Instituto de Física
Universidade de São Paulo

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Chi-Yong. Lin; Hussein, M.S.; de Toledo Piza, A.F.R.
Instituto de Física, Universidade de São Paulo, São Paulo, Brasil

Eddy Timmermans
T-4, Los Alamos National Laboratory, Los Alamos, NM, U.S.A.

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UNIVERSIDADE DE SÃO PAULO
Instituto de Física
Cidade Universitária
Caixa Postal 66.318
05315-970 - São Paulo - Brasil

Collective excitations of mutually coherent condensates*

Chi-Yong. Lin[1,†], M.S. Hussein[1,‡], A. F. R. de Toledo Piza[1], and Eddy Timmermans[2]

[1]Instituto de Física, Universidade de São Paulo, CP 66318

CEP 05389-970, São Paulo, SP, Brazil

[2]T-4, Los Alamos National Laboratory, Los Alamos, NM 87545, U.S.A.

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Abstract

The mutual coherence of atomic-trap Bose-Einstein condensates generalizes the usual symmetry-breaking of boson superfluids. We show how the mutual phase coherence, a remarkable property especially if the condensates consist of particles as distinct as atoms and molecules, implies the existence of massive collective modes. This feature appears as nodeless high frequency modes in atom traps. We develop a Gaussian wavefunction dynamics to describe the structure of these modes, the detection of which could serve as a signal of the long-range mutual phase coherence.

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The atomic trap Bose-Einstein condensates (BEC's) [1] exhibit the long range phase coherence that characterizes superfluidity in matter systems [2]. The corresponding broken symmetry description captures the essence of the superfluid features and, hence, the extent to which the BEC's behavior represents that of other superfluids. Against this backdrop, the fact that available atomic and optical techniques can generalize the usual U(1) symmetry breaking of boson superfluids [3], appears particularly significant. In this letter we discuss the example of mutually coherent condensates and the possibility of observing its off-diagonal long-range order by probing the hydrodynamic collective excitations. This long-range *mutual* phase coherence – a novel quality for boson superfluids – can be achieved by converting n condensate bosons of type 1, to a single distinguishable boson of type 2 (a molecule if $n > 1$). If this process is 'coherent', i.e. if it does not alter the many-body state in any other way, its contribution to the second quantized effective Hamiltonian takes on the form

$$\int d^3r \, \left(J_n \hat{\psi}_2^\dagger(\mathbf{r}) \hat{\psi}_1^n(\mathbf{r}) + h.c. \right) \quad , \tag{1}$$

where $\hat{\psi}_j$ denotes the corresponding field operator, and where we have confined our attention to the s-wave processes that dominate the relevant low energy regime. For $n = 1$, the expression (1) describes a Rabi-like coupling between different states of the same atom. The $n = 2$ case corresponds to the atom-diatomic molecule coupling discussed in the BEC-context for the cases of Feshbach resonances [4]– [5] and coherent photoassociation [6] – [8]. Furthermore, since the individual atoms 'perfectly' overlap in a BEC and high intensity lasers are available, it may be possible to probe coherent transitions to tri-atom molecules ($n = 3$). A simple mean-field argument shows that the interaction (1) in a condensate of bosons 1, $\langle \hat{\psi}_1 \rangle = \psi_1 \neq 0$ creates a second condensate, $\langle \hat{\psi}_2 \rangle = \psi_2 \neq 0$. Indeed, the commutator with (1) contributes a source term $J_n \langle \hat{\psi}_1^n \rangle \neq 0$ to the expectation value of the Heisenberg equation of motion for ψ_2 so that $\dot{\psi}_2 \neq 0$.

Below, we show that, in addition to the usual Goldstone mode with a dispersion that vanishes in the long wavelength limit, $\lim_{k\to 0}\omega_k = 0$, homogeneous mutually coherent condensates support a 'massive' mode of finite gap frequency [7], $\lim_{k\to 0}\omega_k = \omega_{\rm gap} \neq 0$ ($\omega_{\rm gap}$ plays the role of 'mass' in field theory). This massive mode involves coherent Josephson-like population oscillations. The mutual long-range phase coherence gives rise to coherent oscillations of the population balance – a manifestation of macroscopically coherent chemistry. To describe the corresponding collective modes of the *inhomogeneous* trap systems, we generalize the Gaussian wave function dynamics [9] to describe mutually coherent condensates. We find that the essential signature of high-frequency modes with a characteristic frequency dependence on the effective energy ϵ, of bosons 2, remains. The massive modes then provide a promising scheme to detect the mutual-coherence long-range order, especially in the case of the Feshbach resonant experiments that are hampered by significant particle–loss near $\epsilon = 0$.

The emergence of the BEC-order parameters, $\langle\hat{\psi}_j\rangle = \psi_j = \sqrt{\rho_j}\exp(i\theta_j)$ with a well-defined complex phase θ_j breaks the symmetry of the Hamiltonian, $\int d^3r\mathcal{H}$,

$$\mathcal{H} = \psi_1^*\left[\frac{-\hbar^2\nabla^2}{2m} + V_1 + \frac{\lambda_1}{2}\psi_1^*\psi_1\right]\psi_1 + \psi_2^*\left[\frac{-\hbar^2\nabla^2}{2nm} + V_2 + \epsilon + \frac{\lambda_2}{2}\psi_2^*\psi_2\right]\psi_2$$
$$+\lambda\psi_1^*\psi_1\psi_2^*\psi_2 + J_n\left[\psi_2^*\psi_1^n + \psi_2(\psi_1^n)^*\right] \quad . \tag{2}$$

In the above expression, m denotes the single atom mass, V_j represents the external potential experienced by bosons j, and the λ-interaction strengths account for the usual inter-particle scattering. Note that (2) is invariant under the simultaneous global phase transformation $\psi_1 \to \exp(i\varphi)\psi_1$, $\psi_2 \to \exp(in\varphi)\psi_2$. The only dependence on global (i.e. position-independent) phases stems from the J_n-interaction $\sim \cos(\theta_2 - n\theta_1)$, so that $\theta_2^{(0)} - n\theta_1^{(0)} = \pi$ in the ground state. In the Goldstone mode the phases fluctuate around their ground state value, $\delta\theta_j(\mathbf{x},t) = \theta_j(\mathbf{x},t) - \theta_j^{(0)}$, while the system remains locally in the potential minimum, $\theta_2 - n\theta_1 = \pi$, i.e. $\delta\theta_2 = n\delta\theta_1$, implying that the superfluid velocities $\mathbf{v}_j = (\hbar/m_j)\nabla\theta_j$ fluctuate *in phase*, $\delta\mathbf{v}_2 = (\hbar/nm)\nabla(\delta\theta_2) = (\hbar/m)\nabla(\delta\theta_1) = \delta\mathbf{v}_1$.

In the long wave length massive mode, the phases in the (θ_1, θ_2)-plane fluctuate in the perpendicular direction, $n\delta\theta_2 = -\delta\theta_1$, and the superfluid velocities fluctuate *out-of-phase*, $\delta\mathbf{v}_2 = (\hbar/nm)\nabla(\delta\theta_2) = -(\hbar/m)\nabla(\delta\theta_1)/n^2 = -\delta\mathbf{v}_1/n^2$.

We describe the infinite wave length dynamics of massive modes by homogeneous wave-functions, $\psi_j = \sqrt{N_j(t)/\Omega}\exp[i\theta_j(t)]$ that represent condensates j of N_j bosons confined to a macroscopic volume Ω. We obtain their time-evolution from the time-dependent variational principle: Minimization of the action, $\Gamma = \int dt\ L$, where L represents the Lagrangian,

$$L = \int d^3x\ (i\hbar/2)[\psi_1^*\dot{\psi}_1 - \dot{\psi}_1^*\psi_1 + \psi_2^*\dot{\psi}_2 - \dot{\psi}_2^*\psi_2] - E[\psi_1, \psi_2] \tag{3}$$

and E denotes the energy, $E = \int d^3r\ \mathcal{H}$, yields classical equations of motion for (θ_j, N_j). Since the time derivative terms of L reduce to $-N_1\dot{\theta}_1 - N_2\dot{\theta}_2$, the N_j and θ_j–variables are canonically conjugate and the equations of motion are Anderson-like: $\hbar\dot{N}_j = \partial E/\partial\theta_j$, $\hbar\dot{\theta}_j = -\partial E/\partial N_j$. With (2), the energy is equal to

$$E = \epsilon N_2 + \frac{1}{\Omega}\left[\frac{\lambda_1}{2}N_1^2 + \frac{\lambda_2}{2}N_2^2 + \lambda N_1 N_2\right] + \frac{2J_n}{\Omega^{(n-1)/2}}\sqrt{N_2}N_1^{n/2}\cos(\theta_2 - n\theta_1)\ . \tag{4}$$

We extract the massive mode structure by subjecting the phase variables to a rotation alining the new axes with the fluctuation directions of the Goldstone ($\theta_+ = [\theta_1 + n\theta_2]/\sqrt{1+n^2}$) and massive ($\theta_- = [-n\theta_1 + \theta_2]/\sqrt{1+n^2}$) modes. Rotating the conjugate variables similarly to preserve the canonical nature of the variables, new number variables appear: N_+, proportional to the total number of atomic particles ($N = N_1 + nN_2$), $N_+ = N/\sqrt{1+n^2}$ and $N_- = [-nN_1 + N_2]/\sqrt{1+n^2}$, which describes the population imbalance. The rotated variables evolve according to new Anderson equations. The (N_+, θ_+) equations, and specifically $\hbar\dot{N}_+ = \partial E/\partial\theta_+ = 0$ express conservation of atoms. The (N_-, θ_-)–equations describe the dynamics of population imbalance:

$$\begin{aligned}\dot{N}_- &= \frac{E}{\partial\theta_-} \quad ,\\ \dot{\theta}_- &= -\frac{E}{\partial N_-} \quad .\end{aligned} \tag{5}$$

The equilibrium conditions, $\dot{N}_-^{(0)} = \dot{\theta}_-^{(0)} = 0$, determine the relative phase, $\theta_2 - n\theta_1 = 0, \pi$ (π in the state of lowest energy) and the equilibrium fractions, $f_j = N_j^{(0)}/N$. The massive mode corresponds to fluctuations around this equilibrium $N_- = N_-^{(0)} + \delta N_-$, $\theta_- = \theta_-^{(0)} + \delta\theta_-$, which evolve according to a Taylor expansion of (5):

$$\begin{aligned} \hbar\delta\dot{N}_- &= \frac{\partial^2 E}{\partial\theta_-^2}\,\delta\theta_- \quad , \\ \hbar\delta\dot{\theta}_- &= -\frac{\partial^2 E}{\partial N_-^2}\,\delta N_- \quad . \end{aligned} \tag{6}$$

The corresponding frequency, ω ($\delta\ddot{N}_- = -\omega^2\delta N_-$, so that $\hbar^2\omega^2 = (\partial^2 E/\partial\theta_-^2) \times (\partial^2 E/\partial N_-^2)$) is the gap frequency or 'mass'. In the ground state, the $\partial^2 E/\partial\theta_-^2$–factor measures the energy gain due to the J_n–interaction. We express its value, $\partial^2 E/\partial\theta_-^2 = (1 + n^2)(2J_n/\Omega^{(n-1)/2})N_2^{1/2}N_1^{n/2}$ in units of $\epsilon_J = J_n\rho^{(n-1)/2}$, where ρ is the atomic particle density, $\rho = N/\Omega$. Likewise, $\partial^2 E/\partial N_-^2$ is positive for a stable system – a negative $\partial^2 E/\partial N_-^2$-curvature indicates that the system can lower its energy by converting all bosons to the same type. In terms of the equilibrium fractions, f_j, the gap frequency reads

$$\begin{aligned} \hbar^2\omega^2 = {} & 2\epsilon_J f_2^{1/2} f_1^{n/2}\lambda_1\rho\left[n^2 - 2n(\lambda/\lambda_1) + (\lambda_2/\lambda_1)\right] \\ & +\epsilon_J^2\left(\frac{f_1^n}{f_2}\right)\left[1 + 2n^2\frac{f_2}{f_1} - n^3(n-2)\left(\frac{f_2}{f_1}\right)^2\right] \quad . \end{aligned} \tag{7}$$

In the off-resonant limit, $\epsilon >> 0$, $f_1 \to 1$, $f_2 < 1$, so that $\hbar\omega \to \epsilon_J\sqrt{N/N_2} \to \epsilon$ (the latter is a consequence of the off-resonant limit of N_2 that follows from $\partial E/\partial N_2 = 0$, $N_2 \to N(\epsilon_J/\epsilon)^2$). Although at large detunings it may become increasingly difficult to excite and observe the massive mode, this result suggests that an easily observable frequency shift may serve as a signal of the long-range mutual phase coherence at reasonably large detunings. Such detection scheme may be particularly attractive in the Feshbach resonance case where large loss-rates hamper near-resonant experiments. Near-resonance, a measurement of the gap frequency as a function of ϵ reveals information about the boson 2 interaction strengths (specifically, about $-2n\lambda + \lambda_2$), which are largely unknown for molecules.

To study the inhomogeneous trap systems, we generalize the variational Gaussian description of single condensate dynamics [10]. We restrict ourselves to the n=2 case of mutually coherent atomic ($a = 1$) and molecular ($m = 2$) condensates generated by a Feshbach resonance or coherent photoassociation. The Gaussian ansatz wave functions,

$$\psi_a(r,t) = \frac{N_a^{1/2}}{\pi^{3/4} q^{3/2}} \prod_{j=x,\,y,\,z} e^{-\left[\frac{1}{2q_j^2(t)} - i\frac{p_j(t)}{q_j(t)}\right]\left[r_j - r_j^c(t)\right]^2 + i\pi_j^c(t)\left[r_j - r_j^c(t)\right] + i\theta_a(t)} \tag{8}$$

$$\psi_m(r,t) = \frac{N_m^{1/2}}{\pi^{3/4} Q^{3/2}} \prod_{j=x,\,y,\,z} e^{-\left[\frac{1}{2Q_j^2(t)} - i\frac{P_j(t)}{Q_j(t)}\right]\left[r_j - R_j^c(t)\right]^2 + i\Pi_j^c(t)\left[r_j - R_j^c(t)\right] + i\theta_m(t)} \tag{9}$$

contain q_j, p_j, r_j^c, π_j^c (Q_j, P_j, R_j^c, Π_j^c) as variational parameters for the atomic (molecular) condensates and are 'normalized' so that $\int d^3r |\psi_a^*|^2 = N_a$ and $\int d^3r |\psi_m^*|^2 = N_m$ (subject to the constraint $N_a + 2N_m = N$) by the $q = (q_x q_y q_z)^{1/3}$ and $Q = (Q_x Q_y Q_z)^{1/3}$-factors. Thus, $\psi_a(\psi_m)$ is a Gaussian centered at r_j^c (R_j^c) with a width q_j (Q_j) and conjugate momenta π_j^c (Π_j^c) and p_j (P_j) respectively. Accordingly, with (3), the action takes on the form

$$\Gamma = \int dt \left\{ \theta_a \dot{N}_a + \theta_m \dot{N}_m - E + \sum_{j=x,\,y,\,z} \left[N_a \pi_j^c \dot{r}_j^c + \frac{N_a}{4}(p_j \dot{q}_j - \dot{p}_j q_j) + N_m \Pi_j^c \dot{R}_j^c + \frac{N_m}{4}(P_j \dot{Q}_j - \dot{P}_j Q_j) \right] \right\} \tag{10}$$

The energy, E, consists of single condensate contributions, E_a, E_m, and inter-condensate interaction terms, E_λ and E_J. The single condensate Gaussian Gross-Pitaevskii energies are of the usual form [9]: $E_a = N_a \sum_{j=x,y,z} [\frac{1}{2m}\pi_j^2 + \frac{1}{4m} p_j^2 + \frac{\hbar^2}{4m} q_j^{-2} + \frac{m}{2}\omega_{a_j}^2 (r_j^c)^2 + \frac{m}{4}\omega_{a_j}^2 q_j^2] + \frac{\lambda_a N_a^2}{2(2\pi)^{3/2}} q^{-3}$. To obtain E_m, we replace $m \to 2m$, $\pi_j \to \Pi_j$ etc... in E_a and add ϵN_m. The elastic scattering inter-condensate energy, E_λ, reduces to $E_\lambda = \frac{\lambda N_a N_m}{\pi^{3/2}} \{\prod_{j=x,\,y,\,z} exp[-\frac{(r_j^c - R_j^c)^2}{q_j^2 + Q_j^2}](q_j^2 + Q_j^2)^{-1/2}\}$. In performing the integration to evaluate the J_2-contribution, it is useful to group the variables into complex numbers, $B_j = (2Q_j^2)^{-1} + iP_j/Q_j$ and $\beta_j = (2q_j^2)^{-1} + ip_j/q_j$. Finally, the coherent atom-molecule interaction, reads $E_J = \frac{J_2 N_a N_m^{1/2}}{\pi^{3/4} q^3 Q(3/2)} \{e^{i(2\theta_a - \theta_m)} \prod_{j=x,\,y,\,z} exp[\frac{b_j^2}{4a_j} - c_j] a_j^{-1/2} + \text{c.c.}\}$, where we have introduced $a_j = B_j^* + 2\beta_j$, $b_j = -2B_j^* R_j^c - 4\beta_j r_j^c + i(\Pi_j^c - 2\pi_j^c)$ and $c_j = B_j^*(R_j^c)^2 + 2\beta_j (r_j^c)^2 + i(\Pi_j^c R_j^c - 2\pi_j^c r_j^c)$. Denoting all variables by X_j, the Euler-Lagrange equations, $\delta\Gamma = 0$, take the form of first order equations,

$$\sum_k \mathbf{S}_{ik}\, \dot{X}_k = \frac{\partial E}{\partial X_i}\,, \tag{11}$$

giving a simple description of the time evolution. The ground state is determined by (11) with $\dot{X}_k = 0$. The small amplitude oscillation dynamics is described by an expansion around that equilibrium, $X^{(0)}$. Thus, if $X^{(1)}$ is the fluctuation, we have

$$i\omega\, \mathbf{K}\, X^{(1)} = \mathbf{A}\, X^{(1)}\,, \tag{12}$$

where $\mathbf{A}_{ij} = \left.\frac{\partial^2 E}{\partial X_i \partial X_j}\right|_{(0)}$ and $\mathbf{K}$ is an appropriate 26×26 inertia matrix. To obtain the frequencies, we diagonalize the set of 26 coupled equations, which come out in pairs of opposite parity. The resulting frequencies can be grouped into three sets: one corresponds to the center-of-mass motion of the atomic/molecular gases; one describes the dynamics of the spatial frontiers of the atomic/molecular condensates and one describes the Josephson tunneling between the two condensates.

For the sake of definiteness, we consider the center-of-mass modes in an isotropic trap with the same trap frequency [11], ω_a, experienced by atoms and molecule. In the corresponding equilibrium state, the center-of-mass positions are simple

$$r_j^c = \pi_j^c = R_j^c = \Pi_j^c = 0\,. \tag{13}$$

In addition, $\frac{\partial^2 E}{\partial X^c \partial X^w} = 0$ where X^c are generic coordinates of the subspaces corresponding the centers of mass and X^w represent the other degrees of freedom of the variational space. Thus, the matrix A reduces to the form $\begin{pmatrix} C & 0 \\ 0 & W \end{pmatrix}$, where $C_{ik} = \frac{\partial^2 E}{\partial X_i^c \partial X_k^c}$ and $W_{ik} = \frac{\partial^2 E}{\partial X_i^w \partial X_k^w}$. As a consequence, X_c decouple from X_w at X^0 and the dynamics of the centers of mass system is described by two coupled oscillator equations. Eliminating $\pi_c^{(0)}$ and $\Pi_c^{(0)}$ we find

$$N_a\, \ddot{r}_c = -[N_a\, \omega_a^2 + \Delta\omega^2]\, r_c + \Delta\omega^2\, R_c$$

$$2N_m\, \ddot{R}_c = -[2N_m\, \omega_a^2 + \Delta\omega^2]\, R_c + \Delta\omega^2\, r_c \tag{14}$$

where the frequency shift $\Delta\omega$ is produced by the interaction between the atomic and molecular condensates and thus depends on J_2 and ε. The $\Delta\omega$–shift is related to the curvature of $E^i = E_\lambda + E_J$,

$$\Delta\omega^2 = m\omega_a^2\Big(\frac{\partial^2 E^i}{\partial \pi_c^2}\Big) + \frac{1}{m}\Big(\frac{\partial^2 E^i}{\partial r_c^2}\Big) + \Big(\frac{N}{N_a} + \frac{N}{2N_m}\Big)\Big(\frac{\partial^2 E^i}{\partial \pi_c^2}\Big)\Big(\frac{\partial^2 E^i}{\partial r_c^2}\Big) \qquad (15)$$

The two coupled equations (14) can be uncoupled in favor of the normal modes, $\eta \equiv N_a r_c + 2N_m R_c$ and $\xi \equiv r_c - R_c$, whose frequencies are, respectively,

$$\Omega_-^2 = \omega_a^2\,, \qquad \Omega_+^2 = \omega_a^2 + \Delta\omega^2 \frac{N}{2N_m(N-2N_m)} \ . \qquad (16)$$

The Ω_-–mode corresponds to an in-phase motion in which the atomic and molecular centers of mass move together [9]. The Ω_+–mode represents an out-of-phase (massive) dipole oscillation with the atomic and molecular center-of-mass moving in opposite direction. Since $\frac{\partial^2 E^i}{\partial \pi_c^2} \sim N_a N_m$ and $\frac{\partial^2 E_J}{\partial \pi_c^2} \sim N_a N_m^{1/2}$, we find that when $N_m << N$, Ω_+ exhibits the characteristic gap-frequency dependence $\Omega_+ \sim N_m^{-1/2}$. This result is illustrated in Fig.1, where Ω_+ increases significantly with ϵ.

Next, we turn to some of the other low-lying modes, the frequencies of which are shown in Fig. 2. The lowest frequency represents a breathing mode. The other two curves represent massive quadrupole modes in which q and Q oscillate out of phase. In the dashed curve-mode the number of atoms oscillates in phase with q, in the dotted-curve mode, the number of atoms oscillates out-of-phase with q. As in the case of the massive dipole modes, these frequencies strongly depend on the equilibrium state and, consequently, contain interesting information about the many-body structure.

In conclusion, we have pointed out that the symmetry-breaking structure of mutually coherent condensates gives rise to massive modes. In a trap system, the massive modes appear as high frequency ($> \omega_a$) nodeless modes of low angular momentum, the frequency of which increases with detuning ϵ. We have developed a variational Gaussian wave function dynamics to describe such massive modes in a trapped mutually coherent atom-molecule condensate. Using realistic values for the density, trap frequency and J_2 (for the case of a Feshbach resonance), we found massive mode frequencies up to an order of magnitude higher than the trap frequency within a detuning range of a few kHz, suggesting that the observation of these modes may be experimentally feasible.

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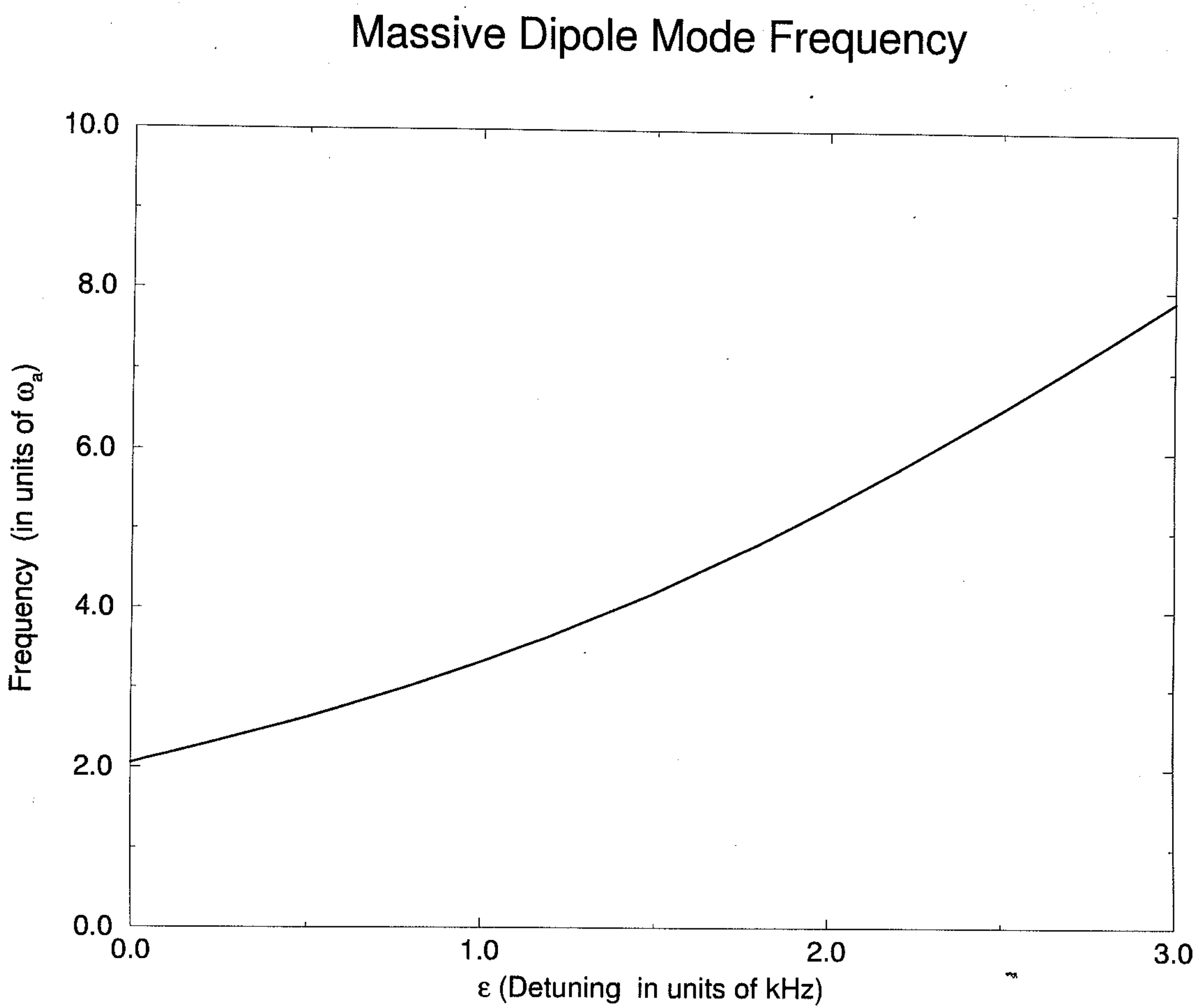
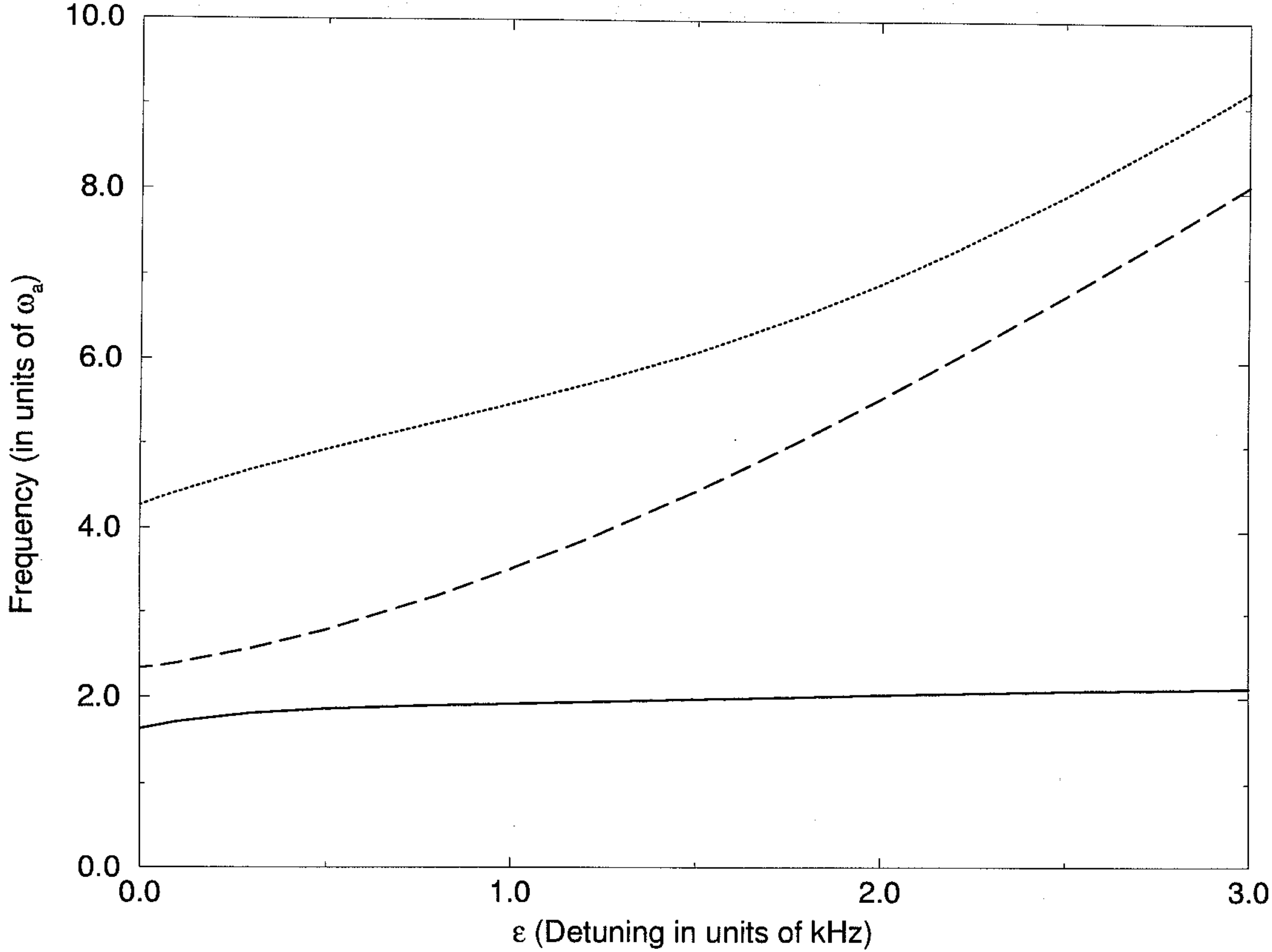
Figure Captions

Figure 1. The massive dipole mode frequency as function of 'detuning', ε, in units of the trap frequency, $\omega_a = 2\pi 50Hz$. The physical parameters were chosen to be realistic: $\lambda_a = \lambda_m = \lambda = 100\hbar(\mu m)^3 s^{-1}$, $N = 10^4$, $V_a = 2\pi * 50Hz$, $\alpha = 200m\mu^{3/2}s$.

Figure 2. Eigenfrequencies of other low-lying modes, calculated with the parameters of Fig.(1). The lower curve represents a breathing mode, the other two curves represent massive quadrupole modes for which q and Q oscillate out-of-phase. N_a fluctuates in phase with q in the mode of the dashed curve, and out-of-phase in the mode represented by the dotted curve.


Massive Dipole Mode Frequency
Frequency (in units of ωa)
10.0
8.0
6.0
4.0
2.0
0.0
0.0
1.0
2.0
3.0
ε (Detuning in units of kHz)



10.0
8.0
6.0
4.0
2.0
0.0
0.0
1.0
2.0
3.0
Frequency (in units of ωa)
ε (Detuning in units of kHz)