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ON THE SPECIFIC HEAT OF THE LIQUID HELIUM

by

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ABSTRACT

We make an attempt to explain the λ - singularity in the specific heat of the liquid He^4. We take into account simultaneously the Bose - Einstein condensation, the atomic interaction and an order-disorder transition.

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For low temperatures[1] ($T < 0.6$ K) it is expected that all the thermal energy is associated with the excitation of longitudinal phonons. In this case, the de Broglie wavelength ℓ is bigger than the mean intermolecular separation a .

As the temperature rises, local atomic motions become relatively more important than the collective excitations: ℓ decreases so that $\ell \lesssim a$. Let us call E_n the energy levels that an atom can assume in these local vibrations: $E_n = 0$ for $n = 0$ and $E_n = \Delta + \varepsilon_n$ for $n = 1,2,\ldots$ with $\varepsilon_1 = 0$. In our approach, Δ is an adjustable parameter and is the minimum value of energy that a particle can assume in local motions: for this value of energy $\ell \sim a$. This Δ must not be confused with the energy gap of the roton spectrum[2]. We have found that $\alpha = \frac{\Delta}{kT_\lambda} = 2.6$, where $T_\lambda = 2.19$ K is the temperature of the λ - point and k is the Boltzmann constant.

Due to the weak interaction between the Helium atoms, we must expect that the energy spectrum ε_n is close to the free particle spectrum.

Since we are not able, up to the present moment, to incorporate in a consistent scheme both phonon excitations and individual atomic motion, we take an additive superposition of the two contributions. The phonon energy can be easily obtained and is given, for instance, in London's book[1].

Let us now calculate the contribution of the individual atomic motion. If N is the total number of Helium atoms we have, using the Bose-Einstein statistics and using the fact that the

spectrum ε_n is quasi-continuum ($\varepsilon_{n+1} - \varepsilon_n << kT$) :

$$N = N_o + N_{exc} = \frac{1}{e^{\alpha} - 1} + kT \int_0^{\infty} \frac{1}{\psi(\varepsilon,T)} \frac{d(\varepsilon/kT)}{e^{\alpha' + \varepsilon/kT} - 1} \qquad (1)$$

where $N_o = \frac{1}{e^{\alpha} - 1}$ is the number of particles in the ground state, $\alpha' = \alpha + \Delta/kT$, and $1/\psi(\varepsilon,T)$ is the number of states in the interval $d\varepsilon$ of energy at a temperature T . We assume that the energy spectrum ε_n changes with the temperature. The function $1/\psi(\varepsilon,T)$ increases, since the energy spectrum ε_n tends to an energy spectrum of free particles, when T increases.

If the particles were free, $1/\psi$ would be proportional to $\varepsilon^{1/2}$ and if they were vibrating harmonically about a center of equilibrium, $1/\psi$ would be equal to $1/h\nu$, where ν is the fundamental frequency of vibration. Thus $1/\psi$ would be independent of ε .

It seems reasonable to expect that $1/\psi \propto \varepsilon^{\delta}$ where δ is more close to $1/2$ than to 0 . As we will see in what follows this dependence of $1/\psi$ with ε^{δ} $(0 \leq \delta \leq 1/2)$ is not important for $T < T_{\lambda}$. Thus we put simply $1/\psi(\varepsilon,T) = \frac{2}{\sqrt{\pi}} \frac{\varepsilon^{1/2}}{\psi(T)}$. In this case, the number of excited particles N_{exc} given in equation (1) becomes[1]:

$$N_{exc} = \frac{2}{\sqrt{\pi}} \frac{(kT)^{3/2}}{\psi(T)} \int_0^{\infty} \frac{x^{1/2}\, dx}{e^{\alpha' + x} - 1} = \frac{(kT)^{3/2}}{\psi(T)} F_{3/2}\left(\alpha + \frac{\Delta}{kT} \right) \qquad (2)$$

As we will see, the choise of $\delta = 1/2$ is important for $T \geq T_{\lambda}$. For $T \geq T_{\lambda}$, the condition $N_{exc} = N$ must be satisfied.

The total energy U is given by:

$$U = \frac{2}{\sqrt{\pi}} \frac{1}{\psi(T)} \int_0^\infty \frac{(\varepsilon + \Delta)\, \varepsilon^{1/2}\, d\varepsilon}{e^{\alpha' + \varepsilon/kT} - 1} = N_{exc} \left[\frac{3}{2} kT \frac{F_{5/2}(\alpha + \Delta/kT)}{F_{3/2}(\alpha + \Delta/kT)} + \Delta \right] \quad (3)$$

Our problem now is to obtain $\psi(T)$. At low temperatures it is observed[3] that, in average, the Helium atoms are disposed in some ordered structure. As the temperature increases, this ordered structure tends to disappear and Fröhlich[4] proposed that there occurs an order-disorder transition in liquid Helium.

As is well known from the order-disorder transition formalism[5], the order parameter X obeys the equation $X = \tanh(\frac{T_c}{T} X)$, where $T_c = T_\lambda$ is the critical temperature. It may seem unrealistic to treat a liquid by a lattice model. This objection is quite valid in general but many properties of the liquids are calculated approximately by using the lattice model[6].

For our purpose it is not necessary that the equation $X = \tanh(\frac{T_\lambda}{T} X)$ describes exactly the order-disorder transition or that it is not perfectly consistent with the equations (1), (2) and (3). We intend to use it only to show that probably there is some relation between the λ - singularity and an order-disorder transition. We are not interested in the exact calculation of the specific heat.

We must expect that the energy spectrum ε_n is the free particle spectrum when the system is completely disordered ($X = 0$ at $T = T_\lambda$). So $\psi(T)$ must decrease when X decreases. Let us assume that $\psi(T) = \eta.(1 + \xi X^\theta)$ where the parameter η is determined using the condition $N_{exc}(T_\lambda) = N$ and θ and ξ are

adjustable parameters.

Now substituting $\psi(T) = \eta.(1 + \xi X^{\theta})$ in formulas (2) and (3) and following the same development seen in ref.1 we obtain the specific heat per unit of mass C_v^- , for $T < T_\lambda$, which is given by:

$$C_v^- = \frac{k}{m} \left(\frac{T}{T_\lambda}\right)^{3/2} \frac{1}{(1+\xi X^{\theta})} \left[\frac{15}{4} + 3\alpha \frac{T_\lambda}{T} + \alpha^2 \left(\frac{T_\lambda}{T}\right)^2 \right] \exp\left[\alpha\left(1 - \frac{T_\lambda}{T}\right)\right]$$

$$+ \frac{k}{m} \left(\frac{T}{T_\lambda}\right)^{1/2} \frac{\theta \xi X^{\theta}}{(1+\xi X^{\theta})^2} \frac{\left[\frac{3}{2} + \alpha \frac{T_\lambda}{T} \right]}{\cosh^2\left(\frac{T_\lambda}{T} X \right) - \frac{T_\lambda}{T}} \exp\left[\alpha\left(1 - \frac{T_\lambda}{T} \right) \right]$$

For $T \geq T_\lambda$ the specific heat per unit of mass C_v^+ is given by $C_v^+ = \frac{3}{2} \frac{k}{m}$, which is the specific heat of an ideal gas.

We see in the figure the theoretical results compared with the experimental results of Keesom and Clusius and Keesom and Keesom which are seen in London's book (the phonons contribution[1], which is very small compared with C_v^- and C_v^+ , has been also taken into account). The best agreement with the experimental results was found putting $\theta = 0.22$, $\alpha = 2.60$ and $\xi = 8.0$. Our result for C_v^- diverges as $(T_\lambda - T)^{-0.89}$ at the λ - point and experimentally it seems to diverge as a $\log|T - T_\lambda|$. A better agreement at T_λ could be obtained by allowing $\psi(T)$ to be a more appropriate function of X .

To obtain C_v^- and C_v^+ , we have substituted $F_{3/2}(\frac{\Delta}{kT})$ and $F_{1/2}(\frac{\Delta}{kT})$ by $\exp(-\frac{\Delta}{kT})$ since $\Delta/kT > 2.6$. The functions $F_\sigma(y)$ $(\sigma \geq 1/2)$ can be approximate by $\exp(-y)$ for $y > 2.0$ with an error of only a few percent[1]. For this reason, our result for

C_V^- is only slightly modified if we put $\delta = 0$ in the function $1/\psi(\varepsilon,T) \propto \varepsilon^{\delta}$ and the agreement with the experimental results is the same as that obtained above with $\delta = 1/2$.

On the other hand for $\delta = 0$, $C_V^+ = k/m$ which is the specific heat of an isolated harmonic oscillator. In this case, the agreement with the experimental results is not as good as $3k/2m$, which we obtained above with $\delta = 1/2$.

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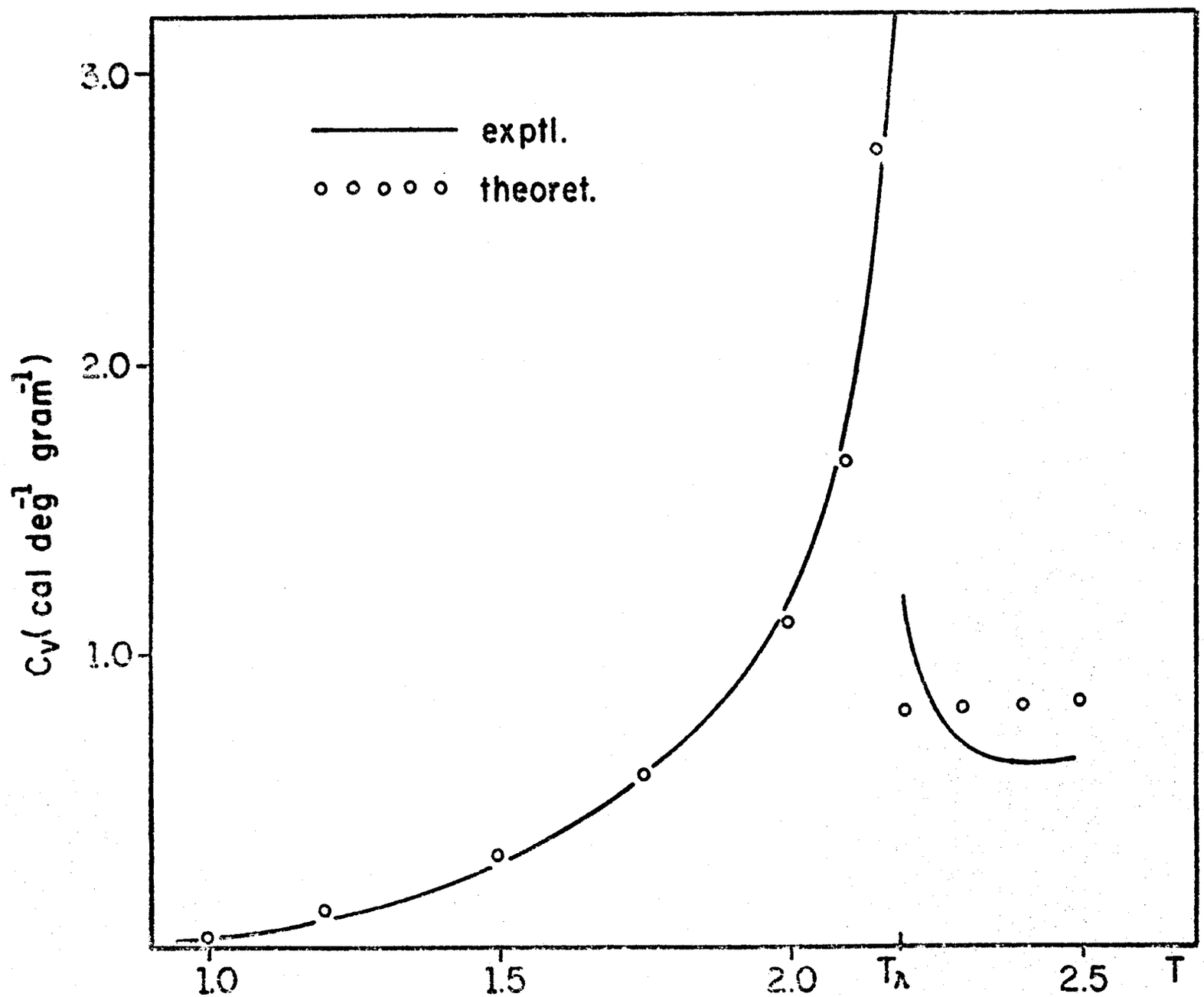
FIGURE CAPTION

The specific heat C_V as a function of T .


3.0
2.0
1.0
C_V(cal deg^{-1} gram^{-1})
exptl.
theoret.
1.0
1.5
2.0
T_λ
2.5
T