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ON NUCLEAR PAIRING THEORY

by

Hsi-Tseng Chen* - Pelletron Laboratory, Instituto de Fisica - Universidade de São Paulo
São Paulo - Brasil

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Pelletron Laboratory, Instituto de Física

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1. INTRODUCTION

The analysis of experimental data from two nucleon transfer reactions [1-4] suggested that the T=1 pairing force plays an important role in the description of strongly populated 0^+ states in medium weight nuclei. These 0^+ states can be understood in terms of pairing vibrations [1,2] and pairing rotations [3]. Recently a more sophisticated collective treatment of the T=1 pairing Hamiltonian has been developed by Bes and others [4]. In this macroscopic model, the pairing deformation is considered to be taking place in a four-dimensional isospin and gauge space [4].

On the other hand, as was shown in our previous work [5], the calculation of the above-mentioned 0^+ states is an exactly solvable many-body problem and can be reduced to the solution of a set of dispersion type energy equations which depend parametrically upon the symmetry constants that characterize the symmetry of the state under exchange among the J=0 pairs. It has been demonstrated that the collective spectrum can be derived from this set of energy equations and that a rotation-vibration model in the T=1 pairing deformed system was thus obtained.

This paper is intended as a review of the existing theories which have been put forward to study the effects of T=1 pairing correlation on 0^+ states in medium weight nuclei. The charge independent pairing Hamiltonian is used as a schematic model Hamiltonian. The interaction in this Hamiltonian is effective between any two nucleons that are coupled to J=0 and therefore to isospin T=1. We will start with the exact treatment with which the author is most familiar and show how this exact theory is related to the BCS approximation and the other approximate

theories of pairing collective motion.

In what follows, section 2 consists of a description of the general properties of the pairing correlation in nuclei. In section 3 we introduce an exact treatment of T=1 pairing Hamiltonian and obtain a set of dispersion type energy equations. In section 4 we will show how the BCS approximation can be derived from this set of equations. The validity and the limitations of this approximation are also explored. In section 5, we have a brief discussion on the properties of pairing vibrations. In section 6, the quasi-particle approximation treatment of neutron-proton pairing is reviewed. Finally section 6 consists of a microscopic description of pairing collective motion of the isospin degree of freedom. The collective spectrum is shown to be obtained from the above mentioned exact equations.

2. PAIRING CORRELATION IN NUCLEI

In 1955, Bardeen [6] observed that the phenomenon of superconductivity can be explained in terms of an energy gap in the single-particle energy spectrum of the conduction electrons in the metal. Cooper [7] showed that the formation of bound electron pairs leads to such an energy gap, because only when a pair is broken up can a single electron be excited. This discovery of the pairing in superconducting metals led to great advances in the general many-body problem. However, historically speaking, pairing correlation was first recognized in the nuclear context. It can be traced back to the invention of the seniority scheme by Racah [8] in 1942 in the studies of atomic spectroscopy. This scheme resulted from diagonalizing a pairing operator and constructing the associated states. The seniority quantum number was then introduced as a purely mathematical tool, and all the properties connected with the seniority were also obtained in a purely mathematical way. Only when nuclear spectroscopy was later developed, did it appear that seniority had a direct physical meaning in nuclei. Jahn [9] first noted that for short-range forces,like the nuclear force, the Hamiltonian is nearly diagonal in Racah's seniority scheme and, in the limiting case of contact forces, seniority is a good quantum number. This observation was also confirmed later by the calculations of energy levels of the nucleons in unfilled shells by Flowers [10] and Edmonds and Flowers [11]. Moreover, the celebrated Mayer-Jensen pairing rule [12] is based on the assumption that nucleons in each j must be paired off in J=0 pairs, and the spin of an odd nucleus is then predicted to be that of the last unpaired nucleon. The success of this last prediction, coupled with

the Mayer-Jensen level order was one of the triumphant arguments in favor of the adoption of the shell model.

In 1959, Bohr, Mottelson and Pines [13] suggested that the energy gap in the spectra of nuclei is a result of pairing effects similar to those present in the superconducting state of metals. According to the data taken from Ref. 14 nearly all even-even nuclei have either one or no excited state less than 0.5 MeV. in energy while the odd nuclei usually have a number of excited states in this energy range. If one eliminates the obviously collective states, then an energy gap is clearly observed in the remaining single-particle spectrum. The lowest single-particle excitation is at least 0.9 MeV. in the even nuclei and is often between 0.1 and 0.3 MeV. in odd nuclei.

The pairing correlation is not only confirmed by the existence of the energy gap in the low-lying nuclear spectra but also by other phenomena such as γ and β transitions and the reduced widths for nucleons or nucleon pairs, etc. For example, the transitions like $Sn^{117}(d,p)Sn^{118}$ are forbidden in the light of the simple shell model. However, it has been pointed out by Yoshida [15] that this is no longer the case if the pairing interaction is taken into account. The experimental evidence for this effect in (d,p) and (d,t) reactions has been observed by Cohen et al [16].

Pairing also plays an important role in the interpretation of empirical moments of inertia. It has proved to be responsible for the observed fact that rotational noments of inertia are appreciably smaller than the values corresponding to rigid rotation[17].

The nuclear phenomenon mentioned above do show an important departure from independent particle motion in nuclei, a departure arising from the residual forces between nucleons. The pairing

correlation is then considered to be the contribution due to the short range part of residual interaction. Imagine a pair of classical particles moving in a central potential and interacting by a short-range attractive force. It is easy to see that for most pairs of orbits the interaction has no effect, as the particles are rarely within its range. The particles interact most strongly if they move in orbit b in opposite directions for they will collide after half a revolution. After each collision they will scatter, but both go into another orbit b' again moving in opposite directions because angular momentum is conserved and initially the total angular momentum is zero.

In order to gain deeper insight into the sources of the pairing force, we consider the effect of very short range two-body forces upon n nucleons moving in degenerate particle states outside the closed shell. We allow the n nucleons to be composed of both neutrons and protons. Assuming charge independence of the nuclear force, we treat protons and neutrons as different states of the same particle, and describe this new degree of freedom by the isospin. In occupation number representation, we denote $a^{+}_{jm\tau}$ ($a_{jm\tau}$) as a creation (annihilation) operator of a nucleon in an orbit with space-spin quantum number jm, isospin projection τ. An operator creating a pair of particles coupled to $JMTM_T$ will be denoted as

$$A^{+}(j_1 j_2 JM\,TM_T) = \frac{1}{\sqrt{2}} \sum_{\substack{m_1 m_2 \\ \tau_1 \tau_2}} \left(\tfrac{1}{2}\tau_1 \tfrac{1}{2}\tau_2 \middle| \tfrac{1}{2}\tfrac{1}{2} T M_T\right) (j_1 m_1 j_2 m_2 | j_1 j_2 JM)\, a^{\dagger}_{j_1 m_1 \tau_1} a^{\dagger}_{j_2 m_2 \tau_2} \qquad (2.1)$$

In terms of these operators a charge independent Hamiltonian within the j orbit can then generally be written as

$$H = \epsilon_j \sum_{m\tau} a^{+}_{jm\tau} a_{jm\tau} + \sum_{JT} \langle j^2 JT | V | j^2 JT \rangle \sum_{MM_T} A^{+}(jjJMTM_T)\, A(jjJMTM_T) \qquad (2.2)$$

where ϵ_j is the single particle energy.

To diagonalize this Hamiltonian, we require a well-defined complete set of many particle wave functions. It is expected that this complete set should reflect some qualitative physical features of the Hamiltonian and that the individual wave functions in the set should be as far as is possible good approximations to the exact eigenfunctions. In expressing the correlation set up by a short range force, a coupling scheme can be provided by defining a pairing operator $\hat{Q} = \sum_{i<k} q_{ik}$ for n particles, where

$$\langle j^2 JMTM_T | q_{12} | j^2 J'M'T'M'_T \rangle = (2j+1)\, \delta_{JJ'}\, \delta_{TT'}\, \delta_{MM'}\, \delta_{M_T M'_T}\, \delta_{J0} \qquad (2.3)$$

then the eigenstates of the pairing operator $\hat{Q}$ will form a physically reasonable complete set of many particle functions. A little Racah algebra gives rise to the formula

$$q_{12} = -\frac{1}{2} - (2\vec{t}_1 \cdot \vec{t}_2) - 2 \sum_{k(odd)} (2k+1) \left(\hat{\mu}^{(k)}_{(1)} \cdot \hat{\mu}^{(k)}_{(2)} \right) \qquad (2.4)$$

where $\hat{\mu}^{(k)}$ is a spherical tensor operator of degree k under rotation whose components are defined by

$$\mu^k_q = \frac{1}{\sqrt{2k+1}} \sum_{m_1 m_2} (jm_1 jm_2 | jjkq)\, (-1)^{j-m_2}\, a^{+}_{jm_1\tau}\, a_{j-m_2 i} \qquad (2.5)$$

and satisfies the following commutation relation

$$[\mu^k_q \, \mu^{k'}_{q'}] = \sum_r \sqrt{2r+1}\,[(-1)^{k+k'+r} - 1]\,(kq\,k'q'|kk'r s)\; W(kk'jj:rj)\,\mu^r_s \qquad (2.6)$$

with k = 0, 1, ..., 2j.

One may see from the last relation that the u^k_q are the (2j+1) infinitesimal operators which generate the unitary group U_{2j+1}. The irreducible representations of the group are provided by the space spanned by $a^\dagger_{jm}$ (m=j,j-1,...,-j). If k and k' are odd then only odd r will appear in (2.6). Hence the set of u^k_q with odd k generates a subgroup of U_{2j+1}. Since

$$\left[\sum_m (-1)^{j-m} a^\dagger_{jm\tau}\, a'_{j-m\tau} \quad \mu^k_q\right] = \frac{2}{\sqrt{2k+1}}\, A^\dagger(j^2 kq \mid M_T) = 0$$

if k is odd, this subgroup can be identified with the symplectic group Sp_{2j+1} which leaves either the J=0 state of two particle or the J=0 pair operator invariant. This transformation property of the group Sp_{2j+1} is very important in connection with the concept of seniority v and reduced isospin t of a state. A function which contains pairs coupled to J=0 will transform exactly like the function obtained by removing the pairs. The quantum numbers v and t may then be interpreted as the number and total isospin of that part of the wave function remaining after all J=0 pairs have been removed. These two numbers specify the irreducible representation of Sp_{2j+1} to which the state belongs.

In the j^n configuration the transformation induced by Sp_{2j+1} are direct products of the transformation in 2j+1 dimensional space spanned by $a^\dagger_{jm\tau}$. The infinitesimal operators in this space are thus the sums of the individual infinitesimal operators. Thus $U^k_q = \sum_{i=1}^{n} u^k_q(i)$ with odd k are the infinitesimal operators

of the representation of Sp_{2j+1} in the space of tensors of the j^n configuration. It is interesting to note that the infinitesimal operators U_q^k are analogous to J_x, J_y, J_z, the infinitesimal generators of R_3, while the relations analogous to the commutation relations $[J_x\ J_y] = iJ_z$ are (2.6). We now construct the Casimir operator of the group considered. It is analogous to the operator $J^2 = J_x^2 + J_y^2 + J_z^2$:

$$\hat{G}_{SP} = 2 \sum_{r(odd)} (2r+1) \left(\hat{U}^{(r)} \cdot \hat{U}^{(r)} \right) \qquad (2.7)$$

Like J^2 commuting with J_x, J_y, J_z, G_{Sp} commutes with all U and is diagonal in the seniority-reduced isospin scheme. Its eigenvalues are given by

$$\langle v\,t \,|\, \hat{G}_{SP} \,|\, v\,t \rangle = \frac{1}{2} v(4j+8-v) - 2t(t+1) \qquad (2.8)$$

With a little algebra we now can express the pairing operator in terms of G_{Sp}:

$$\hat{Q} = \frac{n(n-1)}{4} - 2 \sum_{i<k}^{n} (\vec{t}_i \cdot \vec{t}_k) + n(j+1) + \hat{G}_{SP} \qquad (2.9)$$

Now it is obvious from the last expression that since the Casimir operator of Sp_{2j+1}, $\hat{G}_{Sp}$, if diagonalized in the seniority-reduced isospin scheme so is the pairing operator $\hat{Q}$.

States which are characterized by v and t have the very interesting property of being nearly diagonal for short-range Wigner forces. Consequently, if forces of not too long a range are used, v and t can serve as approximately good quantum numbers. If the ratio of the range of forces to the nuclear radius is much smaller

than one, the proposed classification is very useful. Of course, the nuclear symmetries represented by these two quantum numbers are approximate ones, but due to the complexity of the nuclear many-body problem, the study of approximate symmetries becomes a useful approach in gaining deeper insight into qualitative physical features and serves as a stepping-stone to more realistic calculations.

In general, one can find the spectrum for j^n configuration generated by some given type of central force by setting up the Hamiltonian matrix in the v-t scheme and diagonalizing it. However, it is not easy to study systematics of nuclear spectra in this way. Fortunately, a useful closed formula for the energies of eigenstates of Wigner forces $\sum_{i<j} V_{ij}$ for j^n configuration has been given by the use of group theoretical methods [10]. The diagonal energy matrix element of a central interaction for states in v,t,J,T may be put into the form

$$E(j^n; vt; JT) = \langle \Psi(j^n; vt; JM, TM_T) | H | \Psi(j^n; vt; JM, TM_T) \rangle$$
$$= e_1(n) E_0 + e_2(v,t,T) E_1 + \sum_{i=2}^{j-\frac{1}{2}} e_i(J) E_i \qquad (2.10)$$

where the E_i are certain linear combinations of the Slater integrals F^k, defined by

$$F^k(n_1 l_1, n_2 l_2) = \iint v_k(r_1, r_2) \left| R_{n_1 l_1}(r_1) R_{n_2 l_2}(r_2) \right|^2 dr_1 dr_2 \qquad (2.11)$$

where R(r)/r is the radial wave function appropriate to the shell and the $v_k(r_1,r_2)$ are defined by the expansion

$$V(r_{12}) = \sum_k v_k(r_1, r_2) P_k(\cos\omega_{12}) \qquad (2.12)$$

By the group theoretical analysis, the coeffficients e_i show the following interesting properties:

The coefficient e_0 is a function of n only and gives the main interacting energy of the configuration. The coefficiente e_1 involves in addition to n the quantum numbers T,v,t, but not J or any additional quantum number. The expressions of the remaining e_i are very complicated functions of J. Fortunately, the shorter the range of the forces the smaller the $E_2 \ldots E_{j-\frac{1}{2}}$ will become in comparison to E_0 and E_1. This last statement makes it possible to estimate the level ordering roughly from (v,t) alone without making calculations.

For the j^n configuration in the short range limit we have $F^k = (2k+1)F^o$ and $E_1 = 2E_o = (j+\frac{1}{2})F^o/(j+1)$. In the same limit, E_o and E_1 are much larger than the other $E_2 .. E_{j-\frac{1}{2}}$ ($E_0, E_1 \gtrsim 100E_0$), so that the main ordering in the spectrum is determined by v,t and T, with J-splitting as fine structure. The non-diagonal matrix elements of a Wigner force in the present states do not contain E_0 and E_1 but only $E_2 \ldots E_{j+\frac{1}{2}}$. These elements then are small and the interaction matrix is approximately diagonal in these states. We therefore conclude that v and t are approximately good quantum numbers as long as the range of the force is not too long. For the lowest T values of an even number of neutrons and protons, there occur states of v=0, while for odd n there occur states of v=1. This result ties in very well with the Mayer and Jensen pairing rule[12].

Since the δ-force is almost as difficult to diagonalize as a completely general two-body interaction in a many-particle system, we shall now define an idealization of the δ-force which is called a pairing force. This idealized potential is chosen to

simplify the shell model calculations and to typify the short range components of the force. Recalling that for sufficiently short-range forces, the integrals $E_2 \ldots E_{j+\frac{1}{2}}$ become small in comparison with E_0 and E_1, we further take a fictitious limit such that all E_i in (2.10) vanish except E_0 and E_1. Spelling out the expression for e_0 and e_i, we then have

$$E(j^n; vt; JT) = \frac{n}{2}(n-1)E_0 + \frac{1}{2}\left[Q_0(n,T) + Q_1(n,T,v,t)\right] \qquad (2.13)$$

where

$$Q_0(n,T) = \frac{1}{2}n(\tfrac{1}{2}n+1) - T(T+1) \qquad (2.14)$$

are the eigenvalues of the isospin operator $\sum_{i<k}(\frac{1}{4} - \vec{t}_i.\vec{t}_k)$. This operator counts the number of T=0 pairs: $Q_1(n,T;v,t)$ are the eigenvalues of the pairing operator $\hat{Q} = (2j+1)\sum_{i<j}\delta_{J_{ij}0}$ which are related to the effective number of the J=0, T=1 pairs in the corresponding eigenstates. Q_1 are not simply given by $(2j+1)\frac{1}{2}(n-v)$ but rather by

$$\tfrac{1}{2}(n-v)\left[2j+4-\tfrac{1}{2}(n-v)\right] - T(T+1) + t(t+1) \qquad (2.15)$$

This can also be written as

$$Q_1(n,T;v,t) = \tfrac{1}{2}(n-v)(2j+1) - \tfrac{1}{2}k(n-v)\left(\tfrac{1}{2}(n-v)-1\right) \qquad (2.16)$$

where

$$k = \frac{\frac{n-v}{2}\left(\frac{n-v}{2}-3\right) + T(T+1) - t(t+1)}{\frac{n-v}{2}\left(\frac{n-v}{2}-1\right)} \qquad (2.17)$$

can be understood as the measure of the effect of the Pauli principle. Here we see that the Pauli principle reduces the number of J=0, T=1 pairs by the amount

$$\frac{k}{2j+1}\,\frac{n-v}{2}\,\left(\frac{n-v}{2}-1\right).$$

Since both the operators $\sum_{i<k}\left(\frac{1}{4}-\vec{t}_i\cdot\vec{t}_k\right)$ and $\sum_{i<k} q_{ik}$ are diagonal in the seniority-reduced isospin scheme, we can now define a generalized pairing force which includes both T=0 and T=1 pairing by

$$V_{ij} = -G_0 - G_{J0}\left(\tfrac{1}{4}-\vec{t}_i\cdot\vec{t}_j\right) - \frac{G}{2}\left\{-\frac{1}{2}-2\left(\vec{t}_i\cdot\vec{t}_j\right)-2\sum_{k(odd)}(2k+1)\left(\hat{\mu}^k_{(i)}\cdot\hat{\mu}^k_{(j)}\right)\right. \qquad (2.18)$$

where G_0, G_{J0}, G are force strengths and can be determined by fitting the two nucleon spectra. Defining the barycentric energy of a group of levels of given (v,t) as

$$E(j^2;v,t) = \sum_J (2J+1)\,E(j^2;v,t,J)\Big/\sum_J (2J+1) \qquad (2.19)$$

we then have

$$\begin{aligned} G_0 &= -E(j^2,2,1) \\ G_{J0} &= E(j^2;2,1)-E(j^2,2,0) \\ G &= \frac{2}{2j+1}\left[E(j^2,2,1)-E(j^2;0,0)\right] \end{aligned} \qquad (2.20)$$

The eigenvalues of this simple potential are given by

$$\begin{aligned} E = &-\tfrac{1}{2}n(n-1)G_0 - \tfrac{1}{2}G_{J0}\left[\tfrac{n}{2}\left(\tfrac{n}{2}+1\right)-T(T+1)\right] \\ &-\tfrac{1}{2}G\left[\frac{n-v}{2}\left(2j+4-\frac{n+v}{2}\right)-T(T+1)+t(t+1)\right] \end{aligned} \qquad (2.21)$$

We notice some interesting features of this idealized potential. First, since we are dealing with an oversimplified δ-force, this potential contains essential features of the short range part of the nuclear force and reproduces the energy gap of the spectrum of even-even nuclei. This effect can be easily seen from the spectrum of low-lying states for the configuration $(7/2)^4$ which is obtained according to (2.21) and is plotted in Fig. 1. For a system of identical nucleons, such as Ca^{44}, the J=0 state is indeed well separated from the rest. With neutrons and protons, such as Ti^{44}, the pairing force approximation is not as good as that in the system of identical nucleons. This is due to the presence of the isopairing force. However, we always have the freedom within the seniority-reduced isospin scheme to shift the T=0 states relative to those with T=1. Moreover, it can be seen from (2.21) that the effect of the isopairing force is proportional to the force strength G_{J0} as well as to the number of T=0 pairs. This number is maximum when the total isospin is equal to zero. We therefore expect the isospin pairing force to make an important contribution in light nuclei. On the other hand, the number of T=0 pairs decreases with the increase of T, and consequently the T=1 pairing force becomes more prominent in the higher T levels. Since the ground states of nuclei with larger neutron excess are identified with higher T levels in the T_z=0 nuclei of the same mass number, we would then expect that the pairing force would play an important role in the medium weight nuclei. This observation is well confirmed by recent model calculations in the sd-shell and also by calculations with realistic forces [18].

So far we have only considered the j^n configuration. We now wish to specify the extent to which we will allow the pairing force to scatter particles out of a single shell. Let us note that the definition of the generalized pairing force given by (2.18) is equivalent to the one which is defined as follows by its matrix elements:

$$\langle j^2\ J=0\ T=1 \mid V_{12} \mid j^2\ J=0\ T=1\rangle = -G_0 - \frac{G}{2}(2j+1)\,\delta_{J0}$$

$$\langle j^2\ J=\text{even}\ T=1 \mid V_{12} \mid j^2\ J=\text{even}\ T=1\rangle = -G_{T_0} \qquad (2.24)$$

$$\langle j^2\ J=\text{odd}\ T=0 \mid V_{12} \mid j^2\ J=\text{odd}\ T=0\rangle = -G_{T_0} - G_{J0}\,\delta_{T0}$$

These last expressions suggest that one may arrive at our definition of the pairing force spanning several single particle levels, if one requires that

$$\langle j^2\ J=0, T=1 \mid V_{12} \mid j'^2\ J=0\ T=1\rangle = -G_0\,\delta_{jj'} - \frac{G}{2}\sqrt{(2j+1)(2j'+1)}\,\delta_{J0}$$

$$\langle j^2\ J=\text{even}, T=1 \mid V_{12} \mid j'^2\ J=\text{even}\ T=1\rangle = -G_0\,\delta_{jj'}$$

$$\langle j^2\ J=\text{odd}\ T=0 \mid V_{12} \mid j'^2\ J=\text{odd}\ T=0\rangle = -G_0\,\delta_{jj'} - G_{J0}\,\delta_{T0}$$

The shematic pairing force thus chosen does fulfill the requirements of simplicity in calculation. As the latter part of this paper will show, the calculation of exact states of this pairing Hamiltonian can be reduced to the solutions of a system of coupled equations. Thus the short range part of the nuclear force can be well typified in a sufficiently simple way that complex configuration mixing for many particles may be introduced without heavy computations. By using this method, one learns something of qualitative feature and acquires much insight into real many-body problems. The simplicity of these calculations also makes it possible to study, systematically and in detail, the

variation from nucleus to nucleus of various nuclear properties, such as excitation energies, electromagnetic moments, transition rates, and reaction rates. Furthermore, these schematic calculations lead to advances in technique which could be used with more realistic potentials. They also provide a convenient basis of functions with which more realistic calculations may be done.

3. EXACT TREATMENT OF T=1 PAIRING CORRELATION .

For the 2N-particle seniority-zero states of the charge-independent pairing Hamiltonian, all N pairs of nucleons are coupled to J=0 and therefore T=1, M_T=t. Placing these pairs into a set of M single-particle j-levels, one is able to classify the states according to the unitary group U(M) x U(3). The group chain for this scheme was proposed as follows:[5]

$$\begin{array}{ccccc} & \text{J- space} & & \text{T-space} & \\ & U(M) & \times & U(3) & \\ & & \downarrow & & \\ U(1) \times U(1) \times \ldots \times U(1) & & \times & R(3) & \qquad (3.1) \\ \text{(M factors)} & & & & \end{array}$$

The general rule for the allowable values of the total isospin T which occur for the given symmetry $[\lambda]$ of U(3) is obtained in the same manner as described in Ref. 19. In our case, we perform the reduction of the representation $[\lambda]$ when the group is restricted to the rotational group R_3 in isospin space and we obtain

$$\begin{aligned} T &= K, K+1, K+2, \ldots, K + \max(\sigma_1,\sigma_2) && K \neq 0 \\ &= \max(\sigma_1,\sigma_2), \max(\sigma_1,\sigma_2) - 2, \ldots, 1 \text{ or } 0 && K = 0 \end{aligned}$$

with the integer K taking on the values

$$K = \min(\sigma_1,\sigma_2), \min(\sigma_1,\sigma_2) - 2, \ldots, 1 \text{ or } 0 \qquad (3.2)$$

where $\sigma_1 = \lambda_1 - \lambda_2$ and $\sigma_2 = \lambda_2 - \lambda_3$.

These states look like a series of rotational bands cut off at some maximum value of T and indicate a relation between the

present scheme and the pairing rotational model[3]. The parameter K in (3.2) would correspond to the projection of the isospin on the symmetry axis of the T=1 pairing deformed system.

In the spin-orbital space, the representations of the product group U(1) x U(1) x ... x U(1) are labeled by $[N_1][N_2] \ldots [N_M]$ where N_m are the occupation of the pairs in the j_m level at zero pairing strength. The decomposition of representation of U(M) into representations of U(1) x U(1) x ... x U(1) is obtained by determining all the possible values of $N_1, N_2, \ldots, N_m$ that are permitted by the following representation relation:

$$[\lambda_1\lambda_2\lambda_3] = [N_1] \textcircled{\scriptsize\lambda} [N_2] \textcircled{\scriptsize\lambda} \cdots \textcircled{\scriptsize\lambda} |N_M| \tag{3.3}$$

where $\textcircled{\scriptsize\lambda}$ stands for outer product.

According to the said classification scheme the energy and wave function of the state are thus labeled as

$$\begin{aligned} E &= E([N];[\lambda_1\lambda_2\lambda_3];[N_1][N_2]\ldots[N_M];KT), \\ \Psi &= \Psi(|N|;[\lambda_1\lambda_2\lambda_3]\gamma;[N_1][N_2]\ldots[N_M];KTM_T) \end{aligned} \tag{3.4}$$

We consider a proton-neutron system where all nucleons interact only through a charge independent pairing interaction which spans several single particle states. The Hamiltonian of the system is given by

$$H = \sum_{jm\tau} \varepsilon_j \, a^+_{jm\tau} a_{jm\tau} - g \sum_{t=0,1} \sum_{jj'} \sqrt{\Omega_j \Omega_{j'}} \, b^+_{jt} \cdot b_{j't} \tag{3.5}$$

where $a^+_{jm\tau}$ and $a_{jm\tau}$ are creation and annihilation operators for nucleons in the state with space-spin quantum number jm, isospin

projection τ and the energy ε_j, and b^+_{jt} and b_{jt} are creation and annihilation operators for a pair of nucleons in the spatial state j with J = 0, T = 1 and M_τ = t. The interaction strength is g. Ω_j are pair degeneracies of single particle j-levels.

Instead of using any approximation, the Schrödinger equation for the nuclear system can be solved exactly using group theory and algebraic analysis[5]. The eigenvalues E of the pairing Hamiltonian are then given by the sum of the roots of the following coupled equations in N variables E_i:

$$\frac{1}{g} + \sum_i{}' \frac{k_{i\ell}}{E_i - E_\ell} = \sum_{m=1}^{m=M} \frac{\Omega_m}{2\varepsilon_m - E_\ell} \qquad (3.6)$$

$$\ell = 1, 2, \ldots, N$$

The symmetry constants $k_{i\ell}$ that characterize the charge symmetry $[\lambda]$ of the state can be expressed in terms of the total number of nucleons and the total isospin and are given analytically in Ref. 5 for the states with shapes $[N]$ and $[N-1\ 1]$. In (3.6) the prime on the summation index excludes the values $i=\ell$.

For illustration, we list in table 1 the symmetry constants for the seniority-zero states of a system of eight nucleons

Even for given symmetry and isospin quantum numbers, there exist many solutions to eqs. (3.6). These solutions correspond to the many states of the nucleons that have these quantum numbers. The different solutions may be distinguished in a practical sense and identified with the states of the noninteracting system by considering the pair-energies to be functions of the interaction strength g and requiring that they take on values for g = 0 that

characterize the state of the noninteracting system. These values are determined by the occupations of the single particle levels in this limit. Thus if these occupations are given by the number N_j, then N_j of the pair energies must take on the values $2\varepsilon_j$ in this limit. This provides a unique labeling of the solutions.

Some numerical methods for solving the above-mentioned coupled equations have been discussed and developed in Ref. 20. The algorithm used was such that the calculation of the exact states of the pairing Hamiltonian is made as practical as that of BCS approximation. It is therefore very convenient to carry out the calculations for a series of nuclei.

As the gap equations are to the BCS approximation, so are the equations (3.6) to the present exact treatment. All the relevant physics is contained in (3.6) as it is in the gap equations of BCS theory.

By way of an introduction to the structure of the equations, (3.6), we will now briefly show how two theories known as the Tamm-Dancoff approximation (TDA) (ref. 21) and the modified Tamm-Dancoff approximation (MTDA) (ref. 21) can be obtained from the exact equations. We obtain the TDA by simply ignoring the term involving the pair energies on the left-hand side of the equations. This is tantamount to ignoring the Pauli principle and pairs of nucleons are treated as true bosons. In the MTDA, the Pauli principle is treated in an approximate fashion. In (3.6), this approximation is obtained by replacing the pair energy E_j on the left-hand side of the equations by its value at zero interaction strength, $2\varepsilon_j$. Note that the isospin and symmetry dependence of the energy are contained in the symmetry constants k_{ij}. Thus we see that these two approximate theories are readily obtainable from the exact equations.

In the following sections, we shall further extract from the above equations the usual BCS theory as well as the pairing collective motion.

4. BCS APPROXIMATION

Richardson [22] has beautifully shown that very simple approximation applied to (3.6) yields the results of BCS theory.

Following Richardson, we assume that $g=G/N$ with G of order one and we assume that the degeneracies of the single-particle levels Ω_ν are extensive quantities and therefore of order N. We write the pair energies as $E_i = 2e_i$ and then for charge symmetric states eqs. (3.6) becomes

$$-\frac{2N}{kG} + \sum_j{}' \frac{1}{e_i - e_j} - \frac{1}{k}\sum_j \frac{\Omega_j}{e_i - e_j} = 0 \qquad i = 1, N \tag{4.1}$$

It is very interesting to note that the solution of (4.1) represents a problem in two-dimensional electrostatics, i.e., the equilibrium distribution of a collection of parallel lines of charge. For we can think of the complex numbers e_i as being the locations of N free lines of charge of unit strengh in the complex plane. These free lines of charge are in the presence of a uniform external field of strength -2N/kG and the field of a number of fixed lines of charge of strength $-\Omega_\nu/k$ located at the points ε_ν on the real axis. Eqs. (4.1) then describe the equilibrium of such a system of charges. In order to make use of this analogy, we shall calculate the electrostatic field produced by the charges rather than the charge distribution itself. This field at position ξ in the complex plane is given by

$$h(\xi) = \sum_{i=1}^{N} \frac{1}{\xi - e_i} - \frac{1}{k}\sum_j \frac{\Omega_j}{\xi - e_j} - \frac{2N}{kG} \tag{4.2}$$

Now let us make an educated guess as to what the limiting form of h is in the limit $N \to \infty$. We then show that this limiting form implies all the results of the BCS theory. For the ground state charge distribution, we determine h_o by assuming that, as the number of free lines of charge increases, the lines coalesce to form a sheet of charge. This assumption is suggested by numerical studies which indicate this behavior. The result is that the poles of h arising from the first term of (4.2) merge and form a branch cut. We assume that this branch cut extends from the point a to the point a* since complex roots of eqs. (4.1) occur in complex conjugate pairs. After some complex variable analysis, Richardson suggests

$$h_o(\xi) = -\frac{\sqrt{(\xi-a)(\xi-a^*)}}{K} \sum_\nu \frac{\Omega_\nu}{\eta_\nu (\xi - \epsilon_\nu)} \tag{4.3}$$

where the value of $a = \lambda + i\Delta$ is yet to be determined and

$$\eta_\nu = \sqrt{(\epsilon_\nu - a)(\epsilon_\nu - a^*)} = \sqrt{(\epsilon_\nu - \lambda)^2 + \Delta^2} \tag{4.4}$$

Now since

$$\lim_{\xi \to \infty} h_o(\xi) = -\frac{2N}{KG} \tag{4.5}$$

we finally arrive at

$$\sum_\nu \frac{\Omega_\nu}{\eta_\nu} = \frac{2N}{G} \tag{4.6}$$

which is the gap equation of the BCS theory. By further analysis, one is able to obtain the second BCS equation

$$\frac{1}{k}\sum_{j}\Omega_j\left[1-\frac{\epsilon_j-\lambda}{\eta_j}\right]=N \qquad (4.7)$$

and the usual BCS approximation for the ground energy

$$E_0=\frac{2}{k}\left\{\sum_{j}\Omega_j\epsilon_j\left(1-\frac{\epsilon_j-\lambda}{\eta_j}\right)-\frac{N\Delta^2}{G}\right\} \qquad (4.8)$$

if we put k=2 as is in the case of the system of identical nucleons.

In view of the preceding results, we are able to make the following identification between the parameters of the BCS theory and the location of the branch points in the limiting field h_0:

$$\begin{aligned}\lambda &= \mathrm{Re}\{a\} = \text{chemical potential}\\ \Delta &= \mathrm{Im}\{a\} = \text{energy gap}\end{aligned} \qquad (4.9)$$

Historically, most of the calculations with the Hamiltonian of a pairing force which spans a number of nondegenerate single-particle levels have been done with the approximations of Bardeen-Cooper-Schrieffer (BCS) theory [23] of superconductivity and the equivalent Bogoliubov-Valatin transformation [24]. The applications of these methods to nuclei was pioneered by A. Bohr, B.R.Mottelson [13] and S.Belyaev [25].

For a system of identical nucleons, one would guess with Bayman [26] that the N-particle state (N even) is of the form

$$\Psi_N=\left(\sum_{j}\frac{V_j}{\sqrt{1-V_j^2}}\,a_j^{+}a_{\bar{j}}^{+}\right)^{\frac{N}{2}}|0\rangle. \qquad (4.10)$$

where a_ν^+ and $a_{\tilde{\nu}}^+$ are creation operators for particles in conjugate (time reversed) orbits of energy ε_ν. This expression could be used as a trial wave function in a variational calculation. The parameters V_ν are determined by minimizing the energy. Unfortunately there are considerable practical difficulties in working with this wave function. However, if we give up the number conservation and mix the seniority states of different numbers of particles, we are able to construct a trial wave function which is much easier to work with:

$$\Phi = \left(\prod_{\nu=0} \sqrt{1 - V_\nu^2}\, \exp\left\{ \sum_\nu \frac{V_\nu}{\sqrt{1 - V_\nu^2}}\, a_\nu^+ a_{\tilde{\nu}}^+ \right\} \right) |0\rangle \qquad (4.11)$$

Such a wave function represents an ensemble of seniority-zero states of many nuclei with variable number of particles. Such description of many particle system are common such as the use of grand cannonical ensemble in statistical mechanics.

The non-conservation of the number of particles can be remedied by requiring that the average number of particles is the desired number. This is done by introducing a Lagrange multiplier. Fortunately, with this procedure the spread of numbers of particles in BCS proves to be small, as will be shown below. The Hamiltonian that one minimizes is not the true one H, but rather

$$H' = H - \lambda n \qquad (4.12)$$

where n is the number operator

$$n = \sum_\nu a_\nu^+ a_\nu \qquad (4.13)$$

The chemical potential λ is determined by requiring that the expectation value of n be the correct number of particle N. This procedure has the unfortunate consequences that the BCS approximations really represent a mixture of neighboring even-even nuclei. This means that energies and transition probabilities calculated from such wave functions can at best represent averages taken over several neighboring nuclei. Only if the properties we are trying to calculate vary sufficiently smoothly in going from nucleus to nucleus will such an average be useful. In addition, the fact that states are not eigenstates of the number operator gives rise to the occurrence of spurious states which do not correspond to any state of a single nucleus. This last consequence will be discussed in some detail in the latter part of this section.

In spite of the shortcomings mentioned above, BCS approximation gains compensation from the fact that it replaces the original interacting particles by weakly-interacting "quasi-particles" and thus replaces a complicated many-body problem into a simple one-body problem. This is easily seen by noting that BCS state is the vacuum of a set of quasi-particle operators α_ν^+ and $\alpha_{\tilde{\nu}}^+$ which are related to the physical particle by the Bogolyubov-Valatin transformation

$$\begin{aligned} \alpha_\nu^+ &= U_\nu a_\nu^+ - V_\nu a_{\tilde{\nu}} \\ \alpha_{\tilde{\nu}}^+ &= U_\nu a_{\tilde{\nu}}^+ + V_\nu a_\nu \end{aligned} \qquad (4.14)$$

with $$U_\nu^2 + V_\nu^2 = 1$$

The transformed Hamiltonian then takes the form

$$H = E_0 + \sum_\nu \eta_\nu \alpha_\nu^+ \alpha_\nu + H_{20} + H_{02} + H_{22} + H_{31} + H_{13} + H_{40} + H_{04} \qquad (4.15)$$

with

$$E_0 = \sum_{\nu>0} (2\epsilon_\nu) V_\nu^2 - \frac{\Delta^2}{g} - g \sum_{\nu>0} V_\nu^4 \qquad (4.16)$$

where H_{22} is the Hamiltonian containing two α^+'s and two α's and so on. V's are chosen such that the quasi-particle move independently of each other with the energy

$$\eta_\nu = \sqrt{(\epsilon - \lambda)^2 + \Delta^2} \qquad (4.17)$$

where parameter Δ is half the energy gap and is defined by

$$\Delta = g \langle BCS | \sum_\nu a_\nu^+ a_{\bar{\nu}}^+ | BCS \rangle$$
$$\Delta = g \sum_\nu U_\nu V_\nu \qquad (4.18)$$

the Lagrange multiplier λ and the gap parameter Δ are obtained from the equations

$$\frac{\Delta}{g} = \sum_\nu \frac{\Delta}{2\eta_\nu} \qquad (4.19)$$

$$\sum_{\nu>0} V_\nu^2 = \frac{N}{2} \qquad (4.20)$$

where the occupation probability of the pair state ν, V_ν^2 is given by

$$V_\nu^2 = \frac{1}{2}\left\{1 - \frac{2\epsilon_\nu - 2\lambda}{2E_\nu}\right\} \qquad (4.21)$$

For a degenerate level, one obtains an approximate expression for the energy of a state with v quasi-particle:

$$E_v = \frac{1}{4}g\left[(N-v)(2\Omega-N-v) - v^2\left(1-\frac{N}{2\Omega}\right)^2 + \frac{N^2}{\Omega} + 2v\left(1-\frac{N}{\Omega}\right) - v^2\left(1-\frac{N}{2\Omega}\right)^2 \right] \tag{4.22}$$

This expression is seen to agree with exact expression to within terms of order of $1/\Omega$ as long as $v<N$. In other words, the energies are good to order g/Δ, since $\Delta = g\Omega$ in this case. For a system of non-degenerate levels there is an effective pairing degeneracy

$$\Omega_{eff} = \frac{\Delta}{g} = \frac{1}{2}\sum_{\nu>0} \frac{1}{\sqrt{\left(\frac{\epsilon_\nu - \lambda}{\Delta}\right)^2 + 1}} \tag{4.23}$$

which is a measure of the accuracy of the approximation.

As mentioned before, the BCS wave function represents an ensemble of nuclei with particle numbers slightly different from the average and desired values. The spread in numbers of particles in BCS state is given by

$$(\Delta N)^2 = \langle BCS | (n-N)^2 | BCS \rangle = \sum_{\nu>0} \frac{\Delta^2}{(\epsilon_\nu - \lambda)^2 + \Delta^2} \tag{4.24}$$

In strong coupling limit, this is $\frac{1}{2}N(1-\frac{1}{2}N/\Omega)$ which is $<N/2$ so that

$$\frac{\Delta N}{N} < \sqrt{\frac{1}{2N}} \tag{4.24}$$

During the past few years, there have been numerous efforts to improve the basic approximation of this simple theory which treats the properties of a given nucleus as the average of the

properties of an ensemble of nuclei. Furthermore, there exist exact numerical results, both for spherical [27] and for deformed nuclei [28] against which the accuracy of the BCS theory and its proposed extensions can be measured. As is well known, exact solutions of the pairing problem, both in j-j and L-S coupling, can be given explictly if only one shell is assumed [29]. If two shells are taken into account, the numerical difficulties in finding the sharp seniority solutions are not too great, and various results for this case are known [30]. In the case of more than two shells, however, the numerical complications increases and investigations with a large configuration space have been performed by diagonalization of the Hamiltonian by numerical methods. These have been done only for a few, simple cases where a small number of different single-particle energies ε_ν are involved. These models thus are far from realistic, particularly when considering deformed nuclei.

An improvement in this situation was made by Richardson [31,32] who found that the calculation of the exact eigenstates of the pairing Hamiltonian can be reduced to the solution of equations (3.6) with k=2. The calculation of the solutions of these equations is so straight forward that Richardson was able to obtain exact solutions for configuration spaces of very high dimensions (up to 10^7)[Ref.31]. Furthermore, the exact calculation is no more difficult than the calculation of approximate solutions using any improvement of the BCS theory that gives up the assumption of independent quasi-particles. Therefore, for the simple-pairing force Hamiltonian one is left to choose between doing a BCS calculation and doing an exact calculation. As a result, large errors were discovered in the energies and occupation probabilities calculated by the simple BCS scheme. For example, the

accuracy of BCS solutions of the eigenstates of the pairing force Hamiltonian in the theory of deformed nuclei has been restricted [28] to comparisons of the BCS states with the exact states of this Hamiltonian for systems that are much smaller (8 or fewer interacting particles) than typical nuclear systems. This restriction is usually justified by an appeal to the asymptotic exactness of the BCS ground state in the limit of a large number of particles. It was pointed out by Richardson [31] that the accuracy of the BCS ground state energy does not improve as the number of particles increase from 8 to 32. Furthermore , in this range of particle numbers the accuracy of the BCS excitation energies gets worse with increasing particle number instead of better. This can be easily understood by noting that the hope that the BCS calculation would become more accurate as the size of the system increased is based upon the strong coupling estimate of the errors. This estimate indicates that the percentage errors in the BCS ground-state energy and excitation energies should be proportional to 1/N. However, the exact calculations shows that this is only true for interaction strengths that are much larger than those appropriate for nuclear calculations. Richardson's solutions also provide a valid testing ground for the various approximation techniques in use. In the recent work of Bang and Krumlinde [33], Richardson's method was used to compare BCS and BCS + RPA + Projection approximations with the exact results. In addition, this model is free from anomalous behavior in the cases where the BCS type of approximation collapses. Therefore it is suitable to study some of the interesting regions of the periedic table where 0^+ levels are seen to have prominent properties and where the standard approximation fail.

In concluding this section, let us note that the BCS approximation also introduces a characteristic uncertainty which arises directly from the fact that the wave functions are not eigenfunctions of the number of nuclear particles. This is the introduction of spurious states, and in particular of one spurious spin zero two quasi-particle state. In the degenerate case, the BCS treatment gives Ω distinct two quasi-particle states, whereas the exact solution allows only $(\Omega-1)$ seniority two states. It is easy to see that the spin zero two quasi-particle state,

$$\Psi^{(2)} = \sum_{\nu} \alpha_{\nu}^{+} \alpha_{\tilde{\nu}}^{+} |\mathrm{BCS}\rangle$$

is a state which, although it has two quasi-particles present, is actually a linear combination of states which are all ground states. In the non-degenerate case, a spurious spin zero two quasi-particle state also exists. The independent quasi-particle picture yields M of 0^{+} excited states, which is one more than the right value, where M is the number of single-particle levels. Since the BCS ground state is not an eigenstate of the number operator $\hat{n}$, the state $\hat{n}|\mathrm{BCS}\rangle$ is different from $|\mathrm{BCS}\rangle$ and its components on two-quasi-particle states are spurious; only states orthogonal to it have equivalents in a physical nucleus. But the two-quasi-particle states resulting from an approximate diagonalization of the Hamiltonian are usually not orthogonal to the spurious state, with the result that the spurious states are mixed with various percentage among all the states that one calculates. Sometimes, such spurious components occur nearly all in one state with large total strength, in which case that state must be omitted as spurious. For example, in the quasi-particle spectrum of Pb^{206}, the lowest excited state with spin zero proves [34] to be

entirely spurious. It is interesting to note that inclusion of the particular part of H_{22}, H_{40} and H_{04} in (4.15), namely

$$H' = \frac{g}{4}\left[\sum_{\nu}\left(\alpha_{\nu}^{\dagger}\alpha_{\bar{\nu}}^{\dagger} - \alpha_{\bar{\nu}}\alpha_{\nu}\right)\right]^2$$

leads to a lowering of the spurious spin zero two-quasi-particle state-if systematically treated with the random phase approximation (RPA method)[35,36] - all the way down to the ground state. If we further include that part of H_{22}, H_{40} and H_{04} which is responsible for the pairing vibration [36] in the RPA treatment, then the pairing vibration states can be identified with those non-spurious 0^+ excited states. We will return to the more detailed discussions on these states in the next section.

We should emphasize that the error in BCS theory arises solely from the neglect of H_{res}, or equivalently from the choice of trial wave function. No error is introduced by the Bogolyubov-Valatin transformation. Consequently, if H_{res} were diagonalized on the complete quasi-particle, exact eigenstates, with no particle fluctuation, must result.

5. PAIRING VIBRATION

It is clear that the basic BCS equations (4.19) and (4.20) always have the trivial solution

$$\Delta = 0;\ V_\nu = 1 \quad \text{or} \quad 0, \quad U_\nu = 0 \quad \text{or} \quad 1$$

which is the normal state, corresponding to a sharp Fermi surface. But generally there is a non-trivial superconducting solution of lower energy for which $\Delta \neq 0$. To study the existence of this solution one may draw the curve

$$y = f(x) = \sum_{\nu>0} \frac{1}{2\sqrt{(\varepsilon_\nu - \lambda)^2 + x^2}}$$

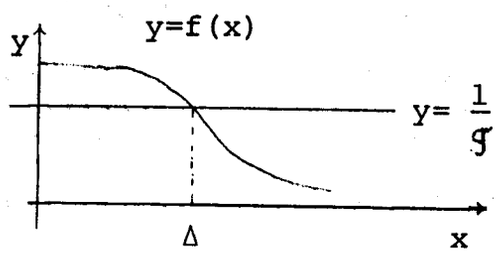

Fig. 2

The value of f(x) at the origion is $\sum_{\nu>0} \frac{1}{2|\varepsilon_\nu - \lambda|}$ which, except λ equal to one of ε_ν, is finite. Thus the superconducting solution may not exist if g is below a certain critical value which can be interpreted as indicating a transition from a "superconducting" to a "normal" ground state. In general for closed shell nuclei all $|\varepsilon_\nu - \lambda|$ are large so that f(0) is small and g is not large enough for the superconducting solution to exist. Alternatively, for a partly filled shell some of the $|\varepsilon_\nu - \lambda|$ are very small so that a superconducting solution is practically always possible. The nonexistence of a superconducting solution does not mean that the pairing interaction has no

effect. It simply means that no improvement over first order perturbation theory for the energy has been achieved in the quasi-particle approximation. In such a case, one must either treat the parts in pairing Hamiltonian that are non-diagonal with respect to the BCS wave function by some linearization method as, e.g., the random phase approximations or solve the whole pairing Hamiltonian exactly. Thus the treatment of the pairing residual interaction not only recaptures the number invariance but also shows the effect of the pairing force in the case where BCS approximation collapses. The diagonalization of H_{res} has been approximated with perturbation theory or with some linearization method as e.g., the random phase approximation, by several authors[35,36]. It is interesting to note that a newly discovered collective mode of vibration appears when the nuclei is sufficiently close to the transition point between the single-particle and a superconducting system.

The pairing vibration state may be considered as genereted by the pairing force elements non-diagonal in the BCS scheme. If we select a particular part of H_{22} and H_{40} in H_{res}, namely

$$H' = -\frac{g}{4} P^2 \tag{5.1}$$

with

$$P = \sum_{\nu} (U_\nu^2 - V_\nu^2)(\alpha_\nu^+ \alpha_{\tilde{\nu}}^+ + \alpha_\nu \alpha_\nu) \tag{5.2}$$

and treat the interaction $H_{qp} + H'$ with the usual random phase approximation [36], we obtain a dispersion type energy equation. This reduces to a very simple form if we assume symmetry between levels above and below the Fermi surface:

$$f(\omega) = \sum_{\nu} \frac{(U_\nu^2 - V_\nu^2)\ 2\eta_\nu}{4\eta_\nu^2 - \omega^2} = \frac{1}{g} \tag{5.3}$$

with the lowest root being the possible collective root

$$\omega = 2\Delta \tag{5.4}$$

Of other roots, there occur one between $2\eta_1$ and $2\eta_2$, one between $2\eta_2$ and $2\eta_3$, etc. where η_1 and η_2 etc. are the smallest and next smallest quasi-particle energies, etc. The general character of the energy spectrum of (5.3) is easily seen from Fig. 3 in which the $f(\omega)$ is sketched as a function of ω. All the solutions of (5.3) are trapped between the energies $2\eta_\nu$. The solution that is not trapped falls a long way below the rest and becomes collective provided a) all $\eta_\nu >> \Delta$; b) many η_ν contribute. A collective state implies the existence of enhancements in the matrix elements corresponding to physical observables. The enhanced operator in this case is the pair transfer operator $\sum_\nu a_\nu^+ a_{\tilde{\nu}}^+$ or its Hermitian conjugate. The operator P in (5.2) can be rewritten as a sum of transfer operators

$$P = \sum_{\nu} (a_\nu^+ a_{\tilde{\nu}}^+ + a_{\tilde{\nu}} a_\nu) \tag{5.5}$$

and is found to be the "specific" operator which is related to the pairing vibrations in the same way as the mass quadrupole operator is connected with quadrupole vibrations. The matrix element of this operator between the ground and the collective state ω is given by an expression valid in the symmetric case [36]

$$\langle \omega | P | 0 \rangle \approx \frac{2}{g} \Big/ \sqrt{\sum_{\nu} \frac{1}{(\varepsilon_\nu - \lambda)^2}} \tag{5.6}$$

Here we see that the matrix element vanishes as one of the single particle levels approaches the Fermi surface. In the

opposite case, in which all the levels are pushed away from the Fermi-energy, both increases in $\varepsilon_\nu - \lambda$ and decreases in Δ tend to enhance the matrix element.

From the equations above it is clear that a favourable case for this particular collective degree of freedom on the whole to be observed-and distinguished from the β-vibrational one - occurs first when all single-particle levels are far away from the Fermi surface.

The above mentioned β-vibrations are the quadrupole vibrations which conserve axial symmetry of deformed nuclei. They are caused by the superposition of a large number of two-quasi-particle states and residual quadrupole-quadrupole interactions. As the force strength χ increases from zero and passes χ_{crit} at which the one phonon state (2^+) becomes a rotational excitation, the two-phonon quadrupole vibration goes over into a beta vibration of the deformed nucleus. As a general rule, the beta-vibration states are low-lying. Their energies are in the 0.8-1.7 MeV range while the pairing vibration states must lie higher than 1.4 MeV. (Cf. Ref. 37).

For those nuclei in which no superconducting solution exists, it is convenient to use the particle-particle and the hole-hole coupling in constructing the bosons. The dispersion relations are obtained as follows:

$$\frac{1}{g} = \sum_p \frac{1}{2\epsilon_p - \omega_p} + \sum_h \frac{1}{2\epsilon_h - \omega_p}$$

$$\frac{1}{g} = \sum_h \frac{1}{2\epsilon_h - \omega_h} + \sum_p \frac{1}{2\epsilon_p + \omega_h} \qquad (5.7)$$

where the quantities e_p and e_h are the magnitudes of the energy separation of the single-particle and hole levels from a zero of energy chosen to lie between the particle and hole states. The structure of Equations (5.7) is shown in Fig.4 by the solid lines together with their reflection (not shown) through the vertical axis.

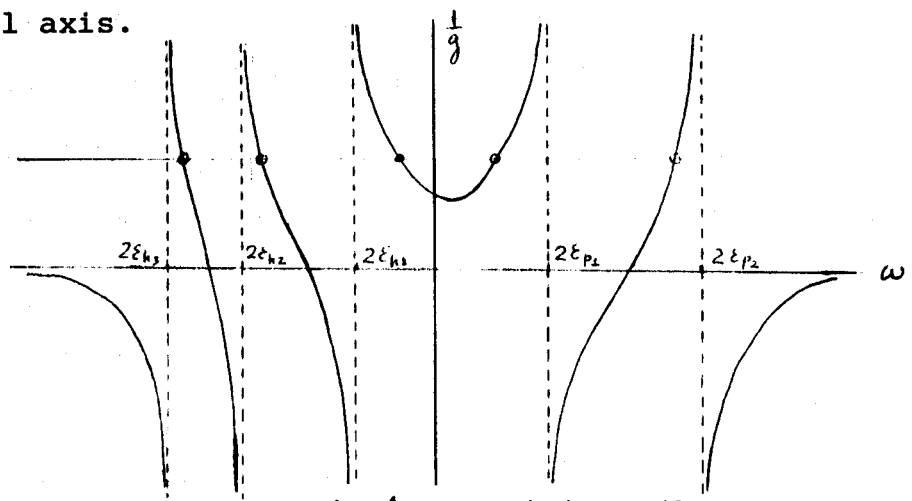

Fig.4 - Pairing Vibrations

For a fixed value of g, the distances of the circles to the right (left) of the vertical axis represent the energies $\omega_p(\omega_h)$ of 0^+ states in the nucleus Z,N+2(Z,N-2). We therefore view the ground state of the nucleus Z,N+2 and the nucleus Z,N-2

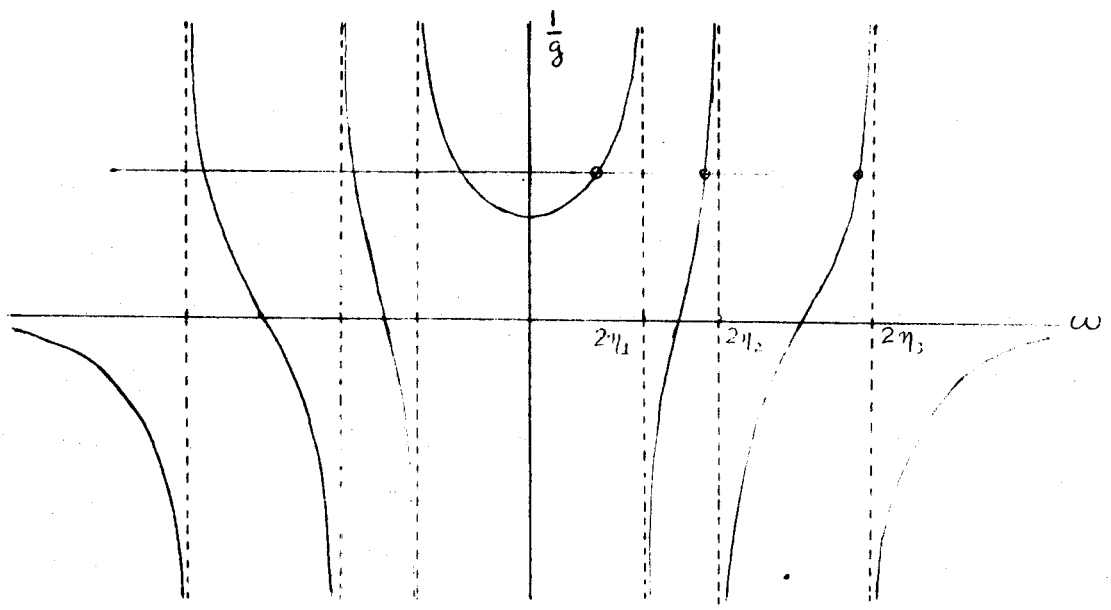

Fig. 3 - Pairing Vibrations for superconducting nuclei.

as collective excitations of the nucleus Z,N. The quanta of excitation of these "pairing vibration" modes are called the pairing phonons and may be labelled by the quantum number, α, the nucleon transfer number. Thus, $\alpha=+2$ for the pair addition mode and $\alpha=-2$ for the pair removal mode. The conservation of the number of particles implies the use of two pairing phonons (one phonon with $\alpha=+2$ plus one phonon with $\alpha=-2$) to obtain an excited state of a nonsuperconducting nucleus. This is not the case with superconducting nuclei where the collective oscillations correspond to only one phonon because the lack of particle conservation.

The excited energy of the lowest 0^+ state in a closed-shell nucleus is lowered from the smallest of the distances $2(\epsilon_p+\epsilon_h)$ to the smallest sum of $\omega_p+\omega_h$. For example, in considering Pb^{208}, the lowest 0^+ excited state is predicted to be lowered from the shell model prediction ($2(\epsilon_{9/2}+\epsilon_{/2}) = 6.82$ MeV) by an amount given by the sum of the pairing binding energies in Pb^{206} and Pb^{210} (1.84 MeV) which gives 4.98 MeV. Experimentally, the lowest excited 0^+ level is found at 4.87 MeV [38].

For a closed-shell nucleus, the pair transfer operator strongly connects (Z,N) to the one phonon states which are the ground states of (Z±2,N) and (Z,N±2). If there are real pairing vibrations there will also be two-phonon states strongly connected to the one phonon states by the pair transfer operator. These will include excited 0^+ states of (Z,N). Thus pairing vibrations are characterized by strong transitions of pair transfer operator from (Z,N-2) to both the ground state and an excited 0^+ state of (Z,N). For example, two transitions to the state with 4.87MeV and the ground state have comparable intensities in the reaction $Pb^{206}(t,p)Pb^{208}$: $\frac{\sigma(4.87)}{\sigma(g.s.)} = 0.5$.

6) TREATMENT OF NEUTRON-PROTON PAIRING WITH QUASI-PARTICLE APPROXIMATION

As viewed in Section 4, considerable progress has been made in understanding the effect of pairing correlations upon nuclear spectra by means of the BCS approximation. But an important limitation in the application of the BCS theory to nuclei is that it can be strictly applied to a system of neutrons or protons alone. In practice one hopes that this situation may be realized in single closed shell nuclei such as the Tin isotopes. However, the nucleon-nucleon interaction is generally assumed to be charge independent; that is, the neutron-proton interaction would be just as strong as the interaction between a pair of like nucleons. This suggests that the neutron-proton pairing correlation should be treated on the same footing as the correlation between like particles. The pairing correlations are usually thought to be confined to the nucleons near the Fermi-surface; thus the neutron-proton pairing would not be expected to play an important role in heavy nuclei where the neutrons and protons are placed in different major shells. Due to the small overlaps between neutron and protons orbitals, the short range part of the neutron-proton force would not be so effective in producing correlations. In this case, one would treat the two kinds of particles separately by the BCS theory and there would be two independent BCS fluids. However, for the medium-weight nuclei in the region $28 \leq Z \leq 50$; $28 \leq N \leq 50$ where neutrons and protons both fill the same major shell, there would be no valid grounds for neglecting the pairing correlations between them. This observation has been confirmed by the calculations of Kisslinger and Sorensen[39] who neglect n-p pairing and

do not get good agreement with experimental data on the low-energy properties for nuclei in this region.

A method based on an obvious generalization of the Bogolyubov-Valatin transformation was proposed by Pal and Goswami [40]. It is a straightforward application of Baranger's [41] adaption to nuclear physics of the Bogolyubov generalized transformation [42]. In this approach, the linear quasi-particle transformation is given by

$$\alpha^{\dagger}_{j\,m\,\sigma} = \sum_{t} \left[U_{\sigma t}\, a^{\dagger}_{j m t} + V_{\sigma t}\,(-)^{j-m}\, a_{j\,-m\,t} \right] \tag{6.1}$$

where σ is a label that distinguishes between the two kinds of quasi-particle which are differenciated by their energies and are obtained by mixing the neutron, the proton, the neutron-hole and proton hole states through the neutron-proton pairing force.

The coefficients $U_{\sigma t}$ and $V_{\sigma t}$ in (6.1) are real and are determined by solving the Hartree-Bogolyubov equations [41]

$$\begin{bmatrix} \mu_{pp} + \epsilon_{jp} - \lambda_p & \mu_{np} & \Delta_{pp} & \Delta_{np} \\ \mu_{np} & \mu_{nn} + \epsilon_{jn} - \lambda_n & \Delta_{np} & \Delta_{nn} \\ \Delta_{pp} & \Delta_{np} & -\mu_{pp} - \epsilon_{jp} + \lambda_p & -\mu_{np} \\ \Delta_{np} & \Delta_{nn} & -\mu_{np} & -\mu_{nn} - \epsilon_{jn} + \lambda_n \end{bmatrix} \begin{bmatrix} U_{\sigma p} \\ U_{\sigma n} \\ V_{\sigma p} \\ V_{\sigma n} \end{bmatrix} = E_{j\sigma} \begin{bmatrix} U_{\sigma p} \\ U_{\sigma n} \\ V_{\sigma p} \\ V_{\sigma n} \end{bmatrix} \tag{6.2}$$

self-consistently. Here $E_{j\sigma}$ are quasi-particle energies. The self energies $\mu_{tt'}$ and the gap parameters $\Delta_{tt'}$ are given by

$$\mu_{tt'} = -\frac{g_{tt'}}{4}\left[\delta_{tt'}\sum_{\nu\sigma} V_{\sigma\nu}(j) + \sum_{\sigma} V_{\sigma t}(j)\, V_{\sigma t'}(j)\right] \tag{6.3}$$

$$\Delta_{tt'} = g_{tt'}\sum_{j\sigma} \Omega_j\, U_{\sigma t}(j)\, V_{\sigma t'}(j) \tag{6.4}$$

The symbols λ_n and λ_p represent the chemical potentials (Fermi energies) for the neutron and proton respectively and are to be determined from the "number" equation:

$$N = \sum_{j\sigma} \Omega_j\, V^2_{\sigma n}(j) \tag{6.5}$$

$$\frac{Z}{2} = \sum_{j\sigma} \Omega_j\, V^2_{\sigma p}(j) \tag{6.6}$$

The quasi-particle vaccum state energy E_o and its corresponding normalized wave function $|\tilde{0}\rangle$ are given by

$$E_o = \sum_{j\sigma}\sum_{tt'}\left[\epsilon_{jt}\,\delta_{tt'} + \mu_{tt'}(j)\right] V_{\sigma t}\, V_{\sigma t'} - \sum_{tt'} \frac{\Delta^2_{tt'}}{g_{tt'}} \tag{6.7}$$

and

$$\begin{aligned}|\tilde{0}\rangle = \prod_{j,m>0}\Big[& C^j_c + C^j_{np}\left(a^\dagger_{j-mp}\, a^\dagger_{jmn} + a^\dagger_{j-mn}\, a^\dagger_{jmp}\right) \\ & + C^j_{nn}(-1)^{j-m}\, a^\dagger_{j-mn}\, a^\dagger_{jmn} \\ & + C^j_{pp}(-1)^{j-m}\, a^\dagger_{j-mp}\, a^\dagger_{jmp} \\ & + C^j_4\, a^\dagger_{j-mp}\, a^\dagger_{jmp}\, a^\dagger_{jmn}\, a^\dagger_{jmn}\Big]\,|0\rangle\end{aligned} \tag{6.8}$$

where

$$C_o^j = (U_{21}V_{21}+U_{22}V_{22})(U_{11}V_{11}+U_{12}V_{12})-(U_{11}V_{21}+U_{12}U_{22})^2$$

$$C_{np}^j = (U_{21}V_{21}+U_{22}V_{22})V_{11}V_{12}+(U_{11}V_{11}+U_{12}V_{12})V_{21}V_{22}$$
$$-(U_{11}V_{21}+U_{12}V_{22})(V_{12}V_{21}+V_{11}V_{22})$$

$$C_{pp}^j = (U_{21}V_{21}+U_{22}V_{22})V_{11}^2+(U_{11}V_{11}+U_{12}V_{12})V_{21}^2$$
$$-2(U_{11}V_{21}+U_{12}V_{22})V_{11}V_{21}$$

$$C_{nn}^j = (U_{21}V_{21}+U_{22}V_{22})V_{12}^2 + (U_{12}V_{12}+U_{11}V_{11})V_{21}^2$$
$$-2(U_{12}V_{22}+U_{11}V_{21})V_{12}V_{22}$$

$$C_4^j = (V_{11}V_{22}-V_{12}V_{21})^2$$

It is clear from the complicated expressions for the quasi-particle energies and energy gaps, etc., shown above that the simplicity of the BCS theory is lost. For a system of nucleons moving in many j shells, one must solve a very complicated set of coupled equations. Numerical solutions are obtained only by long computational work. Furthermore, when mixing neutron and proton operators as well as creation and annihilation operators, not only are the particle numbers not conserved, but the total isospin, T, and its z-component, T_z, are also no longer good quantum numbers. It is therefore necessary to introduce more Lagrangian multipliers in order to adjust the average values of these quantities to their desired values. This procedure, no doubt, makes the generalized quasi-particle transformation even more unreliable than the original Bogolyubov-Valatin transformation.

the results in (6.11) and (6.12). Here again we see the serious limitation of the treatment of n-p pairing with a linear quasi-particle transformation and the necessity for an improved treatment of n-p correlations. As will be shown, this limitation also applies to a system of nucleons interacting through a charge-independent pairing force in many j shells.

Let us introduce with Girocchio and Wenesser [43] a new set of fermion operators, b, which are rotated in isospace:

$$\begin{bmatrix} b^{\dagger}_{jmp} \\ b^{\dagger}_{jmn} \\ (-1)^{j-m} b_{j-mp} \\ (-1)^{j-m} b_{j-mn} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} a^{\dagger}_{jmp} \\ a^{\dagger}_{jmn} \\ (-1)^{j-m} a_{j-mp} \\ (-1)^{j-m} a_{j-mn} \end{bmatrix}$$

It follows the matrix elements of the generalized Bogoliubov transformation in the above two representation are then related by

$$\begin{bmatrix} U'_{\sigma p} \\ U'_{\sigma n} \\ V'_{\sigma p} \\ V'_{\sigma n} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} U_{\sigma p} \\ U_{\sigma n} \\ V_{\sigma p} \\ V_{\sigma n} \end{bmatrix} \tag{6.13}$$

then the transformed Hartree-Bogoliubov matrix takes the form

$$\begin{bmatrix} \Gamma' & \Delta' \\ \Delta' & -\Gamma \end{bmatrix} \qquad (6.14)$$

with

$$\Gamma' = \begin{bmatrix} \Gamma'_{pp} & \Gamma'_{np} \\ \Gamma'_{np} & \Gamma'_{nn} \end{bmatrix} \qquad \Delta' = \begin{bmatrix} \Delta'_{pp} & \Delta'_{np} \\ \Delta'_{np} & \Delta'_{nn} \end{bmatrix} \qquad (6.15)$$

where

$$\Gamma'_{pp} = (\mu_{pp} + \epsilon_{jp} - \lambda_p)\cos^2\phi + (\mu_{nn} + \epsilon_{jn} - \lambda_n)\sin^2\phi - \mu_{np}\sin 2\phi$$

$$\Gamma'_{nn} = (\mu_{pp} + \epsilon_{jp} - \lambda_p)\sin^2\phi + (\mu_{nn} + \epsilon_{jn} - \lambda_n)\cos^2\phi + \mu_{np}\sin 2\phi \qquad (6.16)$$

$$\Gamma'_{np} = \mu_{np}\cos 2\phi - \tfrac{1}{2}(\mu_n - \mu_p + \epsilon_{jn} - \epsilon_{jp} - \lambda_n + \lambda_p)\sin 2\phi$$

$$\Delta'_{pp} = \Delta_{pp}\cos^2\phi + \Delta_{nn}\sin^2\phi - \Delta_{np}\sin 2\phi$$

$$\Delta'_{nn} = \Delta_{pp}\sin^2\phi + \Delta_{nn}\cos^2\phi + \Delta_{np}\sin 2\phi$$

$$\Delta'_{np} = \Delta_{np}\cos 2\phi - \tfrac{1}{2}(\Delta_{nn} - \Delta_{pp})\sin 2\phi$$

One can always make such a transformation which diagonalizes Γ', keeping Δ' unchanged, which amounts to a renormalization of the single-particle energies for pairing type forces. After such a transformation, the vaccum state energy E_0 which is given by (6.7) becomes

$$E_o = \sum_j 2\Omega_j \epsilon_j \left[V_{11}^2(j) + V_{12}^2(j) + V_{21}^2(j) + V_{22}^2(j) \right] - \frac{N_o m |\Delta'|}{g} \qquad (6.17)$$

It is important to note that this energy is invariant under the transformation (6.13), i.e., under rotations in the isospin space.

One can always choose a representation in which $\Delta'_{np}=0$, according to (6.16), by setting

$$\tan 2\phi = \frac{2\Delta_{np}}{\Delta_p - \Delta_n} \qquad (6.18)$$

The ground state expectation values of $\hat{T}_z$ and $\hat{T}_z'$ are simply related

$$T_z = \langle \hat{T}_z \rangle = \langle \hat{T}_z' \rangle \cos 2\phi \qquad (6.19)$$

Further, the expectation value of T^2 can be expressed as $\langle T_z' \rangle^2$ plus a term of an order smaller than $\langle T_z' \rangle^2$ and is therefore approximated by

$$\langle \hat{T}^2 \rangle \approx \langle T_z' \rangle^2 \qquad (6.20)$$

These last two expressions, (6.19) and (6.20), provide the physical interpretation of $\langle T_z' \rangle$ as the isospin quantum number T. The value of T_z can be found by choosing T and ϕ, (6.19), $-T \leq T_z \leq T$. The statement that E_o is independent of ϕ for fixed T is the statement of isospin invariance within BCS approximation. It is, of course, not an exact invariance, since $|\tilde{0}\rangle$ is not an exact eigenfunction of $\hat{T}^2$.

If we pick $\phi = 0$, we have the solution for $T_z=T$, then

$$\Delta'_p = \Delta_p$$

$$\Delta'_n = \Delta_n$$

$$\Delta_{np} = 0,\ C^j_{np} = 0$$

The wave function $|\tilde{0}\rangle$ is then reduced to a BCS product form for the neutrons and protons separately. In the present approximation there is nothing gained from the neutron-proton part of the pairing interaction. However, for the same T but $T>|T_z|$, the neutron-proton pairing does show its effect within the quasi-particle approximation. This is exactly the conclusion draws by Ginocchio and Weneser [43].

We have observed that the general Bogoliubov transformation, which minimizes the ground state energy (note that $T=T_z$ for the ground state), reduces to the product of two ordinary Bogoliubov transformation. This behavior is partly due to the number and isospin fluctuation in the vaccum state wave function. As in the degenerate case, the error in the ground state energy caused from these fluctuations is of the same relative order as the omtted n-p part of the energy to the total energy. The situation can be improved by an isospin projection. The procedure is similar to the construction of rotational wave function by an angular momentum projection from a simple intrinsic function. If we are only interested in energies, the projection may be treated formally without any explicit calculations. We start with the quasi-particle vaccum state wave function (6.8) which now can be written as

$$|\tilde{0}\rangle = \prod_{\delta>0} \left(U_n(\delta) + V_n(\delta)\, a^+_{\delta n p}\, a^+_{\bar{\delta} n p}\right) \qquad (6.21)$$

$$\times \left(U_n(\delta) + V_n(\delta)\, a^+_{\delta n n}\, a^+_{\bar{\delta} n n}\right)|0\rangle$$

and expand the n-p part of the pairing Hamiltonian in normal order in terma of $|\tilde{0}\rangle$

$$H_{np} = -\frac{g}{2}\sum_{jm}\langle a^{\dagger}_{jmp}a_{jmp}\rangle\langle a^{\dagger}_{j-mn}a_{j-mn}\rangle$$
$$-\frac{g}{2}\sum_{jm}\langle a^{\dagger}_{jmp}a_{jmp}\rangle a^{\dagger}_{j-mn}a_{j-mn} + \langle a^{\dagger}_{j-mn}a_{j-mn}\rangle a^{\dagger}_{jmp}a_{jmp} \qquad (6.22)$$
$$+ H_{res}(np)$$

It was observed by Elliot and Lea [44] that the Hamiltonian

$$H_{np}^{(eff)} = -\frac{1}{2}g\sum_{jm} a^{\dagger}_{jmp}a^{\dagger}_{j'm'n}a_{jmp}a_{jmn} =$$
$$= \frac{g}{4}\left(\hat{T}^2 - \hat{T}_z^2 - \frac{1}{2}\hat{n}\right) \qquad (6.23)$$

gives exactly the same expansion as H_{np} apart from differences in H_{res} which can be neglected in the spirit of BCS method. Thus if the component of definite isospin (T,T_o) and particle number $N+Z$, $|\tilde{0}\rangle_{N+Z,TT_o}$ is projected from the wave function $|\tilde{0}\rangle$, i.e.

$$|\tilde{0}\rangle_{N+Z,TT_o} = P_T P_{T_o} P_{N+Z}|\tilde{0}\rangle$$

we would then have

$$H|\tilde{0}\rangle_{N+Z,TT_o} = HP_T P_{T_o} P_{N+Z}|\tilde{0}\rangle$$
$$\approx \left[E_o + \frac{g}{2}\{T(T+1) - \frac{1}{4}(N-Z)^2\right.$$
$$\left. - \frac{1}{2}(N+Z)\}\right] \quad |\tilde{0}\rangle_{N+Z,TT_o} \qquad (6.24)$$

where E_o is the ground state energy obtained from the quasi-particle approximation. For $T=T_Z=0$, the energy is shown to be

lowered by the amount of $-Zg/2$, which is the same size as error made in quasi-particle approximation, from the quasi-particle vaccum state energy. Thus it shows the effect of neutron-proton pairing which the simple quasi-particle approximation failed to explain.

Before concluding this section, let us point out that for the medium weight nuclei, the exact treatment as described in sec. 3 has some advantages over the treatments with quasi-particle approximation. Since the former treats the nucleon number and isospin as good quantum numbers whereas the latter includes number and isospin fluctuations thus making it some what unreliable because of the smallness of the nucleon number. Furthermore, as has been mentioned, for $T=|T_Z|$ the quasi-particle approximation fails to predict any n-p pairing correlation effects in a system of nucleons interacting through charge-independent pairing force.

One should also bear in mind that the wave functions of those charge intermadiate symmetric states and the interplay between isospin and symmetry cannot be treated within the framework of a BCS theory or its extension.

7. PAIRING-ROTATION-VIBRATION MODEL WITH ISOSPIN

As mentioned before the analysis of experimental data from the two nucleon transfer reactions [1-4] suggested that T=1 pairing force plays an important role in the description of strongly populated 0^+ states in medium weight nuclei. These 0^+ states can be understood in terms of pairing vibrations [1,2] and pairing rotations [3]. Recently a more sophisticated collective treatment of T=1 pairing Hamiltonian has been developed by Bes and others[4]. In this macroscopic model, the pairing deformation is considered to be taken place in a four-dimensional isospin and gauge space [4].

In this section, we indicate the microscopic basis of the pairing rotation-vibration model and show how a microscopic description of pairing collective motion is obtained by making approximations to the set of dispersion type energy equation (3.6) which one derives from the exact treatment of T=1 pairing Hamiltonian.

Multiplying the ℓ-th equation in (3.6) by E_ℓ, and then summing over ℓ, we obtain

$$E = \frac{1}{2} kN(N-1)g - N \sum_m \Omega_m g + \sum_i \sum_m \frac{\Omega_m (2\epsilon_m)}{2\epsilon_m - E_i} g \qquad (7.1)$$

with

$$k = \frac{N(N-3) + T(T+1)}{N(N-1)} \qquad (7.2)$$

It is well known that if we set $k_{i\ell} = 0$ in (3.6), we then have the boson approximation in which the T=1 pairs of fermions are approximated by bosons. Eq. (7.1) then becomes

$$E = \sum_\nu N_\nu \mathcal{E}_\nu \qquad (7.3)$$

where the boson energies ξ_ν are simply given by

$$\frac{1}{g} = \sum_m \frac{\Omega_m}{2\epsilon_m - \xi_\nu} \tag{7.4}$$

If we take Ca^{40} as inert core then the lowest two roots of (7.4), ξ_1 and ξ_2, are expected to be collective. This is due to the fact that there is a large energy gap between the first two j-levels. We may therefore view the 0^+ states for the nuclei in the region $40<A<64$ as collective excitations of Ca^{40}. The quanta of excitation are the pairing phonons each of which carries the energy ξ_1 or ξ_2. With these quanta as building blocks, we can construct the pairing vibrational spectrum for the nuclei in the said region and label each pairing vibrational state as

$$(N_1,N_2)\,[\lambda]\,T,K \tag{7.5}$$

The correspondence between the present labelling of the state and the conventional one [2,3] is obvious as is illustrated in Table 3.

In order to consider the anharmonic effects, we either solve (3.6) and (7.1) exactly or make some approximations.

As long as there exist a large gap between the first and the second j-level outside the inert core, as is the case for the nuclei in the above-mentioned region, (7.1) can be approximated by the following desired expressions for the charge totally-symmetric states:

$$E' = E_o + \sum_{\ell=2} b^\ell \left[N(N-3) + T(T+1)\right]^\ell \tag{7.6}$$

where

$$E_o = \sum_\nu N_\nu \varepsilon_\nu + \frac{h^2}{2\mathcal{J}} \left[N(N-3) + T(T+1) \right] \tag{7.7}$$

with

$$\frac{h^2}{2\mathcal{J}} = \frac{1}{N(N-1)} \sum_\nu \left[\frac{1}{2} N_\nu (N_\nu - 1) \overset{o}{g}_\nu - \frac{\varepsilon_\nu}{\varepsilon_1 + \varepsilon_2 - 2\varepsilon_\nu} N_1 N_2 \overset{o}{g}_\nu \right] \tag{7.8}$$

and

$$b_\ell = \frac{1}{N(N-1)} \sum_\nu \left[\frac{1}{2} N_\nu (N_\nu - 1) \overset{o}{g}_\nu - \frac{\varepsilon_\nu}{\varepsilon_1 + \varepsilon_2 - 2\varepsilon_\nu} N_1 N_2 \overset{o}{g}_\nu \right]$$

$$\left[- \frac{1}{N(N-1)} \frac{N_1 N_2 \overset{o}{g}_\nu}{N_\nu (\varepsilon_1 + \varepsilon_2 - 2\varepsilon_\nu)} \right]^{\ell - 1} \tag{7.9}$$

The $\overset{o}{g}_\nu$ in these last expressions are given by

$$\overset{o}{g}_\nu = 1 \Big/ \left[\frac{1}{g} - \sum_m \frac{\Omega_m (2\epsilon_m)}{(2\epsilon_m - \varepsilon_\nu)^2} \right] \tag{7.10}$$

A similar yet more complicated expression for E', which we will not present here, can be obtained for the states of shape $[N-1\ 1]$. One would immediately recognize that the expression (7.6) is very similar to that of rotation-vibration model for deformed nuclei [45]. We are thus led to a description of our nuclear system in terms of a rotation-vibration picture in which $\mathcal{J}$ in (7.7) can be interpreted as the moment of inertia of the T=1 pairing axially symmetric deformed system and the b_ℓ as the rotation-vibration interaction terms. The expression N(N-3) + + T(T+1) in (7.7) can now be attributed to the fact that the pairing deformation is taking place in isospin space as well as in gauge space[4].

So far we have only considered the states with charge sym-

metries $[N]$ and $[N-1\ 1]$ for the nuclei of interest. According to the well known reduction of product representations of permutation groups [46], these states are the only ones that can be reached by pair transfer reactions and therefores are the physically most interesting ones.

As in quadrupole-deformed nuclei, we can group 0^+ states into the ground state, Δ -, and Γ - bands [3,4]. The Δ-oscillations are vibrations of the gap parameter (K=0) which preserve axial symmetry while the Γ-vibrations (K=1) cause deviations from axial symmetry. In the present scheme, the allowed values of K for the charge symmetry $[\lambda_1\lambda_2\lambda_3]$ are given in (3.2). It follows that K=0 for the charge totally-symmetric states and K=1 for the states of shape $[N-1\ 1]$. This suggests that the first excited band of states with the shapes $[N]$ and $[N-1\ 1]$ can be considered as Δ - and Γ - band respectively with band head $\varepsilon_2 - \varepsilon_1$. Thus the correspondence between the classification of states according to the present scheme and that according to the two schemes referred to in Ref. 3 is established.

In order to test the validity of the present approximation and to examine the effects due to vibration-rotation interaction, we have calculated the approximate energies E_o in (7.7), E' in (7.6) and the exact energies E in (7.1) for the low-lying pairing 0^+ states of the nuclei in the region $40 \leq A \leq 64$. As an illustration, we present the calculated results in Table 3 for the various states in the ground state, Δ and Γ rotational band of the A=56, A=60, A=52 and A=48 systems. In general, the present approximation (7.6) is fairly good for the nuclei of interest as is compared with the exact solutions. Our results show that, for nuclei with $A < 56$, the effect due to vibration-rotation interaction is quite small as is illustrated in Table 3.

However, for those with A $\geq$ 56, the contribution from the vibration-rotation coupling becomes significant.

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TABLE CAPTION

Table 3. Energies calculated for the various 0^+ states in the ground state, Δ and Γ rotational band of the A=56, A=60, A=52 and A=48 systems. The second, the third and the fourth column list the labels of the state according to (7.5) and the vibrational and the rotational schemes described in Ref. 3. The approximate energies E_o, E' and the exact energies of these states are calculated according to (7.7), (7.6) and (7.1) respectively for the following given pairing strengths and single particle energies. g = 24/A MeV. The single particle energies are given an A-dependence following Ref. 39 with the values $e_{7/2} = 0.0$, $e_{3/2} = 4.25$, $e_{5/2} = 5.03$, $e_{1/2} = 5.36$ MeV for $A_o = 56$ (See the appendix 2 of Ref. 39 for details). These last values for $2p_{3/2}$, $1f_{5/2}$ and $2p_{1/2}$ level are taken from the spectrum of Ni^{57}. The $1f_{7/2} - 2p_{3/2}$ single-particle splitting is chosen so that the observed first strongly excited 0^+ state of Ni^{56} is reproduced.

TABLE 1

Symmetry constant

for

the seniority-zero states of eight nucleons

	[4]			[31]			[2^2]		[21^2]
T	0	2	4	1	2	3	0	2	1
k_{12}	$\frac{1}{3}$	$\frac{5}{6}$	2	0	0	0	0	0	0
k_{13}	$\frac{1}{3}$	$\frac{5}{6}$	2	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{5}{4}$	0
k_{14}	$\frac{1}{3}$	$\frac{5}{6}$	2	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{5}{4}$	1
k_{23}	$\frac{1}{3}$	$\frac{5}{6}$	2	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{3}{2}$	0	0	0
k_{24}	$\frac{1}{3}$	$\frac{5}{6}$	2	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{5}{4}$	1
k_{34}	$\frac{1}{3}$	$\frac{5}{6}$	2	$-\frac{1}{2}$	2	2	$\frac{1}{2}$	$\frac{5}{4}$	1

TABLE 2

Percentage errors in ground state energy of 7/2 shell nuclei due to the neglect of n-p pairing

Isotope	22Ti22	22Ti24	22Ti26	22Ti28
Errors	11.11%	9.09%	9.09%	11.11%

Isotope	24Cr24	24Cr26	24Cr28
Errors	14.28%	14.28%	16.67%

Isotope	26Fe26	26Fe28	28Ni28
Errors	20%	23.07%	33.33%

TABLE 3

A	State			E_o (MeV)	E' (MeV)	Exact Energy (MeV)
	$(N_1,N_2)\ [\lambda],TK$	$(N_rT_r,N_aT_a)_T$	$(A,n_\Delta,n_\Gamma,K,T)$			
56	$(8,0)\ [8]0,0$	$(00,00)_0$	(56,0,0,0,0)	-7.12	-7.12	-7.13
	$(7,1)\ [8]0,0$	$(11,11)_0$	(56,1,0,0,0)	-0.03	-0.33	-0.43
	$(7,1)\ [8]2,0$	$(11,11)_2$	(56,1,0,0,2)	1.19	0.79	0.70
	$(7,1)\ [71]1,1$	$(11,11)_1$	(56,0,1,1,1)	0.21	-0.15	-0.27
60	$(8,2)\ [10]0,0$	$(00,20)_0$	(60,0,0,0,0)	8.86	7.98	8.17
	$(8,2)\ [10]2,0$	$(00,22)_2$	(60,0,0,0,2)	9.82	8.79	8.96
	$(7,3)\ [10]0,0$	$(11,31)_0$	(60,1,0,0,0)	15.34	14.31	15.22
	$(7,3)\ [10]2,0$	$(11,31)_2$	(60,1,0,0,2)	16.08	14.87	15.85
	$(7,3)\ [91]1,1$	$(11,31)_1$	(60,0,1,1,1)	15.57	14.50	15.57
52	$(6,0)\ [6]0,0$	$(20,00)_0$	(52,0,0,0,0)	-9.20	-9.20	-9.31
	$(5,1)\ [6]0,0$	$(31,11)_0$	(52,1,0,0,0)	-0.85	-0.98	-1.05
	$(5,1)\ [6]2,0$	$(31,11)_2$	(52,1,0,0,2)	0.34	0.13	0.05
	$(5,1)\ [51]1,1$	$(31,11)_1$	(52,0,1,1,1)	-0.62	-0.78	-0.86
	$(5,1)\ [51]2,1$	$(31,11)_2$	(52,0,1,1,2)	0.59	0.39	0.32
48	$(4,0)\ [4]0,0$	$(40,00)_0$	(48,0,0,0,0)	-9.16	-9.16	-9.21
	$(3,1)\ [4]0,0$	$(51,11)_0$	(48,1,0,0,0)	0.62	0.60	0.58
	$(3,1)\ [4]2,0$	$(51,11)_2$	(48,1,0,0,2)	1.66	1.57	1.53
	$(3,1)\ [31]1,1$	$(51,11)_1$	(48,0,1,1,1)	0.83	0.80	0.77
	$(3,1)\ [31]2,1$	$(51,11)_2$	(48,0,1,1,2)	2.19	2.12	2.08

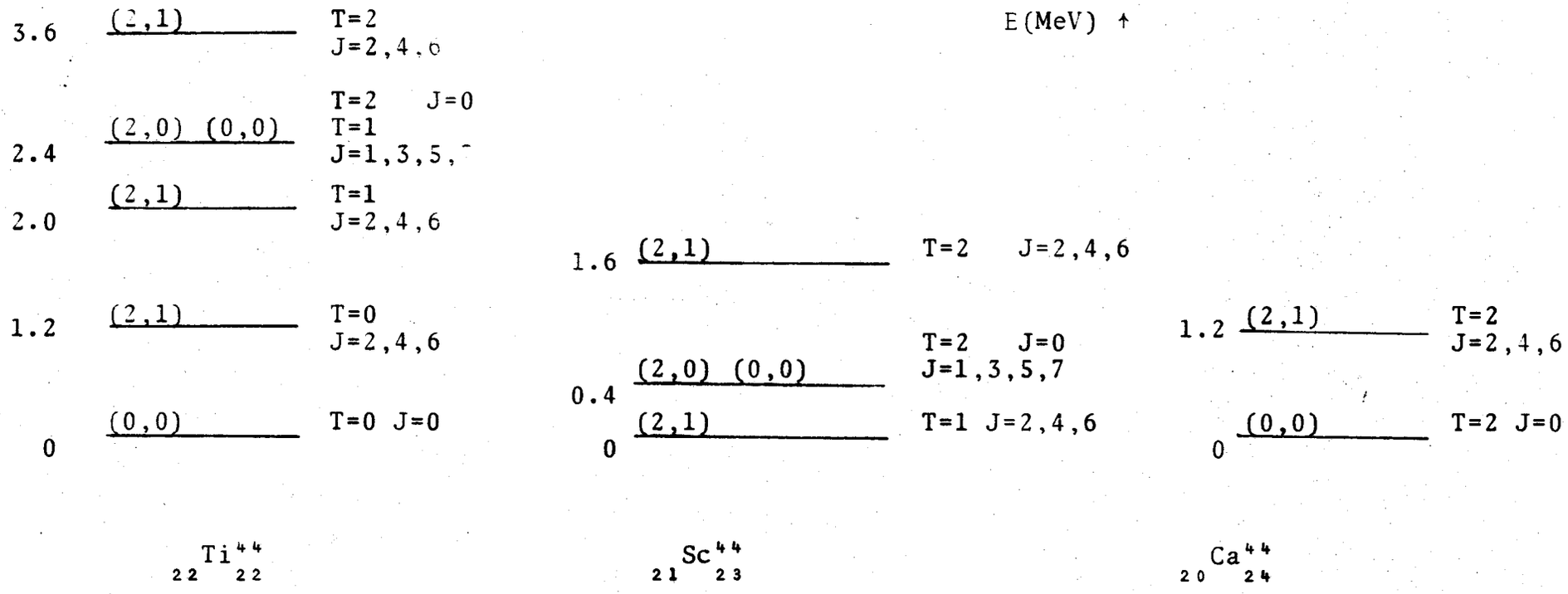

Fig. 1 The Spectrum Of Pairing Plus Isopairing Force For The Configuration $\left(\frac{7}{2}\right)^4$

Note: Each state is labled by two quantum numbers (v,t) where (v,t) represent seniority and reduced-isospin.

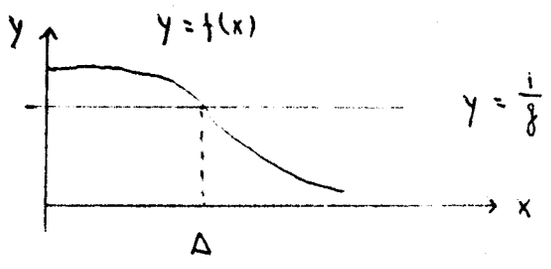

FIG. 2

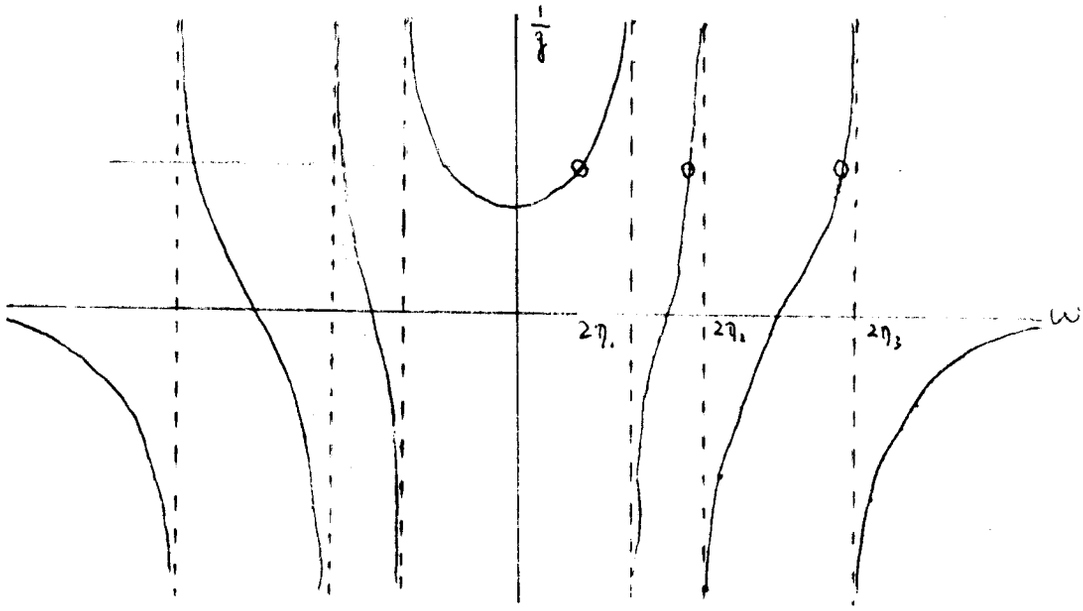

FIG. 3 PAIRING VIBRATIONS FOR SUPERCONDUCTING NUCLEI

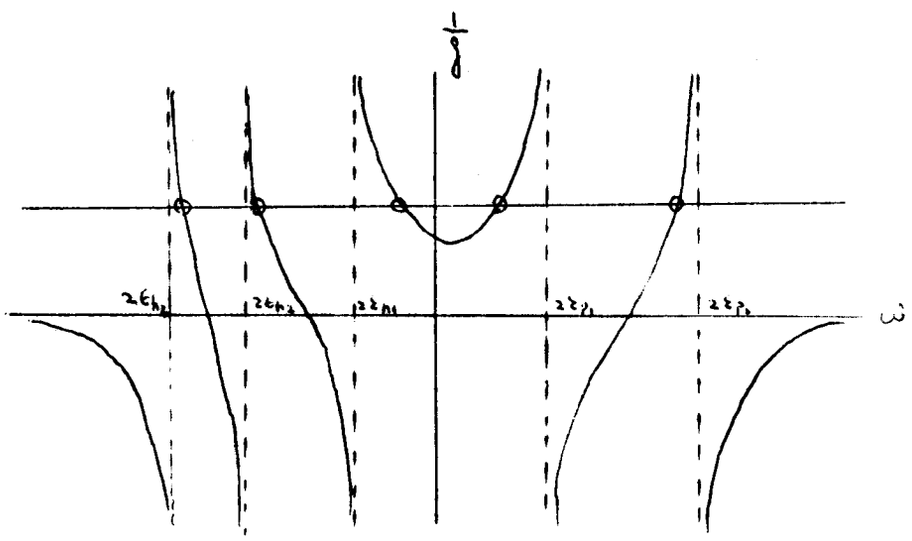

FIG. 4 PAIRING VIBRATIONS FOR NON-SUPERCONDUCTING NUCLEI