Instituto de Física
Universidade de São Paulo

Uniaxial Lifshitz Point at $0(E^2_L)$

Leite, M. M.; Albuquerque, L. C. de
Instituto de Física, Universidade de São Paulo, SP, Brasil

Publicação IF - 1413/2000

IFUSP-P/1413

UNIAXIAL LIFSHITZ POINT AT $O(\epsilon_L^2)$

Luiz C. de Albuquerque [1]
and
Marcelo M. Leite [2]

Instituto de Física, Universidade de São Paulo
C.Postal 66318, 05389-970, S.Paulo, SP, Brazil

June, 2000

Abstract

The critical exponents ν_{L2}, η_{L2} and γ_L of a uniaxial Lifshitz point are calculated at two-loop level using renormalization group and ϵ_L expansion techniques. We found a new constraint involving the loop momenta along the competition axis, which allows to solve the two-loop integrals. The exponent γ_L obtained using our method is in good agreement with numerical estimates based on Monte Carlo simulations and high-temperature series.

[1] e-mail: claudio@fma.if.usp.br
[2] e-mail: leite@fma.if.usp.br

The Lifshitz point occurs in a variety of physical systems and has been extensively studied over the last twenty-five years [1, 2, 3, 4]. In magnetic systems, the uniaxial Lifshitz critical behavior can be described by an axially next-nearest-neighbor Ising model (ANNNI). It consists of a spin-$\frac{1}{2}$ system on a cubic lattice ($d = 3$) with nearest-neighbor ferromagnetic couplings and next-nearest-neighbor antiferromagnetic interactions along a single lattice axis [5]. The competition gives origin to a modulated phase, in addition to the ferromagnetic and paramagnetic ones. In spite of having several modulated phases, it was shown recently that around the Lifshitz critical region, a simple field-theoretic setting can be defined for this ANNNI model [6]. In general, the antiferromagnetic couplings can show up in m directions. In that case, the system possesses the m-fold Lifshitz critical point. Here we are going to focus our attention in the uniaxial case ($m = 1$), as some materials present this type of critical behavior. MnP was studied both theoretically and experimentally and displays this sort of uniaxial behavior [7]. Theoretical studies involving the uniaxial Lifshitz point have been put forward using analytical and numerical tools. Examples of the latter are high-temperature series expansion [8] and Monte Carlo simulations [5]. Conformal invariance calculations in $d = 2$ (in the context of strongly anisotropic criticality) [9] and ϵ-expansion techniques [1] have been the main analytical tools available to dealing with this kind of system.

We report on what we believe to be the first study of critical exponents at two-loop order for the uniaxial Lifshitz point. Using $\lambda\phi^4$ field theory and the expansion in powers of $\epsilon_L = 4.5-d$ in the critical Lifshitz region, we calculate the exponents ν_{L2} and η_{L2}. The exponents ν_{L2} and η_{L2} are associated with the directions perpendicular to the competition axis. The exponents ν_{L4} and η_{L4} are associated with the competition axis. Knowledge of three of these four exponents allows one to get all other exponents using scaling relations. Using ν_{L2} and η_{L2}, we get $\gamma_L = 1.485$ for this ANNNI model $d = 3$.

To begin with, we write down the bare Lagrangian associated with the Lifshitz critical region[3]:

$$L = \frac{1}{2}| \nabla_1^2 \phi |^2 + \frac{1}{2}| \nabla_{(d-1)} \phi |^2 + \delta\frac{1}{2}| \nabla_1 \phi |^2 + \frac{1}{2}t_0\phi^2 + \frac{1}{4!}\lambda\phi^4. \qquad (1)$$

The competition is responsible for the appearance of the quartic term in the free propagator. The Lifshitz critical region is characterized by the value $\delta = 0$. From now on, this is the case which interests us in this work. We follow the conventions and procedure described in [6, 10]. First, we are going to compute the renormalized coupling constant at the fixed point. We absorb in the coupling constant a geometric angular factor, which is $\frac{3\sqrt{2}}{8}S_{d-1}S_1$, where $S_d = [2^{d-1}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})]^{-1}$. The beta function has the same form as the one coming from a pure Isinglike critical behavior, with the loop expansion parameter $\epsilon = 4 - d$ replaced by $\epsilon_L = 4.5 - d$. In practice, we have to calculate the two-loop integrals $I_{4SP} \equiv I_4$, $\frac{\partial}{\partial p^2} I_3|_{SP} \equiv I_3'$ and the three-loop integral $\frac{\partial}{\partial p^2} I_5|_{SP} \equiv I_5'$, in order to find the fixed point at two-loop level and the critical exponents. The subscript SP is used to denote our choice of the subtraction point. (We found convenient to label all loop integrals according to the number of propagators they contain). They are given by:

$$I_3 = \int \frac{d^{d-1}q_1 d^{d-1}q_2 dk_1 dk_2}{(q_1^2 + k_1^4)(q_2^2 + k_2^4)((q_1 + q_2 + p)^2 + (k_1 + k_2)^4)}, \tag{2}$$

$$\begin{aligned} I_5 &= \int \frac{d^{d-1}q_1 d^{d-1}q_2 d^{d-1}q_3 dk_1 dk_2 dk_3}{(q_1^2 + k_1^4)(q_2^2 + k_2^4)(q_3^2 + k_3^4)((q_1 + q_2 - p)^2 + (k_1 + k_2)^4)} \\ &\times \frac{1}{(q_1 + q_3 - p)^2 + (k_1 + k_3)^4}. \end{aligned} \tag{3}$$

$$\begin{aligned} I_4 &= \int \frac{d^{d-1}q_1 d^{d-1}q_2 dk_1 dk_2}{(q_1^2 + k_1^4)((P - q_1)^2 + k_1^4)(q_2^2 + k_2^4)} \\ &\times \frac{1}{(q_1 - q_2 + p_3)^2 + (k_1 - k_2)^4}, \end{aligned} \tag{4}$$

The symmetry point is chosen as follows. In the first two integrals, p is the external momentum,associated with the two-point vertex, and in I_4, $P = p_1 + p_2$, with p_1, p_2, p_3 being the external momenta associated with the four-point vertex. All these momenta are defined in $(d-1)$-directions, perpendicular to the competition axis. Note that the integrals have external momenta parallel to the competition direction set equal to zero. For the four-point vertex, it is defined by $p_i.p_j = \frac{\kappa^2}{4}(4\delta_{ij} - 1)$. We fix the momentum scale of the two-point function through $p^2 = \kappa^2 = 1$.

In order to solve the internal bubbles in the diagrams we demand that the loop momenta in the internal and external bubble at the quartic direction should be related. This is the simplest way to disentangle the two quartic integrals in the loop momenta. In general, one can use $k_1 = -\alpha k_2$, with $\alpha(\neq \pm 1)$ a real number. However, this is not simple enough to handle the remaining integral [11]. To get a result just in terms of the usual Beta function, we have to use a particular value for α ($\alpha = 2$ for I_3 and I_5, and $\alpha = -2$ for I_4). This approximation then yields a well defined ϵ_L-expansion. The upshot of this procedure is that each diagram gets a purely numerical factor in terms of the original ones. This ambiguity can be fixed by identifying the leading singularity in each diagram to the one in the pure $\lambda\phi^4$ theory. This is equivalent to fix the factors of the diagrams. One finds the factor $\frac{7}{4}$ for I_3 and I_5, and 2 for I_4. We can then calculate all the integrals and make the ϵ_L-expansion.

We illustrate our method by calculating the integral I_3, which is the simplest one. First, put $k_1 = -2k_2$ into the integration over k_2. We have:

$$I_3 = \frac{7}{4} \int \frac{d^{d-1}q_1 dk_1 d^{d-1}q_2 dk_2}{(q_1^2 + k_1^4)\,(q_2^2 + k_2^4)\,((q_2 + q_1 + p)^2 + k_2^4)} \tag{5}$$

The internal bubble is always proportional to the one-loop integral [12]

$$I_{SP} \equiv I_2 = \frac{1}{\epsilon_L}(1 + i_2 \epsilon_L), \tag{6}$$

where $i_2 = \frac{1}{3} - \frac{\pi}{4} + \frac{3\ln 2}{2}$. Performing the integral over q_2 , k_2 , we find:

$$I_3 = \frac{7}{4} \int \frac{d^{d-1}q_1 dk_1}{(q_1^2 + k_1^4)\,[(q_1 + p)^2]^{\frac{\epsilon_L}{2}}} I_{SP} \tag{7}$$

In the remaining integral we use Schwinger parameterization again to solve along the quartic direction. We obtain:

$$I_3 = \frac{7}{8}\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) I_2 \int \frac{d^{d-1}q_1}{[q_1^2]^{\frac{3}{4}}[(q_1 + p)^2]^{\frac{\epsilon_L}{2}}}. \tag{8}$$

At this point, one can use Feynman parameters to solve the momentum integrals. One solves the integrals in terms of Gamma functions with non integer arguments. A useful identity involving the expansion of Gamma functions around a small number is given by:

$$\Gamma(a + bx) = \Gamma(a)\left[1 + b\,x\,\psi(a) + O(x^2)\right], \tag{9}$$

where $\psi(z) = \frac{d}{dz} ln\Gamma(z)$. This allows one to get an ϵ_L-expansion when the Gamma functions have non integer arguments. The dependence of the integral in the external momenta is proportional to $(p^2)^{1-\epsilon_L}$. We find :

$$I_3' = -\frac{1}{8\epsilon_L}\Big[1 + \Big(i_2 + \frac{6}{7}\Big)\epsilon_L\Big]. \tag{10}$$

The integrals I_4 and I_5' can be calculated along the same lines. They are:

$$I_4 = \frac{1}{2\epsilon_L^2}\Big[1 + \Big(i_2 + \frac{4}{3}\Big)\epsilon_L\Big], \tag{11}$$

$$I_5' = -\frac{1}{6\epsilon_L^2}\Big[1 + \Big(2i_2 + \frac{8}{7}\Big)\epsilon_L\Big]. \tag{12}$$

With this information we compute the fixed point at two-loop level. In case the order parameter has a $O(N)$ symmetry, we find:

$$u^* = \frac{6}{8+N}\,\epsilon_L + 2\,\epsilon_L^2\,\frac{346 + 77\,N - 3\,i_2\,(N^2 + 36N + 152)}{(8+N)^3}. \tag{13}$$

Therefore, the exponents η_{L2} and ν_{L2} are given by:

$$\eta_{L2} = \frac{1}{2}\epsilon_L^2\,\frac{2+N}{(8+N)^2} + \epsilon_L^3\,\frac{(2+N)(5228 + 1174N + 6N^2 - 3\,i_2\,(7N^2 + 392N + 1680)}{42(8+N)^4}, \tag{14}$$

$$\nu_{L2} = \frac{1}{2} + \frac{1}{4}\epsilon_L\,\frac{2+N}{8+N} + \epsilon_L^2\,\frac{(2+N)(380 + 139N + 3N^2 - 3\,i_2\,(28N + 80))}{24\,(8+N)^3}. \tag{15}$$

Now using Fisher's law $\gamma_L = \nu_{L2}(2 - \eta_{L2})$, we get:

$$\gamma_L = 1 + \frac{1}{2}\epsilon_L\,\frac{2+N}{8+N} + \epsilon_L^2\,\frac{(2+N)\;(356 + 136N + 3\,N^2 - 3\,i_2\,(28N + 80))}{12\;(8+N)^3}. \tag{16}$$

For the ANNNI model, $\gamma_L = 1.4 \pm 0.06$ is the Monte Carlo output [5], whereas the best estimates from the high-temperature series is $\gamma_L = 1.62 \pm 0.12$ [8]. Our two-loop calculation ($N = 1$) in three dimensions yields $\gamma_L = 1.485$. Indeed, as conjectured in [6], even though the ϵ_L parameter is not small, the ϵ_L-expansion is reliable. The agreement between the numerical and analytical results is remarkable. The numerical value obtained here for the correlation length exponent is $\nu_{L2} = 0.753$. The experimental value of this critical index is still lacking. We hope our result sheds some light towards its experimental determination (using neutron scattering measurements, for example).

An interesting open question is the calculation of the critical exponents ν_{L4} and η_{L4} using the ϵ_L-expansion at two-loop level. The approach followed here is not suitable to computing these critical exponents (parallel to the competition axis), since our choice of the symmetry point prevents a proper treatment in this direction. The possibility of devising another symmetry point to deal with these exponents seems to be feasible, and will be reported elsewhere.

Nevertheless, our method can be used to test the consistency of the ratio $\theta \equiv \nu_{L4}/\nu_{L2} = 1/2$ (which is known to be true at $O(\epsilon_L)$, where θ is the anisotropy exponent) at $O(\epsilon_L^2)$. Recently, a renormalization group calculation for an m-fold Lifshitz point ($m \neq 1$) was developed in [13]. There, it was found that $\theta \neq 1/2$ at two-loop level for the cases $m = 2, 6$. In the present uniaxial case we can use Josephson law involving the exponents ν_{L4}, ν_{L2} and α_L to see whether this relation is valid at two-loop order. One should get the exponent α_L consistent with either the experimental value [14], or the high-temperature series result [8]. The relation is not valid at two-loop level, for one obtains $\alpha_L < 0$ in this way. This corroborates the above mentioned studies for $m \neq 1$.

In conclusion, we have found a way to perform two- and three-loop integrals for the uniaxial Lifshitz point, needed to calculate universal properties at directions perpendicular to the competition axis. The constraint in the loop momenta at the competition direction is the key ingredient to carry out the calculations. Topics including the extension of the present method to the region out of the Lifshitz point ($\delta \neq 0$) and two-loop calculations using a modified symmetry point along the competition axis are in development.

LCA would like to thank FAPESP for financial support. MML would like to thank Warren Siegel, Peter van Nieuwenhuizen, C. N. Yang Institute for Theoretical Physics (SUNYSB, where this work has started), Victor O. Rivelles and the Departamento de Física Matemática da Universidade de São Paulo for kind hospitality. He also thanks Nathan Berkovits for useful discussions.

References

[1] R. M. Hornreich, M. Luban and S. Shtrikman, *Phys. Rev. Lett.* **35**, 1678 (1975); R. M. Hornreich, *Journ. Magn. Magn. Mat.* **15-18**, 387. (1980).

[2] For applications of the Lifshitz point in High-T_C superconductivity, see S. M. Hayden *et al., Phys. Rev. Lett.* **66**, 821 (1991); B. Keimer *et al., ibid.* **67**, 1930 (1991); S. Sachdev and J. Ye, *ibid.* **69**, 2411 (1992).

[3] The occurrence of isotropic Lifshitz points ($m = N = d$) in the context of polymer physics can be found in G. H. Fredrickson and S. T. Milner, *Phys. Rev. Lett.* **67**, 835 (1991); F. S. Bates *et al ibid.* **75**, 4429 (1995); P. R. Netz and M.Schick, *ibid.* **77**, 302 (1996); F. S. Bates *et al, ibid.* **79**, 849 (1997).

[4] For Lifshitz points in ferroelectric liquid crystals, see for example, S. B. Rananavare, V. G. K. M. Pisipati and E. W. Wong, *Phys. Rev. Lett.* **72**, 3558 (1994); B. Zalar et al *ibid.* **80**, 4458 (1998).

[5] W. Selke, *Phys. Rep.* **170**, 213 (1988) and references therein. See also W. Selke, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz (Academic Press, London, 1992), vol.15.

[6] M. M. Leite, *Phys. Rev. B* **61**, 14691 (2000).

[7] C. S. O. Yokoi, M. D. Coutinho-Filho and S. R. Salinas, *Phys. Rev. B* **24**, 5430 (1981); Y. Shapira, *J. Appl. Phys.* **53**, 1914 (1982); C. S. O. Yokoi, M. D. Coutinho-Filho and S. R. Salinas, *Phys. Rev. B* **29**, 6341 (1984).

[8] Z. Mo and M. Ferer, *Phys. Rev. B* **43**, 10890 (1991).

[9] M. Henkel, *Phys. Rev. Lett.* **78**, 1940 (1997).

[10] See, e.g., D.J. Amit, in *Field Theory, the Renormalization Group and Critical Phenomena* (World Scientific, Singapore, 1984).

[11] When the internal integral over k_2 and q_2 with generic α is performed, it remains the integral $I = \int_0^1 dx\, x^a\,(1-x)^b\,(1+(1-(1\pm\alpha)^4)\,x\,)^c = B(a+1,b+1)\,F(1+a,-c;2+a+b;-1+(1\pm\alpha)^4)$. The Hypergeometric

function is equal to 1 for the particular values of α mentioned in the text.

[12] The expression (9) in reference 3 has an error of 8.8 percent. This is so because the formula $\Gamma(1.75 - \frac{\epsilon_L}{2})\Gamma(0.25) = \frac{3\sqrt{2}\pi}{4}\Gamma(2 - \frac{\epsilon_L}{2})$ is not an identity, but an approximation. There is an extra contribution of order ϵ_L that is not captured by this approximation. The way to get the right result is to make use of the identity shown in eq.(9).

[13] C. Mergulhão, Jr. and C. E. I. Carneiro, Phys. Rev. B **58**, 6047 (1998); *ibid.* **59**, 13954 (1999).

[14] V. Bindilatti, C. C. Becerra and N. Oliveira *Phys. Rev. B* **40**, 9412 (1989).