IFUSP/P39

TAMM-DANCOFF APPROACH TO THE RELATIVISTIC BOUND STATE PROBLEM FORMULATED ON THE LIGHT-LIKE PLANE $t + z = 0$

Heinz J. Rothe †
Instituto de Física
Universidade de São Paulo
and
Institut für Theoretische Physik
der Universität Heidelberg

Abstract

We discuss a formulation of the relativistic bound state problem for the case of a scalar hermitian field with a ϕ^3 interaction, which resembles very closely the second quantized formulation in a non-relativistic theory, with the light-like plane $t + z = \text{const.}$ replacing the usual equal-time surface, and with $P_o - P_z$ playing the role of the Hamiltonean in this formulation. Choosing as basis the eigenstates of $\vec{P}_\perp = (P_x, P_y)$ and $P_o + P_z$, where P_o is the usual Hamiltonean of the interacting system, one finds that the structure of this space is very similar to the Fock space of a non-relativistic theory, and that in fact non-relativistic Fock space methods can be used to derive coupled integral equations for the n-particle bound state wave functions. All calculations can be carried out with interacting fields. As an example we derive the integral equation for the two-body bound state wave function in the small coupling limit making a Tamm-Dancoff type approximation.

I-INTRODUCTION

There has been considerable interest for the past few years in the formulation of field theories on light-like planes [1-4] as an alternative to the infinite momentum frame technique introduced by Fubini and Furlan in connection with current algebra sum rules [5] and studied first by Weinberg in connection with old fashioned time ordered perturbation theory [6]. In fact it is well known that the current algebra sum rules obtained in the infinite momentum limit, can also be derived without actually going to this limit, if one assumes that the commutators of the current algebra type hold on light like planes $t+z = \text{const.}$ [7]. Furthermore, as has been first pointed out by Chang and Ma [1] , Weinberg's rules for computing the old fashioned time ordered perturbation theory diagrams of a ϕ^3 theory in the infinite momentum limit can be readily derived by reformulating covariant perturbation theory in terms of a new time variable $\tau = (t+z)/\sqrt{2}$ and a new z-variable $Z = (t-z)/\sqrt{2}$. In such a formulation the new rules for computing τ -ordered diagrams are just those derived by Weinberg for time-ordered perturbation theory in the limit $p_z \to \infty$. The point of view of Chang and Ma has been subsequently first used by Kogut and Soper [3] to study the formal foundations of quantum electrodynamics in the infinite momentum frame.

Because of this connection between the infinite momentum frame technique and the approach on light-like planes, one expects that a formulation of the bound state problem on a fixed light-like plane $t+z = \text{const.}$ (replacing the plane $t = \text{const.}$

in a non-relativistic theory) will lead to similar simplifications as those encountered in the infinite momentum limit where the vacuum structure of field theories simplifies considerably [6] and where the dynamical equations acquire a non-relativistic Schrödinger-type structure [8]. In fact it is because of the many special features of the infinite momentum frame, that this frame has proved to be a useful tool for discussing e.g. current algebra sum rules [9], parton models [10], quantum electrodynamical calculations [11] and many other topics of particle physics [12]. Alternatively the light-like plane has also been used extensively for a discussion of these problems [4].

In this paper we wish to take a look at the relativistic bound state problem formulated on the light-like plane $t+z = 0$. We shall show that the problem can be formulated in very close analogy to the non-relativistic case if one choses an appropriate set of basis states in which the bound state is expanded. These basis states are just the eigenstates of $\vec{P}_\perp = (P_x, P_y)$ and P_o+P_z, where P_o is the Hamiltonean of the interacting system, and can be easily constructed on light-like planes. The structure of the space spanned by these states is very similar to the non-relativistic Fock space, and in fact allows non-relativistic Fock space methods to be used for carrying out the calculations. All computations can be done using interacting fields. For simplicity we shall restrict ourselves to the case of a scalar hermitian field with a ϕ^3 interaction. As an example we derive the integral equation for the two-body wave function in the weak coupling limit using a Tamm-Dancoff approach [13]. A similar point of view has been advanced by Feldman, Fulton and Townsend [14] whose work we became aware of only after completion of this paper. These authors however do not place the same emphasis on the non-relativistic analogy, nor do they discuss the details of the calculations including renormalization effects. It is our main purpose to stay as close as possible to the non-relativistic case at every step of the calculations. Of course the price we have to pay for this is that the formulation is not manifestly Poincaré covariant so that the approximations made are frame dependent. This is a consequence of the fact that we shall be dealing with a one-"time" formalism, in contrast to the multi-time Bethe-Salpeter approach [15], which for one thing is very involved and also has the unpleasant feature of involving a relative time variable whose physical significance is unclear. It is our hope that the kind of "non-relativistic" approach to the relativistic bound state problem to be described in this paper will give us further insight into dynamics at small distances, and that it will eventually provide us with a simpler and more economical way for dealing with bound state problems.

The organization of the paper will be as follows: In section II we start with a brief review of the non-relativistic two-body problem in the second quantized formulation. In section III we then construct a set of basis states in close analogy to the non-relativistic Fock space states introduced in section II. The bound state problem for the case of a scalar field with ϕ^3 interaction is then formulated in this space in section IV, and an integral equation for the two-body wave function is derived in the small coupling limit using a Tamm-Dancoff approach. We conclude the paper with some remarks and an appendix where we give the Fock space represenation of the generators of the Poincaré group on the light-like plane and discuss the transformation properties of the fields.

II - THE NON-RELATIVISTIC BOUND STATE PROBLEM

In this section we take a brief look at the non-relativistic bound state problem in the second quantized formulation, placing special emphasis on those features which are peculiar to the non-relativistic theory. In the following sections we then give a relativistic formulation which resembles very much the non-relativistic treatement given here.

Let us consider a non-relativistic two body system interacting via a potential $V(\vec{x}_1-\vec{x}_2)$. The state of the system is then characterized by a wave function $\psi(\vec{x}_1,\vec{x}_2,t)$ describing the instantaneous spacial distribution of the 2 particles. This wave function is covariant with respect to homogeneous Galilei transformations. In the second quantized formulation $\psi(\vec{x}_1,\vec{x}_2,t)$ is given by the matrix element.

$$\psi(\vec{x}_1,\vec{x}_2,t) = \langle 0|\phi(\vec{x}_1,t)\phi(\vec{x}_2,t)|\psi\rangle \tag{2.1}$$

where $|0\rangle$ is the physical vacuum, $|\psi\rangle$ a Heisenberg state with the quantum numbers of the two particle state, and $\phi(x,t)$ is the field operator satisfying the equation of motion (we restrict ourselves to the scalar case):

$$\left(i\frac{\partial}{\partial t} + \frac{\vec{\nabla}^2}{2m} \right) \phi(\vec{x},t) = \mathcal{V}(\vec{x},t)\phi(\vec{x},t) \tag{2.2a}$$

where

$$\mathcal{V}(\vec{x},t) = \int d\vec{x}'\ \phi^{\dagger}(\vec{x}',t)\ \phi\ (\vec{x}',t)\ V\ (\vec{x}-\vec{x}') \tag{2.2b}$$

and where $\phi\ (\vec{x},t)$ satisfies the following equal time commutation relation

$$[\ \phi\ (\vec{x},t),\ \phi^{\dagger}(\vec{x}',t)\ ] = \delta(\ \vec{x}-\ \vec{x}') \tag{2.3}$$

The momentum operator and the Hamiltonean of the system are given by

$$\vec{P} = \int d\vec{x}\ \phi^{\dagger}\ (\vec{x},0)(-i\vec{\nabla})\phi(\vec{x},0) \tag{2.4}$$

and

$$\begin{aligned} H = & \int d\vec{x}\ \phi^{\dagger}(\vec{x},0)\ \left(-\frac{\vec{\nabla}^2}{2m}\right)\ \phi(\vec{x},0)\ + \\ & + \frac{1}{2}\int d\vec{x}'d\vec{x}''\ \phi^{\dagger}(\vec{x}',0)\phi^{\dagger}(\vec{x}'',0)V(\vec{x}'-\vec{x}'')\phi(\vec{x}',0)\phi(\vec{x}'',0) \end{aligned} \tag{2.5}$$

where we have set $t=o$ since the integrals are independent of t. Introducing the Fourier transform of the field $\phi(\vec{x},0)$

$$a(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d\vec{x}\ e^{-i\vec{k}.\vec{x}}\ \phi(\vec{x},0) \tag{2.6}$$

with

$$[\ a(\vec{k}),\ a^{\dagger}(\vec{k}')\ ] = \delta(\ \vec{k}-\vec{k}') \tag{2.7}$$

(2.4) and (2.5) take the form

$$\vec{P} = \int d\vec{k}\ \vec{k}\ a^{\dagger}(\vec{k})\ a(\vec{k}) \tag{2.8}$$

$$H = \int d\vec{k}\ \frac{\vec{k}^2}{2m}\ a^{\dagger}(\vec{k})\ a(\vec{k}) + \frac{1}{2(2\pi)^{3/2}} \int d\vec{k}d\vec{k}'d\vec{k}''a^{\dagger}(\vec{k}')a^{\dagger}(\vec{k}'')a(\vec{k}'-\vec{k}) . .a(\vec{k}''+\vec{k})V(\vec{k}) \tag{2.9}$$

where

$$V(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d\vec{x}\ e^{i\vec{k}.\vec{x}}\ V(\vec{x})$$

A special property of non-relativistic theories is that the bare and physical vacuum states coincide; thus the operator (2.6) annihilates the physical vacuum:

$$a(\vec{k})|\ 0 >= 0 \tag{2.10}$$

Introducing the n-particle Fock space states in the usual way

$$|\vec{k}_1,\ldots,\vec{k}_n> = \frac{1}{\sqrt{n!}}\ a^{\dagger}(\vec{k}_1)\ldots a^{\dagger}(\vec{k}_n)\ |0> \tag{2.11}$$

a two particle eigenstate of $\vec{P}$ and H with eigenvalues $\vec{k}$ and E will have the form

$$|k> = \frac{1}{2!}\int d\vec{k}_1 d\vec{k}_2\ \delta(\vec{k}-\vec{k}_1-\vec{k}_2) f_E(\vec{k}_1,\vec{k}_2) a^{\dagger}(\vec{k}_1) a^{\dagger}(\vec{k}_2)|0> \tag{2.12}$$

(2.4) and (2.5) take the form

$$\vec{P} = \int d\vec{k}\ \vec{k}\ a^{\dagger}\ (\vec{k})\ a\ (\vec{k}) \tag{2.8}$$

$$H = \int d\vec{k}\ \frac{\vec{k}^2}{2m}\ a^{\dagger}\ (\vec{k})\ a\ (\vec{k}) + \frac{1}{2(2\pi)^{3/2}} \int d\vec{k}d\vec{k}'d\vec{k}''a^{\dagger}(\vec{k}')a^{\dagger}(\vec{k}'')a(\vec{k}'-\vec{k}).\ .a(\vec{k}''+\vec{k})V(\vec{k}) \tag{2.9}$$

where

$$V(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d\vec{x}\ e^{i\vec{k}.\vec{x}}\ V(\vec{x})$$

A special property of non-relativistic theories is that the bare and physical vacuum states coincide; thus the operator (2.6) annihilates the physical vacuum:

$$a(\vec{k})|\ 0 >= 0 \tag{2.10}$$

Introducing the n-particle Fock space states in the usual way

$$|\vec{k}_1,\ldots,\vec{k}_n> = \frac{1}{\sqrt{n!}}\ a^{\dagger}(\vec{k}_1)\ldots a^{\dagger}(\vec{k}_n)\ |0> \tag{2.11}$$

a two particle eigenstate of $\vec{P}$ and H with eigenvalues $\vec{k}$ and E will have the form

$$|k> = \frac{1}{2!} \int d\vec{k}_1 d\vec{k}_2\ \delta(\vec{k}-\vec{k}_1-\vec{k}_2)f_E(\vec{k}_1,\vec{k}_2)a^{\dagger}(\vec{k}_1)a^{\dagger}(\vec{k}_2)|0> \tag{2.12}$$

(2.4) and (2.5) take the form

$$\vec{P} = \int d\vec{k}\ \vec{k}\ a^{\dagger}(\vec{k})\ a(\vec{k}) \tag{2.8}$$

$$H = \int d\vec{k}\ \frac{\vec{k}^2}{2m}\ a^{\dagger}(\vec{k})\ a(\vec{k}) + \frac{1}{2(2\pi)^{3/2}} \int d\vec{k}d\vec{k}'d\vec{k}''a^{\dagger}(\vec{k}')a^{\dagger}(\vec{k}'')a(\vec{k}'-\vec{k}).\ .a(\vec{k}''+\vec{k})V(\vec{k}) \tag{2.9}$$

where

$$V(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d\vec{x}\ e^{i\vec{k}.\vec{x}}\ V(\vec{x})$$

A special property of non-relativistic theories is that the bare and physical vacuum states coincide; thus the operator (2.6) annihilates the physical vacuum:

$$a(\vec{k})|\ 0 >= 0 \tag{2.10}$$

Introducing the n-particle Fock space states in the usual way

$$|\vec{k}_1,\ldots,\vec{k}_n> = \frac{1}{\sqrt{n!}}\ a^{\dagger}(\vec{k}_1)\ldots a^{\dagger}(\vec{k}_n)\ |0> \tag{2.11}$$

a two particle eigenstate of $\vec{P}$ and H with eigenvalues $\vec{k}$ and E will have the form

$$|k> = \frac{1}{2!}\int d\vec{k}_1 d\vec{k}_2\ \delta(\vec{k}-\vec{k}_1-\vec{k}_2)f_E(\vec{k}_1,\vec{k}_2)a^{\dagger}(\vec{k}_1)a^{\dagger}(\vec{k}_2)|0> \tag{2.12}$$

where $k = (E,\vec{k})$ [Since the interaction conserves the particle number, the eigenvalue problem may be formulated entirely within the two-particle subspace]. Because of the commutation relation

$$[\, a(\vec{k}),\ \vec{P}\,] = \vec{k}a(\vec{k}) \tag{2.13}$$

the state (2.12) is an eigenstate of $\vec{P}$. In order that it also be an eigenstate of H, i.e

$$H \mid \vec{k},E> = E \mid \vec{k},E> \tag{2.14}$$

$f_E\ (\vec{k}_1,\vec{k}_2)$ must satisfy an integral equation. To derive this equation we consider the matrix element

$$\delta(\vec{k}-\vec{k}_1-\vec{k}_2)f_E(\vec{k}_1,\vec{k}_2) = <0|a(\vec{k}_1)a(\vec{k}_2)|\vec{k},E> \tag{2.15}$$

and make use of eq. (2.14) to obtain the following relation

$$E<0|a(\vec{k}_1)a(\vec{k}_2)\ |\vec{k},E> = <0|[a(\vec{k}_1),H]\ a(\vec{k}_2)\ |\vec{k},E> + + <0|a(\vec{k}_1)[a(\vec{k}_2),\ H]|\ \vec{k},E> \tag{2.16}$$

Substituting for the commutator the expression obtained from eq. (2.9) and (2.7) and making use of the fact that $a(\vec{k})$ annihilates the physical vacuum, one arrives at the following integral equation for the 2-particle wave function:

$$\left(E-\frac{\vec{k}_1^2}{2m}-\frac{(\vec{k}-\vec{k}_1)^2}{2m}\right) f_E(\vec{k}_1,\vec{k}-\vec{k}_1) = \frac{1}{(2\pi)^{3/2}}\int d\vec{k}_1'\ V(\vec{k}_1-\vec{k}_1')f_E(\vec{k}_1',\vec{k}-\vec{k}_1') \tag{2.17}$$

As we shall see in the following sections, the relativistic bound state problem may be treated in a manner very similar to the one just presented with the light like plane $t+z=0$ replacing the surface $t=0$. However, while (2.17) is an exact integral equation for the 2-body Schrödinger wave function we shall of course be forced to make approximations when dealing with relativistic interactions since the particle number is no longer conserved.

III - MATHEMATICAL PRELIMINARIES

In this section we present some mathematical preliminaries which we shall need for a formulation of the relativistic bound state problem in close analogy to the non relativistic case. For simplicity we shall restrict ourselves to a neutral scalar field with a ϕ^3 interaction. The corresponding Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \, \partial_\mu \phi - \frac{1}{2} m_o^2 \, \phi^2 - \frac{\lambda}{3!} \phi^3 . \qquad (3.1)$$

When dealing with a relativistic theory we are immediately confronted with a number of complications which are absent in the non relativistic case:

a) the interaction does not conserve the number of particles, so that the Hilbert space no longer separates into invariant subspaces of definite particle number.

b) the requirement of Galilean covariance of the theory is replaced by Poincaré covariance. Hence a wave function describing the instantaneous spacial distribution of the particles is not a covariant object.

c) creation and annihilation operators analogous to those given by eq. (2.6) and its adjoint cannot be defined on the plane $t = \text{const.}$ if $\phi(x)$ is an interacting field.

Since we are interested in formulating the relativistic bound state problem in as close a way as possible to the non-relativistic case, we shall want to discuss a one-"time" formalism, where "time" however will not be the usual laboratory time. This of course means that the formulation will not be manifestly

Poincaré covariant. Furthermore, we shall want to work with operators analogous to those given by eq. (2.6) defined on the new equal "time" surface, having the property that they annihilate the physical vacuum. Such operators can be readily constructed if one choses as the new "time" coordinate the variable [2]

$$\tau=\frac{t+z}{2} \tag{3.2}$$

To this effect consider the operator

$$a(\vec{k}_\perp,\eta,\tau)=\frac{i}{(2\pi)^{3/2}}\int_{n.x=2\tau} d\sigma\; e^{\,i\eta\xi-i\vec{k}_\perp.\vec{x}_\perp}\, n^\mu \overleftrightarrow{\partial}_\mu \phi(\tau,\xi,\vec{x}_\perp) \tag{3.3a}$$

where the integral extends over the light like plane $\tau=$ const. with normal 4-vector

$$n^\mu = (1,0,0,-1) \tag{3.3b}$$

and surface element

$$d\sigma = d\vec{x}_\perp \; d\xi \tag{3.3c}$$

with

$$\vec{x}_\perp = (x,y) \tag{3.3d}$$

and

$$\xi = \frac{t-z}{2} \tag{3.3e}$$

In fact (3.3a) is (appart from a factor 2η) just the Fourier transform of the field $\phi(x)$ with respect to the variables $\vec{x}_\perp$ and ξ in the plane τ = const., since $\eta^\mu\partial_\mu=\partial/\partial\xi$, so that the derivative acts within the surface.* Thus the operator (3.3a) is defined in a completely analogous way to the non-relativistic operator $a(\vec{k})$ [cf.eq. (2.6)] which is given by the Fourier transform of the field with respect to the usual space variables appropriate to the plane t=const. However whereas $a^+(\vec{k})$ when applied to the physical vacuum creates a state of definite 3-momentum $\vec{k}$, the operator $a^+(\vec{k}_\perp,\eta,\tau)$ acting on the vacuum will yield an eigenstate of $\vec{P}_\perp=(P_x,P_y)$ and P_o+P_z, (where P_o is the total Hamiltonean of the interacting system) with eigenvalues $\vec{k}_\perp$ and η respectively, if the field $\phi(x)$ transforms in the usual way under space-time translations. The analogy with the non-relativistic annihilation operators (2.6) is now completed by noticing that the operator (3.3a) annihilates the physical vacuum for $\eta > 0$ i.e

$$a(\vec{k}_\perp,\eta,\tau)|0\rangle=0 \qquad \text{for} \qquad \eta>0 \tag{3.4}$$

which may be readily verified by making use of the translational invariance of the physical vacuum and the completeness of the physical particle spectrum.

In terms of (3.3a) and its adjoint, the (hermitian) field $\phi(x)$ can be expanded as follows:

$$\phi(\tau,\xi,\vec{x}_\perp)=\frac{1}{(2\pi)^{3/2}}\int d\vec{k}_\perp\int_0^\infty \frac{d\eta}{2\eta}\left[a(\vec{k}_\perp,\eta,\tau)e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}+a^+(\vec{k}_\perp,\eta,\tau)e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\right] \tag{3.5}$$

where $\underset{\sim}{k}\cdot\underset{\sim}{x} = \eta\xi-\vec{k}_\perp\cdot\vec{x}_\perp$. Hence (3.5) gives the decomposition of the interacting field in terms of creation and annihilation operators on the light like plane τ = const. The equal τ - commutation relations among scalar fields with non-derivative coupling have been studied in detail by Chang, Root and Yan [4] using Schwinger's action principle. They find them to be the same as those for free fields restricted to the light like plane:

$$[\phi(x),\ \phi(x')]_{\tau=\tau'} = -\frac{i}{4}\,\varepsilon(\xi-\xi')\ \delta\ (\vec{x}_\perp - \vec{x}_\perp'\ ) \tag{3.6}$$

where

$$\varepsilon(\xi) = +1 \text{ for } \xi>0 \text{ and } \varepsilon(\xi) = -1 \quad \text{for } \xi<0.$$

The corresponding commutator for the operator (3.3a) and its adjoint then can be shown to read

$$[a(\vec{k}_\perp,\eta,\tau),\ a^\dagger\ (\vec{k}_\perp',\eta',\tau)] = 2\eta\ \delta(\eta-\eta')\ \delta\ (\vec{k}_\perp-\vec{k}_\perp') \tag{3.7}$$

with all other commutators vanishing.

Let us define $a(\vec{k}_\perp,\eta) \equiv a(\vec{k}_\perp,\eta,0)$. Written in terms of the variables (3.3) we have that

$$a(\vec{k}_\perp,\eta) = \frac{i}{(2\pi)^{3/2}} \int d\xi d\vec{x}_\perp\ \ e^{\,i\eta\xi - i\vec{k}_\perp\cdot\vec{x}_\perp}\ \overleftrightarrow{\partial}_\xi\ \phi(x) \tag{3.8}$$

where $\partial_\xi = \partial/\partial_\xi$. With the help of these operators we now introduce a set of basis states in the same way as was done in the non-relativistic case:

$$|\vec{k}_{1\perp},\eta_1;\ldots;\vec{k}_{n\perp},\eta_n\ > = \frac{1}{\sqrt{n!}}\ \ a^\dagger(\vec{k}_{1\perp},\eta_1)\ldots a^\dagger(\vec{k}_{n\perp},\eta_n)\,|0> \tag{3.9}$$

These states are eigenstates of $\vec{P}_\perp = (P_x, P_y)$ and

$$P_\xi = P_o + P_Z \tag{3.10}$$

with eigenvalues $\sum \vec{k}_{i_\perp}$ and $\sum \eta_i$ respectively, as follows from the commutation relations

$$\left[a(\vec{k}_\perp,\eta),\, P_\xi\right] = \eta\, a(\vec{k}_\perp,\eta) \tag{3.11}$$

$$\left[a(\vec{k}_\perp,\eta),\, \vec{P}_\perp\right] = \vec{k}_\perp a(\vec{k}_\perp,\eta) \tag{3.12}$$

derived from eqs. (A.10b,c) of the appendix. An important point which should be noted about the states (3.9), is that because they are defined on the plane $\tau = 0$, the space spanned by the vectors for fixed n is only invariant with regard to those transformations which do not change the surface $\tau=0$. The corresponding generators are discussed in the appendix to this paper, where we also give explicit expressions for the generators of the Poincaré group on the plane τ=const. and discuss the transformation properties of the field $\phi(x)$.

Assuming that the states (3.9) together with the vacuum state form a complete set, we now can proceed to the formulation of the relativistic bound state problem for the case of a theory described by the Lagrangian (3.1).

These states are eigenstates of $\vec{P}_\perp = (P_x, P_y)$ and

$$P_\xi = P_o + P_Z \tag{3.10}$$

with eigenvalues $\sum \vec{k}_{i_\perp}$ and $\sum \eta_i$ respectively, as follows from the commutation relations

$$[a(\vec{k}_\perp, \eta), P_\xi] = \eta\, a(\vec{k}_\perp, \eta) \tag{3.11}$$

$$[a(\vec{k}_\perp, \eta), \vec{P}_\perp] = \vec{k}_\perp a(\vec{k}_\perp, \eta) \tag{3.12}$$

derived from eqs. (A.10b,c) of the appendix. An important point which should be noted about the states (3.9), is that because they are defined on the plane $\tau = 0$, the space spanned by the vectors for fixed n is only invariant with regard to those transformations which do not change the surface $\tau=0$. The corresponding generators are discussed in the appendix to this paper, where we also give explicit expressions for the generators of the Poincaré group on the plane τ=const. and discuss the transformation properties of the field $\phi(x)$.

Assuming that the states (3.9) together with the vacuum state form a complete set, we now can proceed to the formulation of the relativistic bound state problem for the case of a theory described by the Lagrangian (3.1).

IV. FORMULATION OF THE RELATIVISTIC BOUND STATE PROBLEM

In this section we wish to solve the eigenvalue equation

$$P^2 | B> = M^2 |B> \tag{4.1}$$

where $P^2 = P_o^2 - \vec{P}^2$, and M is the mass of the composite system characterized by the state $|B>$. Introducing the operator

$$P_\tau = P_o - P_z \tag{4.2}$$

conjugate to the variable $\tau = (t+z)/2$, which hence plays the role of a Hamiltonean in the present formulation, we have that

$$P^2 = P_\tau P_\xi - \vec{P}_\perp^2$$

where P_ξ is defined by eq. (3.10). Hence the eigenvalue problem (4.1) is equivalent to solving the following set of eigenvalue equations:

$$\vec{P}_\perp |\vec{k}_\perp, \eta, \varepsilon> = \vec{k}_\perp |\vec{k}_\perp, \eta, \varepsilon> \tag{4.3}$$

$$P_\xi |\vec{k}_\perp, \eta, \varepsilon> = \eta |\vec{k}_\perp, \eta, \varepsilon> \tag{4.4}$$

$$P_\tau |\vec{k}_\perp, \eta, \varepsilon> = \varepsilon |\vec{k}_\perp, \eta, \varepsilon> \tag{4.5a}$$

where

$$\varepsilon = \frac{k_\perp^2 + M^2}{\eta} \tag{4.5b}$$

The Fock space representations of the generators $\vec{P}_\perp$, P_ξ and P_τ for the case of the Lagrangian (3.1) are readily derived from eqs. (A.7a,b,c) and (A.6) of the appendix:

$$\vec{P}_\perp = \int d\vec{k}_\perp \int_0^\infty \frac{d\eta}{2\eta} \vec{k}_\perp a^\dagger(\vec{k}_\perp,\eta)\, a\,(\vec{k}_\perp,\eta) \qquad (4.6)$$

$$P_\xi = \int d\vec{k} \int_0^\infty \frac{d\eta}{2\eta}\eta\; a^\dagger(\vec{k}_\perp,\eta)\, a\,(\vec{k}_\perp,\eta) \qquad (4.7)$$

$$P_\tau = \int d\vec{k}\int_0^\infty \frac{d\eta}{2\eta}\left(\frac{\vec{k}_\perp^2+m_o^2}{\eta}\right) a^\dagger(\vec{k}_\perp,\eta)\, a\,(\vec{k}_\perp,\eta) + \qquad (4.8)$$

$$+\frac{\lambda}{(2\pi)^{3/2}}\left\{\int d\vec{k}_\perp'\, d\vec{k}_\perp'' \int_0^\infty \frac{d\eta}{2\eta'}\,\frac{d\eta''}{2\eta''}\,\frac{1}{2(\eta'+\eta'')} a^\dagger(\vec{k}_\perp',\eta')a^\dagger(\vec{k}_\perp'',\eta'')\cdot\right.$$

$$\left.\cdot a(\vec{k}_\perp'+\vec{k}_\perp'',\eta'+\eta'')+\text{h.c.}\right\}$$

where "h.c." stands for " hermitian conjugate".

Now the most general eigenstate of $\vec{P}_\perp$, P_ξ and P_τ in the space spanned by the vectors (3.9) can be written in the form

$$|\vec{k}_\perp,\eta,\varepsilon> = \sum_n \frac{1}{n!}\int d\vec{k}_{1_\perp}\dots d\vec{k}_{n_\perp}\int_0^\infty \frac{d\eta_1}{2\eta_1}\dots\frac{d\eta_n}{2\eta_n}\,\delta(\vec{k}_\perp-\sum_{i=1}^n \vec{k}_{i\perp}).$$

$$(4.9)$$

$$.\delta(\eta-\sum_{i=1}^n \eta_i)\; f_n^{(M^2)}(\vec{k}_{1_\perp},\eta_1;\dots;\vec{k}_{n_\perp},\eta_n)\; a^\dagger(\vec{k}_{1_\perp},\eta_1)\dots a^\dagger(\vec{k}_{n_\perp},\eta_n)|0>$$

where the wave functions $f_n^{(M^2)}$ have to satisfy certain integral equations if (4.9) is to be an eigenstate of P_τ. For convenience we have chosen to label these functions by the eigenvalue of P^2, rather than by ε. Let us define

$$F^{(M^2)}_{n;\vec{k}_\perp,\eta}(\vec{k}_{1_\perp},\eta_1;\ldots;\vec{k}_{n_\perp},\eta_n) = \delta(\vec{k}_\perp-\sum_{i=1}^{n}\vec{k}_{i_\perp})\ \delta(\eta-\sum_{i=1}^{n}\eta_i)\ f_n^{(M^2)}(\vec{k}_{1_\perp},\eta_1;$$

$$\ldots;\vec{k}_{n_\perp},\eta_n) = \langle 0\ |a(\vec{k}_{1_\perp},\eta_1)\ldots a(\vec{k}_{n_\perp},\eta_n)\ |\vec{k}_\perp,\eta,\varepsilon\rangle \qquad (4.10)$$

The right hand side of eq. (4.10) actually depends on η and η_i, only through their ratios

$$\frac{\eta_i}{\eta} \equiv x_i \qquad (4.11)$$

This follows directly from the transformation properties of the operators (3.3a) under boosts along the z-direction [see eq. (A.12) of the appendix] and the fact that the corresponding generator annihilates the physical vacuum. Hence we may write

$$\begin{aligned} F^{(M^2)}_{n;\vec{k}_\perp\eta}(\vec{k}_{1\perp},\eta_1;\ldots;\vec{k}_{n\perp},\eta_n) &= F^{(M^2)}_{n,\vec{k}_\perp}(\vec{k}_{1_\perp}, x_1;\ldots;\vec{k}_{n_\perp},x_1) \\ &= \delta(\vec{k}_\perp-\sum_{i=1}^{n}\vec{k}_{i\perp})\ \delta\ (1-\sum_{i=1}^{n} x_i)\ f_n^{(M^2)}(\vec{k}_{1\perp},x_1;\ldots;\vec{k}_{n\perp},x_n) \end{aligned} \qquad (4.12)$$

(for convenience we have used the same symbols for the functions appearing on the right hand side). The equations which the amplitudes (4.10) have to satisfy are derived in exactly the same way as was done in the non-relativistic case. However now we shall obtain an infinite set of coupled equations for the n-particle wave functions since the interaction does not conserve the particle number. Thus proceeding as in Section II we find that (4.5) leads to the following relations:

$$\varepsilon\langle 0|\ a(\vec{k}_{1\perp},\eta_1)\ldots.a(\vec{k}_{n\perp},\eta_n)\ |\vec{k}_\perp,\eta,\varepsilon\rangle$$

$$= \sum_{i=1}^{n} \langle 0|a(\vec{k}_{1\perp},\eta_1)\ldots\ [a(\vec{k}_{i\perp},\eta_i),P_\tau]\ldots.a(\vec{k}_{n\perp},\eta_n)|\vec{k}_\perp,\eta,\varepsilon\rangle \qquad (4.13)$$

From the structure of the commutator

$$[a(\vec{k}_\perp,\eta),\ P_\tau] = \left(\frac{\vec{k}_\perp^{\,2}+m_o^2}{\eta}\right) a(\vec{k}_\perp,\eta)$$

$$+ \frac{2\lambda}{(2\pi)^{3/2}} \int d\vec{k}'_\perp \int_o^\infty \frac{d\eta'}{4\eta'(\eta+\eta')}\ a^\dagger(\vec{k}'_\perp,\eta')\ a\ (\vec{k}_\perp+\vec{k}'_\perp,\eta+\eta'\ ) + \qquad (4.14)$$

$$+ \frac{\lambda}{(2\pi)^{3/2}} \int d\vec{k}'_\perp \int_o^\eta \frac{d\eta'}{4\eta'\ (\eta-\eta')}\ a(\vec{k}'_\perp,\eta')\,a\ (\ \vec{k}_\perp-\vec{k}'_\perp,\eta-\eta')$$

which follows from eqs.(4.8) and (3.7), we see that (4.13) yields an infinite set of coupled equations relating the n-1,n, and n+1particle wave functions.** To solve them we shall therefore have to cut off this chain at some value of n. Since we are interested in two - particle bound states it seems reasonable to break up the set of equations (4.13) at n=3, whereby for n=3 the right hand side of (4.13) is approximated by the contributions coming from the two and three-particle wave functions. But before writing down the closed set of equations obtained in this approximation, one other technical point must be mentioned which we did not have to bother with in the non-relativistic case. Since in solving the coupled set of equations we expect to encounter divergencies which require mass renormalization, we shall introduce the physical mass m of the one particle eigen-states in the usual way by setting

$$m^2 = m_o^2 + \delta m^2 \tag{4.15}$$

and consider the term proportional to δm^2 as part of the interaction

$$V = -\delta m^2 \int d\vec{k}_\perp \int_o^\infty \frac{d\eta}{2\eta} \frac{1}{\eta} a^+(\vec{k}_\perp,\eta) a(\vec{k}_\perp,\eta) + V_I \tag{4.16}$$

where V_I is given by the term proportional to λ in eq.(4.8). With this modification we obtain the following set of coupled equations upon neglecting all contributions with $n>3$, and making use of the relation (4.10) and (4.12):

$$\begin{aligned}(\vec{k}^2+M^2 - \sum_{i=1}^{3} \frac{\vec{k}_i^2+m^2}{x_i}) F_3^{(M^2)} (\vec{k}_1,x_1;\vec{k}_2,x_2;\vec{k}_3,x_3) \\ \approx \frac{\lambda}{(2\pi)^{3/2}} \frac{1}{x_1+x_2} F_2^{(M^2)}(\vec{k}_1+\vec{k}_2,x_1+x_2;\vec{k}_3,x_3) \\ + \frac{\lambda}{(2\pi)^{3/2}} \frac{1}{x_2+x_3} F_2^{(M^2)}(\vec{k}_1,x_1;\vec{k}_2+\vec{k}_3,x_2+x_3) \\ + \frac{\lambda}{(2\pi)^{3/2}} \frac{1}{x_1+x_3} F_2^{(M^2)}(\vec{k}_1+\vec{k}_3,x_1+x_3;\vec{k}_2,x_2)\end{aligned} \tag{4.17}$$

$$(\vec{k}^2+M^2-\sum_{i=1}^{2}\frac{\vec{k}_i^2+m^2}{x_i})F_2^{(M^2)}(\vec{k}_1,x_1;\vec{k}_2,x_2)=\frac{\lambda}{(2\pi)^{3/2}}\frac{1}{x_1+x_2}F_1^{(M^2)}(\vec{k}_1+\vec{k}_2;x_1+x_2)$$

$$+\frac{\lambda}{(2\pi)^{3/2}}\int d\vec{k}'\int_0^{x_1}\frac{dx'}{4x'(x_1-x')}F_3^{(M^2)}(\vec{k}',x';\vec{k}_1-\vec{k}',x_1-x';\vec{k}_2,x_2)$$

$$+\frac{\lambda}{(2\pi)^{3/2}}\int d\vec{k}'\int_0^{x_2}\frac{dx'}{4x'(x_2-x')}F_3^{(M^2)}(\vec{k}',x';\vec{k}_2-\vec{k}',x_2-x';\vec{k}_1,x_1) \tag{4.18}$$

$$-\delta m^2(\frac{1}{x_1}+\frac{1}{x_2})F_2^{(M^2)}(\vec{k}_1,x_1;\vec{k}_2,x_2)$$

and

$$(M^2-m^2)f_1^{(M^2)}(\vec{k})=-\delta m^2 f_1^{(M^2)}(\vec{k})+\frac{\lambda}{(2\pi)^{3/2}}\int d\vec{k}'\int_0^1\frac{dx'}{4x'(1-x')}f_2^{(M^2)}(\vec{k}',x';\vec{k}-\vec{k}',1-x') \tag{4.19}$$

We have chosen to write the last equation in terms of the amplitudes $f_i^{(M^2)}$ rather than $F_i^{(M^2)}$ [cf.eq. (4.12)] and have dropped for simplicity all subscripts Labeling the distribution functions as well as the "transverse" Label on $\vec{k}_i$. Notice that a term proportional to δm^2 has not been included in the approximate equation for the three-particle amplitude since this term is needed to cancel the divergencies which arise when taking the four-particle contributions into account. We like to point out that (4.17) is the only approximation we have made and that the remaining eqs. (4.18) and (4.19) are exact. In deriving these equations use has been made of the fact that the operators $a(\vec{k}_\perp,\eta)$ annihilate the physical vacuum. In figs. 1, 2, and 3 the above equations are displayed in graphical form.

Substituting the expression for $F_3^{(M^2)}$ obtained from eq. (4.17) into the integrand of (4.18), we find that

$$\left(\vec{k}^2+M^2-\sum_{i=1}^{2}\frac{\vec{k}_i^2+m^2}{x_i}\right)F_2^{(M^2)}(\vec{k}_1,x_1;\vec{k}_2,x_2) = \frac{\lambda}{(2\pi)^{3/2}}\frac{1}{x_1+x_2}F_1^{(M^2)}(\vec{k}_1+\vec{k}_2;x_1+x_2)$$

$$+\frac{\lambda^2}{2(2\pi)^3}\left\{\int d\vec{k}'\int_0^{x_1}\frac{dx'}{x'(x_1-x')(x_1+x_2-x')}\frac{F_2^{(M^2)}(\vec{k}_1+\vec{k}_2-\vec{k}',x_1+x_2-x';\vec{k}',x')}{\left[\vec{k}^2+M^2-\frac{\vec{k}'^2+m^2}{x'}-\frac{\vec{k}_2^2+m^2}{x_2}-\frac{(\vec{k}_1-\vec{k}')^2+m^2}{x_1-x'}\right]}\right.$$

$$\left.+\ \vec{k}_1\leftrightarrow\vec{k}_2,\ x_1\leftrightarrow x_2\right\}-$$

$$-\left[\frac{1}{x_1}\left\{\delta m^2-\frac{\lambda^2}{(2\pi)^3}\int d\vec{k}'\int_0^{x_1}dx'\frac{1}{4x'(x_1-x')}\frac{1}{\left[\vec{k}^2+M^2-\frac{\vec{k}'^2+m^2}{x'}-\frac{\vec{k}_2^2+m^2}{x_2}-\frac{(\vec{k}_1-\vec{k}')^2+m^2}{x_1-x'}\right]}\right\}\right.$$

$$\left.+\ \vec{k}_1\leftrightarrow\vec{k}_2,\ x_1\leftrightarrow x_2\right]F_2^{(M^2)}(\vec{k}_1,x_1;\vec{k}_2,x_2)$$

$$(4.20)$$

Notice that the integral within curly brackets multiplying

$F_2^{(M^2)}(\vec{k}_1,x_1;\vec{k}_2,x_2)$ diverges. In fig.(4) we have displayed this equation in graphical form..

We now would like to show that the divergent integral in eq. (4.20) combines with δm^2 to give in fact a finite term proportional to $\lambda^2 K_2$, where

$$K_2 = \vec{k}^2+M^2-\sum_{i=1}^{2}\frac{\vec{k}_i^2+m^2}{x_i} \tag{4.21}$$

and hence may be neglected for sufficiently small λ.***
To see this we determine δm^2 by considering the corresponding eigenvalue problem for the physical one particle state of mass m; the equations to be solved for the n-particle distribution functions $F_n^{(m^2)}$ are of course the same as before with $M=m$, except that now for consistency we shall cut off the infinite set of coupled equations at n=2. This yields the following approximate expression for $f_2^{(m^2)}$ in terms of $f_1^{(m^2)}$:

$$\left(\vec{k}^2+m^2-\frac{\vec{k}_1^2+m^2}{x_1}-\frac{(\vec{k}-\vec{k}_1)^2+m^2}{1-x_1}\right)\quad f_2^{(m^2)}(\vec{k}_1,x_1;\vec{k}-\vec{k}_1,1-x_1)\approx\frac{\lambda}{(2\pi)^{3/2}}f_1^{(m^2)}(\vec{k}) \tag{4.22}$$

where, for the same reason as discussed in connection with the 3-particle bound state wave function no term proportional to δm^2 has been included. Substituting this result into eq. (4.19) with $M=m$, we find that

$$\delta m^2 = \frac{\lambda^2}{(2\pi)^3} \int d\vec{k}' \int_o^1 \frac{dx'}{4x'(1-x')} \frac{1}{\left[\vec{k}^2+m^2-\frac{\vec{k}'^2+m^2}{x'} - \frac{(\vec{k}-\vec{k}')^2+m^2}{1-x'}\right]} \tag{4.23}$$

Upon introducing this divergent integral into eq.(4.20) one finds that the expression within curly bracket multiplying $F_2^{(M^2)}(\vec{k}_1,x_1;\vec{k}_2,x_2)$ is given by $-\lambda^2 x_1 K_2 \pi\ (x_1 K_2)$ where K_2 is defined by(4.21), and

$$\pi(z) = \frac{1}{4(2\pi)^3} \int d\vec{k}' \int_o^1 \frac{x'(1-x')dx'}{[\vec{k}'^2+m^2(1-x'+x'^2)][zx'(1-x')-\vec{k}'^2-m^2(1-x'+x'^2)]} \tag{4.24}$$

Hence for sufficiently small λ , this contribution to (4.20) may be neglected as compared to the left hand side.

Making this approximation and going to a reference frame where $\vec{k}_1=-\vec{k}_2=\vec{q}$ we find immediately that (4.20) may be written in the form

$$\{M^2-\frac{\vec{q}^2+m^2}{x} - \frac{\vec{q}^2+m^2}{1-x}\} f_2^{(M^2)}(\vec{q},x) = \frac{\lambda}{(2\pi)^{3/2}} f_1^{(M^2)}(o)+$$

$$+ \frac{\lambda^2}{2(2\pi)^3} \int d\vec{q}' \int_o^1 \frac{dx'}{x'(1-x')} V(\vec{q},x;\vec{q}',x') f_2^{(M^2)}(\vec{q}',x') \tag{4.25a}$$

where $f^{(M^2)}(\vec{q},x) \equiv f^{(M^2)}(\vec{q},x;-\vec{q},1-x)$ with $f^{(M^2)}$ defined by eq. (4.12), and where

$$V(\vec{q},x;\vec{q}',x') = \frac{\theta(x-x')}{x-x'} \left\{ M^2 - \frac{\vec{q}^{\,2}+m^2}{1-x} - \frac{\vec{q}'^{\,2}+m^2}{x'} - \frac{(\vec{q}-\vec{q}')^2+m^2}{x-x'} \right\}^{-1} +$$

$$+ \frac{\theta(x'-x)}{x'-x} \left\{ M^2 - \frac{\vec{q}^{\,2}+m^2}{x} - \frac{\vec{q}'^{\,2}+m^2}{1-x'} - \frac{(\vec{q}-\vec{q}')^2+m^2}{x'-x} \right\}^{-1} \tag{4.25b}$$

Finally we still have to consider eq. (4.19) coupling the amplitude $f_1^{(M^2)}(\vec{k})$ with the two-particle amplitude.

Substituting in eq. (4.19), with $\vec{k}=o$, the expressions for $f_2^{(M^2)}$ and δm^2 obtained from eqs. (4.25) and (4.23) we find that the divergent part of the integral in (4.19) combines with the term proportional to to δm^2, giving a finite result proportional to $\lambda^2\ f_1^{(M^2)}$ times a finite integral; thus for small λ this contribution may be neglected as compared to the left handside of eq. (4.19), since we are looking for bound state solutions with $M \sim 2m$. Hence we obtain that

$$f_1^{(M^2)} = \left(\frac{\lambda}{2(2\pi)^{3/2}}\right)^3 \frac{1}{M^2-m^2} \int d\vec{q}' \int_o^1 \frac{dx'}{x'(1-x')} \frac{1}{\left[M^2 - \frac{\vec{q}'^{\,2}+m^2}{x'} - \frac{\vec{q}'^{\,2}+m^2}{1-x'}\right]} \cdot$$

$$\cdot \int d\vec{q}'' \int_o^1 \frac{dx''}{x''(1-x'')} V(\vec{q}',x';\vec{q}'',x'')\ f_2^{(M^2)}(\vec{q}'',x'') \tag{4.26}$$

Thus $f_1^{(M^2)}$ is proportional to the third power of the coupling constant. Upon neglecting this contribution in eq. (4.25a) we obtain an integral equation for the 2-particle wave function alone which has a form very similar to that obtained in the non-relativistic case.

Finally we wish to point out that the integral equation just derived in the small coupling limit is equivalent to that obtained by Weinberg for the T-matrix in the infinite momentum frame, after retaining only one meson exchange contribuitions in the interaction kernel [6]. It is also the same as that obtained by the authors of ref. [14], who however did not discuss the details of the approximation scheme including the effects of renormalization, nor have they pointed out the extreme analogy that exists between the formulation of the bound state problem on the-light like plane, and the corresponding non-relativistic formulation on the plane $t=0$.

IV. CONCLUSION

The calculations of the previous sections suggest that the states (3.9) may provide a very convenient set of basis states to work with if one is interested in calculating the invariant mass squared of bound systems. In fact, as has been shown by Feldman, Fulton and Townsend [14] , the eigenvalues of the Wick equation computed using covariant perturbation theory agree with those obtained from an integral equation of the type (4.25). It has been our purpose in this paper to stress the extreme non-relativistic analogy of the formulation on the light-like plane $t+z = 0$, staying as close as possible to the non-relativistic treatment of bound systems at every step of the calculations. Of course we could not avoid those relativistic complications which are connected with the fact that the particle number is not conserved, and that renormalization effects have to be taken into account. On the other hand we have shown that some of the problems one encounters in formulating relativistic problems on the space-like plane $t = \text{const.}$ could be avoided by a proper choice of basis states. In particular we have seen that the Fock space spanned by the eigenvectors of $\vec{P}_{\perp}$ and P_o+P_z, where P_o is the full Hamiltonean of the interacting system, has a structure similar to that in a non-relativistic theory. The use of the light like plane has accomplished a great deal for us by providing us with relativistic operators, analogous to those of eq. (2.6), having the property that they annihilate the physical vacuum. This allowed us to use non-relativistic Fock space methods to derive integral equations for the bound state wave functions which are much easier to handle than those obtained in a manifestly covariant Bethe-Salpeter approach. For scattering problems, on the other hand, the states (3.9) will in general not be a very convenient set to work with, except possibly at high energies. In fact, in the Bjorken limes for deep inelastic electroproduction where the leading contributions to the structure functions are determined by the current commutators restricted to the light-like plane t+z=0, the eigenstates of $\vec{P}_{\perp}$ and P_o+P_z behave like free particle states, coupling pointlike to the electromagnetic field. In low energy scattering processes, however, we do not expect these states to be very useful since their definition involves an integration over a surface which includes dynamical information at all times; their relation to the free particle scattering states is certainly a complicated one. For the same reason, however, these states may in fact provide us with a very economical way of handling bound state problems, as opposed to using scattering states which seems unnatural when dealing with dynamics at short distances. Of course by using the light-like plane we had to pay a price, for we have lost manifest Lorentz covariance. Although the basic formulation does not depend on any particular choice of Lorentz frame, the approximations made are frame dependent, since the space spanned by the states (3.9) for fixed n is only invariant under the kinematical subgroup of the Poincaré group which leaves the surface t+z = 0 unchanged (see Appendix). Hence the cutting off procedure used in deriving the integral equation for the two-body wave function is not an invariant procedure with regard to those transformations which take us out of the light like plane. We hope to come back to this question in a future publication.

ACKNOWLEDGEMENTS

We are very thankful to Prof. B.Stech for many interesting discussions which have led to the present investigation, and for numerous helpful comments made during the progress of this work. We also are very grateful to Prof. G.Jacob and Prof. T.Maris for several very stimulating and fruitful conversations on the topic of this paper and to Prof.J.F.Perez for several useful comments. Finally we would like to thank the Instituto de Física of the University of São Paulo for its hospitality extended during the period in which a great part of this work was done, and the Instituto de Física of the University at Porto Alegre for inviting us to discuss the problems dealt with in this paper.

A substantial part of this work was carried out while the author was a guest of the CNPq. We thank all people who have made this scientific exchange program between the Brazilian Research Council and the KFA - Jülich possible.

ACKNOWLEDGEMENTS

We are very thankful to Prof. B.Stech for many interesting discussions which have led to the present investigation, and for numerous helpful comments made during the progress of this work. We also are very grateful to Prof. G.Jacob and Prof. T.Maris for several very stimulating and fruitful conversations on the topic of this paper and to Prof.J.F.Perez for several useful comments. Finally we would like to thank the Instituto de Física of the University of São Paulo for its hospitality extended during the period in which a great part of this work was done, and the Instituto de Física of the University at Porto Alegre for inviting us to discuss the problems dealt with in this paper.

A substantial part of this work was carried out while the author was a guest of the CNPq. We thank all people who have made this scientific exchange program between the Brazilian Research Council and the KFA - Jülich possible.

APPENDIX

The generators of the Poincaré group written as integrals over the light-like plane $\frac{t+z}{2}=\tau$, have the form[++]

$$P_\mu = \int d\sigma^\nu \, T_{\nu\mu} \tag{A.1}$$

$$M_{\mu\nu} = \int d\sigma^\delta \, m_{\mu\nu\delta}$$

where

$$d\sigma^\nu = \eta^\nu \, d\sigma \tag{A.2}$$

with n^ν and $d\sigma$ defined by eqs. (3.3b) and (3.3c) of the text, and where $T_{\nu\mu}$ and $m_{\mu\nu\delta}$ are given by the following expressions for the case of the Lagrangian (3.1)

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \mathcal{L} \tag{A.3}$$

$$m_{\mu\nu\delta} = x_\mu \, T_{\nu\delta} - x_\nu \, T_{\mu\delta}$$

It is convenient to consider separtely the generators which generate transformations which leave the surface (3.2) invariant and those which take us out of the plane $t+z=$ const. The former ones are given by

$$P_\xi = P_o + P_Z = \int d\sigma\, n^\mu \; T_{\mu\nu}\, n^\nu \qquad \text{(A.4a)}$$

$$P_i = \int d\sigma\; T_{i\mu}\, n^\mu \qquad (i=1,2) \qquad \text{(A.4b)}$$

$$J_3 = M_{12} = \int d\sigma\; (x_1\; T_{2\mu} - x_2\; T_{1\mu})\; n^\mu \qquad \text{(A.4c)}$$

$$M_{oi} - M_{3i} = n^\mu\, M_{\mu i} = \int d\sigma\; (n.x\; T_{i\nu} - x_i\; n^\mu\; T_{\mu\nu})\, n^\nu \qquad \text{(A.4d)}$$

and

$$K_3 = M_{o3} = \frac{1}{2}\int d\sigma (n.x\; m^\mu\; T_{\mu\nu} - m.x\; n^\mu\; T_{\mu\nu}) n^\nu \qquad \text{(A.4e)}$$

where

$$m^\mu = (1,0,0,1) \qquad \text{(A.4f)}$$

The operators (A.4a,b) generate translations <u>in</u> the Light-Like plane, while (A.4c,d) generate rotations <u>in</u> the plane. K_3 generates boosts along the z-direction.
The remaining generators are

$$P_\tau = P_o - P_Z = \int d\sigma\; m^\mu\; T_{\mu\nu} n^\nu \qquad \text{(A.5a)}$$

$$M_{oi} + M_{3i} = m^\mu\; M_{\mu i} = \int d\sigma (m.x\; T_{i\nu} - x_i\; m^\mu\; T_{\mu\nu}) n^\nu, \; (i=1,2) \qquad \text{(A.5b)}$$

where P_τ generates parallel translations of the plane $\frac{t+z}{2} = \tau$,

and $M_{oi} + M_{3i}$ generates rotations of this surface. Thus P_τ plays the role of the Hamiltonean in this formulation.

Expressing the above generators (A.4) and (A.5) in terms of the fields $\phi^{(\pm)}(x)$;

$$\phi^{(+)}(x) = \frac{1}{(2\pi)^{3/2}} \int d\vec{k}_\perp \int_0^\infty \frac{d\eta}{2\eta} \; a^\dagger(\vec{k}_\perp,\eta;\tau)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}$$

$$\phi^{(-)}(x) = \left[\phi^{(+)}(x)\right]^\dagger \qquad \text{(A.6)}$$

where $\underset{\sim}{k}\cdot\underset{\sim}{x} = \eta\xi - \vec{k}_\perp\cdot\vec{x}_\perp$, one finds after making use of the relation

$$\lim_{\eta\to 0} a(\vec{k}_\perp,\eta;\tau) = 0$$

which follows from eq. (3.8), that

$$P_\tau = 2\int d\sigma\, \phi^{(+)}(-\vec{\nabla}_\perp{}^2+m^2)\,\phi^{(-)} + \lambda\int d\sigma(\phi^{(+)}\phi^{(+)}\phi^{(-)}+\phi^{(+)}\phi^{(-)}\phi^{(-)}) \qquad \text{(A.7a)}$$

$$P_\xi = -2\int d\sigma\, \phi^{(+)}\,\partial_\xi\, \phi_\xi^{(-)} \qquad \text{(A-7b)}$$

$$\vec{P}_\perp = 2\int d\sigma\, \phi^{(+)}\,\vec{\nabla}_\perp\, \phi_\xi^{(-)} \qquad \text{(A.7c)}$$

$$J_3 = -2\int d\sigma\, \phi^{(+)}\,(x_1\,\partial_2 - x_2\,\partial_1)\,\phi_\xi^{(-)} \qquad \text{(A.7d)}$$

$$K_3 = \tau\, P_\tau - 2\int d\sigma\, \xi\,\partial_\xi\,\phi^{(+)}\,\partial_\xi\,\phi^{(-)} \qquad \text{(A.7e)}$$

$$M_{oi} - M_{3i} = 2\int d\sigma\, \phi^{(+)}(x_i\,\partial_\xi - 2\tau\,\partial_i)\phi_\xi^{(-)} \qquad \text{(A.7f)}$$

$$M_{oi}+M_{3i} = 2\int d\sigma\, \xi(\partial_i\phi^{(+)}\partial_\xi\,\phi^{(-)} + \partial_\xi\phi^{(+)}\partial_i\phi^{(-)}) - \int d\sigma\, x_i\,\mathcal{P}_\tau(x) \qquad \text{(A.7g)}$$

where

$$\mathcal{P}_\tau(x)=2\vec{\nabla}_\perp\phi^{(+)}\vec{\nabla}_\perp\phi^{(-)}+2m^2\phi^{(+)}\phi^{(-)}+\lambda(\phi^{(+)}\phi^{(+)}\phi^{(-)}+\phi^{(+)}\phi^{(-)}\phi^{(-)}) \tag{A.7h}$$

and $\partial_\xi = \partial/\partial_\xi$, $\partial_i = \partial/\partial x^i$, $\phi_\xi = \partial_\xi\,\phi$. For simplicity we have suppressed the argument of the fields. In writing down the above expressions we have normal ordered for the usual reasons. The corresponding expressions in terms of the operators (3.3a) and its adjoint can also be easily obtained by making use of (A.6). Notice that because of (3.4) all generators annihilate the physical vacuum. Since the above integrals (A.7a-g) are independent of the surface τ, as may be verified by making use of the equation of motion

$$-\partial_\tau\partial_\xi\ \phi(x) = (-\vec{\nabla}_\perp^2+m^2)\phi(x) + \frac{\lambda}{2}\ :\phi^2(x): \tag{A.8}$$

where $:\phi^2:$ denotes the normal product, we may chose as integration surface the plane $\tau = 0$. Since on this plane the generators (A.4) do not depend explicitely on the interaction, the states (3.9) will have simple transformation properties with regard to transformations generated by the operators (A.4). It also should be noticed that the integrals (A.7) do not contain terms constructed only from the operator $\phi^{(+)}$ or $\phi^{(-)}$, which is a special property of the formulation on the light-like plane.

The commutators of the generators with the field operators may now be derived by making use of the commutation relations:

$$[\phi^{(-)}(x),\phi^{(+)}(x')]_{\tau=\tau'} = \frac{1}{2\pi}\,\delta(\vec{x}_\perp-\vec{x}_\perp')\int_0^\infty \frac{d\eta}{2\eta}\,e^{-i\eta(\xi-\xi')}$$

and

$$\left[\phi(x),\ \phi(x')\right]_{\tau=\tau'} = -\frac{i}{4}\,\varepsilon(\xi-\xi')\,\delta(\vec{x}_\perp - \vec{x}_\perp') \tag{A.9}$$

and the equation of motion (A.8). One finds after some algebra that:

$$[\ \phi(x),\ P_\tau] = i\partial_\tau\phi(x) \tag{A.10a}$$

$$[\ \phi(x),\ P_\xi] = i\partial_\xi\phi(x) \tag{A.10b}$$

$$[\ \phi(x),\ P_i] = i\partial_i\phi(x),\quad (i=1,2) \tag{A.10c}$$

$$[\ \phi(x),\ J_3] = i(x_1\partial_2 - x_2\partial_1)\phi(x) \tag{A.10d}$$

$$[\ \phi(x),\ K_3] = i(\tau\partial_\tau - \xi\partial_\xi)\phi(x) \tag{A.10e}$$

$$[\ \phi(x), n^\mu M_{\mu i}] = i(2\tau\ \partial_i - x_i\partial_\xi)\ \phi(x) \tag{A.10f}$$

$$[\ \phi(x), m^\mu M_{\mu i}] = i(2\xi\partial_i - x_i\partial_\tau)\phi(x) \tag{A.10g}$$

These are in fact the usual commutation relations written in terms of the variables appropriate to the light-like plane. Furthermore one may verify after some fairly lengthy calculations that the operators (A.7) satisfy the Poincaré algebra.

Finally we wish to show that the matrix element (4.10) only depends on η and η_i through the ratios $x_i = \eta_i/\eta$.

To this effect we consider the corresponding commutator to (A.10e) for the operator $a(\vec{k}_\perp, \eta)$:

$$[a(\vec{k}_\perp,\eta),\ K_3] = i\eta\ \partial_\eta\ a(\vec{k}_\perp,\eta) \tag{A.11}$$

Since the operators (A.7a-g) form a representation of the Poincaré algebra, (A.11) implies the following transformation Law:

$$e^{i\lambda K_3}\ a(\vec{k}_\perp,\eta)\ e^{-i\lambda K_3} = a(\vec{k}_\perp,\ e^{\lambda}\ \eta) \tag{A.12}$$

Furthermore from the commutators

$$[\ P_\tau,\ K_3\ ] = i\ P_\tau$$

and

$$[\ P_\xi,\ K_3\ ] = -i\ P_\xi$$

we deduce that

$$e^{i\lambda K_3}\ |\vec{k}_\perp,\eta,\ \frac{\vec{k}_\perp^2+M^2}{\eta}\rangle = |\vec{k}_\perp,\ e^{\lambda}\eta\ ,\ \frac{\vec{k}_\perp^2+M^2}{e^{\lambda}\eta}\rangle$$

where $|\ \vec{k}_\perp,\eta,\ \frac{\vec{k}_\perp^2+M^2}{\eta}\rangle$ is an eigenstate of $\vec{P}_\perp$, P_ξ, and P_τ with eigenvalues $\vec{k}_\perp,\eta,$ and $(\vec{k}_\perp^2+M^2)/\eta$, respectively. Thus the matrix element (4.10) is invariant under the replacement $\eta \to e^{\lambda}\eta$, $\eta_i \to e^{\lambda}\eta_i$, and hence depends only on the ratios $x_i = \eta_i/\eta$.

Footnotes and References

+ On leave of absence from the Institut für Theoretische Physik der Universität Heidelberg, Germany

* We have chosen to write the integral in this form in order to preserve the analogy with the free destruction operators written in covariant form over an arbitrary space-like surface, and to which the operators (3.3a), when multiplied by $\exp\{[(\vec{k}_\perp^2+m^2)/\eta]\tau\}$ with $\eta = k_o+k_z$, reduce if $\phi(x)$ is a free field.

** We shall loosely speak of the states (3.9) as being n-particle states. In the Bjorken limes for deep inelastic electroproduction, where the leading contributions to the structure functions are determined by the current commutators on the light-like plane $\tau = 0$, such states would exhibit parton-like behaviour, coupling pointlike to the electromagnetic field.

*** Unfortunately the approximation scheme is only consistent for small coupling.

++ We shall verify below that the operators (A.1) defined on the light-like plane still satisfy the Poincaré algebra as required for the Lorentz invariance of the theory. The reader may also consult the work of e.g. Neville and Rohrlich, and Chan-Ting Chang et. al., ref. |4|.

1) S.-J. Chang and S.-K. Ma, Phys. Rev. 180, 1506 (1969).

2) H. Leutwyler, J.R. Klauder and L. Streit, Nuovo Cimento 66A, 536 (1970).

3) J.B. Kogut and D.E. Soper, Phys. Rev. D1, 2901 (1970).

4) J.Bjorken,J.B.Kogut and D.E.Soper, Phys.Rev.D3, 1382 (1971); R.A.Neville and F.Rohrlich, Nuovo Cimento 1A,625(1971);Phys, Rev.D3, 1692(1971); F.Rohrlich, Acta Phys.Austr.Suppl.8,277 (1971). J.M.Cornwall and R.Jackiw, Phys. Rev. D4, 367 (1971), D.A.Dicus, R.Jackiw and V.L.Teplitz, Phys. Rev.D4.1733(1971); Schau-Jin Chang, R.G.Root and Tung-Mow Yan, Phys.Rev. D7,1133 (1973);Schau-Jin Chang and Tung-Mow Yan, Phys.Rev. D7,1147 (1973); Tung-Mow Yan, Phys. Rev. D7,1760, 1780 (1973);L.Banyai and L.Mezinescu, Phys.Rev. D8, 417 (1973).

5) S.Fubini and G.Furlan, Phys. 1, 229 (1965)

6) S.Weinberg, Phys.Rev. 150, 1313 (1966)

7) For a review see H.Leutwyler in "Procedings of the Summer School for Theoretical Physics", Springer Tracts in Mod.Phys. 50 (1969)

8) L.Susskind, Phys.Rev. 165, 1535 (1968); K.Bardakci and M.B. Halpern, Phys.Rev. 176, 1686 (1968)

9) For a discussion see S.L. Adler and R.Dashen,"Current Algebra" (Benjamin, New York, 1968).

10) S.D.Drell, D.J.Levy and T.M.Yan, Phys.Rev. 187, 2159 (1969); Phys.Rev.D1, 1035,1617 (1970)

11) S.Brodsky and R.Roskies, Phys.Lett. 41B, 517 (1972)

12) For a review of the many applications of the infinite momentum frame technique see J.Kogut and L.Susskind, Phys. Reports 8C, Nº.2 (1973).

13) I.Tamm, J.Phys. U.S.S.R. 9, 449 (1945);S.M.Dancoff,Phys.Rev. 78, 382 (1950)

14) G.Feldman, T.Fulton, and J.Townsend, Phys. Rev. D7, 1814(1973)

15) For a complete set of references on the Bethe-Salpeter equation up to 1969 see N.Nakanishi, Progr.Theoret.Phys.(Kyoto) Suppl. 43, 1 (1969).

Figure Captions

Fig. 1 Tree-graph approximation to the three-particle wave function as given by eq. (4.17).

Fig. 2 Graphical representation of eq. (4.18), which connects the one, two and three-particle amplitudes.

Fig. 3 Graphical representation of eq. (4.19), which connects the one and two-particle amplitudes.

Fig. 4 Graphical representation of eq.(4.20) including self energy contributions as well as the terms proportional to δm^2 needed for mass renormalization. We have chosen to label the diagrams by the variable η rather than x. The rules for computing these diagrams are just Weinberg's infinite momentum rules. An energy denominator for the 3-particle intermediate states has to be included.

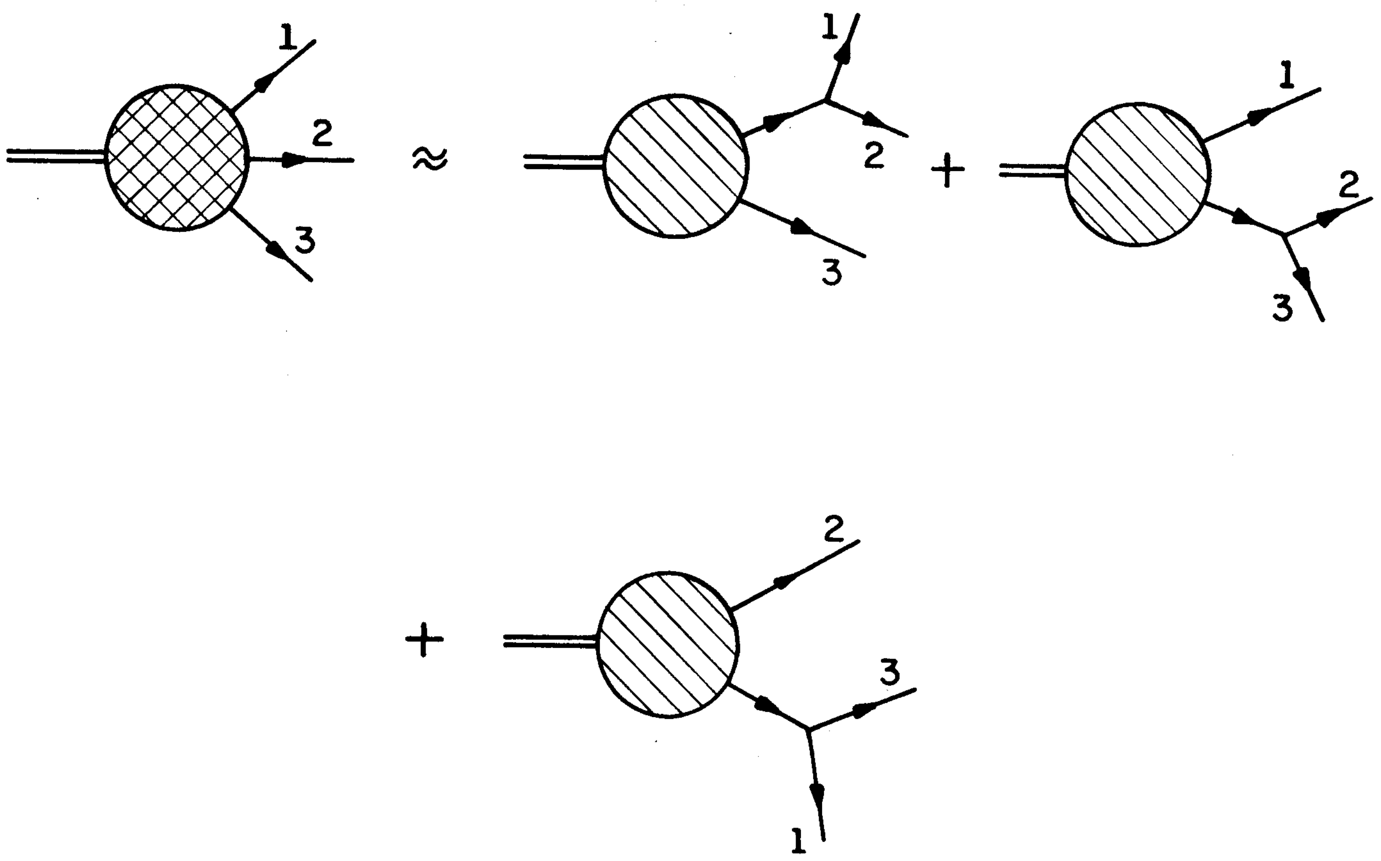

FIG. 1

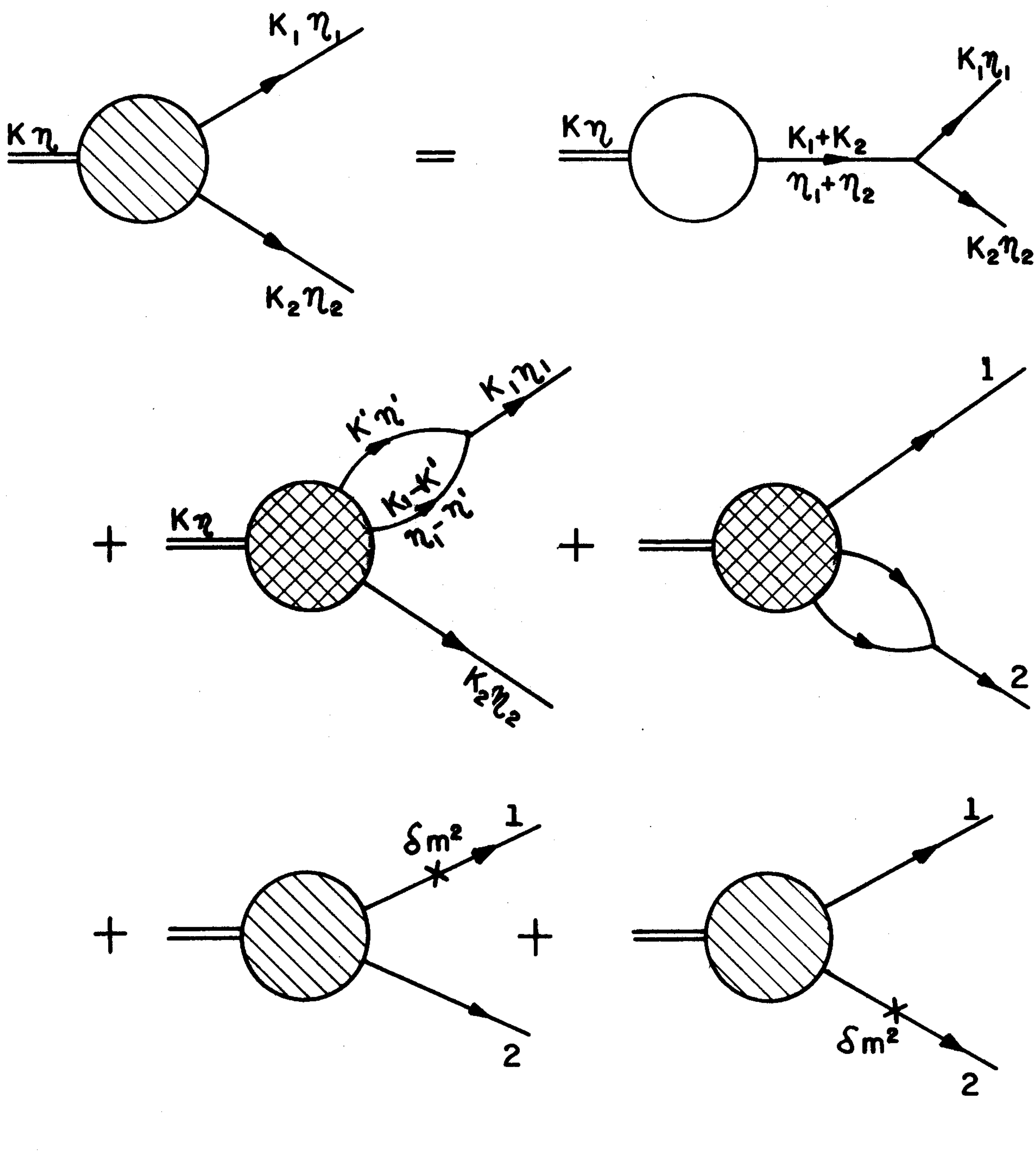

FIG. 2

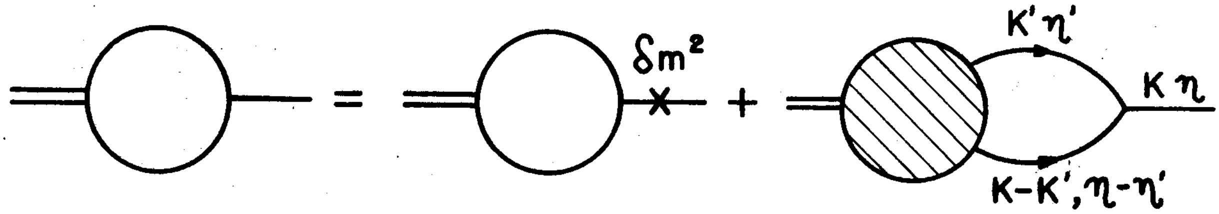

FIG. 3

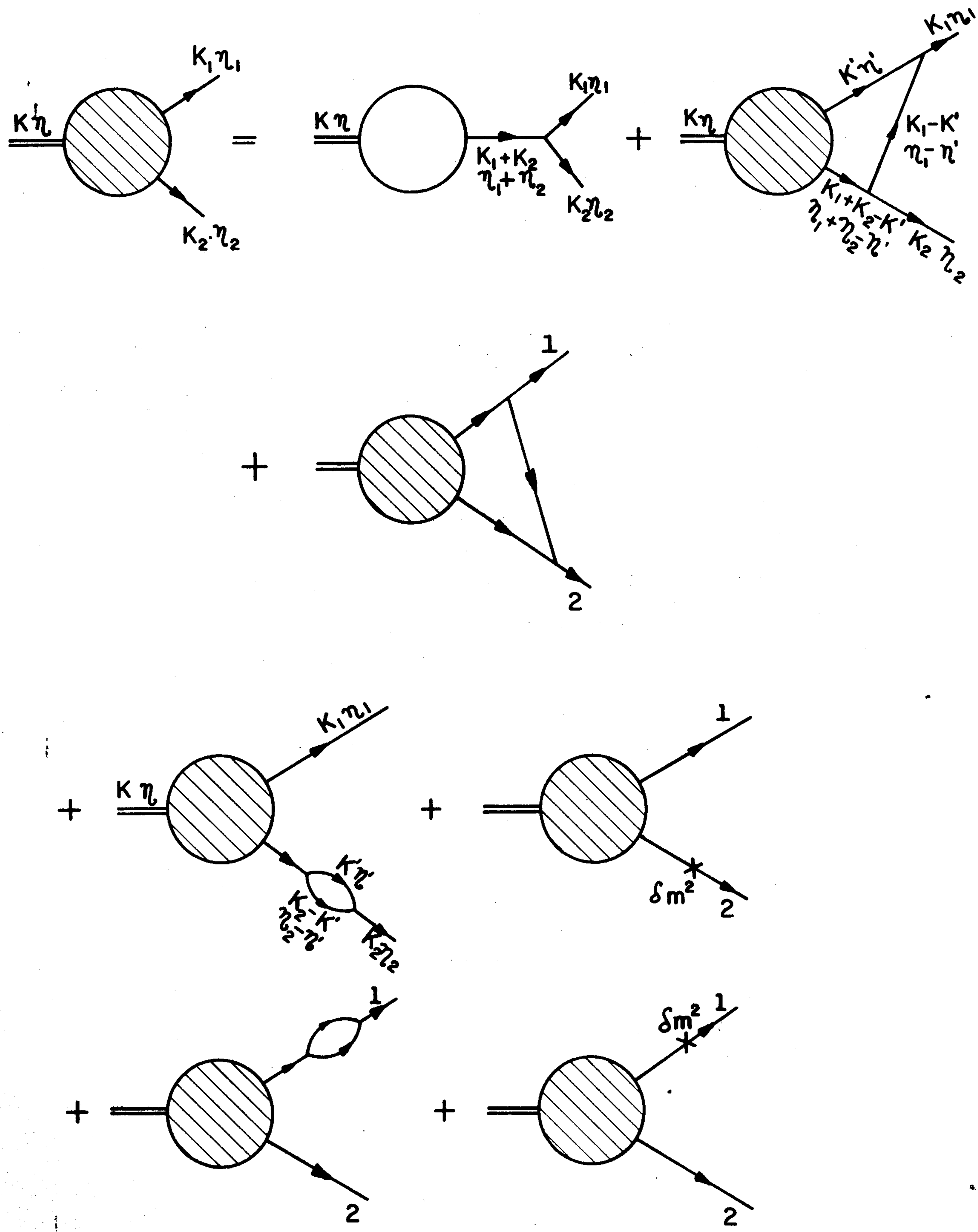

FIG. 4