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Abstract

We give a simple argument for the cancellation of the $\log(-k^2)$ terms (k is the gluon momentum) between the zero-temperature and the temperature-dependent parts of the thermal self-energy.

There have been many studies of thermal Green functions in gauge field theories [1–7], which show that their behavior at finite temperature is rather different from the one at zero temperature. In particular, it was recently pointed out by Weldon [8] that in QED, the logarithmic branch cut singularities cancel to one loop-order, in the thermal self-energy of the electron.

The purpose of this note is to show that in the Yang-Mills theory, a somewhat similar behavior occurs in the full gluon self-energy, which includes finite temperature effects. Of course, in this theory, the massless gluons are quite modified by these effects and the gluon propagator requires the Braaten-Pisarski resummation. Nevertheless, it is interesting to remark that, even before such a procedure is carried out, the one-loop $\log(-k^2)$ terms cancel in the sum of the $T = 0$ and the $T \neq 0$ contributions to the gluon self-energy. As we shall see, this happens because the $\log(-k^2)$ terms appear in the thermal part of the self-energy only in the combination $\log(-k^2/T^2)$. But one can show that the $\log(T^2)$ contributions have the same structure as the ultraviolet divergent terms which occur at zero temperature [9].

Consequently, the $\log(-k^2/T^2)$ terms combine directly with the $\log(-k^2/\mu^2)$ contributions which occur at $T = 0$ (μ is the renormalization scale), so that the $\log(-k^2)$ terms cancel in a simple way in the thermal self-energy of the gluon. The branch cut in the $\log(-k^2)$ contribution at $T = 0$ is associated with the imaginary part of the self-energy, which gives the rate of decay of a time-like virtual gluon into two real gluons. Although this contribution cancels at $T \neq 0$, there appear then additional, temperature-dependent logarithmic branch points. These singularities indicate processes not available at zero temperature, where particles decay or are created through scattering in the thermal bath.

To *one-loop* order, the thermal self-energy of gluons generally depends on three structure functions, Π^T, Π^L and Π^C [10]

$$\Pi^{ab}_{\mu\nu}(k_0, \vec{k}) = g^2 C_G \delta^{ab} \left(\Pi^T P^T_{\mu\nu} + \Pi^L P^L_{\mu\nu} + \Pi^C P^C_{\mu\nu} \right), \tag{1}$$

where the projection operators $P^{T,L}_{\mu\nu}$ are transverse with respect to the external four-momentum k^μ and satisfy: $k^i P^T_{i\nu} = 0$ and $k^i P^L_{i\nu} \neq 0$ [6,7]. Furthermore, the projection operator $P^C_{\mu\nu}$ can be written in the plasma rest frame as follows [10]

$$P^C_{\mu\nu} = \frac{1}{k^2} \left[\frac{k_\nu}{|\vec{k}|} \left(k_0 k_\mu - \eta_{\mu 0} k^2 \right) + \mu \leftrightarrow \nu \right]. \tag{2}$$

Although Π^C vanishes at $T = 0$ because of the Slavnov-Taylor identity, it is in general a non-vanishing function of the temperature, so that $k^\mu \Pi_{\mu\nu} \neq 0$ for the exact self-energy.

We will discuss here, for definiteness, the retarded thermal self-energy of the gluon, which is obtained by the analytic continuation $k_0 \to k_0 + i\epsilon$.(A rather similar analysis can be made in the case of time-ordered self-energy, following the approach presented in reference [11]). In order to illustrate in a simple way the mechanism of the cancellation of the $\log(-k^2)$ contributions, let us first consider the special case of the *Feynman gauge*, where Π^C vanishes even at finite temperature. Then, Π^T and Π^L can be expressed in the plasma rest frame in terms of linear combinations of Π^μ_μ and Π_{00}. After performing the integration over the internal energies q_0, Π^μ_μ and Π_{00} can be written as an integral over internal on-shell momenta $q = (|\vec{q}|, \vec{q})$, as follows

$$\Pi_{\mu}^{\mu\ ab} = g^2 C_G \delta^{ab} \left(\frac{T^2}{3} - 10 k^2 I_0 \right) \tag{3}$$

and

$$\Pi_{00}^{ab} = 2 g^2 C_G \delta^{ab} |\vec{k}|^2 \left(I_0 + 4\, I_1 \right). \tag{4}$$

where (x is the cosine of the angle between $\vec{k}$ and $\vec{q}$).

$$I_{0,1} = \frac{\mu^\epsilon}{(2\pi)^{3-\epsilon}} \int \frac{d^{3-\epsilon}\vec{q}}{2|\vec{q}|} \left(\frac{1}{k^2 + 2q \cdot k} + \frac{1}{k^2 - 2q \cdot k} \right) \left[\frac{\vec{q}^2}{k^2} \left(1 - x^2\right) \right]^{0,1} \left[\frac{1}{2} + N\left(\frac{|\vec{q}|}{T} \right) \right]. \tag{5}$$

The two terms in the last square bracket are associated respectively with the $T = 0$ and the $T \neq 0$ contributions (N is the Bose-Einstein distribution).

In order to express the integrations in (5) in terms of known functions, it is convenient to define the variable

$$K(x) = \frac{1}{4\pi i} \frac{k^2}{k_0 - |\vec{k}| x}. \tag{6}$$

Then it is straightforward to show that

$$I_0 = \frac{i\pi}{|\vec{k}|} \left(\frac{1}{4\pi} \right)^{\frac{3-\epsilon}{2}} \frac{\mu^\epsilon}{\Gamma\left(\frac{3-\epsilon}{2}\right)} \int_{K_-}^{K_+} dK \int_0^\infty d|\vec{q}| \frac{|\vec{q}|^{1-\epsilon}}{|\vec{q}|^2 + (2\pi K)^2} \left[\frac{1}{2} + N\left(\frac{|\vec{q}|}{T} \right) \right], \tag{7}$$

where

$$K_\pm \equiv K(\pm 1) = \frac{k_0 \pm |\vec{k}|}{4\pi i}. \tag{8}$$

The above form shows that the integrals appearing in the calculation of the gluon self-energy can be naturally expressed in terms of the quantities $K_\pm$ (which are proportional to the light-cone momenta $k_0 \pm k_3$, if one chooses, for example, the third axis along $\vec{k}$).

The $|\vec{q}|$ integration of $T = 0$ part of I_0, gives

$$\frac{i}{8\pi|\vec{k}|} \int_{K_-}^{K_+} dK \left[\frac{1}{\epsilon} - \log \frac{2\sqrt{\pi} K}{\mu} - \frac{\gamma}{2} + 1 \right]. \tag{9}$$

Using the fact that $\mathrm{Re} K(x) > 0$, the $|\vec{q}|$ integration of the $T \neq 0$ part of I_0 (where we may set $\epsilon = 0$), yields the result [12]

$$\frac{i}{8\pi|\vec{k}|}\int_{K_-}^{K_+} dK\left[\frac{T}{2K}+\log\frac{K}{T}-T\frac{d}{dK}\log\Gamma\left(1+\frac{K}{T}\right)\right], \tag{10}$$

where the logarithm of the gamma function is analytic when $K \to 0$. Then, the approximation

$$N\left(\frac{|\vec{q}|}{T}\right)=\frac{1}{\exp(|\vec{q}|/T)-1}\simeq\theta(T-|\vec{q}|)\left(\frac{T}{|\vec{q}|}-\frac{1}{2}\right), \tag{11}$$

would simply lead, after performing the $|\vec{q}|$ integration in equation (7), to the first two terms in the exact expression (10). As far as the $\log(K)$ contribution to the $T \neq 0$ part is concerned, one may effectively replace in Eq. (7), for small $|\vec{q}|$, $N(|\vec{q}|/T)$ by $-1/2$. Consequently, this contribution will cancel the $\log(K)$ term associated with the $T = 0$ part of I_0 (this cancellation can also be explicitly verified from Eqs. (9) and (10)).

By itself, the K-integration of the $\log(K/T)$ term in Eq. (10) gives the contribution

$$\frac{i}{16\pi|\vec{k}|}\left[(K_++K_-)\log\frac{K_+}{K_-}+(K_+-K_-)\log\frac{K_+K_-}{T^2}-2\,(K_+-K_-)\right]=$$
$$\frac{1}{32\pi^2}\left[\frac{k_0}{|\vec{k}|}\log\frac{k_0+|\vec{k}|}{k_0-|\vec{k}|}+\log\frac{-k^2}{16\pi^2T^2}-2\right]. \tag{12}$$

The emergence of the $\log(-k^2)$ term in the special combination $\log(-k^2/T^2)$, is a direct consequence of the fact that the integrand in Eq. (10) depends only on the dimensionless ratio K/T. Similarly, the $\log(K/\mu)$ term in Eq. (9) yields a contribution which, apart from sign, can be obtained from Eq. (12) by the replacement $T \to \mu$. Consequently, the $\log(-k^2)$ terms will cancel between the zero-temperature and the temperature-dependent contributions, leaving a net factor proportional to $\log(\mu^2/T^2)$. After calculating the contributions from the first and third terms in Eq. (10), we obtain the following result for the temperature-dependent part of I_0:

$$I_0(T)=\frac{1}{32\pi^2}\log\frac{\mu^2}{T^2}+\frac{iT}{16\pi|\vec{k}|}\log\frac{k_0+|\vec{k}|}{k_0-|\vec{k}|}+\frac{T}{8\pi i|\vec{k}|}\log\frac{\Gamma(1+K_+/T)}{\Gamma(1+K_-/T)}. \tag{13}$$

Next, consider the I_1 integral which can be written as:

$$I_1=\frac{i\pi}{|\vec{k}|^3}\left(\frac{1}{4\pi}\right)^{\frac{3-\epsilon}{2}}\frac{\mu^\epsilon}{\Gamma\left(\frac{3-\epsilon}{2}\right)}\int_{K_-}^{K_+} dK\left[\frac{k^2}{(4\pi\,K)^2}-\frac{i\,k_0}{2\pi\,K}-1\right]$$
$$\times\int_0^\infty d|\vec{q}|\,|\vec{q}|^{1-\epsilon}\left[1-\frac{(2\pi\,K)^2}{|\vec{q}|^2+(2\pi K)^2}\right]\left[\frac{1}{2}+N\left(\frac{|\vec{q}|}{T}\right)\right]. \tag{14}$$

Note that the $T = 0$ contribution, associated with the factor of 1 in the second square bracket, would apparently lead to a quadratically divergent integral, which however vanishes in the dimensional regularization scheme. On the other hand, this factor yields a leading thermal contribution which is quadratic in the temperature

$$I_1^{lead}(T) = \frac{T^2}{24|\vec{k}|^2}\left(1 - \frac{k_0}{2|\vec{k}|}\log\frac{k_0 + |\vec{k}|}{k_0 - |\vec{k}|}\right) \tag{15}$$

The $|\vec{q}|$-integration of the second term in the second square bracket of Eq. (14) is identical to the one which occurs in I_0, so that it gives analogous $\log(K)$ contributions which cancel between the $T = 0$ and thermal parts. As we have seen, only such contributions would give rise, after the K-integration, to individual $\log(-k^2)$ terms. It is possible to evaluate exactly all other temperature-dependent contributions to I_1, in terms of logarithmic functions and of Riemann's zeta functions with arguments $(1+K_\pm/T)$, which are analytic when $K_\pm \to 0$ [11]. Since the complete expression is rather involved, we indicate here, for simplicity, only the logarithmic temperature-dependent contributions to I_1:

$$I_1^{\log}(T) = -\frac{1}{192\pi^2}\log\frac{\mu^2}{T^2} - \frac{iTk^2}{64\pi|\vec{k}|^3}\log\frac{k_0 + |\vec{k}|}{k_0 - |\vec{k}|}. \tag{16}$$

In a general gauge, Π^μ_μ and Π_{00} will have a similar behavior (in particular, the leading T^2 contribution is gauge independent). In this case, the thermal contributions to $\Pi_C = k_\mu \Pi^\mu_0/|\vec{k}|$ are non-vanishing, and can be written as [13]

$$\Pi_C = \frac{(1-\xi)}{(2\pi)^3|\vec{k}|}\int\frac{d^3\vec{q}}{|\vec{q}|}\left[\left(\frac{k^2}{k^2+2k\cdot q} + \frac{1}{2}\frac{d}{dq_0}\frac{k\cdot q}{q_0}\right)\frac{k\cdot q k_0 - k^2 q_0}{k^2 + 2k\cdot q} + q \leftrightarrow -q\right]_{q^2=0} N\left(\frac{|\vec{q}|}{T}\right), \tag{17}$$

where ξ is the gauge parameter ($\xi = 1$ in the Feynman gauge) and the derivative d/dq_0 acts on all terms at its right. Performing the $|\vec{q}|$ integration, the terms involving $\log(K)$ factors turn out to be proportional to

$$\int_{K_-}^{K_+} dK \log K \left[8\pi i K - 3k_0 - \frac{k^2 k_0}{16\pi^2 K^2}\right]. \tag{18}$$

However, the coefficient of the $\log(-k^2)$ term, which is obtained after the K-integration is performed, actually vanishes:

$$\left(2\pi i K_+^2 - \frac{3}{2}k_0 K_+ - \frac{k^2 k_0}{32\pi^2 K_+}\right) - \left(2\pi i K_-^2 - \frac{3}{2}k_0 K_- - \frac{k^2 k_0}{32\pi^2 K_-}\right) = 0 \tag{19}$$

Thus, the full self-energy of the gluon, which includes the thermal effects, does not contain $\log(-k^2)$ contributions. Essentially, these effects replace the zero-temperature $\log(-k^2/\mu^2)$ term by a $\log(T^2/\mu^2)$ contribution. Although this correspondence seems plausible, it is not so obvious. For instance, it would not hold if the thermal contributions would involve individual terms like $\log(k_0^2/T^2)$, $\log(|\vec{k}|^2/k^2)$, etc. To discard this possibility, it is necessary to show that the $\log(-k^2)$ and $\log(T^2)$ contributions appear in the thermal part only in the combination $\log(-k^2/T^2)$. Furthermore, in order to explain the cancellation of the $\log(-k^2)$ terms between the zero-temperature and the temperature-dependent parts, one must also argue [9] that the $\log(T)$ dependence of the self-energy is simply related to its ultraviolet behavior at zero-temperature. Here, these properties of the thermal gluon self-energy have been explicitly verified to one-loop order.

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