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The Effective Astrophysical S–Factor for Non–Resonant Reactions

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Abstract

An analytical method is developed for evaluation of the effective astrophysical S−factor to a very high accuracy. This method, based on the uniform approximation, can easily handle situations involving rather strong energy–dependence in the S–factor. We analyze the reaction $^7\mathrm{Be}(p,\gamma)^8\mathrm{B}$, which is considered the primary source of high–energy solar neutrinos in the solar *pp* chain.

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1. Introduction

In a recent paper, Jennings and Karataglidis [1] investigated approximations to the effective S–factor for the reaction $^7\mathrm{Be}(p,\gamma)^8\mathrm{B}$ using different forms of evaluating the integral over the Gamow peak. Their discussion is based on an asymptotic series that takes into account the energy variation of S–factor up to second order(up to S''). Though their calculation up to S'' is adequate for the above–cited reaction, one may eventually need to include high-order terms in the series. A uniform expansion, which can generate these high-order terms, was given easilier by Anderson *et al.* [2] and by Hussein and Pato [3]. Here we apply the result of Ref. [3] to the reaction $^7\mathrm{Be}(p,\gamma)^8\mathrm{B}$ and assess importance of the term proportinal to the third derivative of the S–factor S'''. For the purpose we first give a short summary of the results of Ref. [3].

2. Calculation of the third derivative term S'''

Our aim here is to supply analytical evaluation of the rate formula valid for a very great accuracy. This allows us to extract an expression for the effective S−factor which is needed in several applications in nuclear astrophysics. The starting point of our discussion is the rate formula

$$R_{12} = 4\pi N_1 N_2 \left(\frac{\mu_m}{2\pi kT}\right)^{3/2} \int_0^\infty \sigma(E) v^3 \; e^{-E/kT} dv \quad , \tag{1}$$

where N_i is the density of nucleus i, μ_m is the reduced mass of the two-nucleus system, kT is the stellar thermal energy and v is the relative velocity. The cross-section $\sigma(E)$ is usually expressed in terms of the astrophysical S−factor $S(E)$

$$\sigma(E) = \frac{S(E)}{E} \, \frac{1}{\exp{(2\pi\eta)} - 1} \approx \frac{S(E)}{E} \; e^{-2\pi\eta} \quad , \tag{2}$$

where η is the Sommerfeld parameter $\eta = Z_1 Z_2 e^2/\hbar v$. For a non–resonant reaction, such as $^7\mathrm{Be}(p,\gamma)^8\mathrm{B}$, $S(E)$ is slowly varying function of E and may be expanded as

$$S(E) = S(0) + \frac{dS(0)}{dE} xkT + \frac{1}{2}\frac{d^2S(0)}{dE^2}(xkT)^2 + \cdots \quad , \tag{3}$$

where $x \equiv E/kT$. Another expansion of $S(E)$

$$S(E) = S(E_0) + \frac{dS(E_0)}{dE}(x - x_0)kT + \frac{1}{2}\frac{d^2S(E_0)}{dE^2}(x - x_0)^2(kT)^2 + \cdots \tag{4}$$

is also used. In the above E_0 is some convenient reference energy. A possible choice is to take E_0 to be the Gamow peak energy. Using the notation $S^{(0)} = S$ and $S^{(n)} = d^nS/dE^n$, we have for the reaction rate R_{12}

$$R_{12} = 2N_1N_2\left(\frac{2}{\pi\mu_m kT}\right)^{1/2}\sum_{n=0}^{\infty} S^{(n)} f_n(a)\frac{(kT)^n}{n!} \quad , \tag{5}$$

where

$$f_n(a) = \int_0^\infty x^n \exp\left[-x - \frac{a}{x^{1/2}}\right]dx \quad , \qquad a = \left[\ \pi\sqrt{2\mu_m}\frac{Z_1Z_2e^2}{\hbar}\ \right]\cdot\frac{1}{\sqrt{kT}} \quad . \tag{6}$$

One can calculate the integral in Eq. (6) to high degree of accuracy using the uniform approximation developed by Dingle [4]. The details of this calculation can be found in Ref. [3] and the appendix. Using $\tau = 3E_0/kT = 3(a/2)^{2/3}$, $f_n(a[\tau])$ in Eq. (6) is given by

$$f_n(\tau) = \frac{2}{3}(\pi\tau)^{1/2}e^{-\tau}Q_n(\tau) \quad , \tag{7}$$

where

$$Q_n(\tau) = \left(\frac{\tau}{3}\right)^n I_n(\tau) = \left(\frac{\tau}{3}\right)^n \sum_{k=0}^{k_M}\frac{P_{2k}(n)}{k!(12)^k\tau^k} \quad . \tag{8}$$

In Eq. (8) $P_{2k}(n)$ is the polynomial of n. Here we give results of $I_n(\tau)$ for $0 \le n \le 3$ up to third order in $1/\tau$. They are written as

$$I_0(\tau) = 1 + \frac{5}{12\tau} + \frac{-35}{2!(12)^2\tau^2} + \frac{665}{3!(12)^3\tau^3} \tag{9}$$

$$I_1(\tau) = 1 + \frac{35}{12\tau} + \frac{385}{2!(12)^2\tau^2} + \frac{-5005}{3!(12)^3\tau^3} \tag{10}$$

$$I_2(\tau) = 1 + \frac{89}{12\tau} + \frac{5005}{2!(12)^2\tau^2} + \frac{85085}{3!(12)^3\tau^3} \tag{11}$$

$$I_3(\tau) = 1 + \frac{167}{12\tau} + \frac{21025}{2!(12)^2\tau^2} + \frac{1616615}{3!(12)^3\tau^3} \quad . \tag{12}$$

¿From Eqs. (5), (7) and (8), we can derive an expression for the effective S−factor

$$S_{eff-MT} = \sum_{n=0}^{n_M} \frac{S^{(n)}(0)}{n!} {E_0}^n I_n(\tau) \quad , \tag{13}$$

where $S^{(0)}(0) = S(0)$ and $S^{(n)}(0) = d^n S(0)/dE^n$. Writing (13) in details up to $n_M = 3$, we have

$$S_{eff-MT} = S(0)\Big[\; I_0(\tau) + \frac{S'(0)}{S(0)} E_0 I_1(\tau) + \frac{S''(0)}{2S(0)} {E_0}^2 I_2(\tau) + \frac{S'''(0)}{6S(0)} {E_0}^3 I_3(\tau)\Big] \quad . \tag{14}$$

The above formula should be compared to that obtained by expanding around $E = E_0$

$$S_{eff-MS} = \sum_{n=0}^{n_M} \frac{S^{(n)}(E_0)}{n!} {E_0}^n \sum_{r=0}^{n} (-)^r \binom{n}{r} I_{n-r}(\tau) \quad . \tag{15}$$

3. Application for ^{7}Be$(p,\gamma)^8$B

In the following we use the above formulae (13) and (15) for the reaction ^{7}Be$(p,\gamma)^8$B. An expression of the fitting curve for the S–factor in the energy region $E < 100$ keV was obtained by Jennings *et al.* [5] and is given by

$$\frac{S(E)}{S(0)} = \frac{0.0409}{0.1375 + E} + 0.703 + 0.343E \quad , \tag{16}$$

where E is in MeV. According to Refs. [1], [6] and [7], the recommended value for $S(0) = S_{17}(0)$ is 19^{+4}_{-2} eV barn. The extrapolated expression (16) is useful as it supplies a mean to calculate its derivatives. These are given up to S''' both for the expansions around $E = 0$ and $E = E_0$ in Table I.

Our results for S_{eff-MS}(expansion around $E = 0$, given by Eq. (13)) and S_{eff-MT}(expansion around $E = E_0$, given by Eq. (15)) are presented in Tables II and III, respectively. We see clearly that whereas S_{eff-MT} converges to the exact value at $n_M = 4$(up to and including S''''), S_{eff-MS} converges faster and only terms up to and including S'' are required. It would seem therefore that S_{eff-MS} is more appropriate for representing non-resonant reactions.

4. Conclusion

We have found in this paper that as far as the reaction ^{7}Be$(p,\gamma)^8$B is concerned, excellent convergence is attained in the rate calculation if S_{eff-MT} is calculated up to and including the fourth-order energy derivative term, while S_{eff-MS} converges faster and one requires including only up to the second derivative term. We believe that this conclusion is valid for non-resonant reactions in general.

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Appendix. Uniform expansion of the thermonuclear reaction rate formula and the effective S–factor for the non-resonant reaction

In this Appendix we show derivation of the effective S–factors S_{eff-MS} and S_{eff-MT} for non-resonant reactions.

A-1. Reaction rate and astrophysical S–factor

The reaction rate in the mixed gas of nuclei 1 and 2 is given by

$$R_{12} = \frac{1}{1+\delta_{12}} N_1 N_2 < \sigma(E) v > \quad , \tag{17}$$

where N_i, $\sigma(E)$ and v are the number density of nucleus i (i = 1, 2), the reaction cross section for the collision between nuclei 1 and 2 at the bombarding energy(in the center of mass) of E and the relative velocity between the nuclei 1 and 2, respectively. $< A >$ is the average value over the Maxwell–Boltzmann distribution

$$< A >= \left(\frac{\mu}{2\pi k_B T}\right)^{3/2} \int d^3v \; A \; \exp\left[-\mu v^2/2k_B T\right] \quad , \tag{18}$$

where μ, k_B and T are the reduced mass $M_1M_2/(M_1+M_2)$, Boltzmann's constant and the temprature of the mixed gas, respectively. Transforming the variable in Eq. (17) from v to E using a relationship $E = \mu v^2/2$, one obtains

$$R_{12} = \frac{N_1 N_2}{1+\delta_{12}} \left(\frac{8}{\pi\mu}\right)^{1/2} \left(\frac{1}{k_B T}\right)^{3/2} \int_0^\infty dE\ S(E)\ \exp\left[-\left(\frac{E}{k_B T} + 2\pi\eta(E)\right)\right] \ , \tag{19}$$

where $\eta(E)$ is the Sommerfeld parameter given by

$$\eta(E) = Z_1 Z_2 e^2/\hbar v \ , \tag{20}$$

where Z_i is the atomic number of the nucleus i. On deriving Eq. (19) we employed the astrophysical S$-$factor, which is defined as

$$\sigma(E) \equiv \frac{S(E)}{E} e^{-2\pi\eta(E)} \ . \tag{21}$$

In nuclear collisions at energies very lower than Coulomb barrier, main energy dependences are $1/E$ and $\exp\left[-2\pi\eta(E)\right]$. They are attributed to the geometrical cross section of a nucleus and the penetrability through the Coulomb barrier, respectively. The energy dependence of the astrophysical S$-$factor is therefore considered to result from the nuclear structure, however, to have a very weak energy-dependence.

In general, the nuclear reaction cross section at energies corresponding to the stellar interior temperature (*e.g.* Temperature in the center region of the sun is about a few keV $\approx 10^7$ K.) is too small to measure using accelerators. One needs to extrapolate from data at high energies($\sim$ a few 10^2 keV) to such a low energy region. In some cases the energy dependence of the astrophysical S$-$factor can be important to calculate the thermonuclear reaction rate.

Furthermore, due to recent development of experimental technique, experimental data with great accuracy have started to be provided. At the same time they have started to consider that the weak dependence of S$-$factor on the energy shoud be taken into account. Formulae shown in this report is for such a case.

Using new variable $x = E/k_B T$ and parameter

$$a = \left[\pi\sqrt{2\mu}\frac{Z_1 Z_2 e^2}{\hbar}\right] \cdot \frac{1}{\sqrt{k_B T}} = \frac{b}{\sqrt{k_B T}} \qquad (>0\ ) \ , \tag{22}$$

Eq. (19) is rewritten as

$$R_{12} = \frac{N_1 N_2}{1+\delta_{12}} \left(\frac{8}{\pi \mu k_B T}\right)^{1/2} \int_0^\infty dx\ S(x k_B T) \exp\left[-\left(x + \frac{a}{\sqrt{x}}\right)\right] \ . \tag{23}$$

In the next subsection we evaluate the integral (23) by using a *conventional* stationary phase approximation.

A-2. Gamow peak energy

On evaluation of Eq. (23), it is useful to introduce the Gamow peak energy E_0 and $\tau = 3E_0/k_B T$. They are calculated in the *conventional* stationary phase approximation.

The phase of exponential in Eq. (23),

$$g(x) = x + \frac{a}{\sqrt{x}} \ , \tag{24}$$

behaves like a parabolic function with a positive curvature. Based on the concept of the stationary phase approximation, we suppose that only a narrow range around the stationary phase position

$$\begin{aligned} \left.\frac{dg(x)}{dx}\right|_{x=x_0} &= 1 - \frac{a}{2}\, x_0^{-3/2} = 0 \\ x_0 &= \left(\frac{a}{2}\right)^{2/3} \end{aligned} \tag{25}$$

contributes to the integration (23). $g(x)$ may be able to be expanded as a Taylor power series in x around x_0 up to the second order(Since $g(x)$ behaves like a quadratic function.)

$$g(x) \approx g(x_0) + \frac{1}{2}\, g''(x_0)(x - x_0)^2 \ , \tag{26}$$

where $g''(x)$ is the second derivative of $g(x)$ with respect to x. Eq. (23) is then approximated to

$$\begin{aligned} R_{12} &\approx \frac{N_1 N_2}{1+\delta_{12}} \left(\frac{8}{\pi \mu k_B T}\right)^{1/2} S(x_0 k_B T) \int_0^\infty \exp\left[-\left(g(x_0) + \frac{g''(x_0)}{2}(x - x_0)^2\right)\right] dx \\ &= \frac{N_1 N_2}{1+\delta_{12}} \left(\frac{8}{\pi \mu k_B T}\right)^{1/2} S(x_0 k_B T) \exp\left[-g(x_0)\right] \left\{\frac{2\pi}{g''(x_0)}\right\}^{1/2} \ . \end{aligned} \tag{27}$$

The Gamow peak energy E_0 corresponds to the stationary phase position x_0 through

$$E_0 = x_0 k_B T \ . \tag{28}$$

$g(x_0)$ and $g''(x_0)$ are given by

$$g(x_0) = x_0 + \frac{a}{\sqrt{x_0}} = \left(\frac{a}{2}\right)^{2/3} + \left(2\cdot\frac{a}{2}\right)\left(\frac{a}{2}\right)^{-1/3} = 3\left(\frac{a}{2}\right)^{2/3}$$
$$= 3x_0 = \frac{3E_0}{k_BT} = \tau \tag{29}$$
$$g''(x_0) = \frac{3a}{4}x_0^{-5/2} = \frac{3}{2}\left(\frac{a}{2}\right)\cdot\left(\frac{a}{2}\right)^{-5/3} = \frac{3}{2x_0} = \frac{9}{2\tau}\ , \tag{30}$$

respectively, where we used Eq. (25). We reach an expression of the *conventinal* formula of the thermonuclear reaction rate

$$R_{12} \approx \frac{1}{1+\delta_{12}} N_1N_2\left(\frac{8}{\pi\mu k_BT}\right)^{1/2} S(E_0)\ \frac{2}{3}(\pi\tau)^{1/2}e^{-\tau}\ , \tag{31}$$

where the Gamow peak energy corresponds to the stationary phase position (the position of the minimum for $g(x)$) and is given by

$$E_0 = x_0k_BT = \left(\frac{bk_BT}{2}\right)^{\frac{2}{3}}\ . \tag{32}$$

On the other hand, τ is the minimum value of $g(x)$ and is given by

$$\tau = \frac{3E_0}{k_BT} = 3\left(\frac{b}{2}\right)^{\frac{2}{3}}\cdot\left(k_BT\right)^{-\frac{1}{3}}\ . \tag{33}$$

Concerning the reaction rate expressed in Eq. (31), three corrections are and will be needed to treat recent and future experimental data of nuclear reactions with great accuracy. One is for a weak but significant energy variation of the astrophysical S−factor. Another is for a significant error resulting from substituting a Gaussian form for the sharply peaked exponential in Eq. (26). The other is for effects of electron screening(See Ref. [6], for example.). We consider only the first two corrections in this paper.

Now let us introduce the *effective S–factor*. ¿From Eqs. (19) and (31), $S(E_0)$ is expressed as

$$S(E_0) \approx \left\{\frac{2}{3}(\pi\tau)^{1/2}e^{-\tau}\right\}^{-1}\frac{1}{k_BT}\int_0^\infty dE\ S(E)\exp\left[-\left(\frac{E}{k_BT}+2\pi\eta(E)\right)\right]$$
$$= \sqrt{\frac{\tau}{4\pi}}\,\frac{e^\tau}{E_0}\int_0^\infty dE\ S(E)\exp\left[-\left(\frac{E}{k_BT}+2\pi\eta(E)\right)\right]\ . \tag{34}$$

The r.h.s. of Eq. (34) is defined as the effective S–factor.

A-3. Evaluation of integration by uniform expansion

In this subsection we evaluate the effective S–factor

$$\begin{aligned} S_{eff} &\equiv \sqrt{\frac{\tau}{4\pi}}\,\frac{e^{\tau}}{E_0}\int_0^{\infty} dE\ S(E)\exp\left[-\left(\frac{E}{k_BT}+2\pi\eta(E)\right)\right] \\ &= \left\{\frac{2}{3}(\pi\tau)^{1/2}e^{-\tau}\right\}^{-1}\int_0^{\infty} dx\ S(xk_BT)\exp\left[-\left(x+\frac{a}{\sqrt{x}}\right)\right] \end{aligned} \tag{35}$$

taking a weak dependence of the astrophysical S−factor on the energy into account. Assuming that $S(E)$ is an analytic function of E, one can expand it as a Taylor power series in E around $E=0$ and $E=E_0$

$$S(E)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\left.\frac{d^nS}{dE^n}\right|_{E=0}\right)E^n \tag{36}$$

$$\begin{aligned} S(E) &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\left.\frac{d^nS}{dE^n}\right|_{E=E_0}\right)(E-E_0)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\left.\frac{d^nS}{dE^n}\right|_{E=E_0}\right)\sum_{r=0}^{n}(-)^r\binom{n}{r}E_0^r\ E^{n-r}\quad , \end{aligned} \tag{37}$$

respectively. Using the power series (36) and (37), one can write Eq. (35) in the form

$$S_{eff-MT}=\left\{\frac{2}{3}(\pi\tau)^{1/2}e^{-\tau}\right\}^{-1}\sum_{n=0}^{\infty}\frac{(k_BT)^n}{n!}\left(\left.\frac{d^nS}{dE^n}\right|_{E=0}\right)f_n(a) \tag{38}$$

$$S_{eff-MS}=\left\{\frac{2}{3}(\pi\tau)^{1/2}e^{-\tau}\right\}^{-1}\sum_{n=0}^{\infty}\frac{(k_BT)^n}{n!}\left(\left.\frac{d^nS}{dE^n}\right|_{E=E_0}\right)\sum_{r=0}^{n}(-)^r\binom{n}{r}x_0^r\ f_{n-r}(a)\quad , \tag{39}$$

respectively, where

$$f_n(a)=\int_0^{\infty}dx\ x^n\exp\left[-\left(x+\frac{a}{\sqrt{x}}\right)\right]\ . \tag{40}$$

Using a new variable $u\equiv x^{n+1}$, Eq. (40) is transformed to

$$f_n(a)=\frac{1}{n+1}\int_0^{\infty}du\ e^{-H_n(a,u)} \tag{41}$$

with

$$H_n(a,u) = u^{\frac{1}{n+1}} + au^{-\frac{1}{2(n+1)}} \quad . \tag{42}$$

In order to evaluate the integration (41), we employ the uniform approximation. Since the phase in Eq. (41), $H_n(a,u)$, behaves like a parabolic function with a positive curvature, we may be able to perform a mapping $u \to t$ which satisfies

$$H_n\Big(a,\ u(t)\Big) = t^2 + A(a) \tag{43}$$

$$u(t=-\infty) = 0\ , \qquad u(t=\infty) = \infty\ , \qquad \frac{du}{dt} > 0\ . \tag{44}$$

The integration (41) thus reduces to

$$f_n(a) = \frac{1}{n+1} e^{-A} \int_{-\infty}^{\infty} dt \left(\frac{du}{dt}\right) e^{-t^2} \quad . \tag{45}$$

du/dt can be also expanded as a Taylor power series in t around the stationary phase position for $t^2 + A(a)$, i.e. $t = 0$, and is written as

$$\frac{du}{dt} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\left.\frac{d^{k+1}u}{dt^{k+1}}\right|_{t=0}\right) \quad , \tag{46}$$

the integration (45) is then described by

$$\begin{aligned} f_n(a) &= \frac{e^{-A}}{n+1} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\left.\frac{d^{k+1}u}{dt^{k+1}}\right|_{t=0}\right) \int_{-\infty}^{\infty} dt\ t^k\ e^{-t^2} \\ &= \frac{2e^{-A}}{n+1} \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left(\left.\frac{d^{2k+1}u}{dt^{2k+1}}\right|_{t=0}\right) \int_0^{\infty} dt\ t^{2k}\ e^{-t^2} \\ &= \frac{\sqrt{\pi}e^{-A}}{n+1} \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \frac{1}{k!} \left(\left.\frac{d^{2k+1}u}{dt^{2k+1}}\right|_{t=0}\right) \quad , \end{aligned} \tag{47}$$

where we used the formula

$$\int_0^{\infty} dx\ e^{-ax^2}\ x^{2n} = \frac{(2n-1)!!}{2^{n+1}} \sqrt{\frac{\pi}{a^{2n+1}}} \quad . \tag{48}$$

We calculate the stationary phase position defined as

$$\left.\frac{\partial H_n(a,u)}{\partial u}\right|_{u=u_n} = 0 \quad , \tag{49}$$

to begin with. From Eq. (42), the kth-order derivative of $H_n(a,u)$ with respect to u is written in the form

$$H_n^{(k)}(a,u) = \frac{\partial^k H_n(a,u)}{\partial u^k}$$
$$= u^{\frac{1}{n+1}-k}\left\{C_k(n) + a(-)^k D_k(n)\, u^{-\frac{3}{2(n+2)}}\right\} \quad (\ k \geq 1\ ) \tag{50}$$
$$C_k(n) = \prod_{j=1}^{k}\left(\frac{1}{n+1} - (j-1)\right) \qquad (\ k \geq 1\ ) \tag{51}$$
$$D_k(n) = \prod_{j=1}^{k}\left(\frac{1}{2(n+1)} + (j-1)\right) \qquad (\ k \geq 1\ )\ . \tag{52}$$

Using Eqs. (50) – (52), $H_n^{(1)}(a,u)$ is given by

$$H_n^{(1)}(a,u) = u^{-\frac{n}{n+1}}\left\{\frac{1}{n+1} - \frac{a}{2(n+1)}u^{-\frac{3}{2(n+1)}}\right\}$$
$$= u^{-\frac{n}{n+1}}\frac{1}{n+1}\left\{1 - \frac{a}{2}u^{-\frac{3}{2(n+1)}}\right\}\ . \tag{53}$$

The stationary phase position u_n is then given by

$$u_n = \left(\frac{a}{2}\right)^{\frac{2(n+1)}{3}} = \left(\frac{\tau}{3}\right)^{n+1}\ . \tag{54}$$

One can also obtain the following relationship from Eq. (54)

$$a = 2\left(\frac{\tau}{3}\right)^{\frac{3}{2}}\ . \tag{55}$$

Since $u_n = u(t=0)$(the property of the mapping $u \to t$), $A(a)$ in Eq. (47) is calculated by

$$A = H_n(a, u(t=0)) = H_n(a, u_n)\ . \tag{56}$$

Substituting Eqs. (54) and (55) into Eq. (42), one can write $A(a)$ as

$$A(a) = H_n(a,u_n) = \left(u^{\frac{1}{n+1}}\right) + 2\left(\frac{\tau}{3}\right)^{\frac{3}{2}}\left(u^{\frac{1}{n+1}}\right)^{-1/2}$$
$$= \left(\frac{\tau}{3}\right) + 2\left(\frac{\tau}{3}\right)^{\frac{3}{2}}\left(\frac{\tau}{3}\right)^{-1/2} = \tau\ . \tag{57}$$

Substituting Eqs. (54) and (55) into Eq. (50), one obtains the following expression

$$H_k = H_n^{(k)}(a,u_n) = \left(\frac{\tau}{3}\right)^{1-k(n+1)}\left\{C_k(n) + 2(-)^k D_k(n)\right\}\ . \tag{58}$$

Using Eq. (58), $H_n^{(2)}(a,u_n)$ is expressed by

$$H_2 = H_n^{(2)}(a, u_n) = \left(\frac{\tau}{3}\right)^{-(2n+1)} \left\{ \frac{1}{n+1}\left(\frac{1}{n+1} - 1\right) + \frac{1}{(n+1)}\left(\frac{1}{2(n+1)} + 1\right) \right\}$$
$$= \left(\frac{\tau}{3}\right)^{-(2n+1)} \frac{3}{2(n+1)^2} \ . \tag{59}$$

We here rewrite Eq. (47) so as to have similar a structure of Eq. (27)

$$f_n(a) = \frac{1}{n+1}\left\{\frac{2\pi}{H_n^{(2)}(a,u_n)}\right\}^{1/2} \sqrt{\frac{H_n^{(2)}(a,u_n)}{2}}\ e^{-H_n(a,u_n)} \sum_{k=0}^{\infty} \frac{1}{2^{2k}}\ \frac{1}{k!}\ \left(\left.\frac{d^{2k+1}u}{dt^{2k+1}}\right|_{t=0}\right)$$
$$= \frac{1}{n+1}\ \left\{2\pi\ \frac{2(n+1)^2}{3}\ \frac{\tau}{3}\left(\frac{\tau}{3}\right)^{2n}\right\}^{1/2}\ e^{-\tau}\sqrt{\frac{H_n^{(2)}(a,u_n)}{2}}\ \sum_{k=0}^{\infty} \frac{1}{2^{2k}}\ \frac{1}{k!}\ \left(\left.\frac{d^{2k+1}u}{dt^{2k+1}}\right|_{t=0}\right)$$
$$= \frac{2}{3}\ (\pi\tau)^{1/2}\ e^{-\tau}\ \left(\frac{\tau}{3}\right)^n\ \sum_{k=0}^{\infty} Q_{2k}(u_n[\tau])\ , \tag{60}$$

where

$$Q_{2k}(u_n[\tau]) = \frac{1}{2^{2k}}\ \frac{1}{k!}\ \sqrt{\frac{H_n^{(2)}(a,u_n)}{2}}\left(\left.\frac{d^{2k+1}u}{dt^{2k+1}}\right|_{t=0}\right)\ . \tag{61}$$

A-4. Calculation of $Q_{2k}(u_n[\tau])$

In this subsection we will show how to calculate $Q_{2k}(u_n)$, i.e. how to derive the kth-order derivative of $u(t)$ at $t = 0$. However, it is so elaborated that we will present the derivation of only the first term of $Q_{2k}(u_n)$.

Firstly, one differentiates Eq. (43) with respect to t. Noted that we don't write the parameter a in H_n explicitly in the following.

$$\frac{dH_n(u(t))}{dt} = \frac{dH_n(u)}{du} \cdot \frac{du}{dt} = 2t\ . \tag{62}$$

If one substitutes u_n and $t = 0$ into Eq. (62), both r.h.s and l.h.s in Eq. (62) vanish so that one cannot obtain $du(t = 0)/dt$. Thus, differentating furthermore Eq. (62) with respect to t, one obtains

$$\frac{d^2H_n}{du^2}\ \left(\frac{du}{dt}\right)^2 + \frac{dH_n}{du}\ \left(\frac{d^2u}{dt^2}\right) = 2 \tag{63}$$

and

$$\left(\frac{du}{dt}\bigg|_{t=0}\right) = \sqrt{\frac{2}{H_n^{(2)}(u_n)}} \quad . \tag{64}$$

¿From Eqs. (61) and (64), one can find easily

$$Q_0(u_n) = 1 \quad . \tag{65}$$

Repeating such differentiations, $d^k u(t=0)/dt^k$ is expressed by a multinomial of $H_n^{(k)}(u_n)$. Fortunately, Dingle [4] provided the multinomial for $Q_k(u_n)$ up to $k = 10$. We here show them in case of $k = 2$, 4 and 6

$$\begin{aligned}
Q_2(u_n) &= \frac{1}{24H_2^3}\left(5H_3^2 - 3H_2H_4\right) \\
Q_4(u_n) &= \frac{1}{1152H_2^6}\left(385H_3^4 - 630H_2H_3^2H_4 + 105H_2^2H_4^2 + 168H_2^2H_3H_5 - 24H_2^3H_6\right) \\
Q_6(u_n) &= \frac{1}{414720H_2^{10}}\left(425425H_3^6 - 1126125H_2H_3^4H_4 + 675675H_2^2H_3^2H_4^2\right. \\
&\quad -51975H_2^3H_4^3 + 360360H_2^2H_3^3H_5 - 249480H_2^3H_3H_4H_5 \\
&\quad +13608H_2^4H_5^2 - 83160H_2^3H_3^2H_6 + 22680H_2^4H_4H_6 \\
&\quad \left.+12960H_2^4H_3H_7 - 1080H_2^5H_8\right) \quad ,
\end{aligned} \tag{66}$$

where

$$H_k = H_n^{(k)}(u_n) \quad . \tag{67}$$

Using Eqs. (58), (51) and (52), Hussein and Pato [3] could succeed in reducing the above expressions to the following simple ones

$$\begin{aligned}
Q_0(u_n(\tau)) &= 1 \\
Q_2(u_n(\tau)) &= \frac{1}{12\tau}\left[\;12n^2 + 18n + 5\;\right] \\
Q_4(u_n(\tau)) &= \frac{1}{2!\;(12)^2\;\tau^2}\left[\;144n^4 + 336n^3 + 84n^2 - 144n - 35\;\right] \\
Q_6(u_n(\tau)) &= \frac{1}{3!\;(12)^3\;\tau^3}\left[\;1728n^6 + 4320n^5 - 4320n^4 - 13320n^3 - 288n^2 + 6210n + 665\;\right] \\
Q_{2k}(u_n(\tau)) &= \frac{1}{k!\;(12)^k\;\tau^k}\;P_{2k}(n) \quad ,
\end{aligned} \tag{68}$$

where $P_{2k}(n)$ are polynomials of n. We can finally obtain the reaction rate formula for the non-resonant reaction

$$R_{12} = \frac{1}{1+\delta_{12}} N_1 N_2 \left(\frac{8}{\pi \mu k_B T}\right)^{1/2} \frac{2}{3} \, (\pi\tau)^{1/2} \, e^{-\tau} \, S_{eff} \tag{69}$$

and two expressions of the effective S–factor

$$S_{eff-MT} = \sum_{n=0}^{\infty} \frac{1}{n!} \, E_0^n \left(\frac{d^n S}{dE^n}\bigg|_{E=0} \right) \sum_{k=0}^{\infty} \frac{P_{2k}(n)}{k! \, (12)^k \, \tau^k} \tag{70}$$

$$S_{eff-MS} = \sum_{n=0}^{\infty} \frac{1}{n!} \, E_0^n \left(\frac{d^n S}{dE^n}\bigg|_{E=E_0} \right) \sum_{r=0}^{n} (-)^r \binom{n}{r} \sum_{k=0}^{\infty} \frac{P_{2k}(n-r)}{k! \, (12)^k \, \tau^k} \tag{71}$$

with

$$P_0(n) = 1 \tag{72}$$

$$P_2(n) = 12n^2 + 18n + 5 \tag{73}$$

$$P_4(n) = 144n^4 + 336n^3 + 84n^2 - 144n - 35 \tag{74}$$

$$P_6(n) = 1728n^6 + 4320n^5 - 4320n^4 - 13320n^3 - 288n^2 + 6210n + 665 \tag{75}$$

$$P_{2k}(n) = \cdots\cdots\cdots\cdots \tag{76}$$

High nth-derivative of $S(E)$ and high-k terms involving τ^{-k} can be negligible practically so that one can set the maximum numbers of n and k, n_M and k_N, respectively. For example, if one chooses $n_M = 2$ and $k_M = 1$, one can obtain the same expression of Eq. (5) in Refs. [1] and [6].

Finally, let us show an example of Eq. (70) in case of $n_M = 5$ and $k_M = 3$

$$\begin{aligned}
S_{eff-MT} &= \sum_{n=0}^{n_M=5} \frac{1}{n!} \, E_0^n \left(\frac{d^n S}{dE^n}\bigg|_{E=0} \right) \sum_{k=0}^{k_M=3} \frac{P_{2k}(n)}{k! \, (12)^k \, \tau^k} \\
&= \sum_{n=0}^{5} \frac{E_0^n}{n!} \left(\frac{d^n S}{dE^n}\bigg|_{E=0} \right) \left\{ 1 + \frac{P_2(n)}{(12)\tau} + \frac{P_4(n)}{2! \, (12)^2 \, \tau^2} + \frac{P_6(n)}{3! \, (12)^3 \, \tau^3} \right\} \\
&= \sum_{n=0}^{5} \frac{E_0^n}{n!} \left(\frac{d^n S}{dE^n}\bigg|_{E=0} \right) I_n(\tau) \qquad (77) \\
&= S(0) I_0(\tau) + E_0 \, S^{(1)}(0) I_1(\tau) + \frac{E_0^2}{2} \, S^{(2)}(0) \, I_2(\tau) + \frac{E_0^3}{6} \, S^{(3)}(0) \, I_3(\tau) \\
&+ \frac{E_0^4}{24} \, S^{(4)}(0) \, I_4(\tau) + \frac{E_0^5}{120} \, S^{(5)}(0) \, I_5(\tau) \ , \qquad (78)
\end{aligned}$$

where

$$S^{(n)}(0) = \left.\frac{d^n S}{dE^n}\right|_{E=0} \tag{79}$$

and

$$I_0(\tau) = 1 + \frac{5}{(12)\tau} + \frac{-35}{2!\,(12)^2\,\tau^2} + \frac{665}{3!\,(12)^3\,\tau^3} \tag{80}$$
$$I_1(\tau) = 1 + \frac{35}{(12)\tau} + \frac{385}{2!\,(12)^2\,\tau^2} + \frac{-5005}{3!\,(12)^3\,\tau^3} \tag{81}$$
$$I_2(\tau) = 1 + \frac{89}{(12)\tau} + \frac{5005}{2!\,(12)^2\,\tau^2} + \frac{85085}{3!\,(12)^3\,\tau^3} \tag{82}$$
$$I_3(\tau) = 1 + \frac{167}{(12)\tau} + \frac{21025}{2!\,(12)^2\,\tau^2} + \frac{1616615}{3!\,(12)^3\,\tau^3} \tag{83}$$
$$I_4(\tau) = 1 + \frac{269}{(12)\tau} + \frac{59101}{2!\,(12)^2\,\tau^2} + \frac{9564065}{3!\,(12)^3\,\tau^3} \tag{84}$$
$$I_5(\tau) = 1 + \frac{395}{(12)\tau} + \frac{133345}{2!\,(12)^2\,\tau^2} + \frac{36159515}{3!\,(12)^3\,\tau^3} \quad . \tag{85}$$

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TABLES

TABLE I. nth-derivative of the astrophysical S−factor, $S^{(n)}(E)$, at the Gamow peak energy E_0 corresponding to the temperature T. $S(E)$ is calculated by $S_0[\, b/(a+E)+c+dE \,]$, where S_0 = 19.0 eV barn, $a = 0.1375$ MeV, $b = 0.0409$ MeV, $c = 0.703$, $d = 0.343$ MeV^{-1} and E is in MeV. These parameters are taken from Ref. [1].

T (10^6 K)	E_0 (keV)	S (eV·barn)	$S^{(1)}$ (10^{-3} barn)	$S^{(2)}$ $\left(\frac{10^{-2}\ \text{barn}}{\text{keV}}\right)$	$S^{(3)}$ $\left(\frac{10^{-1}\ \text{barn}}{\text{keV}^2}\right)$
	$E = 0$	19.01	−34.59	59.79	−130.4
12	15.45	18.54	−26.70	43.44	−85.20
13	16.30	18.52	−26.34	42.72	−83.34
14	17.12	18.49	−25.99	42.04	−81.58
15	17.93	18.47	−25.65	41.39	−79.90
16	18.71	18.45	−25.33	40.77	−78.30
17	19.49	18.43	−25.02	40.17	−76.77

TABLE II. The values of S_{eff-MT}, in eV b, using the fitting function for $S(E)$. E_0 is in keV and T is in 10^6 K. S_{eff-MT} refers to Eq. (13).

	$T = 12$	13	14	15	16	17
E_0	15.45	16.30	17.12	17.93	18.71	19.49
S_{eff}	18.69	18.67	18.65	18.63	18.62	18.60
S_{eff-MT}						
$n_M = 1$ $k_M = 1$	18.62	18.59	18.56	18.54	18.51	18.48
$n_M = 1$ $k_M = 2$	18.61	18.59	18.56	18.53	18.51	18.48
$n_M = 1$ $k_M = 3$	18.61	18.59	18.56	18.53	18.51	18.48
$n_M = 2$ $k_M = 1$	18.70	18.68	18.66	18.65	18.63	18.62
$n_M = 2$ $k_M = 2$	18.70	18.68	18.66	18.65	18.63	18.62
$n_M = 2$ $k_M = 3$	18.70	18.68	18.66	18.65	18.63	18.62
$n_M = 3$ $k_M = 1$	18.69	18.67	18.65	18.63	18.61	18.60
$n_M = 3$ $k_M = 2$	18.69	18.67	18.65	18.63	18.61	18.60
$n_M = 3$ $k_M = 3$	18.69	18.67	18.65	18.63	18.61	18.60
$n_M = 4$ $k_M = 1$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 4$ $k_M = 2$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 4$ $k_M = 3$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 5$ $k_M = 1$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 5$ $k_M = 2$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 5$ $k_M = 3$	18.69	18.67	18.65	18.63	18.62	18.60

TABLE III. The values of S_{eff}, in eV b, using the fitting function for $S(E)$. E_0 is in keV and T is in 10^6 K. S_{eff-MS} refers to Eq. (15).

		$T = 12$	13	14	15	16	17
E_0		15.45	16.30	17.12	17.93	18.71	19.49
S_{eff}		18.69	18.67	18.65	18.63	18.62	18.60
S_{eff-MS}							
$n_M = 1$	$k_M = 1$	18.69	18.67	18.65	18.63	18.61	18.60
$n_M = 1$	$k_M = 2$	18.69	18.67	18.65	18.63	18.61	18.59
$n_M = 1$	$k_M = 3$	18.69	18.67	18.65	18.63	18.61	18.59
$n_M = 2$	$k_M = 1$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 2$	$k_M = 2$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 2$	$k_M = 3$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 3$	$k_M = 1$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 3$	$k_M = 2$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 3$	$k_M = 3$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 4$	$k_M = 1$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 4$	$k_M = 2$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 4$	$k_M = 3$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 5$	$k_M = 1$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 5$	$k_M = 2$	18.69	18.67	18.65	18.63	18.62	18.60
$n_M = 5$	$k_M = 3$	18.69	18.67	18.65	18.63	18.62	18.60

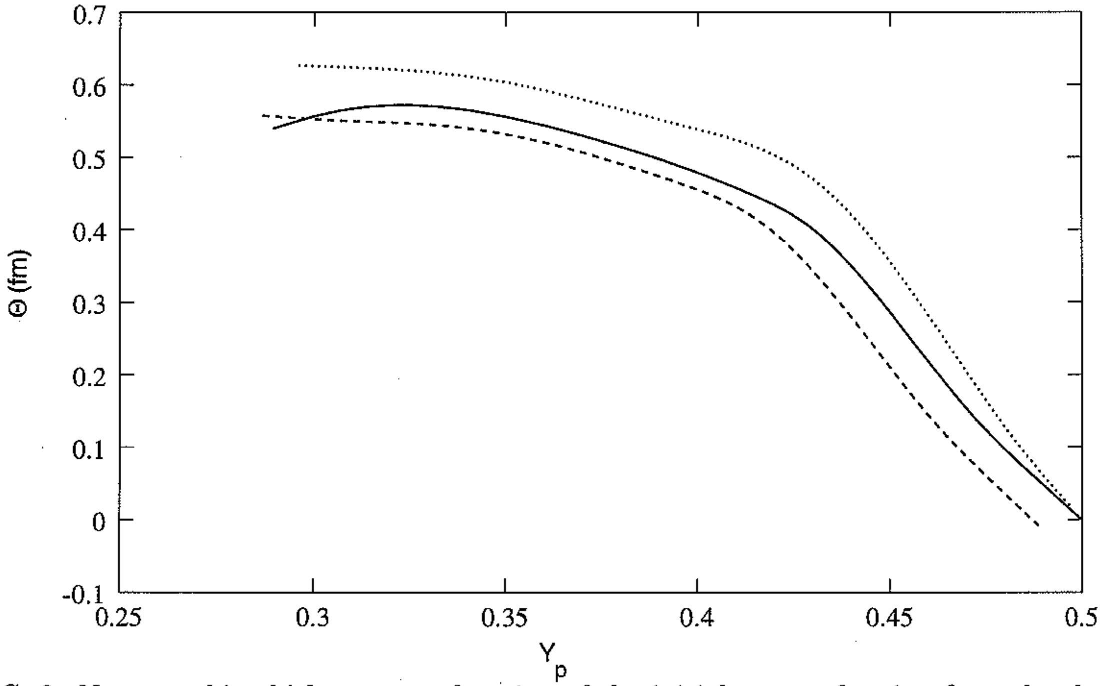

FIG. 8. Neutron skin thickness as a function of the initial proton fraction for a droplet with $A = 20$. The meaning of the lines is the same as in Fig. 4.