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The role of the zero-point fluctuations in the classical stochastic Schrödinger like equation

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Abstract

We study the statistical evolution of a charged particle moving in phase space under the action of the vacuum fluctuations of the zero-point electromagnetic field. Our starting point is the Liouville equation, from which we derive a classical stochastic Schrödinger like equation for the probability amplitude in configuration space. It should be stressed that we are not deriving the Schrödinger wave equation. An equation formally identical to the Schrödinger equation used in Quantum Mechanics is obtained as a particular case of the classical stochastic Schrödinger like equation. An inconsistency appearing in the standard Schrödinger equation, when we take into account the vacuum electromagnetic fluctuations and the radiation reaction, is clearly identified by means of two examples using different sources of electromagnetic noise. The classical stochastic Schrödinger like equation, however, is consistently interpreted within the realm of Stochastic Electrodynamics.

Keywords: Zero-point fluctuations; Stochastic electrodynamics.

I. INTRODUCTION

The classical electromagnetic theory has been largely extended by the program of the Stochastic Electrodynamics (SED) [1,2], due to the inclusion of the effects of the real electromagnetic zero-point radiation. According to the SED picture, there is a clear correspondence between the nonrelativistic Heisenberg equations of motion, for a spinless charged particle interacting with the quantized electromagnetic field of Quantum Electrodynamics (QED), and the classical (Langevin type) Abraham-Lorentz equation with real vacuum fluctuation forces. At the same time, it is well known [3] that for a special class of potentials the dynamical evolution of any Schrödinger wave packet is entirely determined by the laws of Classical Mechanics. Therefore it is desirable to develop a method that is aplicable to any deterministic potential, in order to better clarify the relation between the classical and the quantum theories of the microscopic world.

A mathematical tool widely used in SED is the Fokker-Planck equation, which is derived from the stochastic Liouville equation for describing the Brownian motion of the microscopic charged particles [4]. Unfortunately, however, this method has a restricted use due to the mathematical difficulties for solving the Fokker-Planck equation, mainly in the cases associated with the motion under nonlinear forces.

Our purpose here is to show that the stochastic Liouville equation can be put in a mathematical form that is easier to manipulate even in the case of nonlinear forces. We shall derive a classical Schrödinger like equation from the Liouville equation, using a procedure similar to that introduced by Wigner [5] for describing Quantum Mechanics in phase space. Our approach introduces a free parameter $\hbar'$ in the Wigner type transform [6]. We shall show that this procedure enables us to make a clear distinction between the free parameter

$\hbar'$ and the Planck's constant $\hbar$. Only the vacuum electromagnetic fluctuations will depend on the numerical value of the Planck's constant $\hbar$. We shall see that this distinction will be of great help in order to clarify the physical meaning of the Schrödinger like equation and its interpretation within the realm of a purely classical theory.

II. CONNECTING THE STOCHASTIC LIOUVILLE EQUATION TO A SCHRÖDINGER LIKE CLASSICAL STOCHASTIC EQUATION

The description of classical phenomena by classical statistical mechanics is based on the concept of phase space. The mean value of any dynamical variable $A(\mathbf{x}, \mathbf{p}, t)$ is calculated according to the relation

$$\langle A \rangle = \int A(\mathbf{x}, \mathbf{p}, t)\, W(\mathbf{x}, \mathbf{p}, t)\, \mathbf{d^3x d^3p} \quad , \tag{1}$$

and the probability density distribution in phase space, $W(\mathbf{x}, \mathbf{p}, t)$, evolves in time according to the Liouville equation

$$\frac{\partial W}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial W}{\partial \mathbf{x}} + \dot{\mathbf{p}} \cdot \frac{\partial W}{\partial \mathbf{p}} = 0 \quad , \tag{2}$$

where $\dot{\mathbf{x}}$ and $\dot{\mathbf{p}}$ are obtained from the classical Hamilton's equations of motion.

Consider an ensemble of systems which consist of a nonrelativistic spinless charged particle interacting only with the electromagnetic field. The Hamiltonian which describes the time evolution of the whole system (particle plus field) is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\phi + H_{\text{VF}} \quad , \tag{3}$$

where e and m are the charge and mass of the particle, repectively, $\phi(\mathbf{x},\mathbf{t})$ is the scalar potential, and

$$\mathbf{A}(\mathbf{x},t) = \mathbf{A}_{\text{ext}} + \mathbf{A}_{\text{VF}} + \mathbf{A}_{\text{RR}} \tag{4}$$

is the vector potential. The term $\mathbf{A}_{\text{ext}}$ is an external deterministic disturbance. The term $\mathbf{A}_{\text{VF}}$ is the vector potential associated with the real vacuum fluctuations, and can be written as

$$\mathbf{A}_{\text{VF}}(\mathbf{x},t) = \sum_{\lambda=1}^{2}\sum_{\mathbf{k}}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}}V}}\hat{\epsilon}(\mathbf{k},\lambda)\left[a_{\mathbf{k}\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} + a^{*}_{\mathbf{k}\lambda}\, e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}\right] \quad , \tag{5}$$

where V is the volume containing the particle and the radiation field, $\mathbf{k}$ is the wave vector, $\omega_{\mathbf{k}} = c|\mathbf{k}|$, λ is the polarization index, and $\hat{\epsilon}(\mathbf{k},\lambda)$ are the polarization vectors. The amplitudes $a_{\mathbf{k}\lambda}$ are taken to be random variables. The random character of the field is contained in these variables which are such that $\langle a_{\mathbf{k}\lambda}\rangle = 0$ and $\langle |a_{\mathbf{k}\lambda}|^2\rangle = 1/2$ ($\langle\ \rangle$ denotes the ensemble average). The term $\mathbf{A}_{\text{RR}}$ is the vector potential that describes the radiation reaction [1,2] and H_{VF} is the Hamiltonian of the background radiation field (contains only variables of the field). In the case of zero temperature, H_{VF} can be written as [2]

$$H_{\text{VF}} = \frac{1}{8\pi}\int \mathbf{d}^3\mathbf{r}\ (\mathbf{E}^2_{\text{VF}} + \mathbf{B}^2_{\text{VF}}) \quad , \tag{6}$$

where

$$\mathbf{E}_{\text{VF}} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_{\text{VF}} \quad , \quad \mathbf{B}_{\text{VF}} = \nabla\times\mathbf{A}_{\text{VF}} \quad , \tag{7}$$

so that

$$\langle H_{\rm VF} \rangle = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \quad . \tag{8}$$

The extension to a non-zero temperature T is obtained by introducing the factor $\coth(\hbar\omega_{\mathbf{k}}/2kT)$.

Each particle of the ensemble evolves in time according to the Hamilton's equations

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m}(\mathbf{p} - \frac{e}{c}\mathbf{A}) \quad , \tag{9}$$

and

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\left[\frac{e}{mc}\mathbf{p}\cdot\mathbf{A} - \frac{e^2}{2mc^2}\mathbf{A}^2 - e\phi\right] \quad . \tag{10}$$

Substituting the equations (9), (10) into (2) we get the Liouvillian form of the equation governing the time evolution of the ensemble of particles for each realization of the stochastic field $\mathbf{A}_{\rm VF}$, namely

$$\frac{\partial W}{\partial t} + \frac{1}{m}(\mathbf{p} - \frac{e}{c}\mathbf{A})\cdot\frac{\partial W}{\partial \mathbf{x}} + \frac{\partial W}{\partial \mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{x}}\left[\frac{e}{mc}\left(\mathbf{p} - \frac{e}{2c}\mathbf{A}\right)\cdot\mathbf{A} - e\phi\right] = 0 \quad . \tag{11}$$

It is important to stress that, after obtaining the solution of (11), it is necessary to calculate the ensemble average over all possible realizations of the field $\mathbf{A}_{\rm VF}$ in order to obtain the average distribution $\langle W(\mathbf{x},\mathbf{p},t)\rangle$. This is done by considering the average over the random Gaussian amplitudes $a_{\mathbf{k}\lambda}$ in (5). Notice that in (11) $\mathbf{A} = \mathbf{A}(\mathbf{x},t)$ and $\phi = \phi(\mathbf{x},t)$ are explicit functions of the variables $\mathbf{x}$ and t.

Consider the Fourier transform defined by

$$\widetilde{W}(\mathbf{x},\mathbf{y},t) \equiv \int \mathbf{d^3p}\, W(\mathbf{x},\mathbf{p},t)\, \exp\left(-\frac{2i\mathbf{p}\cdot\mathbf{y}}{\hbar'}\right) \quad , \tag{12}$$

where $\mathbf{y}$ is a point in the configuration space and $\hbar'$ is a free parameter having dimension of action. The meaning of the free parameter $\hbar'$ will be discussed further below. Notice that (12) corresponds to the well known Wigner transform [5] if $\hbar' = \hbar$. Using the definition (12) the Liouville equation (11) assumes the following form

$$\int \mathbf{d^3y} \exp\left(\frac{2i\mathbf{p}\cdot\mathbf{y}}{\hbar'}\right) \left\{ i\hbar' \frac{\partial \widetilde{W}}{\partial t} - \frac{\hbar'^2}{2m} \frac{\partial^2 \widetilde{W}}{\partial \mathbf{x}\cdot\partial \mathbf{y}} - \frac{ie\hbar'}{mc} \mathbf{A}(\mathbf{x},t)\cdot \frac{\partial \widetilde{W}}{\partial \mathbf{x}} \; + \right.$$

$$\left. - \; 2\mathbf{y}\cdot\frac{\partial}{\partial \mathbf{x}} \left[\frac{e}{mc} \left(\mathbf{p} - \frac{e}{2c}\mathbf{A}(\mathbf{x},t) \right) \cdot \mathbf{A}(\mathbf{x},t) - e\phi \right] \widetilde{W} \right\} = 0 \quad . \tag{13}$$

In what follows we will concentrate our attention in the particular case of very small $\hbar'$ ($\hbar' \ll \hbar$). In this case $\widetilde{W}(\mathbf{x},\mathbf{y},t)$ is different from zero only if $|\mathbf{y}|$ is small, as can be seen from equation (12). Therefore, the scalar products appearing in the last term of equation (13) can be replaced by the expressions

$$\mathbf{y}\cdot\frac{\partial}{\partial \mathbf{x}}\phi(\mathbf{x},t) \simeq \phi(\mathbf{x}+\mathbf{y},t) - \phi(\mathbf{x},t) \quad , \tag{14}$$

and

$$\mathbf{y}\cdot\frac{\partial}{\partial \mathbf{x}} A_j(\mathbf{x},t) \simeq A_j(\mathbf{x}+\mathbf{y},t) - A_j(\mathbf{x},t) \quad , \; j = 1,2,3 \quad . \tag{15}$$

For instance, using (15), the term of (13) containing $\mathbf{p}\cdot\mathbf{A}$ can be written as follows

$$\int \mathbf{d^3y}\; 2\mathbf{y}\cdot\frac{\partial}{\partial \mathbf{x}} \left(\frac{e}{mc}\mathbf{p}\cdot\mathbf{A}(\mathbf{x},t) \right) \widetilde{W}(\mathbf{x},\mathbf{y},t)\, \exp\left(\frac{2i\mathbf{p}\cdot\mathbf{y}}{\hbar'}\right) =$$

$$= \int \mathbf{d^3y}\; \widetilde{W}(\mathbf{x},\mathbf{y},t) \frac{e}{mc} \left[\mathbf{A}(\mathbf{x}+\mathbf{y},t) - \mathbf{A}(\mathbf{x}-\mathbf{y},t) \right] \cdot$$

$$\cdot \left(\frac{\hbar'}{2i}\right) \frac{\partial}{\partial \mathbf{y}} \exp\left(\frac{2i\mathbf{p}\cdot\mathbf{y}}{\hbar'}\right) \quad . \tag{16}$$

After rewriting the integrand of (13) using approximations (14) and (15), we perform an integration by parts. The result is the following equation

$$i\hbar' \frac{\partial \widetilde{W}}{\partial t} - \frac{\hbar'^2}{2m} \frac{\partial^2 \widetilde{W}}{\partial \mathbf{x}\cdot\partial \mathbf{y}} - \frac{ie\hbar'}{mc} \left[\frac{\mathbf{A}(\mathbf{x}+\mathbf{y},t)+\mathbf{A}(\mathbf{x}-\mathbf{y},t)}{2}\right] \cdot \frac{\partial \widetilde{W}}{\partial \mathbf{x}} +$$

$$- \frac{ie\hbar'}{2mc} \left[\mathbf{A}(\mathbf{x}+\mathbf{y},t) - \mathbf{A}(\mathbf{x}-\mathbf{y},t)\right] \cdot \frac{\partial \widetilde{W}}{\partial \mathbf{y}} - \frac{ie\hbar'}{2mc} \widetilde{W} \frac{\partial}{\partial \mathbf{y}} \cdot \left[\mathbf{A}(\mathbf{x}+\mathbf{y},t) - \mathbf{A}(\mathbf{x}-\mathbf{y},t)\right] +$$

$$+ \frac{e^2}{2mc^2} \left[\mathbf{A}^2(\mathbf{x}+\mathbf{y},t) - \mathbf{A}^2(\mathbf{x}-\mathbf{y},t)\right] \widetilde{W} + e\left[\phi(\mathbf{x}+\mathbf{y},t) + \phi(\mathbf{x}-\mathbf{y},t)\right] \widetilde{W} = 0 \quad . \tag{17}$$

We shall study the case in which the Fourier transform $\widetilde{W}(\mathbf{x},\mathbf{y},t)$ can be written in the form [5,6]

$$\widetilde{W}(\mathbf{x},\mathbf{y},t) = \psi^*(\mathbf{x}+\mathbf{y},t)\psi(\mathbf{x}-\mathbf{y},t) \equiv \psi^*(\mathbf{r},t)\psi(\mathbf{s},t) \quad . \tag{18}$$

This restriction deserves a comment. A more general expression for $\widetilde{W}(\mathbf{x},\mathbf{y},t)$ is

$$\widetilde{W}(\mathbf{x},\mathbf{y},t) = \sum_k \sum_l A_{kl}(t) G_{kl}(\mathbf{x},\mathbf{y}) \quad , \tag{19}$$

where $\{G_{kl}\}$ is a complete set of orthogonal functions (or states). A differential equation for the coefficients A_{kl} can be obtained from equation (17). Therefore there is no loss of generality in using the hypothesis (18), provided that a complete set of (Fourier transformed) "phase space" states $\{G_{kl}\}$ is introduced in a later stage of the calculation [7].

Substituting (18) in (17) we obtain the following Schrödinger type equation for the functions $\psi(\mathbf{r},t)$

$$i\hbar' \frac{\partial \psi}{\partial t} = \frac{1}{2m}\left(-i\hbar' \frac{\partial}{\partial \mathbf{r}} - \frac{e}{c}\mathbf{A}(\mathbf{r},t)\right)^2 \psi + e\phi(\mathbf{r},t)\psi \quad , \tag{20}$$

and the corresponding equation for $\psi^*(\mathbf{r},t)$, with the vector potential $\mathbf{A}$ as given in (4) and (5).

Therefore, the Schrödinger type equation depends on the Planck's constant only due to its presence in $\mathbf{A}_{\mathrm{VF}}$ defined in (5). In other words, equation (20) has terms which are proportional to $\sqrt{\hbar}$ and $\hbar$. Moreover, the solutions of (20) must be interpreted by considering that the limit $\hbar' \to 0$ must be taken in the end of the calculations.

III. INCOMPATIBILITY OF THE STANDARD SCHRÖDINGER EQUATION WITH THE ZERO-POINT FIELD

The above derivation shows a clear correspondence between the quantum Schrödinger equation for spinless particles, and the classical stochastic Schrödinger like equation given by (20). The case of neutral spinning particle has been already discussed by Dechoum, França and Malta [8].

The limit $\hbar' \to 0$ of the solution of the classical stochastic Schrödinger like equation corresponds, physically, to classical (non-Heisenberg) states of motion as shown by Dechoum and França [6]. Nevertheless, we shall observe several effects, arising from the vacuum fluctuations, which depend non-trivially on the Planck's constant $\hbar$. This is better understood by means of very simple examples. One interesting example, discussed in reference [8], is the derivation of the Pauli-Schrödinger equation in the spinorial form, starting from the Liouville equation. The experimental results of the Stern-Gerlach experiment, and also the Rabi type

molecular beam experiments, were appropriately described and interpreted classically, in the limit $\hbar' \to 0$, that is, in the classical limit where the particles have well-defined trajectory, and also continuous orientation of the spin vector.

The best example, however, is the one-dimensional harmonic oscillator discussed in many details in previous works [6,7]. In order to apply equation (20) to the charged harmonic oscillator, we shall assume that the scalar potential ϕ is the simple static function satisfying $e\phi = (1/2)m\omega_0^2 x^2$, ω_0 being the natural frequency of the oscillator. We have shown in ref. [6] that by introducing the function $\Psi(x,t) \equiv \exp\left[i\frac{ex}{\hbar'}A_x(t)\right]\psi(x,t)$ we obtain for (20) the equivalent equation

$$i\hbar'\frac{\partial\Psi}{\partial t} = \left[-\frac{(\hbar')^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega_0^2 x^2}{2} - ex\left(E_{\rm RR} + E_{\rm VF}\right)\right]\Psi(x,t) \quad , \tag{21}$$

where $E_{\rm VF}$ and $E_{\rm RR}$ depend only on t (dipole approximation). In this equation $-m\omega_0^2 x$ is the harmonic force, $eE_{\rm RR} = -\frac{e}{c}\frac{\partial}{\partial t}(\mathbf{A}_{\rm RR})_x$ is the radiation reaction force, and $eE_{\rm VF} = -\frac{e}{c}\frac{\partial}{\partial t}(\mathbf{A}_{\rm VF})_x$ is the random force. The exact solution of (21), in the form of a coherent state $\Psi_{\rm cs}$, can be easily constructed [6]. It is possible to show that

$$\Psi_{\rm cs}(x,t) = \left(\frac{\hbar'}{m\omega_0}\right)^{-\frac{1}{4}}\exp\left\{\frac{i}{\hbar'}\left[xp_{\rm c}(t) - g(t)\right] - \frac{m\omega_0}{\hbar'}(x - x_{\rm c}(t))^2\right\} \quad , \tag{22}$$

where $p_{\rm c}(t) \equiv m\dot{x}_{\rm c}(t)$, $\quad 2m\dot{g}(t) \equiv p_{\rm c}^2(t) - m^2\omega_0^2 x_{\rm c}^2(t) + m\hbar'\omega_0$, and

$$m\ddot{x}_{\rm c}(t) = -m\omega_0^2 x_{\rm c}(t) + e\left[E_{\rm RR}(t) + E_{\rm VF}(t)\right] \quad , \tag{23}$$

so that $x_{\rm c}(t)$ is the classical stochastic trajectory obtained from the equation of motion (23). At equilibrium (or stationary state) we have $\langle x_{\rm c}(t)\rangle = 0$, and

$$\langle x_{\rm c}^2\rangle = \frac{\hbar}{2m\omega_0} \quad , \tag{24}$$

as is well known [1,2]. However, using the exact solution (22) of the Schrödinger type equation (21), we obtain

$$\langle x^2 \rangle = \langle \int_{-\infty}^{\infty} dx |\Psi_{cs}(x,t)|^2 x^2 \rangle = \frac{\hbar'}{2m\omega_0} + \langle x_c^2 \rangle = \frac{\hbar' + \hbar}{2m\omega_0} \quad . \tag{25}$$

This gives the correct value at zero temperature, namely $\langle x^2 \rangle = \hbar/2m\omega_0$, in the limit $\hbar' \to 0$. Only in this limit the solutions of equations (20) and (21) are physically acceptable. This is an important result that is very easy to understand within the realm of SED, if we recall the derivation (see eqs. (13) to (18)) of the classical Schrödinger like stochastic equation (20). The inevitable conclusion is that the standard Schrödinger equation, namely equation (20) with $\hbar' = \hbar$, does not give consistent results if the zero-point electromagnetic field $\mathbf{E}_{\rm VF}$ is fully considered.

In order to further illustrate the advantages of the Heisenberg picture over the Schrödinger picture, we shall consider the system consisting of a harmonic oscillator (electric dipole) interacting with an anisotropic source of noise as for instance the solenoid of a simple RLC circuit *without battery* [9]. The fluctuating current in the solenoid is modified by the magnetic field ($\mathbf{B}_{\rm dip}$) created by the oscillating dipole, thus generating a random electric field ($\mathbf{E}_{\rm sol}$) that affects the charge oscillating in the x direction as is illustrated in the Fig.1.

Following the steps of the (classical) stochastic calculation presented by Blanco et al. [9], it is possible to obtain the average oscillator energy by taking into account the effects of the zero-point fluctuations acting on the circuit (zero-point Nyquist noise [10]). The result is

$$\begin{aligned} \epsilon &= m\omega_0^2 \langle x_c^2(t) \rangle = \\ &= \frac{\hbar \tau \omega_0^2}{\pi} \int_0^{\infty} \frac{d\omega\ \omega^3\ [1 + \beta(\omega, y)]}{(\omega^2 - \omega_0^2)^2 + \tau^2 \omega^6 [1 + \beta(\omega, y)]^2} \quad , \end{aligned} \tag{26}$$

where $\tau \equiv 2e^2/3mc^3$. The function $\beta(\omega, y)$ is given by (see [9])

$$\beta(\omega, y) = \frac{\frac{3}{2} R \left(\frac{2\pi N a^2}{\ell y}\right)^2}{c|Z(\omega)|^2} \quad , \tag{27}$$

where $Z(\omega)$ is the impedance of the RLC circuit, ℓ is the length of the solenoid and y is the distance from the dipole to the solenoid axis (see Fig. 1). The free space result (see equation (24)) is obtained by taking $y \gg a$ (or $\beta \simeq 0$). In the case $\tau\omega_0 \ll 1$ it is possible to show that the integral in (27) gives [9]

$$\epsilon \simeq \frac{\hbar\omega_0}{2} \quad . \tag{28}$$

The contribution of the Nyquist noise does not affect the oscillator ground state energy. Nevertheles one can show that the lifetime of the oscillator excited states depends on the orientation of the oscillating charge, and are significantly modified by the presence of the solenoid [9]. Similar results were obtained by considering the oscillator between parallel metallic plates [11].

It should be remarked that the theoretical study of these simple systems is rather cumbersome in the conventional Schrödinger picture.

IV. DISCUSSION

Dalibard, Dupont-Roc and Cohen Tannoudji [12], and França, Franco and Malta [13] provided an identification of the contribution of the radiation reaction and the vacuum fluctuation forces to the processes of radiation emission and atomic stability. Using the Heisenberg picture and perturbative QED calculations Dalibard, Dupont-Roc and Cohen Tannoudji [12] have shown that

$$P_{\text{Larmor}}(a) = \frac{4e^2}{3c^3} \sum_{b\ (\epsilon_b < \epsilon_a)} \langle a\,|\ddot{\mathbf{r}}|\,b\rangle \cdot \langle b\,|\ddot{\mathbf{r}}|\,a\rangle \quad . \tag{29}$$

This equation is the quantum generalization of the Larmor formula $(2e^2\ddot{\mathbf{r}}^2)/(3c^3)$ for the rate of radiation emission, including the zero-point field effects, of an electron in the quantum state $|a\rangle$ (Dirac notation) with energy ϵ_a. We see that the inclusion of the zero-point electromagnetic field simply doubles the rate of the radiation emission, being thus very important for obtaining agreement with experiment [11,13]. Dechoum and França [6] extended this result to the SED picture using the harmonic oscillator and the classical stochastic Schrödinger like equation. Further insight on the general connection between SED and QED, for the free electromagnetic fields and for dipole oscillator system, is provided by T.H. Boyer [14] and P.W. Milonni [15].

Nevertheless the Schrödinger picture can be used successfully in many calculations if we consider $\mathbf{A}_{\text{VF}}(t) \equiv 0$, $\mathbf{A}_{\text{RR}} \equiv 0$ and $\hbar' = \hbar$ in equation (20). An interesting example, having a clear classical limit, was provided by Suárez Barnes *et al* [16] that have studied the one-dimensional motion of the electron in the Coulomb field using the simple equation $(\hbar' = \hbar)$

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{e^2}{|x|}\right)\psi(x,t) \quad . \tag{30}$$

The Coulomb potential $V(x)$ was approximated by

$$V(x) = -\frac{e^2}{|x|} \simeq V(q_t) + (x - q_t)V'(q_t) + \frac{(x-q_t)^2}{2}V''(q_t) \quad , \tag{31}$$

where q_t is the *classical trajectory*. A coherent state solution was obtained from (30) and (31).

Equations (30) and (31) allowed Suárez Barnes *et al* to obtain a remarkable reproduction of the hydrogen spectrum, using classical reference trajectories that have a *continuous* energy range. No quantization conditions were imposed on these classical reference trajectories. For the reader's convenience the spectrum calculated in [16] is reproduced in the Fig.2. As far as we know, this constitutes the first accurate *classical* calculation of the atomic spectrum since the advent of Quantum Mechanics. This calculation can be interpreted classically due to the approximation (31). For potentials of this form, the Schrödinger equation is equivalent to the Liouville equation as was pointed out by many authors [1,3,5,17].

Our last remark is concerned with the necessity of the vacuum electromagnetic field within the Heisenberg picture of the quantum theory. A good discussion of this point is given in the book by P. Milonni [15]. Again, the simplest example is the harmonic oscillator that is discussed within chapter 2. The zero-point electromagnetic field, with spectral distribution

$$\rho_0(\omega) = \frac{\hbar\omega^3}{2\pi c^3} \quad , \tag{32}$$

is in fact *necessary* for the formal consistency of the quantum theory. A striking demonstration of this fact is provided by the equal-time commutation relation [15]:

$$\begin{aligned}[x(t), p_x(t)] &= [x(t), m\ \dot{x}(t)] + [x(t), \frac{e}{c}A_x(t)] = \\ &= \frac{i\hbar e^2}{2\pi^2 mc^3}\left(\frac{8\pi}{3}\right)\int_0^\infty \frac{\omega^4\ d\omega}{(\omega^2-\omega_0^2)^2+\tau^2\omega^6} = i\ \hbar \quad .\end{aligned} \tag{33}$$

This result is obtained from the exact solution of the equation of motion for the quantum operators x and p_x. As expected on physical grounds, the quantum equation of motion in the Heisenberg picture is formally identical to our classical stochastic equation (23).

Finally we would like to stress that it would be nice to extend our calculations so as to

reproduce the diffraction pattern observed in many experiments with electron beams.

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Figures Caption and Figures

Figure 1: Schematic picture of the electric dipole at a distance y from the solenoid axis. The relevant electromagnetic fields generated by the solenoid ($\mathbf{E}_{\rm sol}$) and the oscillating dipole ($\mathbf{B}_{\rm dip}$) are indicated.

Figure 2: Spectrum generated by the classical motion in the Coulomb potential, according to the parametric oscillator approximation. The continuous energy is denoted by $-\epsilon$, and the circles correspond to the exact quantum results. The units are such that $e = m = \hbar = 1$.

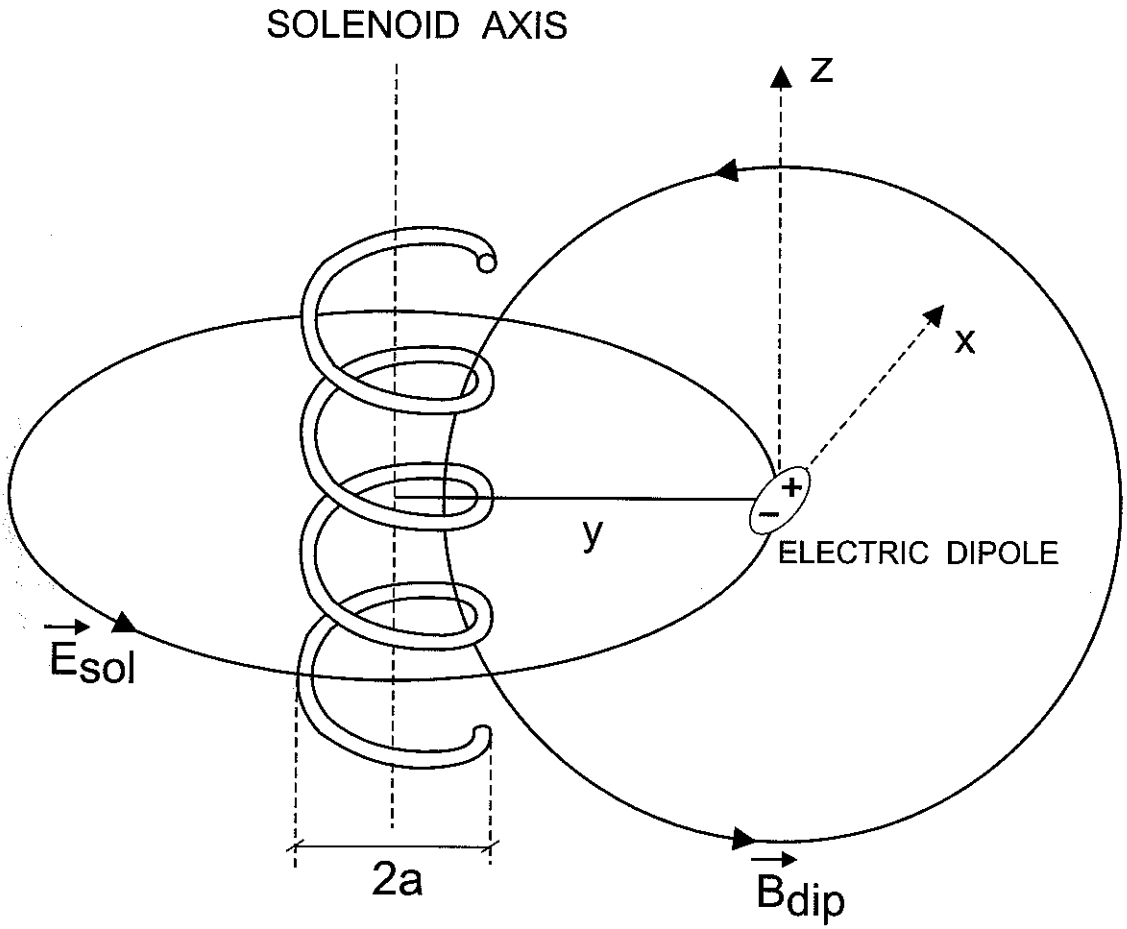

FIGURE 1

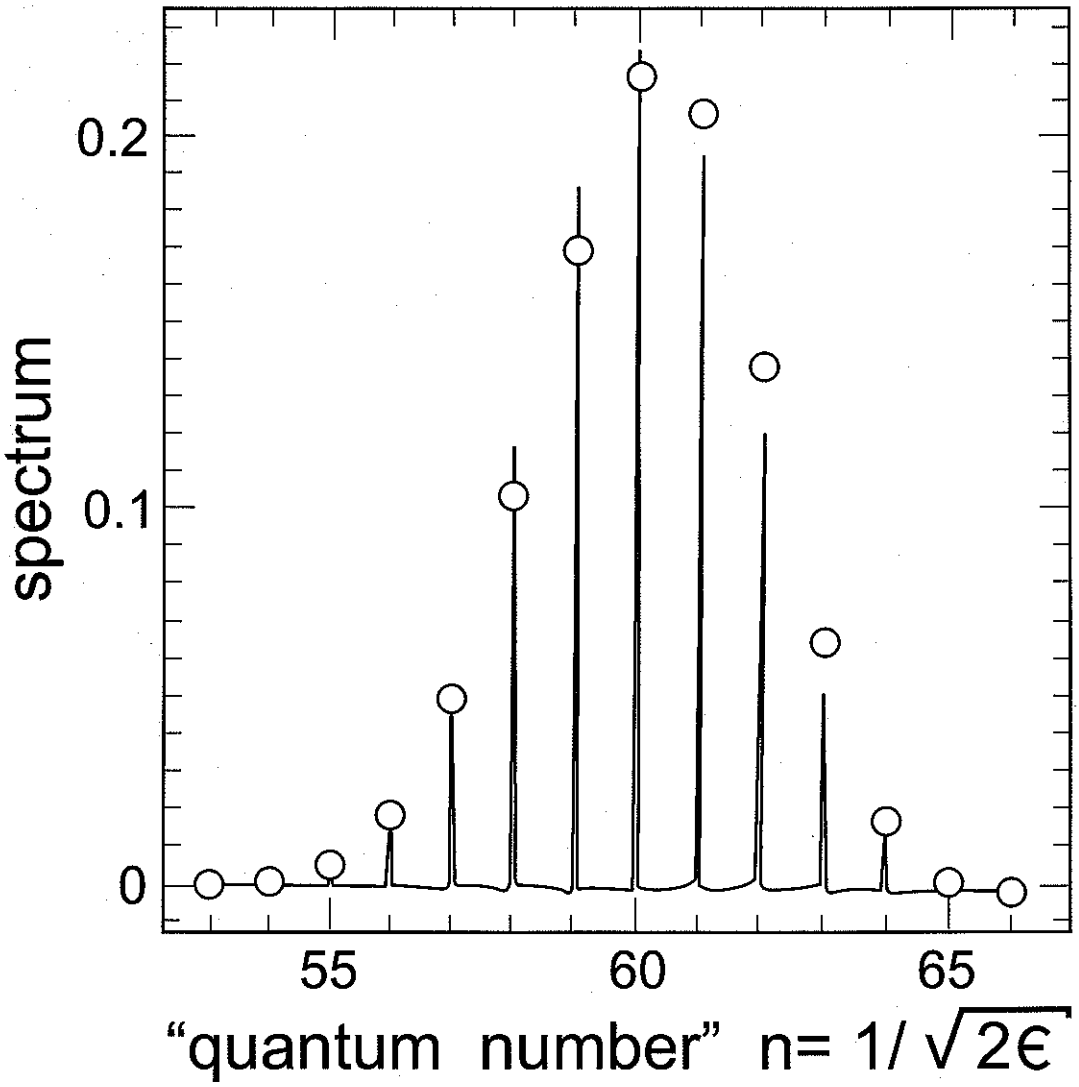

FIGURE 2