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Abstract

We investigate the problems of consistency and causality for the equations of motion describing massive spin two field in external gravitational and massless scalar dilaton fields in arbitrary spacetime dimension. From the field theoretical point of view we consider a general classical action with non-minimal couplings and find gravitational and dilaton background on which this action describes a theory consistent with the flat space limit. In the case of pure gravitational background all field components propagate causally. We show also that the massive spin two field can be consistently described in arbitrary background by means of the lagrangian representing an infinite series in the inverse mass. Within string theory we obtain equations of motion for the massive spin two field coupled to gravity from the requirement of quantum Weyl invariance of the corresponding two dimensional sigma-model. In the lowest order in α' we demonstrate that these effective equations of motion coincide with consistent equations derived in field theory.

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1 Introduction

The purpose of this paper is twofold - first, to describe consistent theory of massive spin 2 field interacting with external gravity from the point of view of classical field theory and, second, to derive effective equations of motion for this field from string theory.

The problem of consistent description of higher spin fields interaction has a long history but is still far from the complete resolution. Within framework of standard field theory it is possible to construct a classical action for the higher spin fields only on specific curved spacetime manifolds [1]–[6][1]. For example, massive integer spins are described by symmetric tensors of corresponding ranks and one can write down the equations of motion and find classical actions for them but only in special spacetimes (e.g. in Ricci flat spaces). It means that in such an appoach gravity field does not feel the presence of higher spins matter through an energy-momentum tensor. So a consistent classical action for the system of dynamical gravity and a higher massive field is still unknown and there are some indications that perhaps it does not exist at all.

One of these indications comes from considering a Kaluza-Klein decomposition of Einstein gravity in $D-$dimensional spacetime into gravity plus infinite tower of massive second rank tensor fields in $(D-1)-$dimensional world, masses being proportional to inverse compactification radius. Then the resulting four dimensional theory of spin 2 fields interacting with gravity and with each other should be consistent as one started from the ordinary Einstein theory and just considers it on a specific manifold. But as was shown in [10], it is impossible to reduce this theory consistently to a finite number of spin 2 fields, i.e. consistency can be achieved only if the whole infinite tower of higher massive fields are present in the theory.

The main problem in higher spin fields theories is that introduction of interaction may excite new unphysical degrees of freedom which lead in general to appearance of negative norm states in the hilbert space and to violation of causality. In free theories these unphysical degrees are absent due to transversality and tracelessness conditions. Lagrangian description of these conditions requires auxiliary fields vanishing on equations of motion [11, 12]. In arbitrary interacting theories the auxiliary fields may become dynamical and to make them vanishing again one has to impose some additional restrictions on the kind of interaction.

As a first step towards a full dynamical theory of interacting higher spin fields one usually tries to describe a single higher spin massive field in an external electromagnetic or gravitational background [3, 5, 6, 13, 14, 15]. In this case one way to achieve consistency is to impose appropriate restrictions on the external background field. For example, it is well known that consistent theories of higher massive spin fields can be easily built in the spacetimes of constant curvature. In the Section 2 we describe in detail the theory of massive spin 2 field in curved spacetime and show how these restrictions on the external

[1]For description of higher spin massless fields on specific background see e.g. [7, 8], the case of a collection of massless spin-2 fields was investigated in [9]

gravitational background arise. As a result we will arrive to a one-parameter family of lagrangian theories which describe consistent propagation of the spin 2 field in arbitrary Einstein spacetime.

There exists another possible way to achieve consistency by constructing equations for higher massive fields in form of infinite series in inverse mass (or, equivalently, in curvature). A recent attempt in this direction was undertaken in [6] where, however, only theories on symmetric Einstein manifolds were considered and consistent equations were derived in the simplest approximation linear in curvature.

In this paper we demonstrate that consistent equations for the spin 2 massive field can, in principle, be constructed as infinite series in inverse mass square in arbitrary gravitational background. These kinds of infinite series arise naturally in string theory which represent another approach for consistent description of higher spins interaction.

String theory contains an infinite number of massive fields with various spins interacting with each other and with a finite number of massless fields. Unfortunately, there are arguments that in string theory a general coordinate invariant effective field action reproducing the correct S-matrix both for massless and massive string states does not exist [16]. The full effective action for all string fields is not general coordinate invariant and general covariance arises only as an approximate symmetry in effective action for massless fields once all the massive fields are integrated out. That effective action for massive fields cannot be covariant follows, for example, from the fact that terms cubic in massive fields can contain only flat metric and there is no terms of higher powers [16].

Influence of massive string modes is negligible at low energies but they become important, for instance, in string cosmology [17] (for a recent review see e.g. [18]) so it would be desirable to have consistent equations describing interaction of the massive string fields with gravity.

And indeed, there exists a possibility to derive from the string theory some covariant parts of equations for massive higher spins fields interacting with background gravity. The aim of our paper is to show explicitly how this procedure works using as an example dynamics of the second rank tensor from the first massive level of open bosonic string.

A convenient method of deriving effective field equations of motion from the string theory is provided by the $\sigma-$model approach [19]–[29]. Within this approach a string interacting with background fields is described by a two dimensional field theory and effective equations of motion arise from the requirement of quantum Weyl invariance. Perturbative derivation of these equations is well suited for massless string modes because the corresponding two dimensional theory is renormalizable and loop expansion corresponds to expansion of string effective action in powers of string length $\sqrt{\alpha'}$.

Inclusion of interaction with massive modes [30]–[39] makes the theory non-renormalizable but this fact does not represent a problem since in string theory one considers the whole infinite set of massive fields. Infinite number of counterterms needed for cancellation of divergences generating by a specific massive field in classical action leads to renormalization of an infinite number of massive fields. The only property of the theory crucial for possibility of derivation of perturbative information is that number of massive fields giving contributions to renormalization of the given field should be finite. As was shown in [33] string theory does fulfill this requirement. To calculate $\beta-$function for any massive field it is sufficient to find divergences coming only from a finite number of other massive fields and so it is possible to derive effective equations of motion for any background fields in any order in α'.

This procedure have serious limitation – it does not allow to derive non-perturbative contributions to the string effective action. It is well known that correct terms cubic in massive fields (quadratic terms in equations of motion) cannot be found within perturbative renormalization of $\sigma-$model. For example, perturbative $\beta-$function of tachyon field is linear in all orders in α' and interacting tachyonic terms are due to non-perturbative (from the point of view of two dimensional field theory) effects [40].

To find these non-perturbative contribution one should use the method exact renormalization group (ERG) [41]–[43]. It is ideally suited for string theory because, first of all, its formulation does not require any *a priori* defined perturbative scheme, and secondly, the equations of exact renormalization group are quadratic in interaction terms thus resembling the cubic structure of exact string field action. But ERG method also has a serious disadvantage as it requires explicit separation of classical action into free and interaction parts and thus leads to non-covariant equations for background fields. From the general point of view this fact does not contradict general properties of string theory. We have already mentioned that exact effective action describing both massless and massive string modes *should* be non-covariant. But non-covariance of the ERG method makes it rather difficult to establish relations between string fields equations and ordinary field theory.

So in this paper we derive covariant equations of motion for massive string fields interacting with gravity by means of ordinary perturbative analysis of quantum Weyl invariance condition in the corresponding $\sigma-$model. Of course, perturbatively we can obtain equations only linear in massive fields. It was noted long ago [16] that one can make such a field redefinition in the string effective action that terms quadratic in massive fields (linear terms in equations of motion) acquire dependence on arbitrary higher powers of massless fields and so may be covariant. In this paper we explicitly obtain these interaction terms in the lowest in α' approximation. We do not get terms quadratic in massive fields which should be non-covariant and arise only non-perturbatively.

As a model for our calculations we use bosonic open string theory interacting with background fields of the massless and the first massive levels. First massive level contains symmetric second rank tensor and so this model provides the simplest example of massive higher spin field interacting with gravity. In order to obtain equations of motion for string fields we build effective action for the corresponding two dimensional theory, perform renormalization of background fields and composite operators and construct the renormalized operator of energy momentum tensor trace.

This rather standard scheme was first developed for calculations in closed string theory with massless background fields [20, 27, 28] and then was generalized for the open string theory [44, 46, 47, 48] and for strings in massive fields [33]. The new feature appearing in open string theory is that the corresponding two dimensional sigma model represents a quantum field theory on a manifold with a boundary. To construct quantum effective action in such a theory we use generalization of Schwinger-De Witt method for manifolds with boundaries developed in the series of papers [49].

Making perturbative calculations we restrict ourselves to string world sheets with topology of a disk. The resulting equations of motion for graviton will not contain dependence on massive fields from the open string spectrum because these fields interact only with the boundary of world sheet and so can not influence the local physics in the bulk. For example, in the case of graviton and massive fields from the open string spectrum one expects that equations of motion for the graviton should look like ordinary vacuum Einstein equations without any matter. Of course, one can obtain contributions from open string background fields to the right hand side of Einstein equations for gravity but it would demand considering of world sheets of higher genus [45, 47].

The organization of the paper is as follows. In the Sec. 2 we describe the most general consistent equations of motion for massive spin 2 field on a specific class of curved manifolds from the point of view of ordinary field theory and generalize them for the interaction with scalar dilaton field in the Sec. 3. In the next section we show how the scheme can be generalized to the case of arbitrary gravitational background by means of either non-lagrangian equations or equations representing infinite series in inverse mass. Sec. 5 contains description of the string model that we use for derivations of effective equations of motion of string massive fields. We calculate divergences of the theory in the lowest order, carry out renormalization of background fields and composite operators and construct the renormalized operator of the energy-momentum tensor trace. Requirement of quantum Weyl invariance leads to effective string fields equations of motion. We compare with each other equations of motion derived in two previous sections and show that string theory gives (at least in the lowest order) consistent equations for the spin 2 massive field interacting with gravity. Conclusion contains summary of the results.

2 Massive spin 2 field coupled to gravity in field theory

In this section we give detailed analysis of the lagrangian formulation for the free spin 2 massive field and then generalize it for the presence of external gravitational field. The first requirement one should impose on such a theory is preservation of the same number of degrees of freedom and constraints as in the flat theory. Another important feature of any consistent relativistic theory is causality, which means that the equations of motion should not describe superluminal propagation [13] (see also [14] for a review). The result obtained in this section is a lagrangian which depends on one arbitrary dimensionless parameter of non-minimal coupling and describes consistent propagation of the spin 2 massive field in arbitrary Einstein spacetime, i.e. it contains the same number of lagrangian constraints as in the flat spacetime and does not violate causality.

In the flat spacetime the massive spin 2 field is described (as follows from the analysis of irreducible representations of 4-dimensional Poincare group) by symmetric transversal and traceless tensor of the second rank $H_{\mu\nu}$ satisfying mass-shell condition:

$$\left(\partial^2 - m^2\right) H_{\mu\nu} = 0, \qquad \partial^\mu H_{\mu\nu} = 0, \qquad H^\mu{}_\mu = 0. \tag{1}$$

In higher dimensional spacetimes Poincare algebras have more than two Casimir operators and so there are several different spins for $D > 4$. Talking about spin 2 massive field in arbitrary dimension we will mean, as usual, that this field by definition satisfies the same equations (1) as in $D = 4$. After dimensional reduction to $D = 4$ such a field will describe massive spin two representation of $D = 4$ Poincare algebra plus infinite tower of Kaluza-Klein descendants.

The most convinient approach to building interacting field theories is a lagrangian one and, in fact, the general point of view is that any consistent equations of motion should follow from some classical action. In the case of the free massive spin 2 field it is well known that all the equations (1) can be derived from the Fierz-Pauli action [11]:

$$\begin{aligned} S &= \int d^D x \Big\{ \frac{1}{4}\partial_\mu H \partial^\mu H - \frac{1}{4}\partial_\mu H_{\nu\rho}\partial^\mu H^{\nu\rho} - \frac{1}{2}\partial^\mu H_{\mu\nu}\partial^\nu H + \frac{1}{2}\partial_\mu H_{\nu\rho}\partial^\rho H^{\nu\mu} \\ &\qquad - \frac{m^2}{4} H_{\mu\nu}H^{\mu\nu} + \frac{m^2}{4}H^2 \Big\} \end{aligned} \tag{2}$$

where $H = \eta^{\mu\nu}H_{\mu\nu}$.

The general scheme of calculating the complete set of constraints in lagrangian formalism [50] is equivalent to the Dirac-Bergmann procedure in hamiltonian formalism and in the case of second class constraints (which is relevant for massive higher spin fields) consists in the following steps. If in a theory of some set of fields $\phi^A(x)$, $A = 1, \ldots, N$ the original lagrangian equations of motion define only $r < N$ of the second time derivaties ("accelerations") $\ddot{\phi}^A$ then one can build $N - r$ primary constraints, i.e. linear combinations of the equations of motion that does not contain accelerations. Requirement of conservation in time of the primary constraints either define some of the missing accelerations or lead to new (secondary) constraints. Then one demands conservation of the secondary constraints and so on, until all the accelerations are defined and the procedure closes up.

Applying this procedure to the action (2) one can see that the equations of motion

$$\begin{aligned} E_{\mu\nu} &= \partial^2 H_{\mu\nu} - \eta_{\mu\nu}\partial^2 H + \partial_\mu\partial_\nu H + \eta_{\mu\nu}\partial^\alpha\partial^\beta H_{\alpha\beta} - \partial_\sigma\partial_\mu H^\sigma{}_\nu - \partial_\sigma\partial_\nu H^\sigma{}_\mu \\ &\qquad - m^2 H_{\mu\nu} + m^2 H \eta_{\mu\nu} = 0 \end{aligned} \tag{3}$$

contain D primary constraints (expressions without second time derivatives $\ddot{H}_{\mu\nu}$):

$$E_{00} = \Delta H_{ii} - \partial_i\partial_j H_{ij} - m^2 H_{ii} \equiv \varphi_0^{(1)} \approx 0 \tag{4}$$

$$E_{0i} = \Delta H_{0i} + \partial_i \dot{H}_{kk} - \partial_k \dot{H}_{ki} - \partial_i\partial_k H_{0k} - m^2 H_{0i} \equiv \varphi_i^{(1)} \approx 0. \tag{5}$$

The remaining equations of motion $E_{ij} = 0$ allow to define the accelerations $\ddot{H}_{ij}$ in terms of $\dot{H}_{\mu\nu}$ and $H_{\mu\nu}$. The accelerations $\ddot{H}_{00}$, $\ddot{H}_{0i}$ cannot be expressed from the equations directly.

Conditions of conservation of the primary constraints in time $\dot{E}_{0\mu} \approx 0$ lead to D secondary constraints. On-shell they are equivalent to

$$\varphi_\nu^{(2)} = \partial^\mu E_{\mu\nu} = m^2 \partial_\nu H - m^2 \partial^\mu H_{\mu\nu} \approx 0 \tag{6}$$

Conservation of $\varphi_i^{(2)}$ defines $D-1$ accelerations $\ddot{H}_{0i}$ and conservation of $\varphi_0^{(2)}$ gives another one constraint. It is convenient to choose it in the covariant form by adding suitable terms proportional to the equations of motion:

$$\varphi^{(3)} = \partial^\mu \partial^\nu E_{\mu\nu} + \frac{m^2}{D-2} \eta^{\mu\nu} E_{\mu\nu} = H m^4 \frac{D-1}{D-2} \approx 0 \tag{7}$$

Conservation of $\varphi^{(3)}$ gives one more constraint on initial values

$$\varphi^{(4)} = -\dot{H}_{00} + \dot{H}_{kk} = \dot{H} \approx 0 \tag{8}$$

and from the conservation of this last constraint the acceleration $\ddot{H}_{00}$ is defined.

Altogether there are $2D+2$ constraints on the initial values of $\dot{H}_{\mu\nu}$ and $H_{\mu\nu}$. The theory contains the same local dynamical degrees of freedom as the system (1) and describes traceless and transverse symmetric tensor field of the second rank.

Now if we want to construct a theory of massive spin 2 field on a curved manifold first of all we should provide the same number of propagating degrees of freedom as in the flat case. It means that new equations of motion $E_{\mu\nu}$ should lead to exactly $2D+2$ constraints and in the flat spacetime limit these constraints should reduce to their flat counterparts. In addition to consistency with the flat space limit any field theory should possess one more crucial property connecting with causal propagation [13, 14]. In general case interaction with external fields changes light cones describing propagation of spin 2 massive field [1] so causality may give another restrictions on the theory.

Generalizing (2) to curved spacetime we should substitute all derivatives for the covariant ones and also we can add non-minimal terms containing curvature tensor with some dimensionless coefficients in front of them. As a result, the most general action for massive spin 2 field in curved spacetime quadratic in derivatives and consistent with the flat limit should have the form [1]:

$$\begin{aligned} S = \int d^D x \sqrt{-G} \Big\{ & \frac{1}{4} \nabla_\mu H \nabla^\mu H - \frac{1}{4} \nabla_\mu H_{\nu\rho} \nabla^\mu H^{\nu\rho} - \frac{1}{2} \nabla^\mu H_{\mu\nu} \nabla^\nu H + \frac{1}{2} \nabla_\mu H_{\nu\rho} \nabla^\rho H^{\nu\mu} \\ & + \frac{a_1}{2} R H_{\alpha\beta} H^{\alpha\beta} + \frac{a_2}{2} R H^2 + \frac{a_3}{2} R^{\mu\alpha\nu\beta} H_{\mu\nu} H_{\alpha\beta} + \frac{a_4}{2} R^{\alpha\beta} H_{\alpha\sigma} H_\beta{}^\sigma + \frac{a_5}{2} R^{\alpha\beta} H_{\alpha\beta} H \\ & - \frac{m^2}{4} H_{\mu\nu} H^{\mu\nu} + \frac{m^2}{4} H^2 \Big\} \end{aligned} \tag{9}$$

where $a_1, \ldots a_5$ are so far arbitrary dimensionless coefficients, $R^\mu{}_{\nu\lambda\kappa} = \partial_\lambda \Gamma^\mu_{\nu\kappa} - \ldots$, $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$.

Equations of motion

$$
\begin{aligned}
E_{\mu\nu} &= \nabla^2 H_{\mu\nu} - G_{\mu\nu}\nabla^2 H + \nabla_\mu\nabla_\nu H + G_{\mu\nu}\nabla^\alpha\nabla^\beta H_{\alpha\beta} - \nabla_\sigma\nabla_\mu H^\sigma{}_\nu - \nabla_\sigma\nabla_\nu H^\sigma{}_\mu \\
&+ 2a_1 R H_{\mu\nu} + 2a_2 G_{\mu\nu} R H + 2a_3 R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} + a_4 R_\mu{}^\alpha H_{\alpha\nu} + a_4 R_\nu{}^\alpha H_{\alpha\mu} \\
&+ a_5 R_{\mu\nu} H + a_5 G_{\mu\nu} R^{\alpha\beta} H_{\alpha\beta} - m^2 H_{\mu\nu} + m^2 H G_{\mu\nu} \approx 0 \qquad (10)
\end{aligned}
$$

contain second time derivatives of $H_{\mu\nu}$ in the following way:

$$
\begin{aligned}
E_{00} &= (G^{mn} - G_{00}G^{00}G^{mn} + G_{00}G^{0m}G^{0n})\nabla_0\nabla_0 H_{mn} + O(\nabla_0), \\
E_{0i} &= (-G_{0i}G^{00}G^{mn} + G_{0i}G^{0m}G^{0n} - G^{0m}\delta_i^n)\nabla_0\nabla_0 H_{mn} + O(\nabla_0), \\
E_{ij} &= (G^{00}\delta_i^m\delta_j^n - G_{ij}G^{00}G^{mn} + G_{ij}G^{0m}G^{0n})\nabla_0\nabla_0 H_{mn} + O(\nabla_0). \qquad (11)
\end{aligned}
$$

So we see that accelerations $\ddot{H}_{00}$ and $\ddot{H}_{0i}$ again (as in the flat case) do not enter the equations of motion while accelerations $\ddot{H}_{ij}$ can be expressed through $\dot{H}_{\mu\nu}$, $H_{\mu\nu}$ and their spatial derivatives.

There are D linear combinations of the equations of motion which do not contain second time derivatives and so represent primary constraints of the theory:

$$
\varphi_\mu^{(1)} = E^0{}_\mu = G^{00}E_{0\mu} + G^{0j}E_{j\mu} \qquad (12)
$$

Now one should calculate time derivatives of these constraints and define secondary ones. In order to do this in a covariant form we can add to the time derivative of $\varphi_\mu^{(1)}$ any linear combination of equations of motion and primary constraints. So we choose the secondary constraints in the following way:

$$
\begin{aligned}
\varphi_\mu^{(2)} &= \nabla^\alpha E_{\alpha\mu} = \dot{\varphi}_\mu^{(1)} + \partial_i E^i{}_\mu + \Gamma^\alpha_{\alpha 0}\varphi_\mu^{(1)} + \Gamma^\alpha_{\alpha i}E^i{}_\mu - \Gamma^\sigma_{\mu 0}\varphi_\sigma^{(1)} - \Gamma^\sigma_{\mu i}E^i{}_\sigma \\
&= (2a_1 R - m^2)\nabla^\mu H_{\mu\nu} + (2a_2 R + m^2)\nabla_\nu H + 2a_3 R^{\mu\alpha}{}_\nu{}^\beta\nabla_\mu H_{\alpha\beta} + a_4 R^{\mu\alpha}\nabla_\mu H_{\alpha\nu} \\
&+ (a_4 - 2)R^\alpha{}_\nu\nabla^\mu H_{\alpha\mu} + a_5 R^{\alpha\mu}\nabla_\nu H_{\alpha\mu} + (a_5+1)R^\alpha{}_\nu\nabla_\alpha H \\
&+ (2a_1 + \frac{a_4}{2})H_{\alpha\nu}\nabla^\alpha R + (2a_2 + \frac{a_5}{2})H\nabla_\nu R \\
&+ H_{\alpha\beta}\Big[(2a_3 + a_5 + 1)\nabla_\nu R^{\alpha\beta} + (a_4 - 2a_3 - 2)\nabla^\alpha R^\beta{}_\nu\Big] \qquad (13)
\end{aligned}
$$

At the next step conservation of these D secondary constraints should lead to one new constraint and to expressions for $D-1$ accelerations $\ddot{H}_{0i}$. This means that the constraints (13) should contain the first time derivatives $\dot{H}_{0\mu}$ through the matrix with the rank $D-1$:

$$
\begin{aligned}
\varphi_0^{(2)} &= A\,\dot{H}_{00} + B^j\dot{H}_{0j} + \ldots \\
\varphi_i^{(2)} &= C_i\dot{H}_{00} + D_i{}^j\dot{H}_{0j} + \ldots \qquad (14)
\end{aligned}
$$

$$
\text{rank}\ \hat{\Phi}_\mu{}^\nu \equiv \text{rank}\begin{Vmatrix} A & B^j \\ C_i & D_i{}^j \end{Vmatrix} = D - 1 \qquad (15)
$$

In the flat spacetime we had the matrix

$$
\hat{\Phi}_\mu{}^\nu = \begin{Vmatrix} 0 & 0 \\ 0 & m^2\delta_i^j \end{Vmatrix} \qquad (16)
$$

In the curved case the explicit form of this matrix elements in the constraints (13) is:

$$\begin{aligned} A &= RG^{00}(2a_1+2a_2)+R^{00}(a_4+a_5)+R^0{}_0G^{00}(a_4+a_5-1) \\ B^j &= m^2G^{0j}+RG^{0j}(2a_1+4a_2)+2a_3R^{0j}{}_0{}^0+R^j{}_0G^{00}(a_4-2) \\ &\quad +R^{0j}(a_4+2a_5)+R^0{}_0G^{0j}(a_4+2a_5) \\ C_i &= R^0{}_iG^{00}(a_4+a_5-1) \\ D_i{}^j &= -m^2G^{00}\delta_i^j+2a_1RG^{00}\delta_i^j+2a_3R^{0j}{}_i{}^0+a_4R^{00}\delta_i^j \\ &\quad +(a_4-2)R^j{}_iG^{00}+(a_4+2a_5)R_i^0G^{0j} \end{aligned} \tag{17}$$

At this stage the restrictions that consistency imposes on the non-minimal couplings and on the external gravitational field reduce to the requirements that the above matrix elements give $\det\hat{\Phi}=0$ while $\det D_i{}^j\neq 0$.

One way to fulfill these requirements is to impose the following restriction on the external gravitational fields:

$$R_{\mu\nu}=\frac{1}{D}G_{\mu\nu}R\,. \tag{18}$$

It means that one considers only Einstein spacetimes [53] representing solutions of vacuum Einstein equations with cosmological constant. In these spacetimes the scalar curvature R is constant as follows from the Bianchi identity $\nabla^\mu R_{\mu\nu}=\frac{1}{2}\nabla_\nu R$ but the Weyl tensor part of the curvature tensor can be arbitrary.

If the Einstein equation (18) for external gravity is fulfilled the coefficients a_4, a_5 in the lagrangian (9) are absent and the matrix $\hat{\Phi}$ takes the form:

$$\hat{\Phi}_\mu{}^\nu=\left\|\begin{array}{c|c} RG^{00}(2a_1+2a_2-\frac{1}{D}) & RG^{0j}(2a_1+4a_2)+2a_3R^{0j}{}_0{}^0+m^2G^{0j} \\ \hline 0 & 2a_3R^{0j}{}_i{}^0+RG^{00}\delta_i^j(2a_1-\frac{2}{D})-m^2G^{00}\delta_i^j \end{array}\right\| \tag{19}$$

The simplest way to make the rank of this matrix to be equal to $D-1$ is provided by the following choice of the coefficients:

$$2a_1+2a_2-\frac{1}{D}=0,\qquad a_3=0,\qquad 2R\Big(a_1-\frac{1}{D}\Big)-m^2\neq 0. \tag{20}$$

As a result, we have one-parameter family of theories:

$$\begin{aligned} &a_1=\frac{\xi}{D},\quad a_2=\frac{1-2\xi}{2D},\quad a_3=0,\quad a_4=0,\quad a_5=0 \\ &R_{\mu\nu}=\frac{1}{D}G_{\mu\nu}R,\qquad \frac{2(1-\xi)}{D}R+m^2\neq 0. \end{aligned} \tag{21}$$

with ξ an arbitrary real number.

The action in this case takes the form

$$\begin{aligned} S=\int d^Dx\sqrt{-G}\Big\{&\frac{1}{4}\nabla_\mu H\nabla^\mu H-\frac{1}{4}\nabla_\mu H_{\nu\rho}\nabla^\mu H^{\nu\rho}-\frac{1}{2}\nabla^\mu H_{\mu\nu}\nabla^\nu H+\frac{1}{2}\nabla_\mu H_{\nu\rho}\nabla^\rho H^{\nu\mu} \\ &+\frac{\xi}{2D}RH_{\mu\nu}H^{\mu\nu}+\frac{1-2\xi}{4D}RH^2-\frac{m^2}{4}H_{\mu\nu}H^{\mu\nu}+\frac{m^2}{4}H^2\Big\}. \end{aligned} \tag{22}$$

and the corresponding equations of motion are

$$E_{\mu\nu} = \nabla^2 H_{\mu\nu} - G_{\mu\nu}\nabla^2 H + \nabla_\mu\nabla_\nu H + G_{\mu\nu}\nabla^\alpha\nabla^\beta H_{\alpha\beta} - \nabla_\sigma\nabla_\mu H^\sigma{}_\nu - \nabla_\sigma\nabla_\nu H^\sigma{}_\mu$$
$$+\frac{2\xi}{D}RH_{\mu\nu} + \frac{1-2\xi}{D}RHG_{\mu\nu} - m^2 H_{\mu\nu} + m^2 HG_{\mu\nu} = 0 \tag{23}$$

The secondary constraints built out of them are

$$\varphi^{(2)}_\mu = \nabla^\alpha E_{\alpha\mu} = (\nabla_\mu H - \nabla^\alpha H_{\mu\alpha})\Big(m^2 + \frac{2(1-\xi)}{D}R\Big) \tag{24}$$

and the matrix $\hat{\Phi}$ looks like

$$\hat{\Phi}_\mu{}^\nu = \Big(m^2 + \frac{2(1-\xi)}{D}R\Big)\left\|\begin{array}{c|c} 0 & G^{0j} \\ \hline 0 & -G^{00}\delta^j_i \end{array}\right\| \tag{25}$$

Just like in the flat case, in this theory the conditions $\dot{\varphi}^{(2)}_i \approx 0$ define the accelerations $\ddot{H}_{0i}$ and the condition $\dot{\varphi}^{(2)}_0 \approx 0$ after excluding $\ddot{H}_{0i}$ gives a new constraint, i.e. the acceleration $\ddot{H}_{00}$ is not defined at this stage.

To define the new constraint in a covariant form we use the following linear combination of $\dot{\varphi}^{(2)}_\mu$, equations of motion, primary and secondary constraints:

$$\begin{aligned}\varphi^{(3)} &= \frac{m^2}{D-2}G^{\mu\nu}E_{\mu\nu} + \nabla^\mu\nabla^\nu E_{\mu\nu} + \frac{2(1-\xi)}{D(D-2)}RG^{\mu\nu}E_{\mu\nu} = \\ &= H\frac{1}{D-2}\Big(\frac{2(1-\xi)}{D}R + m^2\Big)\Big(\frac{D+2\xi(1-D)}{D}R + m^2(D-1)\Big) \approx 0.\end{aligned} \tag{26}$$

Requirement of its conservation leads to one more constraint

$$\dot{\varphi}^{(3)} \sim \dot{H} \quad\Longrightarrow\quad \varphi^{(4)} = \dot{H} \approx 0. \tag{27}$$

The last acceleration $\ddot{H}_{00}$ is expressed from the condition $\dot{\varphi}^{(4)} \approx 0$.

Using the constraints for simplifying the equations of motion we see that the original equations are equivalent to the following system:

$$\begin{aligned}&\nabla^2 H_{\mu\nu} + 2R^\alpha{}_\mu{}^\beta{}_\nu H_{\alpha\beta} + \frac{2(\xi-1)}{D}RH_{\mu\nu} - m^2 H_{\mu\nu} = 0, \\ &H^\mu{}_\mu = 0, \qquad \dot{H}^\mu{}_\mu = 0, \qquad \nabla^\mu H_{\mu\nu} = 0, \\ &G^{00}\nabla_0\nabla_i H^i{}_\nu - G^{0i}\nabla_0\nabla_i H^0{}_\nu - G^{0i}\nabla_i\nabla_0 H^0{}_\nu - G^{ij}\nabla_i\nabla_j H^0_\nu - 2R^{\alpha0\beta}{}_\nu H_{\alpha\beta} \\ &\qquad - \frac{2(\xi-1)}{D}RH^0{}_\nu + m^2 H^0{}_\nu = 0.\end{aligned} \tag{28}$$

The last expression represents D primary constraints.

For any values of ξ the theory describes the same number of degrees of freedom as in the flat case - the symmetric, covariantly transverse and traceless tensor. D primary constraints guarantees conservation of the transversality conditions in time.

Now we turn to the discussion of causality for the spin 2 field equations. Analogous problem with dynamical gravity was investigated in [1] (see also [13, 14] for causality problem in electromagnetic background). In general, when one has a system of differential equations for a set of fields ϕ^B (to be specific, let us say about second order equations)

$$M_{AB}{}^{\mu\nu}\partial_\mu\partial_\nu\phi^B + \ldots = 0, \qquad \mu,\nu = 0,\ldots,D-1 \tag{29}$$

the following definitions are used. A characteristic matrix is the matrix function of D arguments n_μ built out of the coefficients at the second derivatives in the equations: $M_{AB}(n) = M_{AB}{}^{\mu\nu}n_\mu n_\nu$. A characteristic equation is $\det M_{AB}(n) = 0$. A characteristic surface is the surface $S(x) = const$ where $\partial_\mu S(x) = n_\mu$.

If for any n_i ($i = 1,\ldots,D-1$) all solutions of the characteristic equation $n_0(n_i)$ are real then the system of differential equations is called hyperbolic and describes propagation of some wave processes. The hyperbolic system is called causal if there is no timelike vectors among solutions n_μ of the characteristic equations. Such a system describes propagation with a velocity not exceeding the speed of light. If there exist timelike solutions for n_μ then the corresponding characteristic surfaces are spacelike which violates causality.

In the flat spacetime the equations for the spin 2 field (1) lead to the characteristic equation

$$\det M(n) = (n^2)^{D(D+1)/2} \tag{30}$$

which has 2 multiply degenerate roots:

$$-n_0^2 + n_i^2 = 0, \qquad n_0 = \pm\sqrt{n_i^2}. \tag{31}$$

The solutions for n_μ are real and null hence the equations are hyperbolic and causal.

Now consider the curved spacetime generalization. If we tried to use the equations of motion in the original lagrangian form (23) then the characteristic matrix

$$M_{\mu\nu}{}^{\lambda\kappa}(n) = \delta_{(\mu\nu)}{}^{(\lambda\kappa)}n^2 - G_{\mu\nu}G^{\lambda\kappa}n^2 + G^{\lambda\kappa}n_\mu n_\nu + G_{\mu\nu}n^\lambda n^\kappa - \delta_\nu^{(\kappa}n^{\lambda)}n_\mu - \delta_\mu^{(\kappa}n^{\lambda)}n_\nu \tag{32}$$

would be degenerate. This fact can be seen from the relation

$$n^\mu M_{\mu\nu}{}^{\lambda\kappa}(n) \equiv 0 \tag{33}$$

which means that any symmetric tensor of the form $n_{(\mu}t_{\nu)}$ (with t_ν an arbitrary vector) represents a "null vector" for the matrix $M(n)$ and therefore $\det M = 0$.

After having used the constraints we obtain the equations of motion written in the form (28) and the characteristic matrix becomes non-degenerate:

$$M_{\mu\nu}{}^{\lambda\kappa}(n) = \delta_{\mu\nu}{}^{\lambda\kappa}n^2, \qquad n^2 = G^{\alpha\beta}n_\alpha n_\beta. \tag{34}$$

The characteristic cones remains the same as in the flat case. At any point x_0 we can choose locally $G^{\alpha\beta}(x_0) = \eta^{\alpha\beta}$ and then

$$n^2\Big|_{x_0} = -n_0^2 + n_i^2 \tag{35}$$

Just like in the flat case the equations are hyperbolic and causal.

So we demonstrated that in Einstein spacetimes spin 2 massive field can be consistently described by a one-parameter family of theories (22). For any value of the parameter the corresponding equations describe the correct number of degrees of freedom which propagate causally. Our lagrangian for the spin 2 field in curved spacetime is the most general known so far, in all previous works only the theories with specific values of the parameter ξ were considered [3, 5].

The next natural step would consist in building a theory describing dynamics of both gravity and massive spin 2 field. In such a theory in addition to dynamical equations for the massive spin 2 field one would have dynamical equations for gravity with the energy-momentum tensor constructed out of spin 2 field components. The analysis of consistency then changes and one needs to have correct number of constraints and causality for both fields interacting with each other [1].

The only known consistent system of a higher spin field interacting with dynamical gravity is the theory of massless helicity 3/2 field, i.e. supergravity [51] (see also the book [52]). In that case consistency with dynamical gravity requires four-fermion interaction. If a consistent description of spin 2 field interacting with dynamical gravity exists it may also require some non-trivial modification of the lagrangian. At least, it is known that lagrangians quadratic in spin 2 field do not provide such a consistency [1]. In the Section 4 we will describe a possible way of consistent description of the spin 2 field on arbitrary gravitational background which is given by representation of the lagrangian in the form of infinite series in inverse mass.

3 Coupling to background scalar field

Now we investigate a possibility to generalize the above analysis for the case of spin 2 massive field interacting not only with background gravity but also with a scalar dilaton field. This set of fields arises naturally in string theory which contains dilaton field $\phi(x)$ as one of its massless excitations.

Writing a general action similar to (22) for this system one should take into account all possible new terms with derivatives of $\phi(x)$ and also containing arbitrary factors $f(\phi)$ without derivatives of the dilaton fields. For example, string effective action can contain in various terms the factors $e^{k\phi}$, $k = const$. We will consider here the class of actions for the field $H_{\mu\nu}$ where all these factors can be absorbed to the metric $G_{\mu\nu}$ by a conformal rescaling.

The most general action of this type is

$$S = S_G + S_\phi, \tag{36}$$

where S_G is the general action without dependence on scalar field (9) and S_ϕ can contain (up to total derivatives) ten new terms:

$$
\begin{aligned}
S_\phi \;=\; & \int d^Dx\sqrt{-G}\Big\{\frac{c_1}{2}H^{\mu\nu}\nabla_\alpha H_{\mu\nu}\nabla^\alpha\phi+\frac{c_2}{2}H\nabla_\alpha H\nabla^\alpha\phi+\frac{c_3}{2}H^{\mu\nu}\nabla_\mu H_{\alpha\nu}\nabla^\alpha\phi \\
& +\frac{c_4}{2}H\nabla^\mu H_{\alpha\mu}\nabla^\alpha\phi+\frac{c_5}{2}H_{\mu\nu}\nabla^\nu H\nabla^\mu\phi+\frac{c_6}{2}H_{\alpha\beta}\nabla^\mu H_\mu{}^\beta\nabla^\alpha\phi+\frac{c_7}{2}H_{\mu\alpha}H_\nu{}^\alpha\nabla^\mu\phi\nabla^\nu\phi \\
& +\frac{c_8}{2}HH_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi+\frac{c_9}{2}H_{\mu\nu}H^{\mu\nu}(\nabla\phi)^2+\frac{c_{10}}{2}H^2(\nabla\phi)^2\Big\} \qquad (37)
\end{aligned}
$$

We consider ϕ as a background field on the same footing with the metric $G_{\mu\nu}$.

What values of coupling parameters $c_1,\ldots,c_{10}$ are permissible and how does the condition on the background (18) change in the presence of ϕ? To answer these questions one should repeat the analysis of the previous section for the action (36) calculating all constraints of the theory.

The equations of motion are:

$$
\begin{aligned}
E_{\mu\nu} \;=\; & \nabla^2 H_{\mu\nu}-G_{\mu\nu}\nabla^2 H+\nabla_\mu\nabla_\nu H+G_{\mu\nu}\nabla^\alpha\nabla^\beta H_{\alpha\beta}-\nabla_\sigma\nabla_\mu H^\sigma{}_\nu-\nabla_\sigma\nabla_\nu H^\sigma{}_\mu \\
& +2a_1RH_{\mu\nu}+2a_2G_{\mu\nu}RH+2a_3R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta}+a_4R_\mu{}^\alpha H_{\alpha\nu}+a_4R_\nu{}^\alpha H_{\alpha\mu} \\
& +a_5R_{\mu\nu}H+a_5G_{\mu\nu}R^{\alpha\beta}H_{\alpha\beta}+\frac{c_3-c_6}{2}(\nabla_\mu H_{\alpha\nu}\nabla^\alpha\phi+\nabla_\nu H_{\alpha\mu}\nabla^\alpha\phi) \\
& +\frac{c_6-c_3}{2}(\nabla^\alpha H_{\alpha\nu}\nabla_\mu\phi+\nabla^\alpha H_{\alpha\mu}\nabla_\nu\phi)+\frac{c_5-c_4}{2}(\nabla_\nu H\nabla_\mu\phi+\nabla_\mu H\nabla_\nu\phi) \\
& +(c_4-c_5)G_{\mu\nu}\nabla^\beta H_{\alpha\beta}\nabla^\alpha\phi-\frac{c_3+c_6}{2}(H_{\alpha\nu}\nabla^\alpha\nabla_\mu\phi+H_{\alpha\mu}\nabla^\alpha\nabla_\nu\phi) \\
& -c_1H_{\mu\nu}\nabla^2\phi-c_2G_{\mu\nu}H\nabla^2\phi-c_4H\nabla_\mu\nabla_\nu\phi-c_5G_{\mu\nu}H_{\alpha\beta}\nabla^\alpha\nabla^\beta\phi \\
& +c_7(H_{\nu\alpha}\nabla_\mu\phi\nabla^\alpha\phi+H_{\mu\alpha}\nabla_\nu\phi\nabla^\alpha\phi)+c_8H\nabla_\mu\phi\nabla_\nu\phi+c_8G_{\mu\nu}H_{\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi \\
& +2c_9H_{\mu\nu}(\nabla\phi)^2+2c_{10}G_{\mu\nu}H(\nabla\phi)^2-m^2H_{\mu\nu}+m^2HG_{\mu\nu}=0 \qquad (38)
\end{aligned}
$$

Introducing of background dilaton does not change the second derivatives terms in the equations and so $\varphi_\mu^{(1)}=E^0{}_\mu\approx 0$ are again primary constraints. Conditions of their conservation should give D secondary constraints:

$$
\begin{aligned}
\varphi_0^{(2)} &= \nabla^\mu E_{\mu 0}=\frac{1}{2}(c_3-c_6+c_4-c_5)G^{00}\nabla_k\phi(G^{0k}\ddot H_{00}+2G^{kj}\ddot H_{0j})+\ldots, \\
\varphi_i^{(2)} &= \nabla^\mu E_{\mu i}=-\frac{1}{2}(c_3-c_6+c_4-c_5)G^{00}\nabla_i\phi(G^{00}\ddot H_{00}+2G^{0j}\ddot H_{0j})+\ldots \qquad (39)
\end{aligned}
$$

Hence we should impose the restriction

$$
c_3-c_6+c_4-c_5=0. \qquad (40)
$$

in order to cancel second time derivatives in $\varphi_\mu^{(2)}$. Note that without dilaton couplings these expressions do not contain second derivatives at all and represent constraints for any values of non-minimal couplings with gravity (12).

We will restrict ourselves to even simpler particular class of the theories with

$$
c_3=c_6, \qquad c_4=c_5. \qquad (41)
$$

Then the secondary constraints contain first time derivatives of $H_{0\mu}$ with the following coefficients (14):

$$
\begin{aligned}
A &= G^{00}\Big[2(a_1+a_2)R-(c_1+c_2)\nabla^2\phi+2(c_9+c_{10})(\nabla\phi)^2\Big] \\
&\quad + (a_4+a_5)R^{00}-(c_3+c_4)\nabla^0\nabla^0\phi+(c_7+c_8)\nabla^0\phi\nabla^0\phi \\
&\quad + G^{00}\Big[(a_4+a_5-1)R^0{}_0-(c_3+c_4)\nabla_0\nabla^0\phi+(c_7+c_8)\nabla_0\phi\nabla^0\phi\Big] \\
B^j &= 2a_3R^{0j}{}_0{}^0+G^{0j}\Big[m^2+(2a_1+4a_2)R-(c_1+2c_2)\nabla^2\phi+(2c_9+4c_{10})(\nabla\phi)^2\Big] \\
&\quad + (a_4+2a_5)R^{0j}-(c_3+2c_4)\nabla^0\nabla^j\phi+(c_7+2c_8)\nabla^0\phi\nabla^j\phi \\
&\quad + G^{0j}\Big[(a_4+2a_5)R^0{}_0-(c_3+2c_4)\nabla_0\nabla^0\phi+(c_7+2c_8)\nabla_0\phi\nabla^0\phi\Big] \\
&\quad + G^{00}\Big[(a_4-2)R^j{}_0-c_3\nabla_0\nabla^i\phi+c_7\nabla_0\phi\nabla^i\phi\Big] \\
C_i &= G^{00}\Big[(a_4+a_5-1)R^0{}_i-(c_3+c_4)\nabla_i\nabla^0\phi+(c_7+c_8)\nabla_i\phi\nabla^0\phi\Big] \\
D_i{}^j &= 2a_3R^{0j}{}_i{}^0+G^{00}\delta_i^j\Big[-m^2+2a_1R-c_1\nabla^2\phi+2c_9(\nabla\phi)^2\Big] \\
&\quad + \delta_i^j\Big[a_4R^{00}-c_3\nabla^0\nabla^0\phi+\nabla^0\phi\nabla^0\phi\Big] \\
&\quad + G^{00}\Big[(a_4-2)R^j{}_i-c_3\nabla_i\nabla^j\phi+c_7\nabla_i\phi\nabla^j\phi\Big] \\
&\quad + G^{0j}\Big[(a_4+2a_5)R_i^0-(c_3+2c_4)\nabla_i\nabla^0\phi+(c_7+2c_8)\nabla_i\phi\nabla^0\phi\Big] \qquad (42)
\end{aligned}
$$

The simplest way to make the rank of the matrix $\hat{\Phi}$ (15) with the elements (42) to be equal to $D-1$ consists in the choice

$$
\begin{aligned}
&R_{\mu\nu}=\frac{1}{D}RG_{\mu\nu}\,, \qquad 2a_1+2a_2-\frac{1}{D}=0\,, \qquad a_3=a_4=a_5=0\,, \\
&c_1=-c_2\,, \qquad c_3=-c_4\,, \qquad c_7=-c_8\,, \qquad c_9=-c_{10}\,. \qquad (43)
\end{aligned}
$$

supplemented by (41)

In this case consistent action for the spin 2 field interacting with background gravitational and scalar fields contains five arbitrary constant parameters:

$$
\begin{aligned}
S=\int d^Dx\sqrt{-G}\Big\{&\frac{1}{4}\nabla_\mu H\nabla^\mu H-\frac{1}{4}\nabla_\mu H_{\nu\rho}\nabla^\mu H^{\nu\rho}-\frac{1}{2}\nabla^\mu H_{\mu\nu}\nabla^\nu H+\frac{1}{2}\nabla_\mu H_{\nu\rho}\nabla^\rho H^{\nu\mu} \\
&+\frac{\xi}{2D}RH_{\mu\nu}H^{\mu\nu}+\frac{1-2\xi}{4D}RH^2+\frac{\zeta_1}{2}\nabla^\alpha\phi\nabla_\alpha H_{\mu\nu}H^{\mu\nu}-\frac{\zeta_1}{2}\nabla^\alpha\phi\nabla_\alpha HH \\
&+\frac{\zeta_2}{2}\nabla^\alpha\phi\nabla_\mu H_{\alpha\nu}H^{\mu\nu}-\frac{\zeta_2}{2}\nabla^\alpha\phi\nabla^\mu H_{\alpha\mu}H-\frac{\zeta_2}{2}\nabla^\mu\phi\nabla^\nu HH_{\mu\nu}+\frac{\zeta_2}{2}\nabla^\alpha\phi\nabla^\mu H_\mu{}^\beta H_{\alpha\beta} \\
&+\frac{\zeta_3}{2}\nabla^\mu\phi\nabla^\nu\phi H_{\mu\alpha}H_\nu{}^\alpha-\frac{\zeta_3}{2}\nabla^\mu\phi\nabla^\nu\phi H_{\mu\nu}H+\frac{\zeta_4}{2}(\nabla\phi)^2H_{\mu\nu}H^{\mu\nu}-\frac{\zeta_4}{2}(\nabla\phi)^2H^2 \\
&-\frac{m^2}{4}H_{\mu\nu}H^{\mu\nu}+\frac{m^2}{4}H^2\Big\} \qquad (44)
\end{aligned}
$$

and conservation conditions for the secondary constraints $\varphi_\mu^{(2)}=\nabla^\nu E_{\mu\nu}$ can be used for building one new constraint:

$$
\varphi^{(3)} = \nabla^\mu\nabla^\nu E_{\mu\nu}+(\zeta_3\nabla^\mu\phi\nabla^\nu\phi-\zeta_2\nabla^\mu\nabla^\nu\phi)E_{\mu\nu}
$$

$$
\begin{aligned}
&+\Big(m^2+\frac{2(1-\xi)}{D}R-(\zeta_3+2\zeta_4)(\nabla\phi)^2+(\zeta_1+\zeta_2)\nabla^2\phi\Big)\frac{1}{D-2}G^{\mu\nu}E_{\mu\nu}\\
=\;&\nabla_\alpha H_{\mu\nu}\big[-2\zeta_2R^{\mu\alpha\nu\beta}\nabla_\beta\phi+2\zeta_3\nabla^\mu\nabla^\nu\phi\nabla^\alpha\phi-2\zeta_3\nabla^\mu\nabla^\alpha\phi\nabla^\nu\phi\big]\\
&+(\nabla_\nu H-\nabla^\mu H_{\mu\nu})\Big[2(\zeta_1+\zeta_2)\nabla^\nu\nabla^2\phi+\frac{2\xi_2}{D}R\nabla^\nu\phi-2\zeta_3\nabla^\nu\phi\nabla^2\phi\\
&-2(\zeta_3+4\zeta_4)\nabla^\nu\nabla^\alpha\phi\nabla_\alpha\phi\Big]+H_{\mu\nu}f^{\mu\nu}(\phi)+Hf(\phi).
\end{aligned}
\tag{45}
$$

Here $f^{\mu\nu}$ and f are functions of the scalar field ϕ and its derivatives, we will not write the explicit form of them here. The important fact about this constraint is that it does not contain the time derivative of the field component H_{00}. Hence, the conservation condition $\dot\varphi^{(3)}\approx 0$ does not contain the acceleration $\ddot H_{00}$ and leads to another one constraint $\varphi^{(4)}$ providing thus the correct total number of constraints in the theory. The acceleration $\ddot H_{00}$ is defined only from the condition $\dot\varphi^{(4)}\approx 0$. For any values of the coupling parameters the theory (44) describes correct number of degrees of freedom.

Causality in presence of dilaton may be violated even in the flat spacetime. It follows from the fact that in general the constraints $\varphi^{(2)}_\mu$ and $\varphi^{(3)}$ cannot be solved algebraically with respect to the trace H and the longitudinal part $\nabla^\mu H_{\mu\nu}$ and used for cancelling the corresponding terms in the equations of motion. As a result, the characteristic matrix differs signifigantly from its flat counterpart and the characteristic equation may possess new non-trivial solutions spoiling causality for some values of the parameters ζ, ξ. We postpone a detailed study of the causality in the presence of dilaton for a separate publication.

In the rest of the paper we will be considering only pure gravitational background when the scalar field ϕ is absent.

4 Consistent equations in arbitrary background

In previous sections we analyzed a possibility of consistent description of the spin 2 field on arbitrary Einstein manifold. Now we will describe another possibility which allows to remove any restrictions on the external gravitational background by means of considering a lagrangian in the form of infinite series in inverse mass m. Existence of dimensionful mass parameter m in the theory let us construct a lagrangian with terms of arbitrary orders in curvature multiplied by the corresponding powers of $1/m^2$:

$$
\begin{aligned}
S_H=\int d^Dx\sqrt{-G}\Big\{&\frac14\nabla_\mu H\nabla^\mu H-\frac14\nabla_\mu H_{\nu\rho}\nabla^\mu H^{\nu\rho}-\frac12\nabla^\mu H_{\mu\nu}\nabla^\nu H+\frac12\nabla_\mu H_{\nu\rho}\nabla^\rho H^{\nu\mu}\\
&+\frac{a_1}{2}RH_{\alpha\beta}H^{\alpha\beta}+\frac{a_2}{2}RH^2+\frac{a_3}{2}R^{\mu\alpha\nu\beta}H_{\mu\nu}H_{\alpha\beta}+\frac{a_4}{2}R^{\alpha\beta}H_{\alpha\sigma}H_\beta{}^\sigma+\frac{a_5}{2}R^{\alpha\beta}H_{\alpha\beta}H\\
&+\frac{1}{m^2}(R\nabla H\nabla H+RH\nabla\nabla H+RRHH)+\frac{1}{m^4}(RR\nabla H\nabla H+RRH\nabla\nabla H\\
&+R\nabla RH\nabla H+R\nabla\nabla RHH+RRRHH)+O\Big(\frac{1}{m^6}\Big)
\end{aligned}
$$

$$-\frac{m^2}{4}H_{\mu\nu}H^{\mu\nu}+\frac{m^2}{4}H^2\Big\} \tag{46}$$

Actions of this kind are expected to arise naturally in string theory where the role of mass parameter is played by string tension $m^2 = 1/\alpha'$ and perturbation theory in α' will give for background fields effective actions of the form (46) [2].

Possibility of constructing consistent equations for massive higher spin fields as series in curvature was recently studied in [6] where such equations were derived in particular case of symmetrical Einstein spaces in linear in curvature order.

Here we will demonstrate that requirement of consistency with the flat spacetime limit can be fulfilled perturbatively in $1/m^2$ for arbitrary gravitational background at least in the lowest order. We will use the same general scheme of calculating lagrangian constraints as in the previous sections. The only difference is that each condition will be considered perturbatively and can be solved separately in each order in $1/m^2$.

Primary constraints in the theory described by the action (46) should be given by the equations $E^0{}_\mu \approx 0$. Requirement of absence of second time derivatives in these equations will give some restrictions on coefficients in higher orders in $1/m^2$, for example, in terms like $R\nabla H\nabla H$.

Secondary constraints in the lowest order in $1/m^2$ were already calculated in the Section 2 (13). Consistency with the flat spacetime limit requires existence of one additional constraint among conservation conditions of these secondary constraints. So we should calculate the rank of the matrix

$$\hat{\Phi}_\mu{}^\nu = \left\| \begin{array}{cc} A & B^j \\ C_i & D_i{}^j \end{array} \right\| \tag{47}$$

with the elements in the lowest order given by (17). The advantage of having a theory in the form of infinite series consists in the possibility to calculate the determinant of the above matrix perturbatively in $1/m^2$. Assuming that the lower right subdeterminant of the matrix is not zero (it is not zero in the flat case) one has

$$\det\hat{\Phi} = (A - BD^{-1}C)\det D\ , \qquad \det D \neq 0 \tag{48}$$

Converting the matrix D perturbatively

$$D^{-1} = -\frac{1}{m^2 G^{00}}\delta_i^j + O\left(\frac{1}{m^4}\right) \tag{49}$$

we get

$$A - BD^{-1}C = RG^{00}2(a_1+a_2) + R^{00}(2a_4+2a_5-1) + O\left(\frac{1}{m^2}\right) \tag{50}$$

So consistency with the flat limit imposes at this order in m^2 two conditions on the five non-minimal couplings in the lagrangian (46) and we are left with a three parameters

[2] In fact, string theory should lead to even more general effective actions than (46) since in the higher α' corrections higher derivatives of all background fields should appear.

family of theories:

$$a_1 = \frac{\xi_1}{2}, \quad a_2 = -\frac{\xi_1}{2}, \quad a_3 = \frac{\xi_3}{2}, \quad a_4 = \frac{1}{2} - \xi_2, \quad a_5 = \xi_2. \tag{51}$$

The action (46) then takes the form:

$$\begin{aligned} S_H = \int d^D x \sqrt{-G} \Big\{ & \frac{1}{4} \nabla_\mu H \nabla^\mu H - \frac{1}{4} \nabla_\mu H_{\nu\rho} \nabla^\mu H^{\nu\rho} - \frac{1}{2} \nabla^\mu H_{\mu\nu} \nabla^\nu H + \frac{1}{2} \nabla_\mu H_{\nu\rho} \nabla^\rho H^{\nu\mu} \\ & + \frac{\xi_1}{4} R H_{\alpha\beta} H^{\alpha\beta} - \frac{\xi_1}{4} R H^2 + \frac{1-2\xi_2}{4} R^{\alpha\beta} H_{\alpha\sigma} H_\beta{}^\sigma + \frac{\xi_2}{2} R^{\alpha\beta} H_{\alpha\beta} H \\ & + \frac{\xi_3}{2} R^{\mu\alpha\nu\beta} H_{\mu\nu} H_{\alpha\beta} - \frac{m^2}{4} H_{\mu\nu} H^{\mu\nu} + \frac{m^2}{4} H^2 + O\Big(\frac{1}{m^2}\Big) \Big\} \end{aligned} \tag{52}$$

In this case the rank of the matrix $\hat{\Phi}$ is equal to $D-1$ and one can construct from the conservation conditions for the secondary constraints

$$\nabla_0 \varphi_\nu^{(2)} = \nabla_0 \nabla^\mu E_{\mu\nu} = \hat{\Phi}_\nu{}^\mu \ddot{H}_{0\mu} + \ldots \tag{53}$$

one linear combination which does not contain acceleration $\ddot{H}_{0\mu}$:

$$\begin{aligned} \Big[G^{00} - \frac{1}{m^2} R^{00} + O\Big(\frac{1}{m^4}\Big)\Big] \nabla_0 \varphi_0^{(2)} + \Big[G^{0i} - \frac{1}{m^2} R^{0i} + O\Big(\frac{1}{m^4}\Big)\Big] \nabla_0 \varphi_i^{(2)} \\ = G^{0\nu} \nabla_0 \nabla^\mu E_{\mu\nu} - \frac{1}{m^2} R^{0\nu} \nabla_0 \nabla^\mu E_{\mu\nu} + O\Big(\frac{1}{m^4}\Big) \end{aligned} \tag{54}$$

After excluding the accelerations $\ddot{H}_{ij}$ by means of equations of motion this expression will represent a new constraint restricting the trace of the field H. To make it covariant we can add spatial derivatives of the secondary constraints $\sim \nabla_i \nabla^\mu E_{\mu\nu}$:

$$\begin{aligned} \varphi^{(3)} \approx\; & \nabla^\mu \nabla^\nu E_{\mu\nu} - \frac{1}{m^2} R^{\alpha\nu} \nabla_\alpha \nabla^\mu E_{\mu\nu} + O\Big(\frac{1}{m^4}\Big) \\ =\; & (m^2 - \xi_1 R)(\nabla^2 H - \nabla^\alpha \nabla^\beta H_{\alpha\beta}) \\ & + \xi_2 R^{\alpha\beta} (\nabla^2 H_{\alpha\beta} + \nabla_\alpha \nabla_\beta H - \nabla^\nu \nabla_\alpha H_{\beta\nu} - \nabla^\nu \nabla_\beta H_{\alpha\nu}) \\ & + \xi_3 R^{\mu\alpha\nu\beta} \nabla_\mu \nabla_\nu H_{\alpha\beta} + (1 + 2\xi_2 + 2\xi_3) \nabla^\mu R^{\alpha\beta} (\nabla_\mu H_{\alpha\beta} - \nabla_\alpha H_{\mu\beta}) \\ & + (\frac{1}{2} - 2\xi_1 + \xi_2) \nabla^\alpha R (\nabla_\alpha H - \nabla^\mu H_{\alpha\mu}) + (-\xi_1 + \frac{\xi_2}{2}) H \nabla^2 R \\ & + H_{\alpha\beta} \Big[(1 + \xi_2 + \xi_3) \nabla^2 R^{\alpha\beta} + (-\frac{1}{2} + \xi_1 - \xi_2 - \frac{\xi_3}{2}) \nabla^\alpha \nabla^\beta R \\ & - \xi_3 R^{\alpha\mu} R^\beta{}_\mu + \xi_3 R^{\alpha\mu\beta\nu} R_{\mu\nu}\Big] + O\Big(\frac{1}{m^2}\Big) \end{aligned} \tag{55}$$

Derivatives of the field $H_{\mu\nu}$ enter this expression in such a way that it does not contain the accelerations $\ddot{H}_{00}$, $\ddot{H}_{0\mu}$ and the velocity $\dot{H}_{00}$. This velocity does not appear in (55) after excluding the accelerations $\ddot{H}_{ij}$ because the equations of motion do not contain $\dot{H}_{00}$ either. It means that just like in the flat case the conservation condition $\dot{\varphi}^{(3)} \approx 0$ leads to another new constraints $\varphi^{(4)}$ and the last acceleration $\ddot{H}_{00}$ is defined from $\dot{\varphi}^{(4)} \approx 0$. The total number of constraints coincides with that in the flat spacetime.

The constraints $\varphi^{(3)}$ and $\varphi^{(2)}_\mu$ can be solved perturbatively in $1/m^2$ with respect to the trace and the longitudinal part of $H_{\mu\nu}$:

$$\varphi^{(3)} \sim H + O\Big(\frac{1}{m^2}\Big), \qquad \varphi^{(2)}_\mu \sim \nabla^\mu H_{\mu\nu} + O\Big(\frac{1}{m^2}\Big) \tag{56}$$

and used for simplifying the form of the original equations of motion. It is convenient to take these equations in the following linear combination which does not contain the term $m^2 H$:

$$\begin{aligned}
&E_{\mu\nu} - \frac{1}{D-1} G_{\mu\nu} G^{\alpha\beta} E_{\alpha\beta} = \\
&= \nabla^2 H_{\mu\nu} + \nabla_\mu \nabla_\nu H - \nabla_\mu \nabla^\sigma H_{\sigma\nu} - \nabla_\nu \nabla^\sigma H_{\sigma\mu} + \frac{1}{D-1} G_{\mu\nu} (\nabla^\alpha \nabla^\beta H_{\alpha\beta} - \nabla^2 H) \\
&\quad - m^2 H_{\mu\nu} + \xi_1 R H_{\mu\nu} + (\xi_3 + 2) R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} - (\frac{1}{2} + \xi_2)(R_\mu{}^\alpha H_{\alpha\nu} + R_\nu{}^\alpha H_{\alpha\mu}) \\
&\quad + \xi_2 R_{\mu\nu} H - \frac{\xi_2}{D-1} G_{\mu\nu} R H + \frac{\xi_2 - \xi_3 - 1}{D-1} G_{\mu\nu} R^{\alpha\beta} H_{\alpha\beta} + O\Big(\frac{1}{m^2}\Big).
\end{aligned} \tag{57}$$

Adding to these equations a suitable combination of the constraints (56) we obtain that in the lowest order in $1/m^2$ the spin 2 massive field is described by the conditions

$$\begin{aligned}
&\nabla^2 H_{\mu\nu} - m^2 H_{\mu\nu} + \xi_1 R H_{\mu\nu} + (\xi_3 + 2) R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} - (\frac{1}{2} + \xi_2)(R_\mu{}^\alpha H_{\alpha\nu} + R_\nu{}^\alpha H_{\alpha\mu}) \\
&\qquad + \frac{\xi_2 - \xi_3 - 1}{D-1} G_{\mu\nu} R^{\alpha\beta} H_{\alpha\beta} + O\Big(\frac{1}{m^2}\Big) = 0, \\
&H + O\Big(\frac{1}{m^2}\Big) = 0, \qquad \nabla^\mu H_{\mu\nu} + O\Big(\frac{1}{m^2}\Big) = 0
\end{aligned} \tag{58}$$

and also by the D primary constraints $E^0{}_\mu$. We see that even in this lowest order in m^2 not all non-minimal terms in the equations are arbitrary. Consistency with the flat limit leaves only three arbitrary parameters while the number of different non-minimal terms in the equations is four.

However, if gravitational field is also subject to some dynamical equations of the form $R_{\mu\nu} = O(1/m^2)$ then the system (58) contains only one non-minimal coupling in the lowest order

$$\begin{aligned}
&\nabla^2 H_{\mu\nu} - m^2 H_{\mu\nu} + (\xi_3 + 2) R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} + O\Big(\frac{1}{m^2}\Big) = 0, \\
&H + O\Big(\frac{1}{m^2}\Big) = 0, \qquad \nabla^\mu H_{\mu\nu} + O\Big(\frac{1}{m^2}\Big) = 0, \\
&R_{\mu\nu} + O\Big(\frac{1}{m^2}\Big) = 0
\end{aligned} \tag{59}$$

and is consistent for any its value.

Requirement of causality does not impose any restrictions on the couplings in this order. The characteristic matrix of (58) is non-degenerate, second derivatives enter in the same way as in the flat spacetime, and hence the light cones of the field $H_{\mu\nu}$ described by (58) are the same as in the flat case. Propagation is causal for any values of ξ_1, ξ_2, ξ_3. In higher orders in $1/m^2$ situation becomes more complicated and we expect that requirement of causality may give additional restrictions on the non-minimal couplings.

Concluding this section we would like to stress once more that the theory (52) admits any gravitational background and so no inconsistencies arise if one treats gravity as dynamical field satisfying Einstein equations with the energy - momentum tensor for the field $H_{\mu\nu}$. The action for the system of interacting gravitational field and massive spin 2 field and the Einstein equations for it are:

$$S = S_E + S_H, \qquad S_E = -\frac{1}{\kappa^{D-2}} \int d^D x \sqrt{-G} R,$$
$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = \kappa^{D-2} T^H_{\mu\nu}, \qquad T^H_{\mu\nu} = \frac{1}{\sqrt{-G}} \frac{\delta S_H}{\delta G^{\mu\nu}} \tag{60}$$

with S_H given by (52). However, making the metric dynamical we change the structure of the second derivatives by means of nonminimal terms $\sim RHH$ which can spoil causal propagation of both metric and massive spin 2 field [1]. This will impose extra restrictions on the parameters of the theory. Also, one can consider additional requirements the theory should fulfill, e.g. tree level unitarity of graviton - massive spin 2 field interaction [4].

5 Open string theory in background of massive spin 2 field

In this section we will consider sigma-model description of an open string interacting with two background fields – massless graviton $G_{\mu\nu}$ and second rank symmetric tensor field $H_{\mu\nu}$ from the first massive level of the open string spectrum. We will show that effective equations of motion for these fields are of the form (59) and explicitly calculate the coefficient ξ_3 in these equations in the lowest order in α'.

Classical action has the form

$$S = S_0 + S_I = \frac{1}{4\pi\alpha'} \int_M d^2 z \sqrt{g} g^{ab} \partial_a x^\mu \partial_b x^\nu G_{\mu\nu} + \frac{1}{2\pi\alpha' \mu} \int_{\partial M} edt\ H_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{61}$$

Here $\mu, \nu = 0, \ldots, D-1$; $a, b = 0, 1$ and we introduced the notation $\dot{x}^\mu = \frac{dx^\mu}{edt}$. The first term S_0 is an integral over two-dimensional string world sheet M with metric g_{ab} and the second S_I represents a one-dimensional integral over its boundary with einbein e. We work in euclidian signature and restrict ourselves to flat world sheets with straight boundaries. It means that both two-dimensional scalar curvature and extrinsic curvature of the world sheet boundary vanish and we can always choose such coordinates that $g_{ab} = \delta_{ab}$, $e = 1$.

Theory has two dimensionful parameters. α' is fundamental string length squared, D-dimensional coordinates x^μ have dimension $\sqrt{\alpha'}$. Another parameter μ carries dimension of inverse length in two-dimensional field theory (61) and plays the role of renormalization scale. It is introduced in (61) to make the background field $H_{\mu\nu}$ dimensionless. In fact, power of μ is responsible for the number of massive level to which a background field belongs because one expects that open string interacts with a field from n-th massive level through the term

$$\mu^{-n}(\alpha')^{-\frac{n+1}{2}}\int_{\partial M} edt\, \dot{x}^{\mu_1}\dots\dot{x}^{\mu_{n+1}}H_{\mu_1\dots\mu_{n+1}}(x)$$

The action (61) is non-renormalizable from the point of view of two-dimensional quantum field theory. Inclusion of interaction with any massive background produces in each loop an infinite number of divergencies and so requires an infinite number of different massive fields in the action. But massive modes from the $n-$th massive level give vertices proportional to μ^{-n} and so they cannot contribute to renormalization of fields from lower levels. Of course, this argument assumes that we treat the theory perturbatively defining propagator for X^μ only by the term with graviton in (61). Now we will use such a scheme to carry out renormalization of (61) dropping all the terms $O(\mu^{-2})$.

Varying (61) one gets classical equations of motion with boundary conditions:

$$\begin{aligned} &g^{ab}D_a\partial_b x^\alpha \equiv g^{ab}(\partial_a\partial_b x^\alpha + \Gamma^\alpha_{\mu\nu}(G)\partial_a x^\mu \partial_b x^\nu) = 0, \\ &G_{\mu\nu}\partial_n x^\mu|_{\partial M} - \frac{2}{\mu}\mathcal{D}_t^2 x^\mu H_{\mu\nu} + \frac{1}{\mu}\dot{x}^\mu\dot{x}^\lambda(\nabla_\nu H_{\mu\lambda} - \nabla_\mu H_{\nu\lambda} - \nabla_\lambda H_{\mu\nu}) = 0 \end{aligned} \qquad (62)$$

where $\partial_n = n^a\partial_a$, n^a is unit inward normal vector to the world sheet boundary and $\mathcal{D}_t^2 x^\mu = \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}(G)\dot{x}^\nu\dot{x}^\lambda$ is the derivative covariant with respect to D-dimensional diffeomorphisms.

To calculate divergencies of one-loop effective action $\Gamma^{(1)}_{div}$ we need to expand (61) up to the second order in normal coordinates

$$x^\mu = \bar{x}^\mu + \xi^\mu - \frac{1}{2}\Gamma^\mu_{\alpha\beta}\xi^\alpha\xi^\beta + O(\xi^3) \qquad (63)$$

around a solution $\bar{x}^\mu$ of classical equations of motion (62):

$$\begin{aligned} \Gamma^{(1)} &= \frac{1}{2}\mathrm{Tr}\ln\left(S_{0\alpha\beta} + \frac{1}{2\pi\alpha'\mu}V_{\alpha\beta}\right) \\ &= \frac{1}{2}\mathrm{Tr}\ln S_{0\alpha\beta} + \frac{1}{2\mu}\mathrm{Tr}V_{\gamma\alpha}G_0^{\gamma\beta} + O(\mu^{-2}) \\ S_{0\alpha\beta} &= \frac{\delta^2 S_0}{\delta\xi^\alpha\delta\xi^\beta}; \qquad \frac{1}{2\pi\alpha'\mu}V_{\alpha\beta} = \frac{\delta^2 S_I}{\delta\xi^\alpha\delta\xi^\beta} \end{aligned} \qquad (64)$$

Here Green function $G_0^{\gamma\beta}$ of the operator $S_{0\alpha\gamma}$ is defined as

$$2\pi\alpha' S_{0\alpha\gamma}G_0^{\gamma\beta} = \delta^\beta_\alpha$$

The terms $O(\mu^{-2})$ in (64) contribute to renormalization of the second and higher massive levels only and will be omitted from here on.

Explicit calculations give the following expressions for $V_{\alpha\beta}$ and $S_{0\alpha\beta}$

$$
\begin{aligned}
S_{0\alpha\beta}(z,z') &= -\frac{1}{2\pi\alpha'}\left(G_{\alpha\beta}g^{ab}D_aD_b + g^{ab}\partial_a x^\mu \partial_b x^\nu R_{\alpha\mu\beta\nu}\right)\delta(z,z') - \\
&\quad - \frac{1}{2\pi\alpha'}\delta_{\partial M}(z)G_{\alpha\beta}n^a D_a \delta(z,z') \\
V_{\alpha\beta}(z,z') &= \delta_{\partial M}(z)\sum_{k=0}^{2} V^{(k)}_{\alpha\beta}(z)(\mathcal{D}_t)^k\delta(z,z') \\
V^{(0)}_{\alpha\beta} &= -2\nabla_\beta H_{\mu\alpha}\mathcal{D}_t^2 x^\mu + \dot{x}^\mu\dot{x}^\nu\left(\nabla_\alpha\nabla_\beta H_{\mu\nu} - 2\nabla_\beta\nabla_\nu H_{\mu\alpha} - 2H_{\alpha\lambda}R^\lambda{}_{\mu\beta\nu}\right) \\
V^{(1)}_{\alpha\beta} &= 2\dot{x}^\mu\left(\nabla_\alpha H_{\mu\beta} - \nabla_\beta H_{\mu\alpha} - \nabla_\mu H_{\alpha\beta}\right) \\
V^{(2)}_{\alpha\beta} &= -2H_{\alpha\beta}
\end{aligned}
\tag{65}
$$

Here delta-function of the boundary $\delta_{\partial M}(z)$ is defined as

$$
\int_M \delta_{\partial M}(z)V(z)\sqrt{g(z)}d^2z = \int_{\partial M} V|_{z\in\partial M}e(t)dt \tag{66}
$$

Using dimensional regularization and following the procedure [49] one gets divergences of the Green function and its derivatives:

$$
\text{Tr }\ln S_{0;\alpha\beta}|^{div} = \frac{\mu^{-\varepsilon}}{2\pi\varepsilon}\int_M d^{2+\varepsilon}z\sqrt{g}g^{ab}\partial_a x^\mu\partial_b x^\nu R_{\mu\nu}
$$

$$
G^\alpha_{0\beta}\Big|^{div}_{\partial M} = -\frac{\mu^{-\varepsilon}}{\pi\varepsilon}\delta^\alpha_\beta \tag{67}
$$

$$
\mathcal{D}_t G^\alpha_{0\beta}\Big|^{div}_{\partial M} = 0 \tag{68}
$$

$$
\mathcal{D}_t^2 G^\alpha_{0\beta}\Big|^{div}_{\partial M} = -\frac{\mu^{-\varepsilon}}{2\pi\varepsilon}\, g^{ab}\partial_a x^\mu\partial_b x^\nu R^\alpha{}_{\mu\nu\beta}\Big|_{\partial M} \tag{69}
$$

As a result, divergent part of the one loop effective action has the form

$$
\begin{aligned}
\Gamma^{(1)}_{div} &= \frac{\mu^{-\varepsilon}}{4\pi\varepsilon}\int_M d^{2+\varepsilon}z\sqrt{g}g^{ab}\partial_a x^\mu\partial_b x^\nu R_{\mu\nu} \\
&\quad - \frac{\mu^{-\varepsilon-1}}{2\pi\varepsilon}\int_{\partial M} dt e(t)\dot{x}^\mu\dot{x}^\nu\left(\nabla^2 H_{\mu\nu} - 2R_\mu{}^\alpha H_{\alpha\nu} + R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta}\right)
\end{aligned}
\tag{70}
$$

which leads to one-loop renormalization of the background fields:

$$
\begin{aligned}
\overset{\circ}{G}_{\mu\nu} &= \mu^\varepsilon G_{\mu\nu} - \frac{\alpha'\mu^\varepsilon}{\varepsilon}R_{\mu\nu} \\
\overset{\circ}{H}_{\mu\nu} &= \mu^\varepsilon H_{\mu\nu} + \frac{\alpha'\mu^\varepsilon}{\varepsilon}\left(\nabla^2 H_{\mu\nu} - 2R^\sigma{}_{(\mu}H_{\nu)\sigma} + R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta}\right)
\end{aligned}
\tag{71}
$$

with circles denoting bare values of the fields. We would like to stress once more that higher massive levels do not influence the renormalization of any given field from the lower massive levels and so the result (71) represents the full answer for perturbative one-loop renormalization of $G_{\mu\nu}$ and $H_{\mu\nu}$.

Now to impose the condition of Weyl invariance of the theory at the quantum level we should calculate the trace of energy momentum tensor in $d = 2 + \varepsilon$ dimension:

$$T(z) = g_{ab}(z)\frac{\delta S}{\delta g_{ab}(z)} = \frac{\varepsilon\mu^{-\varepsilon}}{8\pi\alpha'}g^{ab}(z)\partial_a x^\mu \partial_b x^\nu G_{\mu\nu} - \frac{\mu^{-1-\varepsilon}}{4\pi\alpha'}H_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\delta_{\partial M}(z) \qquad (72)$$

performing one-loop renormalization of the composite operators $g^{ab}(z)\partial_a x^\mu \partial_b x^\nu G_{\mu\nu}$ and $H_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$.

Divergencies in vacuum expectations values of the operators

$$< H_{\mu\nu}\dot{x}^\mu\dot{x}^\nu >= \frac{\int \mathcal{D}x\, e^{-S[x]} H_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{\int \mathcal{D}x\, e^{-S[x]}} \qquad (73)$$

are calculated by standard background field method with the use of normal coordinates (63) as quantum fields.

In the one loop approximation it is enough to expand the composite operator and the action in (73) up the to second order in normal coordinates and we neglect the terms more than linear in $H_{\mu\nu}$ as giving only contributions to renormalization of higher massive levels:

$$\begin{aligned} < \dot{x}^\mu\dot{x}^\nu H_{\mu\nu} > \;&=\; \dot{\bar{x}}^\mu\dot{\bar{x}}^\nu H_{\mu\nu}(\bar{x}) + < \xi^\alpha\xi^\beta >_0 \dot{\bar{x}}^\mu\dot{\bar{x}}^\nu\Big(\frac{1}{2}\nabla_\alpha\nabla_\beta H_{\mu\nu}(\bar{x}) + R^\sigma{}_{\alpha\beta\mu}H_{\sigma\nu}(\bar{x})\Big) \\ &+ < \mathcal{D}_t\xi^\mu\mathcal{D}_t\xi^\nu >_0 H_{\mu\nu}(\bar{x}) + < \mathcal{D}_t\xi^\nu\xi^\alpha >_0 2\dot{\bar{x}}^\mu\nabla_\alpha H_{\mu\nu}(\bar{x}) \end{aligned} \qquad (74)$$

Here

$$< \xi^\alpha(z)\xi^\beta(z') >\equiv \frac{\int \mathcal{D}\xi\, e^{-\frac{1}{2}S_{0;\mu\nu}[\bar{x}]\xi^\mu\xi^\nu}\xi^\alpha(z)\xi^\beta(z')}{\int \mathcal{D}\xi\, e^{-\frac{1}{2}S_{0;\mu\nu}[\bar{x}]\xi^\mu\xi^\nu}} = 2\pi G_0^{\alpha\beta}(z,z') \qquad (75)$$

is Green function for the fields ξ^μ.

Using (69) one gets:

$$\begin{aligned} < \dot{x}^\mu\dot{x}^\nu H_{\mu\nu} > \;&=\; \mu^{-\varepsilon}\dot{\bar{x}}^\mu\dot{\bar{x}}^\nu\Big(H_{\mu\nu} + \frac{2}{\varepsilon}R^\alpha{}_\mu H_{\alpha\nu} - \frac{1}{\varepsilon}\nabla^2 H_{\mu\nu}\Big) \\ &- \frac{\mu^{-\varepsilon}}{\varepsilon}g^{ab}\partial_a\bar{x}^\mu\partial_b\bar{x}^\nu R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} + (fin) \end{aligned} \qquad (76)$$

Renormalized composite operator should have finite expectation value and so the renormalization should have the form (we use minimal subtraction scheme):

$$\begin{aligned} \Big(\dot{x}^\mu\dot{x}^\nu H_{\mu\nu}\Big)_0 \;&=\; \mu^{-\varepsilon}\Big[\dot{x}^\mu\dot{x}^\nu\Big(H_{\mu\nu} + \frac{2}{\varepsilon}R^\alpha{}_\mu H_{\alpha\nu} - \frac{1}{\varepsilon}\nabla^2 H_{\mu\nu})\Big] \\ &- \frac{\mu^{-\varepsilon}}{\varepsilon}\Big[\dot{x}^\mu\dot{x}^\nu R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta}\Big] + (fin) \end{aligned} \qquad (77)$$

where $(\ldots)_0$ and $[\ldots]$ stands for bare and renormalized operators respectively. Expressing bare values of background field $\overset{\circ}{H}_{\mu\nu}$ in terms of the renormalized ones (71) we finally see that in the lowest order in α' the operator does not receive any renormalization:

$$(\dot{x}^\mu \dot{x}^\nu \overset{\circ}{H}_{\mu\nu})_0 = \mu^{-\varepsilon}[\dot{x}^\mu \dot{x}^\nu H_{\mu\nu}] \tag{78}$$

After the same but more tedious calculations we get the following renormalization of another composite operator:

$$\begin{aligned}(g^{ab}\partial_a x^\mu \partial_b x^\nu \overset{\circ}{G}_{\mu\nu})_0 &= \mu^\varepsilon\Big[g^{ab}\partial_a x^\mu \partial_b x^\nu (G_{\mu\nu} - \frac{\alpha'}{\varepsilon}R_{\mu\nu})\Big] \\ &+ \frac{\alpha' \mu^{-1+\varepsilon}}{\varepsilon}\Big[H_\alpha{}^\alpha \delta''_{\partial M}(z) + \mathcal{D}_t^2 x^\mu (\nabla_\mu H_\alpha{}^\alpha - 4\nabla^\alpha H_{\alpha\mu})\delta_{\partial M}(z) \\ &+ \dot{x}^\mu \dot{x}^\nu (\nabla_\mu \nabla_\nu H_\alpha{}^\alpha - 4\nabla^\alpha \nabla_{(\mu} H_{\nu)\alpha} + 2\nabla^2 H_{\mu\nu} - 2R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta})\delta_{\partial M}(z)\Big]\end{aligned} \tag{79}$$

and build the renormalized operator of the energy momentum tensor trace:

$$\begin{aligned}8\pi[T] &= -\Big[g^{ab}\partial_a x^\mu \partial_b x^\nu E^{(0)}_{\mu\nu}(x)\Big] + \frac{2}{\mu}\delta_{\partial M}(z)\Big[\dot{x}^\mu \dot{x}^\nu E^{(1)}_{\mu\nu}(x)\Big] \\ &+ \frac{1}{\mu}\delta_{\partial M}(z)\Big[\mathcal{D}_t^2 x^\mu E^{(2)}_\mu(x)\Big] + \frac{1}{\mu}\delta''_{\partial M}(z)\Big[E^{(3)}(x)\Big]\end{aligned} \tag{80}$$

where

$$\begin{aligned}E^{(0)}_{\mu\nu}(x) &= R_{\mu\nu} + O(\alpha') \\ E^{(1)}_{\mu\nu}(x) &= \nabla^2 H_{\mu\nu} - \nabla^\alpha \nabla_\mu H_{\alpha\nu} - \nabla^\alpha \nabla_\nu H_{\alpha\mu} \\ &\quad - R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} + \frac{1}{2}\nabla_\mu \nabla_\nu H_\alpha{}^\alpha - \frac{1}{\alpha'}H_{\mu\nu} + O(\alpha') \\ E^{(2)}_\mu(x) &= \nabla_\mu H_\alpha{}^\alpha - 4\nabla^\alpha H_{\alpha\mu} + O(\alpha') \\ E^{(3)}(x) &= H_\alpha{}^\alpha + O(\alpha')\end{aligned} \tag{81}$$

Terms of order $O(\alpha')$ arise from the higher loops contributions.

The requirement of quantum Weyl invariance tells that all $E(x)$ in (81) should vanish and so they are interpreted as effective equations of motion for background fields. They contain vacuum Einstein equation for graviton (in the lowest order in α'), curved spacetime generalization of the mass shell condition for the field $H_{\mu\nu}$ with the mass $m^2 = (\alpha')^{-1}$ and $D+1$ additional constraints on the values of this fields and its first derivatives. Taking into account these constraints and the Einstein equation we can write our final equations arising from the Weyl invariance of string theory in the form:

$$\begin{aligned}&\nabla^2 H_{\mu\nu} + R_\mu{}^\alpha{}_\nu{}^\beta H_{\alpha\beta} - \frac{1}{\alpha'}H_{\mu\nu} + O(\alpha') = 0, \\ &\nabla^\alpha H_{\alpha\nu} + O(\alpha') = 0, \qquad H^\mu{}_\mu + O(\alpha') = 0, \\ &R_{\mu\nu} + O(\alpha') = 0.\end{aligned} \tag{82}$$

They coincide with the equations found in the previous section (59) with the value of non-minimal coupling $\xi_3 = -1$.

In fact, Einstein equations should not be vacuum ones but contain dependence on the field $H_{\mu\nu}$ through its energy - momentum tensor $T^H_{\mu\nu}$. Our calculations could not produce this dependence because such dependence is expected to arise only if one takes into account string world sheets with non-trivial topology and renormalizes new divergencies arising from string loops contribution [45, 47].

$$R_{\mu\nu} + O(\alpha') = T^H_{\mu\nu} - \frac{1}{D-2} T^{H\alpha}{}_{\alpha}, \tag{83}$$

where explicit form of the lowest contributions to the energy-momentum tensor $T^H_{\mu\nu}$ can be determined only from sigma model on world sheets with topology of annulus.

In order to determine whether the equations (82) can be deduced from an effective lagrangian (and to find this lagrangian) one would need two-loop calculations in the string sigma-model. Two-loop contributions to the Weyl anomaly coefficients $E^{(i)}$ are necessary because the effective equations of motion (81,82) are not the equations directly following from a lagrangian but some combinations of them similar to (58). In order to reverse the procedure of passing from the original lagrangian equations to (58) one would need the next to leading contributions in the conditions for $\nabla^\mu H_{\mu\nu}$ and $H_{\mu\nu}$ (82).

6 Conclusion

Let us summarize the obtained results. We investigated the problem of consistency of the equations of motion for spin-2 massive field in curved spacetime and found that two different description of this field are possible. First, for gravitational background satisfying vacuum Einstein equations (18) one can build an action leading to consistent equations including the tracelessness and transversality conditions. Of course, such gravitational backgrounds include all popular vacuum solutions of Einstein equations such as constant curvature spacetimes, Schwarzschild solution, plane waves etc. It would be interesting to investigate properties of the massive spin 2 field dynamics on these specific exact solutions.

Another possibility (naturally arising in string theory) consists in building the theory as perturbation series in inverse mass. In the lowest order no problems of consistency with the flat space limit and causality arise and equations of motion have the form (58).

Then we calculated the equations for the massive spin-2 background field arising in sigma model approach to string theory from the condition of quantum Weyl invariance in the lowest order in α'. The explicit form of the derived equations (82) appears to be a particular case of the general equations in field theory (58). We expect that in general in each order in α' the situation remains the same and it is possible to construct the part of string effective action quadratic in massive background field which should lead to generalized mass-shell, tracelessness and transversality conditions.

To determine this part of the bosonic string effective action completely one should also consider other massless background field including the dilaton $\phi(x)$ and antisymmetric tensor $B_{\mu\nu}(x)$. Inclusion of dilaton will require investigation of strings with curved world sheets and with non-vanishing extrinsic curvature on the boundary which complicates the sigma-model calculations. Interaction with massless antisymmetric tensor will be especially interesting in presence of a D-brane because such a system is a source of non-commutative geometry in string theory (see [54] and references therein). In the limit when components of $B_{\mu\nu}$ along the brane are large there should arise some non-commutative counterpart of the spin two massive field theory and we hope to derive its explicit form in the future.

Also it would be interesting to repeat our analysis in the case of closed string which contains the fourth rank tensor at the lowest massive level. From the field theoretical point of view investigation of such higher spin fields interacting with gravity will require to generalize analysis made in the Section 2 to the case of arbitrary spin fields whose dynamics is governed by more complex lagrangians with auxiliary fields [12]. We leave this investigation for the future work.

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