}_1)$, $(z^2-\bar{z}_2)$. Correspondingly, at $p_0=m$ any function, characterized by $n_1-n_2=-2s$ is the solution of the equation (5.38). We also fix the spin projection by the equation

$$\hat{S}^3 f(x,z) = s^3 f(x,z), \quad \hat{S}^3 = \frac{1}{2}(z^1\partial_1 + \bar{z}_{\dot{1}}\bar{\partial}^{\dot{1}} - z^2\partial_2 - \bar{z}_{\dot{2}}\bar{\partial}^{\dot{2}}), \tag{5.39}$$

that in the set of polynomial of the power $2s$ in $z,\bar{z}$ up to normalization factor define desired function

$$f_{m,s,s^3}(x,z) = C_1 e^{imx^0}(z^1+\bar{z}_1)^{s+s^3}(z^2+\bar{z}_2)^{s-s^3} + C_2 e^{-imx^0}(z^1-\bar{z}_1)^{s+s^3}(z^2-\bar{z}_2)^{s-s^3}. \tag{5.40}$$

Explicit calculation gives $\hat{W}^2 f_{m,s,s^3}(x,z) = -m^2 s(s+1) f_{m,s,s^3}(x,z)$. Therefore, s in the equation (5.38) really defines spin.

At fixed mass m and spin s there are $2s+1$ independent positive-frequency solutions, and $2s+1$ independent negative-frequency solutions.

Notice, that only four-dimensional IR of the de Sitter group, corresponding to spin 1/2, remains irreducible at the reduction on the improper Lorentz group. For spin one 10-dimensional IR split on 6+4 (antisymmetric tensor and four-vector), for spin 3/2 20-dimensional IR split on 8+12, and so on.

Consider plain wave solutions, corresponding to a particle, moving along x^3. They can be obtained from the solutions in the rest frame (5.40) by means of Lorentz transformation

$$P = UP_0U^\dagger, \quad \text{where } P_0 = \pm\,\mathrm{diag}\{m, m\}, \quad U = \mathrm{diag}\{e^{-a}, e^{a}\} \in SL(2, C),$$

where the sign correspond to the sign of p_0,

$$p_\mu = k_\mu \mathrm{sign}\ p_0, \quad k_0 = m\cosh 2a, \quad k_3 = m\sinh 2a, \quad e^{\pm a} = \sqrt{(k_0 \pm k_3)/m}. \tag{5.41}$$

Thus it follows that

$$\begin{aligned} f'_{m,s,s^3}(x,z) = {} & C_1 e^{ik_0x^0 + k_3x^3}(z^1e^a + \bar{z}_1e^{-a})^{s+s^3}(z^2e^{-a} + \bar{z}_2e^{a})^{s-s^3} + \\ & C_2 e^{-ik_0x^0 - k_3x^3}(z^1e^a - \bar{z}_1e^{-a})^{s+s^3}(z^2e^{a} - \bar{z}_2e^{-a})^{s-s^3}. \end{aligned} \tag{5.42}$$

In ultrarelativistic case at $a \to +\infty$ it is convenient to rewrite (5.42) in the form

$$\begin{aligned} & f_{m,s,s^3}(x,z) = \left(\frac{k_0 + k_3}{m}\right)^s \times \\ & \left(\left(C_1 e^{ik_0x^0 + k_0x^3} + C_2(-1)^{s-s^3} e^{-ik_0x^0 - k_0x^3}\right)(z^1)^{s+s^3}(\bar{z}_2)^{s-s^3} + O\left(\frac{k_0 - k_3}{k_0 + k_3}\right)^{\frac{1}{2}}\right) \end{aligned} \tag{5.43}$$

Passing to limit, we obtain the states with certain chirality $\lambda = j_1 - j_2 = s^3$ (correspondingly, at $a \to -\infty$ with chirality $\lambda = j_1 - j_2 = -s^3$). In ultrarelativistic case the main term in (5.43) correspond to functions, transforming under IR $(\frac{s+\lambda}{2}\ \frac{s-\lambda}{2})$, of the Lorentz group. The contribution of other IR $(\frac{s+\lambda'}{2}\ \frac{s-\lambda'}{2})$ are damped by factor $(\frac{p_0-p_3}{p_0+p_3})^{|\lambda-\lambda'|}$. In particular, in the limit IR $(s\,0)\oplus(0\,s)$ of the Lorentz group correspond to the states, characterized by $s^3 = \pm s$.

At $m = 0$ (5.36) split on $2s + 1$ independent equations on the functions $f_{j_1j_2}(x, z)$,

$$f_s(x,z) = \sum_{\lambda=-s}^{s} f_{j_1j_2}(x,z), \quad \text{where } s = j_1 + j_2, \ \ \lambda = j_1 - j_2, \tag{5.44}$$

corresponding to $2s$+1 values of chirality λ. At $m \neq 0$ equation (5.36) in chiral representation has the form

$$\hat{p}_\mu \hat{\Gamma}^\mu \begin{pmatrix} f_{s,0} \\ f_{s-\frac{1}{2},\frac{1}{2}} \\ \dots \\ f_{0,s} \end{pmatrix} = \begin{pmatrix} \hat{p}_\mu \hat{V}^\mu_{21} f_{s-\frac{1}{2},\frac{1}{2}} \\ \hat{p}_\mu \hat{V}^\mu_{12} f_{s,0} + \hat{p}_\mu \hat{V}^\mu_{21} f_{s-1,1} \\ \dots \\ \hat{p}_\mu \hat{V}^\mu_{12} f_s^{-s+1} \end{pmatrix} = ms \begin{pmatrix} f_{s,0} \\ f_{s-\frac{1}{2},\frac{1}{2}} \\ \dots \\ f_{0,s} \end{pmatrix}. \tag{5.45}$$

This equation bind 1+$[s]$ IR of improper Lorentz group and allow one to express components, corresponding to IR $(s\ 0)$ in terms of components, corresponding to IR $(s - \frac{1}{2}\ \frac{1}{2})$ and so on. That, in turn, at s = 1, 3/2, 2 allow one to transit from the first order equations on the reducible representation to second order equations on IR of improper Poincaré group. For example, at $s = 1$, excluding $f_{1,0}$ and $f_{0,1}$, we obtain

$$m^2 f_{\frac{1}{2}\frac{1}{2}}(x,z) = \{\hat{p}_\mu \hat{V}^\mu_{12}, \hat{p}_\nu \hat{V}^\nu_{21}\}_+ f_{\frac{1}{2}\frac{1}{2}}(x,z). \tag{5.46}$$

In general case one also can to transit from the system of first order equations (5.45) on the reducible representation to higher order equations on IR, for example, to the equations of $1+[s]$ order on the components, transforming under IR $(\frac{s}{2}\,\frac{s}{2})$ or $(\frac{(s+1)}{2}\,\frac{(s-1)}{2}) \oplus (\frac{(s-1)}{2}\,\frac{(s+1)}{2})$ at the cases of integer or half-integer spin correspondingly.

Let us consider some particular cases.

1. $s = j_1 + j_2 = 1/2$, the Dirac equation.

$$f_{\frac{1}{2}}(x,z) = \chi_\alpha(x) z^\alpha + \psi^{\dot\alpha}(x)\bar{z}_{\dot\alpha}. \tag{5.47}$$

If we substitute (5.47) into the equation (5.36) and compare the coefficients at z^α and at $\bar{z}_{\dot\alpha}$ in the left and right side, then obtain

$$\hat{p}_\mu \gamma^\mu \Psi_D(x) = m\Psi_D(x), \quad \Psi_D = \begin{pmatrix} \chi_\alpha(x) \\ \overset{*}{\psi}{}^{\dot\alpha}(x) \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{5.48}$$

The complex conjugated function correspond to charge conjugated state,

$$\overset{*}{f}_{1/2}(x,z) = -\psi_\alpha(x)\tilde{z}^\alpha - \overset{*}{\chi}{}^{\dot\alpha}(x)\overset{*}{z}_{\dot\alpha},$$

(the sign reverse because of anticommutation of spinors, $\psi_\alpha z^\alpha = -z_\alpha \psi^\alpha$) or in the matrix form

$$\Psi^c_D = -\begin{pmatrix} \psi_\alpha(x) \\ \overset{*}{\chi}{}^{\dot\alpha}(x) \end{pmatrix} = i\sigma^2 \begin{pmatrix} \psi^\alpha(x) \\ -\overset{*}{\chi}_{\dot\alpha}(x) \end{pmatrix}. \tag{5.49}$$

The matrix $\gamma^5 = \mathrm{diag}\{\sigma^0, -\sigma^0\}$ correspond to chirality operator (5.18).

The real function $f_{1/2}(x,z) = \overset{*}{f}_{1/2}(x,z)$, describing Majorana particle, depend on the elements of Z_M, and correspondingly $\psi^{\dot\alpha}(x) = -\chi^{\dot\alpha}(x) = i\sigma^2 \chi_{\dot\alpha}(x)$.

2. $s = j_1 + j_2 = 1$, the Duffin-Kemmer equation.

$$f_1(x,z) = \chi_{\alpha\beta}(x) z^\alpha z^\beta + \phi_\alpha^{\dot\beta}(x) z^\alpha \bar{z}_{\dot\beta} + \psi^{\dot\alpha\dot\beta}(x)\bar{z}_{\dot\alpha}\bar{z}_{\dot\beta}. \tag{5.50}$$

Substituting (5.50) into equation (5.36), we obtain

$$\begin{aligned} m\psi^{\dot\alpha\dot\beta}(x) &= \frac{1}{2}\hat{p}_\mu \bar{\sigma}^{\mu\dot\alpha\gamma}\phi_\gamma^{\dot\beta}(x), \quad m\chi_{\alpha\beta}(x) = \frac{1}{2}\hat{p}_\mu \sigma^\mu{}_{\dot\gamma\alpha}\phi_\beta^{\dot\gamma}(x), \\ m\phi_\alpha^{\dot\beta}(x) &= \hat{p}_\mu(\bar{\sigma}^{\mu\dot\beta\gamma}\chi_{\alpha\gamma}(x) + \sigma^\mu_{\alpha\dot\alpha}\psi^{\dot\alpha\dot\beta}(x)), \end{aligned} \tag{5.51}$$

or, after the transition to vector indices, $F_{\mu\nu}(x) = \frac{1}{2}m\sigma_{\mu\alpha\dot\alpha}\sigma_{\nu\beta\dot\beta}(\epsilon^{\dot\alpha\dot\beta}\chi^{\alpha\beta}(x) + \epsilon^{\alpha\beta}\psi^{\dot\alpha\dot\beta}(x))$, $\Phi_\mu(x) = \frac{1}{2}\bar{\sigma}_\mu{}^{\dot\beta\alpha}\phi_{\alpha\dot\beta}(x)$,

$$F_{\mu\nu}(x) = \partial_\mu \Phi_\nu(x) - \partial_\nu \Phi_\mu(x), \qquad m^2\Phi_\nu(x) = \partial^\mu F_{\mu\nu}(x). \tag{5.52}$$

The Duffin-Kemmer equation in the form (5.52) is the equation on IR $T_{[20]}$ of de Sitter group $Sp(4,R)$, and, thus, on the reducible representation of the Lorentz group $(1\,0)\oplus(\frac{1}{2}\,\frac{1}{2})\oplus(0\,1)$. This representation contain both four-vector $\Phi_\mu(x)$ and antisymmetric tensor $F_{\mu\nu}(x)$, which correspond to chiralities $\lambda=0$ and $\lambda=\pm1$. One can exclude components $F_{\mu\nu}(x)$ and consider second order system only for the components $\Phi_\mu(x)$, transformed under IR $(\frac{1}{2}\frac{1}{2})$ of the Lorentz group (equations for spin one in the Proca form),

$$(\hat{p}^2-m^2)\Phi_\mu(x)=0, \qquad \hat{p}^\mu\Phi_\mu(x)=0. \tag{5.53}$$

Neutral field of spin 1 one may describe, in particular, by real function of the elements of Majorana z-spinor, $f_1(x,z)=\overset{*}{f}_1(x,z)$. Then $\chi_{\alpha\beta}=\overset{*}{\psi}_{\alpha\beta}$, $\phi_{\alpha\dot\beta}=\overset{*}{\phi}_{\dot\beta\alpha}$, or $\overset{*}{\Phi}_\mu=\Phi_\mu$, $\overset{*}{F}_{\mu\nu}=F_{\mu\nu}$. In this case it is possible instead of complex variables z^α and $\overset{*}{z}_{\dot\alpha}$ to use real variables

$$\begin{aligned} q^\mu &= \frac{1}{2}\sigma^\mu{}_{\alpha\dot\beta}z^\alpha\overset{*}{z}{}^{\dot\beta}, \quad q_\mu q^\mu=0,\\ q^{\mu\nu} &= -q^{\nu\mu}=\frac{1}{2}\sigma_{\mu\alpha\dot\alpha}\sigma_{\nu\beta\dot\beta}(\epsilon^{\dot\alpha\dot\beta}z^\alpha z^\beta+\epsilon^{\alpha\beta}\overset{*}{z}{}^{\dot\alpha}\overset{*}{z}{}^{\dot\beta}). \end{aligned} \tag{5.54}$$

One can rewrite (5.50) in the form

$$f(x,q)=\Phi_\mu(x)q^\mu+\frac{1}{2}F_{\mu\nu}(x)q^{\mu\nu}, \tag{5.55}$$

and, applying operator

$$\hat{p}_\mu\hat{\Gamma}^\mu=-i(\hat{p}_\mu q^{\mu\nu}\partial q^\nu+\hat{p}^\mu\partial q^{\mu\nu}q^\nu), \tag{5.56}$$

obtain (5.52). Such transition to vector indices is possible under considering of any integer spin.

3. $s=j_1+j_2=3/2$, equations for spin 3/2 particle. Equations (5.45) allow one to write the components $(\frac{3}{2}\,0)\oplus(0\,\frac{3}{2})$ in terms of $(1\,\frac{1}{2})\oplus(\frac{1}{2}\,1)$. Then one can to transit to the second order system on the components, transformed under IR $(1\,\frac{1}{2})\oplus(\frac{1}{2}\,1)$ of improper Lorentz group,

$$\begin{aligned} (3m/2)^2 f_{1,\frac{1}{2}}(x,z) &= \hat{p}_\mu\hat{p}_\nu\hat{V}^\mu_{12}\hat{V}^\nu_{21}f_{1,\frac{1}{2}}(x,z)+(3m/2)\hat{p}_\mu\hat{V}^\mu_{21}f_{\frac{1}{2},1}(x,z),\\ (3m/2)^2 f_{\frac{1}{2},1}(x,z) &= \hat{p}_\mu\hat{p}_\nu\hat{V}^\mu_{21}\hat{V}^\nu_{12}f_{\frac{1}{2},1}(x,z)+(3m/2)\hat{p}_\mu\hat{V}^\mu_{12}f_{1,\frac{1}{2}}(x,z). \end{aligned} \tag{5.57}$$

4. $s=j_1+j_2=2$, equations for spin 2 particle.

Equations (5.45) allow one to write other components in terms of $(\frac{3}{2}\,\frac{1}{2})$, $(\frac{1}{2}\,\frac{3}{2})$ and to obtain second order system on the components, transformed under IR $(\frac{3}{2}\,\frac{1}{2})\oplus(\frac{1}{2}\,\frac{3}{2})$ of the Lorentz group,

$$(2m)^2 f_{\frac{3}{2},\frac{1}{2}}(x,z)=\{\hat{p}_\mu\hat{V}^\mu_{12},\hat{p}_\nu\hat{V}^\nu_{21}\}_+f_{\frac{3}{2},\frac{1}{2}}(x,z)+\hat{p}_\mu\hat{p}_\nu\hat{V}^\mu_{21}\hat{V}^\nu_{21}f_{\frac{1}{2},\frac{3}{2}}(x,z), \tag{5.58}$$

$$(2m)^2 f_{\frac{1}{2},\frac{3}{2}}(x,z)=\{\hat{p}_\mu\hat{V}^\mu_{12},\hat{p}_\nu\hat{V}^\nu_{21}\}_+f_{\frac{1}{2},\frac{3}{2}}(x,z)+\hat{p}_\mu\hat{p}_\nu\hat{V}^\mu_{12}\hat{V}^\nu_{12}f_{\frac{3}{2},\frac{1}{2}}(x,z). \tag{5.59}$$

For higher spin it is also possible to write the higher order system on IR of improper Poincare group, but the system will be cumbersome and it is easier to consider corresponding linear equations on the reducible representation.

For the cases $s = 1/2$ and $s = 1$ first equation of the system (5.35) (Klein-Gordon equation) is the consequence of second equation. In other cases the solutions of (5.36) have spin and mass spectrum, $s_i = \{s,\ s-1, \ldots 1\}$ or $s_i = \{s,\ s-1, \ldots 1/2\}$, $m_i = ms/s_i$. Thus, for higher spin the Klein-Gordon equation is independent condition, allowing to exclude from spin spectrum all spins, except maximal s.

The cases $s = 1/2$ and $s = 1$ are also the exceptions in sense of simplicity of labelling the components by spinor and vector indices. The number of the indices of symmetric spin-tensors, necessary for labelling higher spin components, increase in spite of the fact, that for labelling of the states, belonging to symmetric IR of the de Sitter group, it is sufficient to use only three operators and correspondingly only three numbers.

In particular, for spin 3/2 particle there exist four kinds of components, namely $\psi_{\alpha\beta\gamma}$, $\phi^{\dot\alpha\dot\beta}_{\gamma}$, $\omega^{\dot\alpha}_{\beta\gamma}$, $\chi^{\dot\alpha\dot\beta\dot\gamma}$, corresponding to 4 possible chiralities. For the spin 2 particle the representation, analogous to (5.55), is also cumbersome,

$$f_2(x,q) = \Phi_{\mu\nu}(x)q^\mu q^\nu + \frac{1}{2}F_{\mu\nu,\rho}(x)q^{\mu\nu}q^\rho + \frac{1}{4}F_{\mu\nu,\rho\sigma}(x)q^{\mu\nu}q^{\rho\sigma} \tag{5.60}$$

with the necessity to fix independent components by means of relations $q_\mu q^\mu = 0$, $q_{\mu\nu}q^\mu + q_{\mu\nu}q^\nu = 0$ and so on.

Thus, beginning from the spin 3/2, it is convenient to use the universal notations, connected with the decomposition over monomial chiral basis,

$$f_s(x,z) = \sum_{\lambda=-s}^{s} \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} f^{m_1 m_2}_{j_1 j_2}(x) \times$$
$$\left(\frac{(2j_1)!}{(j_1+m_1)!(j_1-m_1)!}\frac{(2j_2)!}{(j_2+m_2)!(j_2-m_2)!}\right)^{\frac{1}{2}} z_1^{j_1+m_1} z_2^{j_1-m_1} \bar z_1^{j_2+m_2} \bar z_2^{j_2+m_2}, \tag{5.61}$$

where $s = j_1 + j_2$, $\lambda = j_1 - j_2$. These notations are also suitable for infinite-dimensional representations. Two indices j_1, j_2 label spin and chirality and two indices m_1, m_2 label independent components inside IR of the Lorentz group.

By analogy with 2+1 case, one can find plain wave solutions of the system (5.35)-(5.36) for the any spin s in general form without using matrix representation. The states, corresponding to the particle, moving along x^3, are eigenstates of the operator $\hat p_i \hat S^i$ with eigenvalues $|p|\sigma$, where $\sigma = s^3 \mathrm{sign}\, p_3$ is the helicity. These states have the form

$$\begin{aligned} f_{m,s,\sigma}(x,z) = {} & C_1 e^{ik_0x^0 + k_3x^3}(z^1e^a + \bar z_1 e^{-a})^{s+\sigma}(z^2e^{-a} + \bar z_2 e^a)^{s-\sigma} + \\ & C_2 e^{-ik_0x^0 - k_3x^3}(z^1e^a - \bar z_1 e^{-a})^{s-\sigma}(z^2e^a - \bar z_2 e^{-a})^{s+\sigma}, \end{aligned} \tag{5.62}$$

where e^a is given by (5.41). For the rest particle one can obtain the general solution, characterized by the spin projection s' on the direction $\mathbf{n}$, from (5.40) by the rotation $z'_\beta = U_\beta{}^\alpha z_\alpha$, $U \subset SU(2)$. For particle, characterized by momentum direction $\mathbf{n}$ and helicity σ one can obtain the solution by the analogous rotation starting from the state (5.62).

D. Relativistic wave equations, invariant under improper Poincare group.

Equations on a few scalar functions

Above we have considered the linear equations on **one** scalar function on the group. The condition of the presence of symmetry with respect to space reflection led us to the system (5.35)-(5.37) for particle with spin $s = j_1 + j_2$ and mass m.

For the construction of invariant wave equations one may also use the operators $\hat{p}_\mu \hat{V}^\mu_{ik}$, which are not invariant under space reflections. It is possible to restore the invariance under space reflections, using a few scalar functions $f(x, z)$.

Let us consider systems of the form (5.26),(5.27), connecting several scalar functions with different j_1, j_2. The equations of this system interlock the representation (j_1, j_2) with at least one of the representations $(j_1 \pm 1, j_2 \mp 1)$, $(j_1 \pm 1, j_2 \pm 1)$, that allow one to identify this system with Gel'fand-Yaglom equations [41,26], which can be written in the matrix form as

$$(\alpha^\mu \hat{p}_\mu - \varkappa)\psi = 0, \quad [S^{\lambda\mu}, \alpha^\nu] = i(\eta^{\mu\nu}\alpha^\lambda - \eta^{\lambda\nu}\alpha^\mu) \tag{5.63}$$

In the present approach the latter relation is a consequence of the commutation relations $[\hat{S}^{\lambda\mu}, \hat{V}^\nu_{ik}] = i(\eta^{\mu\nu}\hat{V}^\lambda_{ik} - \eta^{\lambda\nu}\hat{V}^\mu_{ik})$. This relation is necessary for Poincaré invariance of the equation [4,26].

Finite-component equations of the form (5.63), supplemented by commutation relations $[\alpha^\mu, \alpha^\nu] = S^{\mu\nu}$, are known as Bhabha equations [42], although for the first time was systematically considered by Lubanski [43]. These equations are classified according to the finite-dimensional IR of the de Sitter group $SO(3,2)$. Other possible commutation relations of matrices α^μ are discussed in [44].

The equation (5.36) on a scalar function, considered above, is the particular case of Bhabha equations, connected with symmetrical IR $T_{[2s\,0]}$ of the de Sitter group.

In general case the Bhabha equations characterized by finite number of different m and s. Therefore, this equations connect the fields, transforming under nonequivalent IR of Poincaré group.

If one use the operators $\hat{p}_\mu \hat{V}^\mu_{11}$ and $\hat{p}_\mu \hat{V}^\mu_{22}$, then the equations either describe at least two different spins s, or the condition $s = j_1 + j_2$, connecting spin s with a highest weight of IR of Lorentz group, is not valid.

Cite as an example the system, interlocking IR (00) and $(\frac{1}{2}\frac{1}{2})$ of the Lorentz group, $f_{00}(x, z) = \psi(x)$, $f_{\frac{1}{2}\frac{1}{2}}(x, z) = \psi^{\dot\beta}_\alpha(x) z^\alpha \bar{z}_{\dot\beta}$:

$$\hat{p}_\mu \hat{V}^\mu_{11} f_{00}(x, z) = \varkappa_1 f_{\frac{1}{2}\frac{1}{2}}(x, z), \quad \hat{p}_\mu \hat{V}^\mu_{22} f_{\frac{1}{2}\frac{1}{2}}(x, z) = \varkappa_2 f_{00}(x, z), \tag{5.64}$$

or in component-wise form $\hat{p}_\mu \psi = 2\varkappa_1 \psi_\mu$, $\hat{p}_\mu \psi^\mu = \varkappa_2 \psi$. In the rest frame one may obtain $\varkappa_2 = 2\varkappa_1 = m$. Thus, the system (5.64) is equivalent to Duffin equation for scalar particles, which correspond to five-dimensional vector IR $T_{[01]}$ of $SO(3,2)$ group.

Using the operators $\hat{p}_\mu \hat{V}^\mu_{ik}$, one may construct the equations of higher order, for example, equations, interlocking IR $(s\,0)$ and $(0\,s)$ of the Lorentz group,

$$(\hat{p}_\mu \hat{V}^\mu_{12})^{2s} f_{s0}(x, z) = \varkappa f_{0s}(x, z), \quad (\hat{p}_\mu \hat{V}^\mu_{21})^{2s} f_{0s}(x, z) = \varkappa f_{s0}(x, z).$$

E. Relativistic wave equations, invariant under improper Poincare group. Equations for the particles with composite spin

Many-particle systems are described by the functions of the sets of variables $x_{(i)}, z_{(i)}, \bar{z}_{(i)}$. But here we will consider not many-particle systems in usual sense, but some objects, corresponding to functions $f(x^\mu, z^\alpha_{(1)}, \bar{z}_{\dot{\alpha}(1)}, \dots, z^\alpha_{(n)}, \bar{z}_{\dot{\alpha}(n)})$ (or, briefly, $f(x, \{z_{(i)}\})$), i.e. to functions of one set of x and several sets of z. These objects one may interpret as particles with composite spin.

Let us consider linear symmetric functions of $z_{(1)}, \dots, z_{(n+l)}$,

$$f_{\frac{n}{2}\frac{l}{2}}(x, \{z_{(i)}\}) = \psi^{\dot{\alpha}_1,\dots,\dot{\alpha}_l}_{\beta_1,\dots,\beta_n}(x) \sum z^{\beta_1}_{(1)} \dots z^{\beta_n}_{(n)} \bar{z}_{(n+1)\dot{\alpha}_1} \dots \bar{z}_{(n+l)\dot{\alpha}_l} \tag{5.65}$$

where symmetric spinors $\psi^{\dot{\alpha}_1,\dots,\dot{\alpha}_l}_{\beta_1,\dots,\beta_n}(x)$ transforming under IR $(n/2, l/2)$ and all permutations of $1, \dots, n+l$ are summed over. For this functions one may write the equations, analogous to (5.26)-(5.27).

One may obtain Dirac-Fierz-Pauli equations [45,46], acting by the operators $\hat{V}^\mu_{12(k)}$ and $\hat{V}^\mu_{21(k)}$ on the functions (5.65), which are transforming under the IR $(\frac{n}{2}+\frac{1}{2}, \frac{l}{2})$ and $(\frac{n}{2}, \frac{l}{2}+\frac{1}{2})$ correspondingly,

$$\begin{aligned} \hat{V}^\mu_{12(k)} f_{\frac{n}{2}+\frac{1}{2},\frac{l}{2}}(x, \{z_{(i)}\}) &= \varkappa f_{\frac{n}{2},\frac{l}{2}+\frac{1}{2}}(x, \{z_{(i)}\}), \\ \hat{V}^\mu_{21(k)} f_{\frac{n}{2},\frac{l}{2}+\frac{1}{2}}(x, \{z_{(i)}\}) &= \varkappa f_{\frac{n}{2}+\frac{1}{2},\frac{l}{2}}(x, \{z_{(i)}\}), \end{aligned} \tag{5.66}$$

or in component-wise form

$$\begin{aligned} \partial_\mu \bar{\sigma}^{\mu\dot{\alpha}\beta} \psi^{\dot{\alpha}_1,\dots,\dot{\alpha}_l}_{\beta,\beta_1,\dots,\beta_n}(x) &= i\varkappa \psi^{\dot{\alpha},\dot{\alpha}_1,\dots,\dot{\alpha}_l}_{\beta_1,\dots,\beta_n}(x) \\ \partial_\mu \sigma_{\beta\dot{\alpha}}{}^{\mu} \psi^{\dot{\alpha},\dot{\alpha}_1,\dots,\dot{\alpha}_l}_{\beta_1,\dots,\beta_n}(x) &= i\varkappa \psi^{\dot{\alpha}_1,\dots,\dot{\alpha}_l}_{\beta,\beta_1,\dots,\beta_n}(x) \end{aligned} \tag{5.67}$$

These equations, interlocking two scalar functions, allow parity transformation only at $n = l$.

Considering the functions, which depend on several sets of spin variables, one may obtain the equations, often joined with the Dirac-Fierz-Pauli equations by the name "equations with subsidiary conditions", namely Bargmann-Wigner equations [47], equations for massive tensor fields and Rarita-Schwinger equations [48].

In contrast to the system (5.35)-(5.36) on the functions of one set of space and spin coordinates, these equations suppose the conditions both on the whole system and its spin subsystems separately. Naturally, these conditions can be found inconsistent under interaction. The general scheme of construction is as following.

One impose an equation $(\hat{p}_\mu \hat{\Gamma}^\mu_{(k)} - \varkappa) f = 0$ on each spin subsystem, described by IR $(\frac{1}{2}0) \oplus (0\frac{1}{2})$ or $(\frac{1}{2}\frac{1}{2})$ of Lorentz group. Then one require the wave function to be symmetric with respect to the permutations of indices (k). (That cut the IR $(j_1 + j_1', j_2 + j_2')$ from the direct product $(j_1, j_2) \otimes (j_1', j_2')$.) Finally, if it is necessary, one impose an supplementary conditions to exclude redundant components.

Let us return to the symmetric linear on $z_{(i)}$ functions $f(x, \{z_{(i)}\})$ (5.65) and impose the condition on each spin subsystem

$$(\hat{p}_\mu \hat{\Gamma}^\mu_{(k)} - m/2) f(x, z_{(1)}, \dots, z_{(2j)}) = 0, \tag{5.68}$$

Rewriting this equations in four-component form, we obtain Bargmann-Wigner equations

$$(\hat{p}_\mu \gamma^\mu_{(k)} - m)_{\alpha_k\beta_k}\psi_{\beta_1\ldots\beta_k\ldots\beta_{2j}}(x) = 0. \tag{5.69}$$

As a consequence (5.68) one may obtain a system, including only equations for whole system,

$$\begin{aligned}&(\hat{\mathbf{p}}^2 - m^2)f(x, z_{(1)}, \ldots, z_{(2j)}) = 0,\\ &(\hat{p}_\mu\hat{\Gamma}^\mu - ms)f(x, z_{(1)}, \ldots, z_{(2j)}) = 0, \quad \hat{\Gamma}^\mu = \sum \hat{\Gamma}^\mu_{(k)}, \quad s = j,\end{aligned} \tag{5.70}$$

which are analogous to the system (5.35)-(5.36). Notice, that the connection of the the Rarita-Schwinger and Bargmann-Wigner equations with Bhabha equations was also investigated in [49].

One may obtain the equations for massive tensor field, characterized by integer spin n, considering functions $f(x^\mu, \{q^\mu_{(i)}, q^{\mu\nu}_{(i)}\})$, $i = 1, \ldots n$, which are linear on $q_{(i)}$ and symmetric with respect to permutations of $q_{(i)}$ (where $q_{(i)}$ is given by (5.54)). Let us write n Duffin-Kemmer equations, $k = 1, \ldots n$,

$$(\hat{p}_\mu\hat{\Gamma}^\mu_{(k)} - m)f(x, \{q_{(i)}\}) = 0, \tag{5.71}$$

where $\hat{p}_\mu\hat{\Gamma}^\mu_{(k)}$ is given by (5.56). At $k = 1$ we obtain (dots replace the indices, connected with $q_{(i)}$, $i > 1$)

$$F_{\mu\nu\ldots}(x) = \partial_\mu\Phi_{\nu\ldots}(x) - \partial_\nu\Phi_{\mu\ldots}(x), \qquad m^2\Phi_{\nu\ldots}(x) = \partial^\mu F_{\mu\nu\ldots}(x). \tag{5.72}$$

Excluding $F_{\mu\nu\ldots}(x)$ and taking into account $[\partial_\mu, \partial_\nu] = 0$, we obtain

$$(\hat{p}^2 - m^2)\Phi_{\mu\ldots}(x) = 0, \quad \partial^\mu\Phi_{\mu\ldots}(x) = 0,$$

where functions $\Phi_{\mu_1\mu_2\ldots\mu_n}(x)$ are transformed under representation $(\frac{n}{2} - l\ \frac{n}{2} - l)$, $l = 0, \ldots, [n/2]$ and correspond to the terms $\Phi_{\mu_1\mu_2\ldots\mu_n}(x)q^{\mu_1}_{(1)}q^{\mu_2}_{(2)}\cdots q^{\mu_3}_{(n)}$ in the decomposition of $f(x, q_{(i)})$. One should impose the subsidiary condition $\Phi_{\mu\mu\ldots\mu_n}(x) = 0$ to obtain IR $(\frac{n}{2}\ \frac{n}{2})$. (Notice, that there is not necessary to impose subsidiary conditions, using only one set of spin variables, since $q_\mu q^\mu = 0$ (see (5.54)) and redundant components are not contained in $f(x, q)$.) As a result one may obtain

$$(\hat{p}^2 - m^2)\Phi_{\mu_1\mu_2\ldots\mu_n}(x) = 0, \quad \partial^\mu\Phi_{\mu\mu_2\ldots\mu_n}(x) = 0, \quad \Phi_{\mu\mu\ldots\mu_n}(x) = 0. \tag{5.73}$$

Repeating the previous arguments for functions $f(x^\mu, \{q^\mu_{(i)}, q^{\mu\nu}_{(i)}\}, z^\alpha, \bar{z}_{\dot{\alpha}})$ linear on $q_{(i)}$ and z, one can obtain a system

$$(\hat{p}_\mu\gamma^\mu - m)\Phi_{\mu_1\mu_2\ldots\mu_n}(x) = 0, \quad \partial^\mu\Phi_{\mu\mu_2\ldots\mu_n}(x) = 0, \quad \Phi_{\mu\mu\ldots\mu_n}(x) = 0, \tag{5.74}$$

where $\Phi_{\mu_1\ldots\mu_n}(x)$ is a four-component column, consisting of $\Phi_{\mu_1\ldots\mu_n\alpha}(x)$ and $\Phi_{\mu_1\ldots\mu_n}{}^{\dot{\alpha}}(x)$. This system is one of the standard forms of Rarita-Schwinger equations. The transition to another form is described, for example, in [7].

Let us consider more detail the case $s = 3/2$. A particle are described by function

$$f(x^\mu, q^\mu, q^{\mu\nu}, z^\alpha, \bar{z}_{\dot{\alpha}}) = F_{\mu\nu\alpha}(x)q^{\mu\nu}z^\alpha + F_{\mu\nu}{}^{\dot{\alpha}}(x)q^{\mu\nu}\bar{z}_{\dot{\alpha}} + \Phi_{\mu\alpha}(x)q^\mu z^\alpha + \Phi_\mu{}^{\dot{\alpha}}(x)q^\mu z_{\dot{\alpha}}.$$

Spin subsystems obey the Dirac and Duffin-Kemmer equations correspondingly,

$$(\hat{p}_\mu \hat{\Gamma}^\mu_{(1)} - m/2) f(x,q,z) = 0, \quad (\hat{p}_\mu \hat{\Gamma}^\mu_{(2)} - m) f(x,q,z) = 0, \tag{5.75}$$

where operator $\hat{\Gamma}^\mu_{(1)}$ act only on z, and $\hat{\Gamma}^\mu_{(2)}$ act only on q. The sum of these equations is an equation of the form (5.36) or (5.70), describing with Klein-Gordon equation (that also is a consequence of (5.75)) the system with mass m and spin 3/2 as a whole,

$$\left(\hat{p}_\mu (\hat{\Gamma}^\mu_{(1)} + \hat{\Gamma}^\mu_{(2)}) - 3m/2\right) f(x,q,z) = 0, \quad (\hat{p}^2 - m^2) f(x,q,z) = 0. \tag{5.76}$$

On the other hand, writing equations (5.75) in component-wise form, taking into account the relation $[\partial_\mu, \partial_\nu] = 0$ and excluding $F_{\mu\nu\alpha}$ and $F_{\mu\nu}{}^{\dot{\alpha}}$, one can obtain standard form of Rarita-Schwinger equations for spin 3/2. In contrast to the system (5.76) both these equations and initial system (5.75) contain the subsidiary conditions, which fix the states of spin subsystems.

Apart from the equations with subsidiary conditions, the Ivanenko-Landau-Kähler (or Dirac-Kähler) equation [50,51] also arises as an equation for particles with composite spin. Let us write some linear on $z_{(1)}$ and $z_{(2)}$ scalar function $f(x, z_{(1)}, z_{(2)})$ in the form

$$f(x, z_{(1)}, z_{(2)}) = Z_{(1)} \Psi(x) Z^\dagger_{(2)} = \sum_{i,j=1}^{4} Z_{(1)i} \Psi_{ij}(x) Z^*_{(2)j}, \tag{5.77}$$

where $Z = Z_D = (z^1\, z^2\, \bar{z}_1\, \bar{z}_2)$, and $\Psi(x)$ is a 4×4 matrix with a transformation rule

$$\Psi'(x') = \check{U} \Psi(x) (\check{U})^\dagger, \quad \check{U} = \mathrm{diag}\{U, (U^{-1})^\dagger\},$$

in contrast to the transformation rule $\Psi'_D(x') = \check{U} \Psi_D(x)$ of Dirac spinor (5.48). Let us impose the equation on the first ("left") spin subsystem,

$$(\hat{p}_\mu \hat{\Gamma}^\mu_{(1)} - m/2) f(x, z_{(1)}, z_{(2)}) = 0, \tag{5.78}$$

and do not impose any conditions on the second ("right") spin subsystem (imposing the same equation, we obtain the Bargmann-Wigner equations for spin 1). Writing (5.78) in component-wise form, one can obtain Ivanenko-Landau-Kähler equation in spinor matrix representation,

$$(\hat{p}_\mu \gamma^\mu - m) \Psi(x) = 0. \tag{5.79}$$

According to (5.79), 16 components $\Psi_{ij}(x)$ obey Klein-Gordon equation, therefore mass is equal to m. Spin of both subsystems is equal two 1/2. The spin of system is indefinite, and there are both spin 0 and spin 1 components.

The consideration of this equation mainly connected with the attempts to describe fermions by the antisymmetric tensor fields (see, for example, [52–54] and also [55] as a good introduction). In contrast to papers concerning equations with subsidiary conditions, in latter case the spin subsystems ("left-spin" and "right-spin", [52,54]) and its characteristics are investigated.

Considering the functions of one set of space coordinates x and several sets of spin coordinates z, we have obtained the equations, which are widely discussed in bibliography. However, it is not clear, which type of objects are described by these equations. At least, that is not elementary particles in usual sense, because of the existence of supplementary conditions on spin subsystems. Moreover, these equations are inconsistent or lead to acasuality when minimal electromagnetic interaction is introduced (of course, excluding the cases, when spin is not composite, exhausted by $s = 1/2$ and $s = 1$, and Ivanenko-Landau-Kähler equation, that is the equation only on a subsystem).

From mathematical point of view there is some interest to consider equations on $f(x_{(1)}, \ldots, x_{(n)}, z)$ for particles, which are elementary in spin space and composite in usual space.

F. Discussion

We have obtained the system (5.35)-(5.37), allowing the parity transformation and describing a particle with fixed mass m and spin s, as the resalt of group-theoretical classification of scalar functions $f(x, z)$ on the Poincaré group. One may obtain the same system in multicomponent matrix form using both traditional approaches to the theory of relativistic wave equations.

It is well known [56,57] that earlier attempts at formulating theory for higher spin followed two distinct lines.

Within the first approach, going backwards to [45,46], one considers symmetric spinors and tensors. Dirac or Proca equations are imposed on every index **separately**, that one may treat as fixing of the states of spin subsystems. In the approach, considered in this paper, that correspond to particles with composite spin. The Dirac-Fiertz-Pauli equations, the Rarita-Schwinger equations and the Bargmann-Wigner equations are examples of this kind. In free case, considered in detail in [7], these equations describe particles of given mass m and spin s. But the number of equations exceeds the number of field components. The presence of "subsidiary conditions" which supplement the field equations is an essential feature of these theories. One therefore has an overdetermined set of equations which, although consistent in the free-field case, invariably becomes self-contradictory or leads to acasuality when interaction is introduced [56–62].

Within the other approach, going backwards to [43,42,63,41], see also [26], one considers Poincaré-invariant equations, linear on $\hat{p}^{\mu}$. The problem of minimal electromagnetic coupling for Bhabha equations are considered in papers of Krajcik and Nieto (see [64]; it contains references to the six earlier papers). The theory is casual with minimal electromagnetic coupling [64], but in general case $s > 1$ describe multi-mass systems, $m_i s_i = ms$.

The system (5.35)-(5.37) lie on a halfway between these lines.

In the first place, as it was shown above, one may extract this system from Bargmann-Wigner equations, leaving only conditions on a particle as whole. As before, after the rejection of the conditions on spin subsystems the equations describe fixed mass and spin. Moreover, one can rewrite these equations as higher order equations on the variables, transforming under IR of the Poincaré group.

In the second place, one may consider a Bhabha equation, which corresponds to symmetric representation $T_{[2s\,0]}$ of the de Sitter group $Sp(4, R) \sim SO(3, 2)$ and is the direct generalization of Dirac and spin 1 Duffin-Kemmer equations for higher spin. Supplementing this equation by the condition of mass irreducibility, namely by Klein-Gordon equation, one may obtain the system (5.35)-(5.37).

Which conditions do we neglect in both cases? Excluding subsidiary conditions on the spin subsystems, we neglect the requirement, that equations must be imposed on every index of symmetric tensor or spinor separately. Supplementing the symmetric Bhabha equation by the condition of mass irreducibility, we neglect the requirement, that all equations of the system must be first-order equations on $\hat{p}^\mu$. It is difficult to bind this requirements with some physical conditions. Traditionally used in one the approach, these requirements are not imposed in another one.

Thus, the comparison with both traditional lines once more show, that the system (5.35)-(5.37) is the minimal set of equations, fixing mass and spin and allowing the parity transformation, with the true number of independent components.

The solutions of the system (5.35)-(5.37) have the components, transformed under $2S+1$ IR (j_1, j_2), $j_1 + j_2 = s$, of the Lorentz group. But these components, corresponding to chiralities $\lambda = j_1 - j_2$, are not independent. In contrast to left generators of Poincaré group, operators $\hat{\Gamma}_\mu$ do not commute with chirality operator and combine $2s+1$ representations of the Lorentz group into one IR of the de Sitter group $SO(3,2)$. One may express all components in terms of the components, corresponding to chiralities $\pm\lambda$ and rewrite the system of first order equations on a reducible representation as higher order system on IR of Poincaré group. The condition of spin irreducibility (5.21) is a consequence of (5.35)-(5.37).

Let us briefly consider the problem of equivalence of the different relativistic wave equations. In the case of free field, using the relation

$$[\partial_\mu, \partial_\nu] = 0, \tag{5.80}$$

one can establish the equivalence of wide class of relativistic wave equations, for example, the Rarita-Schwinger equations and the Bargmann-Wigner equations [7] (all solutions of the system are also solutions of another one). Subsidiary conditions on spin subsystems in free case are consequences of the equations, which describe system as whole, and the equations (5.35)-(5.37) are also equivalent to previous ones.

But, for systems with interaction the subsidiary conditions on spin subsystems lead to independent equations. It is obvious, that the coupling, which is minimal for one system, is not minimal for another "equivalent" system, if one use the relation (5.80) to prove this equivalence in the free case. These equations will differ by the terms proportional to the commutator of covariant derivatives

$$[D_\mu, D_\nu] = igF_{\mu\nu}. \tag{5.81}$$

Therefore, when an interaction is introduced, the system of equations can be found inconsistent, if some equations are the consequences of another taking account of (5.80). This situation is typical for the equations with subsidiary conditions as one can see on the example of Bargmann-Wigner equations for spin 1 and 3/2 [7].

The system (5.35)-(5.37) do not include some subsidiary conditions on spin subsystems, but describe a particle with definite mass and spin, in contrast to Bhabha equations. Thus, one may hope to construct the consistent theory for higher spin with minimal coupling by this way.

VI. THE EQUATIONS FOR FIXED SPIN AND MASS: GENERAL FEATURES

Consider the general properties of the obtained equations, describing a particle with definite mass m and spin s, in two dimensions

$$\hat{p}^2 f(x,\alpha) = m^2 f(x,\alpha), \tag{6.1}$$
$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,\alpha) = msf(x,\alpha), \tag{6.2}$$

in three dimensions

$$\hat{p}^2 f(x,z) = m^2 f(x,z), \tag{6.3}$$
$$\hat{p}_\mu \hat{S}^\mu f(x,z) = msf(x,z), \tag{6.4}$$
$$\hat{\mathbf{S}}^2 f(x,z) = S(S+1) f(x,z), \tag{6.5}$$

in four dimensions

$$\hat{p}^2 f(x,z) = m^2 f(x,z), \tag{6.6}$$
$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,z) = msf(x,z), \tag{6.7}$$
$$\hat{S}_{ab}\hat{S}^{ab} f(x,z) = 4j(j+2) f(x,z). \tag{6.8}$$

In all cases first equation (condition of the mass irreducibility) is the equation on the eigenvalues of Casimir operator of the Poincaré group. But the other equations, although seem similar, has distinct origin in even and odd dimensions. That connected with the distinct role of space inversion.

In 2+1 dimensions other equations (6.4)-(6.5) are the equations on the eigenvalues of Casimir operator of the Poincaré group and the Lorentz subgroup.

In even dimensions space inversion combine two equivalent IR of proper Poincaré group, labelled by chiralities $\pm\lambda$, into IR of improper Poincaré group. The system, fixing the eigenvalues of the Casimir operators of Poincaré and Lorentz group, in general case is not symmetric with respect to space inversion. If one reject equations, which fix chirality, (in 3+1 that correspond to the transition to the system (5.20)-(5.22)), then in the rest frame it is easy to see an abundant number of independent components. Thus, it is necessary to construct equation, connecting the states with different chiralities. Since generators of Poincaré group commute with chirality operator, then it is necessary to use supplementary operators $\hat{\Gamma}^\mu$, which extend Lorentz group $SO(N-1,1)$ up to $SO(N-1,2)$ group with the maximal compact subgroup $SO(N-1)\otimes SO(2)$. Operator $\hat{\Gamma}^0$ is the generator of compact $SO(2)$-subgroup.

Third equation of the system fix the homogeneity power $2S$ or $2j$ of the functions $f(x,z)$ on z, defining IR of the Lorentz group in 2+1 dimensions or of the de Sitter group in 3+1 dimensions. (In 1+1 dimensions there exist analogous equation $\hat{\Gamma}_a\hat{\Gamma}^a f(x,\alpha) = s(s+1)f(x,\alpha)$, but this equation connected with the definition of $\hat{\Gamma}^\mu$.)

A positive (half-)integer $S = s$ or $j = s$ correspond to the space of polynomials of the power $2s$ on z, transforming under finite-dimensional nonunitary IR of Lorentz (or de Sitter) group.

A negative $S = -s$ correspond to infinite-dimensional unitary IR. The unitary property allow to combine probability amplitude interpretation and relativistic invariance (the desirability of this combination was stressed by Dirac in the paper "Relativity and quantum mechanics" [28]). Thus, equations under consideration allow two approaches for the description of the same spin, by means of both finite-dimensional nonunitary and infinite-dimensional unitary IR.

In 1+1 and 2+1 dimensions there is the possibility of the existence of particles with fractional spin, since the group $SO(2,1)$ does not contain compact Abelian subgroup. However, the description of massive particles can by given only in terms of infinite-dimensional IR of the group $SO(2,1)$. That is another reason to consider infinite-dimensional IR.

Fixing the IR of the Lorentz or de Sitter group with the help of the third equation of the system, one can transit to usual multicomponent matrix description. This transition is realized by the separation of space and spin variables, $f(x,z) = \sum \psi_k(x)\phi_k(z)$, where $\phi_k(z)$ form the basis of the representation space of the Lorentz (or de Sitter) group. Thus, depending on the choice of the solution of the third equation, second equation in matrix representation is either finite-component equation or infinite-component equation of Majorana type.

For fundamental spinor IR the action of differential operators $2\hat{S}^\mu$ in 2+1 dimensions and $2\hat{\Gamma}^\mu$ in 1+1 and 3+1 dimensions in the space of functions $f(x,z)$ on the Poincaré group can be rewritten in terms of action of corresponding γ-matrices on the functions $\psi(x)$.

Differential operators $\hat{\Gamma}^\mu$ and matrices $\gamma^\mu/2$ obey the same commutation relations,

$$[\hat{\Gamma}^\mu, \hat{\Gamma}^\nu] = -i\hat{S}^{\mu\nu}, \quad [\hat{S}^\mu, \hat{S}^\nu] = -i\epsilon^{\mu\nu\rho}\hat{S}_\rho.$$

In 3+1 dimensions operators $\hat{\Gamma}^\mu$ and $\hat{S}^{\mu\nu}$ obey the commutation relations of generators of $SO(3,2)$ group, see (5.33).

Anticommutation relations for operators $\hat{S}^\mu$ in 2+1 and $\hat{\Gamma}^\mu$ in 1+1 and 3+1 dimensions analogous with relations for γ-matrices,

$$\{\hat{S}^\mu, \hat{S}^\nu\}_+ = \frac{1}{2}\eta^{\mu\nu}, \quad \{\hat{\Gamma}^\mu, \hat{\Gamma}^\nu\}_+ = \frac{1}{2}\eta^{\mu\nu},$$

are valid only for fundamental spinor IR. That is group-theoretical property, connected with the fact, that in these IR the double action of lowering or raising operators on any state give zero as a result. (Anticommutation relations along with spinor IR of orthogonal groups also take place for fundamental N-dimensional IR of $Sp(N)$ and $SU(N)$ groups [65].)

At $s = 1/2$ and $s = 1$ first equation of the system (condition of mass irreducibility) is the consequence of (6.4) or (6.7). At $s > 1$ it is necessary to consider both equations.

Consider some characteristics of the equations, connected with finite-dimensional IR of the Lorentz group. For the second equation of the system (without the condition of mass irreducibility) the component j^0 of the current vector is positive definite only at $s = 1/2$ and the energy density $-T^{00}$ (see (4.26)) is positive definite only at $s = 1$. However, for the system as a whole the component j^0 of the current vector is positive definite for any half-integer spin and energy density is is positive definite for any integer spin.

For the case of infinite-dimensional equations, considered in 2+1 dimensions, energy is positive definite for any spin and j^0 is positive or negative definite in accordance with the sign of charge.

VII. CONCLUSION

In present paper we have considered scalar field on the Poincaré group.

Transformations of the left GRR of the Poincaré group

$$T(g)f(g_0) = f(g^{-1}g_0), \quad g_0 = (x, z), \tag{7.1}$$

where x are coordinates on Minkowski space and z are coordinates on the Lorentz group, one can treat as coordinate transformation on Poincaré group as a whole, induced by coordinate transformation in Minkowski space,

$$T(g)f(x) = f(g^{-1}x).$$

By the other hand, (7.1) can be rewritten as

$$f'(x', z') = f(x, z), \quad x' = gx, \;\; z' = gz, \;\; g \in M(N-1, 1). \tag{7.2}$$

which define a scalar field in the extended space. The equivalence of the concepts of the left GRR and scalar field on the group allow to combine the powerful mathematical method of harmonic analysis on a group with physical interpretation.

The consideration of the functions $f(x, z)$ guarantee the possibility to describe any spin, because any IR of a group is equivalent to one of sub-representation of GRR. Thus, scalar field on the Poincaré group is an uniform field, containing all masses and spins.

As a consequence of this uniformity, we have:

1. The uniformity of spin operators. The spin projection operators are differential operators on z.

2. For this scalar field and, thus, for arbitrary spin, discrete transformations C, P, T are defined as automorphisms of Poincaré group.

3. Relativistic wave equations arise under classification of the functions on the group by eigenvalues of invariant operators and have the same form for arbitrary spin.

The transition to the usual multicomponent description by functions $\psi_n(x)$ correspond to the separation of space-time and spin variables, $f(x, z) = \sum \phi_n(z)\psi_n(x)$, where $\phi_n(z)$ and $\psi_n(x)$ transformed under contragredient representations of the Lorentz group. Many-particle systems are described by functions $f(x_{(1)}, z_{(1)}, \ldots, x_{(n)}, z_{(n)})$.

General scheme of analysis of fields on the Poincaré in different dimensions, developed above, allow to obtain equations without subsidiary conditions, describing particle with fixed mass and spin. These equations characterized by uniform basic properties in 2,3,4 dimensions.

The properties of higher spin equations, corresponding to finite-dimensional representations of the Lorentz group, are similar to the same for spin 1/2 and spin 1. In particular, for particles with half-integer spin the component j^0 of a current vector is positive definite, and for particles with integer spin the energy density (defined in terms of energy-momentum tensor) is positive definite.

The approach also give the possibility for regular construction of the positive energy wave equations, connected with infinite-dimensional representations of the Lorentz group and allowing probability amplitudes interpretation. In present paper we consider such the equations in 2+1 dimensions.

Besides the equations on one scalar function on the Poincaré group, considered above, the present approach allow to reproduce practically all known relativistic wave equations. However, in general case these equations either interlock several functions $f(x,z)$ (as Gel'fand-Yaglom equations), or describe objects with composite spin, corresponding to functions $f(x, z_{(1)}, \ldots z_{(n)})$ of one set of space-time coordinates x and several sets of spin coordinates z.

In particular, equations which in general case are inconsistent or noncasual with minimal electromagnetic coupling (or equations with subsidiary conditions, namely the Dirac-Fiertz-Pauli equations, the Rarita-Schwinger equations and the Bargmann-Wigner equations) arise as equations for systems with composite spin. The latter explain the difficulties in attempts to describe interacting elementary particles by these equations.

The consideration of the field on the Poincaré group allows to ensure also essential progress in the problem of practical computations for multicomponent equations.

As was noted in [66], the general investigation of Gel'fand-Yaglom equations "revealed a number of interesting features, but ... the use of such equations (or more accurately, systems of a large or infinite number of equations) for any practical computations is not possible". It was suggested in [67] for the solution of this problem to consider equations for functions, dependent not only on coordinates x, but also on some other quantities (see also [66]). In particular, in [68,69] one-component functions of x^μ and q^μ was used for the construction of positive energy wave equations. Recently the functions, depending on q^μ, Majorana or Dirac z-spinors are considered in the main not in the context of relativistic wave equations, but in context of models of spinning particles [13–19].

In the present approach due to use of spin differential operators instead of finite or infinite-dimensional matrices there is no essential distinction in the considering of the equations, connected with the representations of the Lorentz group of different dimensions. Therefore, the present approach is adequate to work with higher spin and positive energy wave equations, connected with infinite-dimensional representations of the Lorentz group.

Thus, the scalar field on the Poincaré group allows to give an uniform description of arbitrary spin.

VIII. APPENDIX. BASES OF IR AND S^μ MATRICES OF 2+1 LORENTZ GROUP

Spin projection operators $\hat{S}^\mu$, which act in the space of the functions $f(x,z)$ on $x = (x^\mu)$ and two complex variables $z^1 = z_2$, $z^2 = -z_1$, $|z_1|^2 - |z_2|^2 = 1$, have the form

$$\hat{S}^\mu = \frac{1}{2}(z\gamma^\mu\partial_z - \overset{*}{z}\overset{*}{\gamma}{}^\mu\partial_{\overset{*}{z}}), \quad z = (z_1\; z_2),\; \partial_z = (\partial/\partial z_1\; \partial/\partial z_2)^T, \tag{8.1}$$

where $\gamma^\mu = (\sigma_3, i\sigma_2, -i\sigma_1)$.

The polynomials of the power $2S$ on z, which correspond to finite-dimensional IR T^0_S of 2+1 Lorentz group, can be written in the form

$$T^0_S: \quad f_S(x,z) = \sum_{n=0}^{2S} \phi_n(z)\psi^n(x), \quad \phi_n(z) = (C^n_{2S})^{1/2}(z_1)^{2S-n}(z_2)^n, \quad s^0 = S - n, \tag{8.2}$$

where s^0 is eigenvalue of $\hat{S}^0$, and C_{2S}^n are binomial coefficients. The quasipolynomials of the power $2S \leq -1$, which correspond to infinite-dimensional unitary IR $T_S^{\pm}$ of 2+1 Lorentz group, can be written in the form

$$T_S^+ : \quad f_S(x,z) = \sum_{n=0}^{\infty} \phi_n(z)\psi^n(x), \quad \phi_n(z) = (C_{2S}^n)^{1/2} (\overset{*}{z}_1)^{2S-n} (\overset{*}{z}_2)^n, \quad s^0 = -S + n,$$

$$T_S^- : \quad f_S(x,z) = \sum_{n=0}^{\infty} \phi_n(z)\psi^n(x), \quad \phi_n(z) = (C_{2S}^n)^{1/2} (z_1)^{2S-n} (z_2)^n, \quad s^0 = S - n, \qquad (8.3)$$

$$C_{2S}^n = \left(\frac{(-1)^n \Gamma(n-2S)}{n! \Gamma(-2S)} \right)^{1/2}.$$

There is a correspondence between the action of differential operators $\hat{S}^\mu$ on the functions $f(x,z) = \phi(z)\psi(x)$ and the multiplication of matrices $\hat{S}^\mu$ by columns $\psi(x)$, composed of $\psi^n(x)$, $\hat{S}^\mu f(x,z) = \phi(z) S^\mu \psi(x)$. For the finite-dimensional representations T_S^0 we have $(S^0)^\dagger = S^0$, $(S^k)^\dagger = -S^k$,

$$\begin{aligned}
(S^0)^n{}_{n'} &= \delta_{nn'}(S-n), \qquad n = 0, 1, \ldots, 2S, \\
(S^1)^n{}_{n'} &= -\frac{i}{2} \left(\delta_{n\; n'+1} \sqrt{(2S-n+1)n} + \delta_{n+1\; n'} \sqrt{(2S-n)(n+1)} \right), \\
(S^2)^n{}_{n'} &= -\frac{1}{2} \left(\delta_{n\; n'+1} \sqrt{(2S-n+1)n} - \delta_{n+1\; n'} \sqrt{(2S-n)(n+1)} \right),
\end{aligned} \qquad (8.4)$$

Matrices S^μ satisfy the condition $(S^\mu)^\dagger = \Gamma S^\mu \Gamma$, where Γ is a diagonal matrix, $(\Gamma)_{nn'} = (-1)^n \delta_{nn'}$. The substitution $z \to \overset{*}{z}$ in (8.2) changes only signs of S^0 and S^2. For representations T_S^+ of discrete positive series is valid $(S^\mu)^\dagger = S^\mu$,

$$\begin{aligned}
(S^0)^n{}_{n'} &= \delta_{nn'}(-S+n), \qquad n = 0, 1, 2, \ldots, \\
(S^1)^n{}_{n'} &= -\frac{1}{2} \left(\delta_{n\; n'+1} \sqrt{(n-1-2S)n} + \delta_{n+1\; n'} \sqrt{(n-2S)(n+1)} \right), \\
(S^2)^n{}_{n'} &= \frac{i}{2} \left(\delta_{n\; n'+1} \sqrt{(n-1-2S)n} - \delta_{n+1\; n'} \sqrt{(n-2S)(n+1)} \right),
\end{aligned} \qquad (8.5)$$

For T_S^- matrices S^1 have the same form, whereas S^0, S^2 change only their signs.

The case of representations of principal series, which is not bounded by the highest (lowest) weight, was considered in [8].

---

[1] D. Zhelobenko and A. Schtern, *Representations of Lie Groups* (Nauka, Moscow, 1983).

[2] N. Vilenkin, *Special Functions and Theory of Group Representations* (Nauka, Moscow, 1965).

[3] N. Vilenkin and A. Klimyk, *Representations of Lie Groups and Special Functions* (Kluwer Acad. Publ., Dordrecht, 1991), Vol. 1.

[4] A. Barut and R. Raczka, *Theory of Group Representations and Applications* (PWN, Warszawa, 1977).

[5] E. Wigner, Ann. Math. **40**, 149 (1939).
[6] G. Mackey, *Induced Representations of Groups and Quantum Mechanics* (Benjamin, New Iork, 1968).
[7] Y. Ohnuki, *Unitary representations of the Poincaré group and relativistic wave equations* (World Scientific, Singapore, 1988).
[8] D. Gitman and A. Shelepin, J. Phys. A **30**, 6093 (1997).
[9] I. Benn and R. Tucker, *An Introduction to Spinors and Geometry with Applications in Physics* (Adam Hilger, Bristol, 1988).
[10] F. Berezin, *The Method of Second Quantization* (Academic, New York, 1966).
[11] L. Landau and E. Lifschitz, *Quantum Mechanics* (Pergamon, Oxford, 1977), Vol. 3 of Course of Theoretical Physics.
[12] E. Wigner, in *Theoretical Physics*, edited by A. Salam (IAEA, Trieste, 1963), p. 59.
[13] Y. Kim and E. Wigner, J. Math. Phys. **28**, 1175 (1987).
[14] L. Biedenharn, H. Braden, P. Truini, and H. Van Dam, J. Phys. A **21**, 3593 (1988).
[15] Z. Hasiewicz and P. Siemion, Int. J. Mod. Phys. A **7**, 3979 (1992).
[16] S. Kuzenko, S. Lyakhovich, and A. Segal, Int. J. Mod. Phys. A **10**, 1529 (1995).
[17] S. Lyakhovich, A. Segal, and A. Sharapov, Phys. Rev. D **54**, 5223 (1996).
[18] R. Jackiw and V. Nair, Phys. Rev. D **43**, 1933 (1991).
[19] M. Plyushchay, Int. J. Mod. Phys. A **7**, 7045 (1992).
[20] J. Sucher, J. Math. Phys. **4**, 17 (1963).
[21] K. Samarov, Sov. Phys. – Dokl. **29**, 909 (1984).
[22] H.-J. Briegel, B.-G. Englert, and Süssmann, Z. Naturforsch., A **46**, 933 (1991).
[23] J. Smith, Second Quantization of the Square-Root Klein-Gordon Operator, Microscopic Causality, Propagators, and Interaction, Preprint UCD/II RPA 93-13, University of California, Davis, 1993.
[24] A. Shelepin, Phys. At. Nucl. **60**, 206 (1997).
[25] E. Majorana, Nuovo Cimento **9**, 335 (1932).
[26] I. Gel'fand, R. Minlos, and Z. Shapiro, *Representations of the rotation and Lorentz groups and their applications* (Pergamon press, Oxford, 1963).
[27] D. Stoyanov and I. Todorov, J. Math. Phys. **9**, 2146 (1968).
[28] P. Dirac, Fields and Quanta **3**, 139 (1972).
[29] D. L'vov, A. Shelepin, and L. Shelepin, Phys. At. Nucl. **57**, 1083 (1994).
[30] M. Hamermesh, Ann. Phys. (N.Y.) **9**, 518 (1960).
[31] J.-M. Levy-Leblond, J. Math. Phys. **4**, 776 (1963).
[32] B. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).
[33] B. Binegar, J. Math. Phys. **23**, 1511 (1982).
[34] C. Aragone, S. Deser, and Z. Yang, Ann. Phys. (N.Y.) **179**, 76 (1987).
[35] M. Vasiliev and I. Tyutin, Teor. Mat. Fiz. **113**, 1244 (1997).
[36] A. Vshivtsev, B. Magnitskii, V. Zhukovskii, and K. Klimenko, Phys. Part. Nucl. **29**, 523 (1998).
[37] P. Townsend, K. Pilch, and P. Van Nieuwenhuizen, Phys. Lett. B **136**, 38 (1984).
[38] S. Deser and R. Jackiv, Phys. Lett. B **139**, 371 (1984).
[39] I. Gel'fand, M. Graev, and N. Vilenkin, *Generalized Functions* (Academic Press, New York, 1966), Vol. 5.
[40] A. Kihlberg, Ark. Fys. **28**, 121 (1964).

[41] I. Gel'fand and A. Yaglom, Zh. Ehksp. Teor. Fiz. **18**, 703 (1948).
[42] H. Bhabha, Rev. Mod. Phys. **17**, 200 (1945).
[43] J. Lubanski, Physica **9**, 310 (1942).
[44] L. Castell, Nuovo Cimento A **50**, 945 (1967).
[45] P. Dirac, Proc. R. Soc. London, Ser. A **155**, 447 (1936).
[46] M. Fierz and W. Pauli, Proc. R. Soc. London, Ser. A **173A**, 211 (1939).
[47] V. Bargmann and E. Wigner, Proc. Nat. Acad. USA **34**, 211 (1948).
[48] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).
[49] R.-K. Loide, I. Ots, and R. Saar, J. Phys. A **30**, 4005 (1997).
[50] D. Ivanenko and L. Landau, Z. Phys. **48**, 340 (1928).
[51] E. Kähler, Rend. Math. **21**, 425 (1962).
[52] I. Benn and R. Tucker, Commun. Math. Phys. **89**, 341 (1983).
[53] J. Bullinaria, Ann. Phys. (N.Y.) **168**, 301 (1986).
[54] Y. Obukhov and S. Solodukhin, Theor. Math. Phys. **94**, 198 (1993).
[55] D. Ivanenko, Y. Obukhov, and S. Solodukhin, On antisymmetric tensor representation of the Dirac equation, Preprint IC/85/2, ICTP, Trieste, 1985.
[56] A. Wightman, in *Lecture Notes in Physics* (Springer-Verlag, Berlin, 1978), Vol. 73, pp. 1–101.
[57] W.-K. Tung, Phys. Rev. **156**, 1385 (1967).
[58] G. Velo and D. Zwanziger, Phys. Rev. **186**, 1337 (1969).
[59] G. Velo and D. Zwanziger, Phys. Rev. **188**, 2218 (1969).
[60] G. Velo, Nucl. Phys. B **43**, 389 (1972).
[61] D. Zvanziger, in *Lecture Notes in Physics* (Springer-Verlag, Berlin, 1978), Vol. 73, pp. 143–164.
[62] A. Capri and R. Kobes, Phys. Rev. D **22**, 1967 (1980).
[63] Harish-Chandra, Phys. Rev. **71**, 793 (1947).
[64] R. Krajcik and M. Nieto, Phys. Rev. D **15**, 445 (1977).
[65] D. Gitman and A. Shelepin, Kratk. Soob. Fiz. (Lebedev Inst.) **11**, 21 (1998).
[66] V. Ginzburg, Acta Phys. Pol. **15**, 163 (1956).
[67] V. Ginzburg and I. Tamm, Zh. Ehksp. Teor. Fiz. **17**, 227 (1947).
[68] P. Dirac, Proc. R. Soc. London, Ser. A **322**, 435 (1971).
[69] P. Dirac, Proc. R. Soc. London, Ser. A **328**, 1 (1972).