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New renormalization group equations in a spontaneously broken gauge theory.

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ABSTRACT

A study of the new renormalization group equations is made, using the dimensional regularization scheme. It is shown, in a model-independent way, that these equations can be derived simply by writing dimensional analysis for the regularized Green functions and next reexpressing the result in terms of renormalized parameters and fields. The solution, valid for arbitrary momenta, involves effective momentum-dependent masses and coupling constants. For spontaneously broken gauge theories the solution involves as well effective momentum-dependent vacuum expectation values and gauge parameters. We exemplify these features in the context of a spontaneously broken abelian model where the roles of these effective momentum-dependent parameters are explicitly discussed.

INTRODUCTION

The equations of Callan and Symanzik [1,2] and the closely related renormalization group equations of Gell-Mann and Low [3] have been extensively used in the study of the asymptotic behavior of Green functions as the momenta go into the deep Euclidean region. However, in deriving this large momentum limit of the equations, one has to make certain assumptions, such as the absence of mass terms in the asymptotic region etc. In addition, it is difficult to use these equations to obtain non-leading terms that are important for weak and electromagnetic corrections to the hadronic symmetries. Moreover, computations are usually done within models which involve some internal symmetry group, under the assumption that the symmetry is preserved. In the real world, however, most of these symmetries are broken, and the question of the validity of high momenta solutions of the Callan-Symanzik equation arises. New renormalization group equations have been proposed [4,5], which can be solved before passing to the high energy limit and where the above mentioned questions can be explicitly studied. The solutions of these new renormalization group equations for general, non asymptotic momenta involve, in addition to momentum-dependent effective coupling constants, also momentum-dependent effective masses. Also, in the solutions of spontaneously-broken gauge theories in a general renormalizable gauge will appear in addition to a momentum-dependent effective vacuum expectation values, momentum-dependent effective gauge parameters.

In this paper we study these new equations in renormalizable field theories regularized with the continuous dimension method [6,7]. One of the compelling reasons for using dimensional regularization is that the symmetry properties of the Lagrangian, in particular the scaling properties, are preserved by the renormalized theory. The method is remarkably clear and powerful to study the response the Green functions to the scale transformations $p \to \lambda p$. The Ward identities so obtained for regularized but unrenormalized Green functions away from poles at rational values of the space-time dimension n, are true relations between finite quantities. We show, for arbitrary renormalizable field theories, that the content of these Ward identities is just dimensional analysis in n dimensions. Furthermore, by rewriting these dimensional analysis equations in terms of renormalized parameters and Green functions, we are able to derive the new renormalization-group equations above mentioned, for gauge theories with or without spontaneous breakdown of gauge symmetry. To illustrate these ideas we consider in detail an abelian gauge model with spontaneous symmetry breakdown, in the one-loop approximation. We show that in this case, where the effective coupling constants are not asymptotically small, the assumptions made in the usual renormalization group equation will give in general an incorrect result, as the effective momentum-dependent mass, vacuum expectation value and gauge parameter do not vanish at high momentum.

The paper is organized as follows: in sect 2 we derive the " new " renormalization group equation. In sect. 3 the model is discussed and we determine the counterterms needed to solve the equation, in the one-loop approximation. In order to obtain an explicit solution we choose a renormalization procedure with mass independent counterterms [8,9,10.] In sect.4 the solution is discussed. Finally, we show in the appendix that by rescaling fields and parameters in such a way that the counterterms are mass independent, all the vertices are made indeed finite. The analysis makes use of the Lee's identities for the proper vertices of the theory [11,12].

Our space-time metric is $\delta_{\mu\nu}$, 4-vectors being assigned imaginary fourth components. Thus the momentum 4-vector is $K=(\vec{K},iK_o)$ with norm squared $K^2=K_\mu K_\mu=\vec{K}^2-K_o^2$.

2. The new renormalization group equations.

Consider a renormalizable theory with a set ϕ_i (i=1,...s) of fields and where { $C^{\dagger}_a$,Ca; a=1,...t} denotes the set of Fadeev-Popov ghosts. We wish to study the response of the regularized but unrenormalized Green functions to an infinitesimal scale transformation of the form:

$$\phi_i'(x) = \phi_i(x) + \varepsilon(x.\partial + d_n)\phi_i(x) \qquad (1)$$

$$C_a'(x) = C_a(x) + \varepsilon\left[x.\partial+\frac{1}{2}(n-2) \right]C_a(x) \qquad (2)$$

where ε is an infinitesimal parameter and d_n is $\frac{1}{2}(n-1)$ for fermions. (The above assignments of d_n are necessary for the action in n dimensions to be dimensionless).

The resulting Ward identities are conveniently studied in terms of the generating functional for the one-particle-irreducible regularized Green functions $\Gamma\,[\phi]$ defined by

$$\Gamma\,[\,\Phi\,] = \sum_n \frac{1}{n!} \prod_{i=1}^{n} dx_i\ \Phi(x_i)\ \Gamma\,(x_1,\ldots x_i). \qquad (3)$$

It has been shown [13] that the following Ward identities hold for any (gauge) theory:

$$\frac{\delta\Gamma}{\delta\phi_i}\left[x\cdot\partial+d_n \right]\left[\frac{1}{i}\frac{\delta}{\delta J} + \phi \right]_i + \frac{\delta\Gamma}{\delta C_a}\left[x\cdot\partial+\frac{1}{2}(n-2)\right]\left[\frac{1}{i}\frac{\delta}{\delta K^+}+C\right] + \qquad (4)$$

$$\begin{bmatrix} C_a \to C^{\dagger}_a \\ K^{\dagger}_a \to K_a \end{bmatrix} = i\,\Gamma_{\Delta}\,(0,\Phi).$$

Here J, K and K^+ are sources and Δ represents the divergence of the current associated with the scale transformation. $\Gamma_\Delta(0,\Phi)$ is defined analogously to $\Gamma(\phi)$ and represents the insertion, with zero external momentum, of Δ in the Green function. Now, by taking functional derivative of Γ with respect to ϕ at $\phi = 0$ we obtain the Ward identity satisfied by the function $\Gamma\,(x_1,\ldots x_n)$.

It is more convenient to study the resulting Ward identity in momentum space. We obtain (repeated indices imply, summation and the suffix o indicates unrenormalized quantities):

$$\left(p_i \frac{\partial}{\partial p_i} - D_\Gamma \right) \Gamma_o^K \,(p_1,\ldots p_k) = i\, \Gamma_{o\Delta}^K \,(0,p_1,\ldots p_K). \qquad (5)$$

On the other hand the regularized Green functions depend also on a set of masses m_i^o $(i=1,\ldots r)$, a set of coupling constants g_i^o $(i=1,\ldots u)$ with dimensions d_i^o and, for gauge theories, also on a set of dimensionless gauge parameters $\varepsilon_i^o(i=1,..v)$. Then dimensional analysis tells us that

$$\left(p_i \frac{\partial}{\partial p_i} + m_i^o \frac{\partial}{\partial m_i^o} + d_i^o \; g_i^o \frac{\partial}{\partial g_i^o} - D_\Gamma \right) \Gamma_o = 0, \qquad (6)$$

and comparison with (5) shows that the insertion of one zero-momentum Δ is soft, in the sense that

$$\Gamma_{o\Delta}^K \,(o,p_1,\ldots p_K) = i \left(m_i^o \frac{\partial}{\partial\, m_i^o} + d_i^o\; g_i^o \frac{\partial}{\partial g_i^o} \right) \Gamma_o^K(p_1,\ldots p_K). \quad (7)$$

This relation can, of course, be explicitly checked in each particular model.

Until now we have been treating the case with no symmetry breakdown. To include spontaneous breaking we must allow for the possibility of some of m_i^{2o} being negative. Equation (4) is not affected in this case. To obtain the Ward identity for the regularized Green functions with $\langle\phi^o\rangle = v_i^o$ we write $\phi^o = \tilde{\phi}^o + v^o$ and note that

$$\int dx\, \phi_i^o(x) \frac{\delta}{\delta\phi_i^o(x)} \Gamma_o(\phi^o = \tilde{\phi}^o + v^o) = \left[\int dx\, \tilde{\phi}_i^o(x) \frac{\delta}{\delta\tilde{\phi}_i^o(x)} + v_i^o \frac{\partial}{\partial v_i^o} \right] \Gamma_o \quad (8)$$

Proceeding along exactly the same steps as before, we obtain the following Ward identity, that exhibits the response of the regularized Green functions to the scale transformations (1): (dimensional analysis in n dimensions)

$$\{ p_i \frac{\partial}{\partial p_i} + m_i^o \frac{\partial}{\partial m_i^o} + \tilde{d}_i^o\, v_i^o \frac{\partial}{\partial\, v_i^o} + d_i^o\, g_i^o \frac{\partial}{\partial\, g_i^o} - D_\Gamma \} \Gamma_o^K(p_1, \ldots p_K) = 0 \quad (9)$$

where $\tilde{d}_i$ is the dimension of the field ϕ_i^o in a space-time of n dimensions. In this space-time the dimensions of the bare parameters depend on n. We would like that the renormalized parameters keep their physical dimensions, for each n. This is done through the introduction of a mass parameter μ [5,8] that absorbs the n-dependent part of the dimensions of the bare parameters in such a way that the renormalized parameters end up with dimensions, for any value of n, that coincide with the dimensions of the corresponding bare parameters at n=4. By first writing Γ_o in terms of the renormalized parameters and then reexpressing the derivatives $\frac{\partial}{\partial m_i^o}$, $\frac{\partial}{\partial v_i^o}$, $\frac{\partial}{\partial g_i^o}$ in terms of renormalized quantities with help of the chain rule we obtain

$$\{p_i \frac{\partial}{\partial p_i} + (Dm_i)\frac{\partial}{\partial m_i} + (Dg_i)\frac{\partial}{\partial g_i} + (D\xi_i)\frac{\partial}{\partial \xi_i} + (Dv_i)\frac{\partial}{\partial v_i} - D_\Gamma\}\Gamma_o^K(p_1,\ldots p_K)=0 \tag{10}$$

where $\quad D\equiv m_K^o \frac{\partial}{\partial m_K^o} + \bar{d}_K^o\, v_K^o \frac{\partial}{\partial v_K^o} + d_K^o\, g_K^o \frac{\partial}{\partial g_K^o}$

By dimensional analysis,

$$Dm_i = \left(1-\mu\frac{\partial}{\partial\mu} \right) m_i$$

$$Dv_i = \left(1-\mu\frac{\partial}{\partial\mu} \right) v_i$$

$$Dg_i = \left(d_i-\mu\frac{\partial}{\partial\mu} \right) g_i$$

where d_i is the physical dimension of the coupling constant g_i,

$$D\xi_i = -\mu\frac{\partial}{\partial\mu}\ \xi_i$$

Finally, using the fact that

$$\{\mu\frac{\partial}{\partial\mu} + p_i\frac{\partial}{\partial p_i} + m_i\frac{\partial}{\partial m_i} + v_i\frac{\partial}{\partial v_i} + d_K\, g_K\frac{\partial}{\partial g_K} - D_\Gamma\}\Gamma_o(p_i,m_j,v_i,g_i,\xi_i)=0$$

we obtain:

$$\{\mu\frac{\partial}{\partial\mu} + \left(\mu\frac{\partial m_i}{\partial\mu}\right)\frac{\partial}{\partial m_i} + \left(\mu\frac{\partial v_i}{\partial\mu}\right)\frac{\partial}{\partial v_i} + \left(\mu\frac{\partial g_i}{\partial\mu}\right)\frac{\partial}{\partial g_i} + \left(\mu\frac{\partial \xi_i}{\partial\mu}\right)\frac{\partial}{\partial \xi_i}\}\ \Gamma_o=0 \tag{11}$$

This equation, which can also be written as $\mu \frac{d}{d\mu}\Gamma_o=0$ is obvious, as Γ_o, the unrenormalized Green function, does not depend on μ. It is often taken as the starting point for the derivation of the "new" renormalization group equation. What is not obvious is the connection between it and scale transformations, i.e., dimensional analysis, in n dimensions.

By rewriting the unrenormalized Green function in terms of the renormalized one:

$$\Gamma_o(p_i,\ldots) = Z_\Gamma\ \Gamma(p_i,m_i,v_i,g_i,\xi_i,\mu) \tag{12}$$

we obtain the "new" renormalization group equation

$$\{\mu\frac{\partial}{\partial\mu} + \beta_K \frac{\partial}{\partial g_K} + \delta_K \frac{\partial}{\partial m_K} + \rho_K \frac{\partial}{\partial v_K} + \alpha_K \frac{\partial}{\partial \xi_K} + \gamma\}\ \Gamma(p_i,g_i,m_i,v_i,\xi_i,\mu)=0 \tag{13}$$

where

$$\beta_K=\mu\frac{\partial g_K}{\partial\mu}\ ;\ \delta_K=\mu\frac{\partial m_K}{\partial\mu}\ ;\ \rho_K=\mu\frac{\partial v_K}{\partial\mu}$$

$$\alpha_K = \mu\frac{\partial\xi_K}{\partial\mu} \quad \text{and} \quad \gamma=\mu\frac{\partial}{\partial\mu}\ \log\ Z_\Gamma \tag{14}$$

Remark that in deriving the above equation we have considered v_i,m_i,g_i,ξ_i as independent parameters. Our ultimate interest is, of course, in those values of v_i that satisfy the spontaneous breakdown condition $\left.\frac{\delta\Gamma}{\delta\phi_i}\right)_{v_i}=0$, at least in the tree approximation. This condition will lead to a relationslup among v and the other renormalized parameters of the theory.

In order to be able to find a useful solution to the renormalization group equation it is however important that the coefficients $\beta_K,\ldots\gamma$ be explicitly independent on μ. The mass independent renormalization procedure discussed below will ensure the fulfilment of this condition.

3. The Abelian Model

The abelian model discussed in this paper is described by the Lagrangian

$$L= -\frac{1}{4} (\partial_\mu A_\nu^o - \partial_\nu A_\mu^o)^2 - [(\partial_\mu - ie_o A_\mu^o) \phi_o]^2 + \frac{M_o^2}{2} (\phi_o^* \phi_o) - h_o(\phi_o^* \phi_o)^2 \quad (15)$$

where ϕ_o is a complex scalar field and the suffix o indicates that the quantities are not yet renormalized. We take $Mo^2>0$, $h_o>0$ so that ϕ_o must develop a vacuum expectation value to make the physical masses of the scalar bosons non-negative. We will write accordingly ,

$$\phi_o = \frac{1}{\sqrt{2}} (v_o + \psi_o + i X_o) \quad (16)$$

The Lagangian in Eq. (15) is invariant under the gauge transformation

$$\begin{aligned} \phi_o &\rightarrow e^{i\theta} \phi_o \\ A_\mu^o &\rightarrow A_\mu^o + \frac{1}{e_o} \partial_\mu \theta \end{aligned} \quad (17)$$

We choose a general class of gauge conditions given by

$$F (\phi_o, A_o) = \sqrt{\xi_o} (\partial_\mu A_\mu^o + \frac{\lambda_o}{\xi_o} X_o) = 0 \quad (18)$$

where ξ_o is the unrenormalized gauge parameter and $\lambda_o = \lambda_o(\xi_o, v_o,$

$m_o, g_o, h_o)$ is a parameter to be identified later.

The gauge condition (18) implies that the Fadeev-Popov effective Lagangian will contain also the term $\bar{c}_o\, M_o\, c_o$, with

$M_o = \partial^2 + \frac{\lambda_o}{\xi_o}\, e_o\, (\psi_o + v_o)$. The full unrenormalized Lagangian is given by

$$Lo = -\frac{1}{4}\left(\partial_\mu A^o_\nu - \partial_\mu A^o_\mu\right)^2 - \left|(\partial_\mu - ie_o A^o_\mu)\,\phi_o\right|^2 + \frac{Mo^2}{2}(\phi_o{}^*\phi_o) -$$

$$- h_o\,(\phi_o{}^*\phi_o)^2 - \frac{1}{2}\xi_o\left(\partial_\mu A^o_\mu + \frac{\lambda_o}{\xi_o}\chi_o\right)^2 + \bar{c}_o\left|\partial^2 + \frac{\lambda_o}{\xi_o} e_o(\psi_o + v_o)\right| C_o. \tag{19}$$

Notice that in n-dimensions the fields A^o_μ and ϕ_o will have dimensions $\frac{n-2}{2}$; e_o will have dimension $\frac{4-n}{2}$ and h_o will have $4-n$.

A simple way of generating the necessary renormalization counterterms is by performing the following scaling transformations on the quantities of Eq. (19):

$$A^o_\mu = Z_3^{\frac{1}{2}}\, A_\mu \quad ; \quad \phi_o = Z^{\frac{1}{2}}\,\phi \tag{20}$$

with Z_3 and Z dimensionless and

$$\mu^{\frac{n-4}{2}}\, e_o = Z_3^{-\frac{1}{2}}\, e \quad ; \quad \mu^{n-4}\, h_o = \frac{Z_h}{Z^2}\, h \ ; \quad M_o^2 = \frac{Z_M}{Z}\, M^2 \tag{21}$$

where now the renormalized coupling constants e and h are dimensionless for all n.

In addition, the parameters ξ_o and λ_o will be renormalized so that the gauge fixing term remain invariant. This requires

$$\xi_o = Z_3^{-1}\,\xi \qquad \text{and} \qquad \lambda_o = Z_3^{-\frac{1}{2}}\, Z^{-\frac{1}{2}}\,\lambda \tag{22}$$

and, consequently,

$$\sqrt{\xi_o}\,(\partial_\mu\, A_\mu^o + \frac{\lambda_o}{\xi_o}\; X_o) = \sqrt{\xi}\,(\partial_\mu\, A_\mu + \frac{\lambda}{\xi}\; X) \tag{23}$$

Strictly speaking there is still another counterterm, associated with the Fadeev Popov ghost term. As we are not going beyond the one-loop level, it will not enter our discussion.

We require the field ϕ not to have (divergent) vacuum expectation value, so that a further rescaling of v_o is needed. The entire renormalization of v_o reads:

$$\mu^{\frac{4-n}{2}}\; v_o = Z^{\frac{1}{2}}\; Z_v^{\frac{1}{2}}\; v \tag{24}$$

Notice that v (the renormalized vacuum expectation value) has always dimension 1.

The dimensionless counterterms introduced above are defined to be precisely those needed to subtract the poles in the corresponding Feynman integrals, and thus give a finite result. Clearly, the requirement that the Green functions should be finite at n=4 does not fix the counterterms, and one must add normalization conditions to eliminate this ambiguity. Together with dimensional regularization we adopt a normalization prescription which requires that the couterterms contain only poles at $n=4$. This also guaranties that the renormalized theory will have the same symmetry properties as are formally present in the Lagrangian [14].With this prescription the parameters that appear in L do not coincide with the physical masses and coupling constants. They are however perfectly suitable renormalized parameters in terms of which the Green functions are regular in the limit $n\to 4$.

It can be shown [8] that the counterterms generated as indicated above are independent of the mass and of the parameter μ. This is essentially due to the fact that, with dimensional regularization, the residues of the poles are polynomials in the masses and in the momenta. Since μ only appears as a power of μ^{4-n} (when the Laurent expansion of each Feynman integral is made, only logarithms of μ will appear) and since the Z's are dimensionless, they must be independent of the masses and μ.

Performing the substitutions indicated above we obtain for the renormalized Lagrangian the expression:

$$L = -\frac{1}{4} Z_3(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2} Z[(\partial_\mu - ie\mu^{\frac{4-n}{2}} A_\mu)(Z_v^{\frac{1}{2}} \mu^{\frac{(n-4)}{2}} v+\psi+i\chi]^2$$

$$+ \frac{1}{4} Z_\mu M^2 [Z_v^{\frac{1}{2}} \mu^{\frac{n-4}{2}} v+\psi+iX]^2 - \frac{1}{4} Z_h \mu^{4-n} h [\mu^{\frac{n-4}{2}} Z_v^{\frac{1}{2}} v+\psi+iX]^4 \quad (25)$$

$$- \frac{1}{2} \xi (\partial_\mu A_\mu + \frac{\lambda}{\xi} X)^2 + \bar{C} [\partial^2 + \mu^{\frac{4-n}{2}} \frac{\lambda}{\xi} e(\psi + \mu^{\frac{n-4}{2}} Z_v^{\frac{1}{2}} v)] C$$

We determine λ by requiring that the cross term which couples the longitudinal component of the vector boson to χ be absent in zeroth order of perturbation theory. This yields $\lambda=-ev$.

The Feynman rules are presented in Fig.1. As mentioned before, v is kept as an independent variable. Only after we solve equation (13) will we put $v^2 = \frac{M^2}{2h}$ as a relationship among renormalized quantities, making $\langle\phi\rangle_0$ vanish in the tree approximation. We shall determine $Z_v \equiv 1+z_v$ by requiring that $\langle\phi\rangle_0$ be finite in higher orders. The one-loop tadpole terms are shown in fig.2, while the necessary counterterms can be read from the Lagrangian (25). We obtain, therefore,

$$\left[h\, v^2\, z_h - M^2 z_M + \frac{1}{2}\left(3hv^2 - M^2\right) z_v \right] = \tag{26}$$

$$= e^2 Z_0 \left[3(ev)^2 + 3\frac{h}{e^2}\left(3\, hv^2 - M^2\right) + \frac{h}{e^2}\left(hv^2 - M^2\right) + \frac{hv^2}{\xi} \right]$$

where $Z_h \equiv 1+z_h$, $Z_M \equiv 1+z_M$ and $Z_0 = \frac{1}{8\pi^2}\,\frac{1}{4-n}$

Let us now look at the vector boson self-energy:

$$\Pi_{\mu\nu}\left(K^2\right) = A\left(K^2\right)\delta_{\mu\nu} + B\left(K^2\right)\left(K^2\delta_{\mu\nu} - K_\mu K_\nu\right). \tag{27}$$

We shall determine the counterterms so that $\Pi_{\mu\nu}$ be finite as $n \to 4$. The relevant diagrams are shown in fig.3. We then find:

$$\Pi^{pole}_{\mu\nu} = (z+z_v)\,(ev)^2\,\delta_{\mu\nu} + z_3\left(K^2\,\delta_{\mu\nu} - K_\mu K_\nu\right) + \tag{28}$$

$$+ e^2 Z_0 \left\{ -\left(3+\frac{1}{\xi}\right)(ev)^2\,\delta_{\mu\nu} + \frac{1}{3}\left(K^2\delta_{\mu\nu} - K_\mu K_\nu\right) \right\} = 0$$

Therefore we must have

$$z+z_v = \left(3+\frac{1}{\xi} \right) e^2 Z_o \tag{29}$$

$$z_3 = -\frac{1}{3} e^2 Z_o$$

Note that since e_o and Z_3 are gauge independent, e will also be gauge independent. We can determine Z_h by requiring the four-point function of the scalar meson ψ to be finite. From the Feynman diagrams shown in fig. 4 we find:

$$r\, z_h = \left[10\, r^2 - \frac{2}{\xi} r + 3 \right] e^2 Z_o \tag{30}$$

where $r = \frac{h}{e^2}$.

A particularly convenient way to determine Z_v is by the use of Lee's identities (see appendix). We obtain

$$z_v = \frac{2\, e^2}{\xi} Z_o \tag{31}$$

(To check this result an independent determination of z_v was made by the requirement of regularity of the 2-point function of the scalar meson ψ at n=4).

Therefore, from (26),(29),(30) and (31) we find

$$\begin{aligned} z &= \left(3+\frac{1}{\xi}\right) e^2 Z_o \\ z_M &= \left\{-\frac{1}{\xi} + 4r\right\} e^2 Z_o \end{aligned} \tag{32}$$

Note that $\frac{Z_h}{Z^2}$ is gauge independent and therefore h will also be gauge independent, as h_o does not depend on the gauge either. Similarly, due to the fact that M_o^2 and $\frac{Z_M}{Z}$ are gauge independent, M^2 will also be gauge independent.

These considerations are necessary if we want to write equation where v, g, h, M, ξ are considered as independent parameters. We observe that indeed the counterterms are mass independent and do not explicitly depend on μ.

Finally, it remains to be shown that all other vertices are made finite by the above renormalization procedure. This is discussed in the appendix.

4. Solution of the new renormalization-group equation.

In our model, the renormalization-group equation satisfied by a one-particle irreducible Green function involvilng n_s external scalar particles and n_v external vector bosons becomes:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_e\frac{\partial}{\partial e}+\beta_h\frac{\partial}{\partial h}+\delta\frac{\partial}{\partial M}+\rho\frac{\partial}{\partial v}+\alpha\frac{\partial}{\partial\xi}+\gamma\right)\Gamma^{n_s,n_v}(p_i, M, v, g, h, \xi, \mu) = 0 \tag{33}$$

Using, the definitions (14) and the counterterms determined in equations (29-32) we obtain in lowest non-trivial order:

$$\beta_e = \frac{n-4}{2}\, e + \frac{e^3}{48\,\pi^2} \tag{34}$$

$$\beta_h = (n-4)\, h + \frac{1}{8\pi^2}\,(10h^2 - 6\, e^2 h + 3\, e^4) \tag{35}$$

$$\alpha = \frac{1}{24\pi^2}\, e^2\xi \quad ; \quad \gamma = \frac{e^2}{16\pi^2}\left[\left(\frac{1}{\xi}-3\right) n_s + \frac{1}{3}\, n_v\right] \tag{36}$$

$$\delta = \frac{1}{16\pi^2}\, M\, e^2[-3+4r] \quad ; \quad \rho = \frac{4-n}{2}\, v + \frac{1}{16\pi^2}\, e^2\left(\frac{1}{\xi}+3\right)v. \tag{37}$$

Notice the important fact that the coefficients $\beta_e, \ldots \gamma$ are independent of μ. In order to find a solution to this equation we first make a change of variable. Calling Q^2 an arbitrary quadratic function of the momenta p_i, introduce $t = \frac{1}{2}\log\frac{Q^2}{\mu^2}$. We see that the parameter μ, which is intrinsically connected with the dimensional regularization scheme, effectively fixes the scale of the momentum dependence of Green functions. Then the equation becomes:

$$\left(-\frac{\partial}{\partial t}+ \beta_e\frac{\partial}{\partial e} +\beta_h\frac{\partial}{\partial h}+\delta\frac{\partial}{\partial m}+\rho\frac{\partial}{\partial v}+\alpha\frac{\partial}{\partial \xi}+\gamma\right) \Gamma \left(p_i,e,h,M,v,t,\xi\right)= 0 \qquad (38)$$

The solution of this equation is obtained in two steps [15]. First one defines momentum-dependent effective coupling constants $\bar{g}$, $\bar{h}$, mass $\bar{M}$, vacuum expectation value $\bar{v}$ [16], gauge parameters $\bar{\xi}$, which are solutions of the following set of differential equations:

$$\frac{\partial \bar{e}}{\partial t} = \beta_e(\bar{e},\bar{h},\bar{\xi}) \quad ; \quad \frac{\partial \bar{h}}{\partial t} = \beta_h(\bar{e},\bar{h},\bar{\xi})$$

$$\frac{\partial \bar{M}}{\partial t} = \delta (\bar{e},\bar{h},\bar{\xi}) \quad ; \quad \frac{\partial \bar{v}}{\partial t} = \rho (\bar{e},\bar{h},\bar{\xi},\bar{v}) \qquad (39)$$

$$\frac{\partial \bar{\xi}}{\partial t} = \alpha (\bar{e},\bar{h},\bar{\xi})$$

with the boundary conditions

$$\bar{e} (t=0) = e \quad ; \quad \bar{h} (t=0) = h \; ; \; \bar{M} (t=0) = M \quad ; \; \bar{v} (t=0) = v;$$

$$\bar{\xi} (t=0) = \xi. \qquad (40)$$

Then the solution will be:

$$\Gamma (p_i,M,v,e,h,t,\xi)= \Gamma (p_i,\bar{M},\bar{v},\bar{e},\bar{h},\xi,0)\exp\int_0^t \gamma (\bar{e},\bar{h},\bar{\xi})\, dt' \qquad (41)$$

The main achievement of this solution is that the explicit t dependence of Γ is transfered to an implicit dependence through $\bar{e},\bar{M},\bar{v},\bar{h},\bar{\xi}$, which are independent of the particular Green function considered. Furthermore, it is important to notice that this solution is valid for any choise of the momenta p_i. Wherever $\bar{e}$ and $\bar{h}$ (t) get small compared to unity, we can relay on perturbation theory. For this domain, using eqts. (39),(40) and (34-37) we find, as $n \to 4$,

$$\frac{\partial \bar{e}}{\partial t} = \frac{\bar{e}^3}{48\pi^2} \tag{42}$$

and

$$\frac{\partial \bar{h}}{\partial t} = \frac{1}{8\pi^2} \bar{e}^4 \left[10\bar{r}^2 - 6\bar{r} + 3\right], \tag{43}$$

where $\bar{r} = \frac{\bar{h}}{\bar{e}^2}$.

The solution of (42) is

$$\bar{e}^2 = \frac{e^2}{1 - \frac{e^2 t}{24\pi^2}} \qquad \text{with} \quad \frac{e^2 t}{24\pi^2} < 1 \tag{44}$$

Equation (43) is most conveniently writhen in terms of $\bar{r}$, becoming

$$\frac{\partial \bar{r}}{\partial t} = \frac{1}{8\pi^2} \bar{e}^2 \left[10\bar{r}^2 - \frac{19}{3}\bar{r} + 3\right] \tag{45}$$

which can be solved by quadratures. Observe that, the discriminant $q^2 > 0$, so that in this region $\bar{r}$ increases with t, that is to say, $\bar{h}$ increases faster than $\bar{e}^2$. From (45) one then gets

$$\bar{r} = \frac{aG}{1+bG}, \tag{46}$$

where $G = -\tan\left[3q \log\left(1 - \frac{e^2 t}{24\pi^2}\right)\right]$ (47)

q being the square root of the discriminant, and

$$a = \frac{20 - \frac{19}{3} r}{q} \quad ; \quad b = \frac{\frac{19}{3} - 6r}{q} \tag{48}$$

The above solution requires

$$-\frac{\pi}{6q} < \log\left(1-\frac{e^2 t}{24\pi^2}\right) < \frac{\pi}{6q} \tag{49}$$

so that

$$\left|\frac{e^2 t}{24\pi^2}\right| \lesssim \frac{1}{3q} \tag{50}$$

This means, using (44), that e^2 it self must be small, that is, the perturbative calcularion makes sense only for a theory of weak and electromagnetic interactions. For these, e^2 is typically of order 10^{-2}, so that, by (49) our solution is valid for a very large range of momenta in the region $\frac{Q^2}{\mu^2} \lesssim e^{1000}$. In this region we can effectively solve the system of equations (39), finding

$$\frac{\bar{M}}{M} = \exp\left\{\frac{6}{q}\frac{a}{1+b^2}\left[b\alpha t - \log(\cos\alpha t + b\sin\alpha t)\right]\right. \quad \text{with} \quad \alpha = \frac{qe^2}{8\pi^2}$$

$$\frac{\bar{v}}{v} = \left(1+\frac{3e^2}{16\pi^2}t\right)\exp\frac{e^2}{16\pi^2}\frac{1}{\xi}t \tag{51}$$

$$\frac{\bar{\xi}}{\xi} = \left(1+\frac{e^2}{24\pi^2}t\right) \quad ; \quad \bar{\gamma} = \frac{e^2}{16\pi^2}\left[\frac{n_s}{\xi}+\frac{1}{1-\frac{e^2 t}{24\pi^2}}\left(-3n_s+\frac{1}{3}n_v\right)\right].$$

Perturbation theory cannot tell us anything about the limit $t\to\infty$. We will assume here the existence of the limit and define

$$M_\infty = \lim_{t\to\infty}\bar{M}(t)$$

$$v_\infty = \lim_{t\to\infty}\bar{v}(t) \tag{52}$$

Note that, except for the cases when $M_\infty = v_\infty = 0$, one is not allowed to neglect the mass and vacuum expectation value terms in equation (38). This means therefore that one does not, in general, obtain the correct solution by starting with the usual renormalization group equation.

Appendix.

We have determined all the necessary counterterms by taking a definite set of vertices and requiring them to be finite in the limit $n\to 4$. In order show that these counterterms make all the remaining vertices finite we will make use of Lee's identities for the renormalized one-particle irreducible vertices [11,12], which are a reflection of the invariance of the renormalized theory under a set of gauge tranformations of second kind. As was shown by Lee [12], these identities can be used to show, by induction, the renormalizability of the theory to an arbitrary order in perturbation theory. Therefore, we may restrict our explicit computation to the one-loop order.

To this order and in our model the identities read (we send the reader to ref. [12] for more details):

$$\partial_\mu \frac{\delta}{\delta A_\mu}(L^o+L^1) + \mu^{\frac{4-n}{2}} e\,[\chi-\gamma_\psi(\phi)]\frac{\delta}{\delta\psi}(L^o+L^1) - \\ -[\mu^{\frac{4-n}{2}} e\,\psi + Z_v^{\frac{1}{2}} ev + \mu^{\frac{4-n}{2}} e\gamma(\phi)_\chi]\frac{\delta}{\delta\chi}(L^o+L^1) = 0 \tag{A1}$$

Here L_o is the renormalized Lagrangian in the tree approximation without the inclusion of the gauge-fixing part. L^1 is the effective Lagrangian which represents corrections to L^o resulting from one-loop graphs, and the $\gamma(\phi)$ are related to the generating functional of proper vertices with two ghost lines.

When we expand $\gamma_\chi(\phi)$ around the vacuum expectation values of the fields, that is, $\langle A_\mu\rangle=\langle\psi\rangle= \langle\chi\rangle= 0$, only the first term, denoted by γ_χ, is divergent, in the limit $n\to 4$. This term is represented in fig.5. The other terms, which represent the proper vertices of two ghosts and one field are finite in our model. Similarly, when we expand γ_ψ around the vacuum expectation values of fields, the first term is identically zero,while the others will be finite. We will be interested only in the divergent part, as $n\to 4$, of the above identity. We get

$$\left(\frac{1}{2} z_v\ ev + \mu^{\frac{4-n}{2}} e\gamma_\chi\right)\frac{\delta L^o}{\delta\chi} + \left(\mu^{\frac{4-n}{2}} e\psi + ev\right)\frac{\delta L^1}{\delta\chi} - \tag{A2}$$
$$-\mu^{\frac{n-4}{2}} \chi \frac{\delta L^1}{\delta\psi} - \partial_\mu \frac{\delta}{\delta A\mu} L^1 = 0.$$

On the other hand, the unrenormalized Lagrangian is invariant under the gauge transformation (17). This invariance may be formulated as:

$$-\partial_\mu \frac{\delta L_o}{\delta A^o_\mu} + e_o(\psi_o + v_o)\frac{\delta L_o}{\delta\chi_o} - e_o\chi_o \frac{\delta L_o}{\delta\psi_o} = 0 \tag{A3}$$

After renormalizing fields and parameters according to (20, 21,22,24) the Lagrangian L_o becomes L_o+Lct, Lct representing the counterterms, so that the above equation may be written as

$$\frac{1}{2} z_v\ ev \frac{\delta L_o}{\delta\chi} + \left(\mu^{\frac{4-n}{2}} e\psi + ev\right)\frac{\delta Lct}{\delta\chi} - \mu^{\frac{n-4}{2}} e\chi\frac{\delta Lct}{\delta\psi} - \partial_\mu\frac{\delta}{\delta A_\mu} Lct = 0 \tag{A4}$$

Adding (A2) and (A4) we get

$$-\partial_\mu\frac{\delta}{\delta A_\mu}(L^1 + Lct) + \left(\mu^{\frac{n-4}{2}} e\psi + ev\right)\frac{\delta}{\delta\chi}(L^1 + Lct) - \mu^{\frac{4-n}{2}} e\chi\frac{\delta}{\delta\psi}(L^1 + Lct) + \tag{A5}$$
$$+ \left(z_v\ ev + \mu^{\frac{4-n}{2}} e\gamma_\chi\right) L^o = 0.$$

Therefore, if the theory is renormalizable we must have

$$z_v = \text{pole part of } \left(-\mu^{\frac{4-n}{2}} \frac{\gamma_\chi}{v}\right) = \frac{2\,e^2}{\xi} Z_o \qquad (A6)$$

The relations above can, therefore, be written

$$-\partial(L^1+Lct)_A + (e\psi+ev)(L^1+Lct)_\chi - e\chi(L^1+Lct)_\psi = 0 \qquad (A7)$$

where we have dropped the μ term, as we are interested only in the pole part of the above equation and

$$\partial[L^1+Lct]_A \equiv \partial_\mu \frac{\delta}{\delta A_\mu}(L^1+Lct) \quad ; \quad (L^1+L_{ct})_\chi \equiv \frac{\delta}{\delta\chi}(L_1+Lct);$$

$$(L_1+Lct)_\psi \equiv \frac{\delta}{\delta\psi}(L_1+Lct) \qquad (A8)$$

Taking the functional derivative of this equation with respect to A_μ,χ,ψ we obtain respectively $(\Gamma \equiv L_1+Lct)$:

$$-\partial\Gamma_{AA_\mu} + (e\,\psi + e\,v)\,\Gamma_{A_\mu\chi} - e\chi\,\Gamma_{A\mu\psi} = 0 \qquad (A9)$$

$$-\partial\Gamma_{Ax} + (e\psi + ev)\,\Gamma_{\chi\chi} - e\chi\,\Gamma_{\chi\psi} = 0 \qquad (10)$$

$$-\partial\Gamma_{A\psi} + (e\psi + ev)\,\Gamma_{\psi\chi} + e\Gamma_\chi - e\chi\,\Gamma_{\psi\psi} = 0 \qquad (A11)$$

From the first equation, evaluated at $A=\psi=\chi=0$, we see that by choosing the counterterms such that the vector boson self-energy is finite, the vertex $\Gamma A_\mu\chi$ will be finite. With this result it follows from equation (A10) that the scalar meson χ 2-point function is also finite. By taking the functional derivatives of equations (A9,A10,A11) with respect to A_ν,ψ and χ we obtain six independent relations:

$$-\partial\Gamma_{AA_\mu A_\nu}+(e\psi+ev)\ \Gamma_{A_\mu\ A\chi\chi}-e\chi\ \Gamma_{A_\mu A_\nu\psi}=0 \tag{A12}$$

$$-\partial\Gamma\chi AA_\mu+(e\psi+ev)\ \Gamma_{\chi\chi A_\mu}-\ e\chi\ \Gamma\chi A_\mu\psi\ \ -2\Gamma_{A\mu\psi}\ =\ 0 \tag{A13}$$

$$-\partial\Gamma_{\psi AA_\mu}+(e\psi+ev)\ \Gamma_{\psi A_\mu\chi}+e\ \Gamma_{A_\mu\chi}\ -e\ \chi\Gamma_{A_\mu\psi\psi}\ =\ 0 \tag{A14}$$

$$-\partial\Gamma_{\chi\chi A}\ +(e\psi+ev)\ \Gamma_{\chi\chi\chi}\ -e\chi\Gamma_{\chi\chi\psi}\ -2\ e\Gamma_{\chi\psi}\ =\ 0 \tag{A15}$$

$$-\partial\Gamma_{\psi\chi A}\ +(e\psi+ev)\ \Gamma_{\psi\chi\chi}\ +e\Gamma_{\chi\chi}\ -e\chi\Gamma_{\chi\psi\psi}\ -e\Gamma_{\psi\psi}=0 \tag{A16}$$

$$-\partial\Gamma_{A\psi\psi}\ +(e\psi+ev)\ \Gamma_{\psi\psi\chi}\ +2e\Gamma_{\psi\chi}-e\chi\Gamma_{\psi\psi\psi}\ =\ 0 \tag{A17}$$

From eq. (A14) follows that if the chosen conterterms make $\Gamma\psi A_\mu A_\nu$ finite, then $\Gamma_{\psi A_\nu\chi}$ will also be finite. Together with the previous results, eq. (A16) shows that if the counterterms are chosen to make $\Gamma_{\psi\psi}$ finite, $\Gamma_{\psi\chi\chi}$ will also be finite.

Power counting arguments show that graphs with more than 4 external lines are convergent. Using this fact and taking the appropriate functional derivatives of the equations (A12-A17) it is straight-forward to show that $\Gamma_{\chi\chi\chi A_\mu}$, $\Gamma_{\psi\psi\chi A_\mu}$, and $\partial\Gamma_{A_\mu A_\nu A_\alpha A_\beta}$ are finite. This must be so, since there are no counterterms avilable for the above vertices and the countributions of the 1-loop terms must be finite. Finally, using these results and eqts. (A12-A17) it is easy to show that if we choose conterterms such that $\Gamma_{\psi\psi\psi}$ is finite, then the four-point vertices $\Gamma_{AA\chi\chi}$, $\Gamma_{AA\psi\psi}$, $\Gamma_{\psi\psi\chi\chi}$ and $\Gamma_{\chi\chi\chi\chi}$ will be finite. But we have determined the counterterms already, without the necessty of considering $\Gamma_{\psi A_\mu A_\nu}$ and $\Gamma_{\psi\psi\psi}$. These vertices are not directly related to the vertices Γ_{AA}, $\Gamma_{\psi\psi}$, $\Gamma_{\psi\psi\psi\psi}$, which determine all the counterterms. This is due to the fact that Lee's identities only relate vertices which are linked by gauge transformations. It can be shown, [17] however, that the counterterms already determined make finite $\Gamma_{\psi AA}$ and $\Gamma_{\psi\psi\psi}$. Finally it remains to show that $\Gamma_{AAAA} = L^1_{AAAA}$ is finite, as there are no counterterms available. The graphs contributing to the vector meson-vector meson scattering are shown in fig. 6. A detailed calculation shows that the vertex Γ_{AAAA} is indeed finite. We can understand this result as follows: since the vector mesons satisfy the Bose statistics, $L^1_{A_\mu A_\nu A_\alpha A_\beta}(K_1,K_2,K_3,K_4)$ should be invariant if we interchange $\alpha\leftrightarrow\beta; K_3\leftrightarrow K_4; \mu\leftrightarrow\nu; K_1\leftrightarrow K_2$, etc. Now, by power counting, $L^1_{A_\mu A_\nu A_\alpha A_\beta}$ can at most be logarithmically divergent, so that the pole part is independent of the momenta. Therefore L^1 must be a symmetric tensor in indices $\mu\nu\alpha\beta$. It must then have the form

$$L^{1(\text{pole})}_{\mu\nu\alpha\beta} = C(\delta_{\mu\nu}\,\delta_{\alpha\beta} + \delta_{\mu\beta}\,\delta_{\nu\alpha} + \delta_{\mu\alpha}\,\delta_{\nu\beta}).$$

Now, use of Lee's identities has shown that gauge invariance implies $\partial_\mu\, L^{1(\text{pole})}_{\mu\nu\alpha\beta} = 0$, so that we must have $C=0$

REFERENCES

1. C.G.Callan, Phys.Rev. D2, 1541(1970).
2. K.Symanzik, Comm.Math.Phys. 18, 227 (1970)
3. M.Gell-Mann,F.Low, Phys. Rev. 95, 1300 (1954)
4. S.Weinberg, Phys.Rev. D10,3497(1974).
5. G.'t Hooft, Nucl.Phys. B61, 455 (1973)
6. G.'t Hooft,M.Veltman, Nucl.Phys. B44,189 (1972).
7. C.Bollini, J.Giambiagi, Nuovo Cimento B12,20(1972).
8. J.C.Collins, Phys.Rev. D10, 1201 (1974).
9. M.J.Holwerda, W.L.Van Neerven, R.P.Van Royen,Nucl.Phys.B75,302 (1974)
10. J.C.Collins, Ward Identities, the Renormalization Group and the Callan-Symanzik Equation, Cambridge preprint.
11. B.W.Lee, Phys.Lett. 46B, 214 (1973).
12. B.W.Lee, Phys.Rev. D9, 933 (1974).
13. S. Sarkar, Nucl.Phys.B83, 108 (1974)
14. M.Tonin, Lett. Nuovo Cimento 9, 541 (1974)
15. S.Coleman, Dilatations, in A.Zichichi (ed.),Properties of Fundamental Interactions, Part A,pp.359, Academic Press (1973).
16. B.W.Lee, W.I.Weisberger, Phys.Rev. D10, 2530 (1974).
17. T.Appelquist, J.Carazzone, T.Goldman and H.Quinn, Phys.Rev. D8, 1747 (1973).

Figure captions.

Fig.1 - Feynman rules in the ξ gauge. The wavy, straight, and dashed lines stand for, respectively, the A_μ, ψ and χ lines. The dashed line with the arrow represent the Fadeev-Popov ghost and appears only in closed loops. The Feynman rules for the graphs a)...n) are given respectively, by the following expressions:

a) $\frac{1}{i}\ \frac{1}{K^2+(ev)^2}\left[\delta_{\alpha\beta}+\frac{(1-\xi)K_\alpha K_\beta}{(ev)^2+\xi K^2}\right]$

b) $\frac{1}{i}\ \frac{1}{K^2+hv^2-\frac{M^2}{2}+\frac{(ev)^2}{\xi}}$

c) $\frac{1}{i}\ \frac{1}{K^2+3hv^2-\frac{M^2}{2}}$

d) $\frac{1}{i}\ \frac{1}{K^2+\frac{(ev)^2}{\xi}}$

e) $\frac{1}{i}\ \mu^{(4-n)/2}\ 6hv$

f) $\frac{1}{i}\ \mu^{(4-n)/2}\ 2hv$

g) $\frac{1}{i}\ \mu^{(4-n)/2}2\ e^2v\ \delta_{\alpha\beta}$

h) $\frac{1}{i}\ \mu^{(4-n)/2}\ \frac{1}{\xi}\ e^2v$

i) $\mu^{(4-n)/2}(K+K')_\alpha\ e$

j) $\frac{1}{i}\ \mu^{4-n}\ 2h.$

k) $\frac{1}{i}\ \mu^{(4-n)}6h$

l) $\frac{1}{i}\ \mu^{4-n}\ 6h$

m) $\frac{1}{i}\ \mu^{4-n}2e^2\ \delta_{\alpha\beta}$

n) $\frac{1}{i}\ \mu^{4-n}\ 2e^2\ \delta_{\alpha\beta}$

Fig.2. - One loop tadpole graphs.

Fig.3. One loop graphs for the vector-meson self energy.Crosed diagrams are always to be understood.

Fig. 4. One loop graphs for the scalar meson ψ four-point function.

Fig. 5. One loop diagrams for $^{\gamma}\chi$. The vertex denoted by $\otimes$ is equal to 1.

Fig. 6 One loop graphs for the vector meson four-point function.

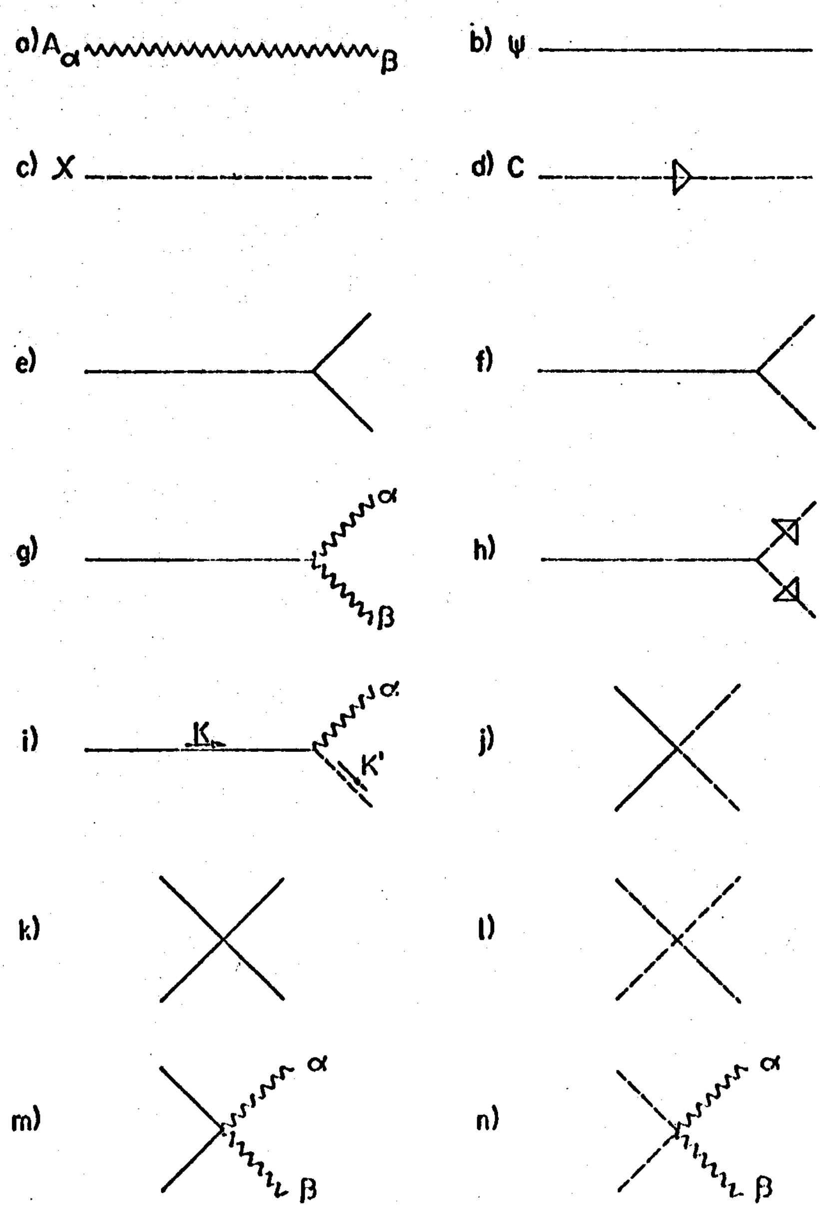
a) A_α
β
b) ψ
c) X
d) C
e)
f)
g)
α
β
h)
i)
K
α
K'
j)
k)
l)
m)
α
β
n)
α
β

Fig. 1

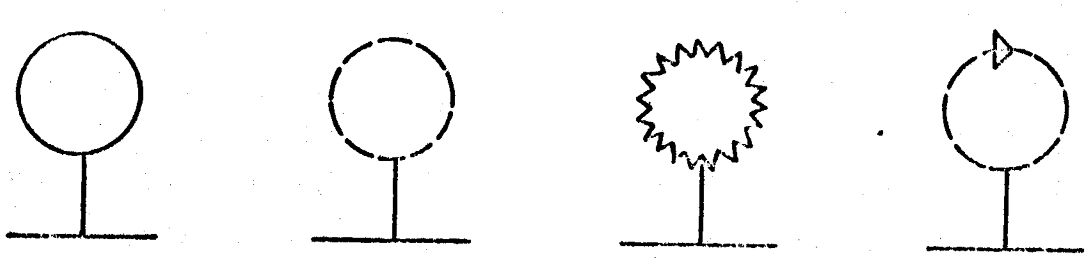

Fig. 2

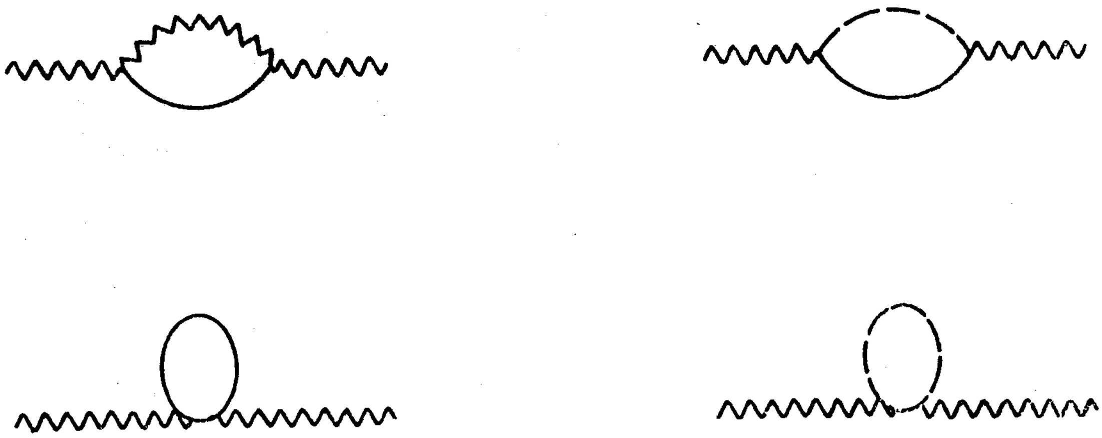

Fig. 3

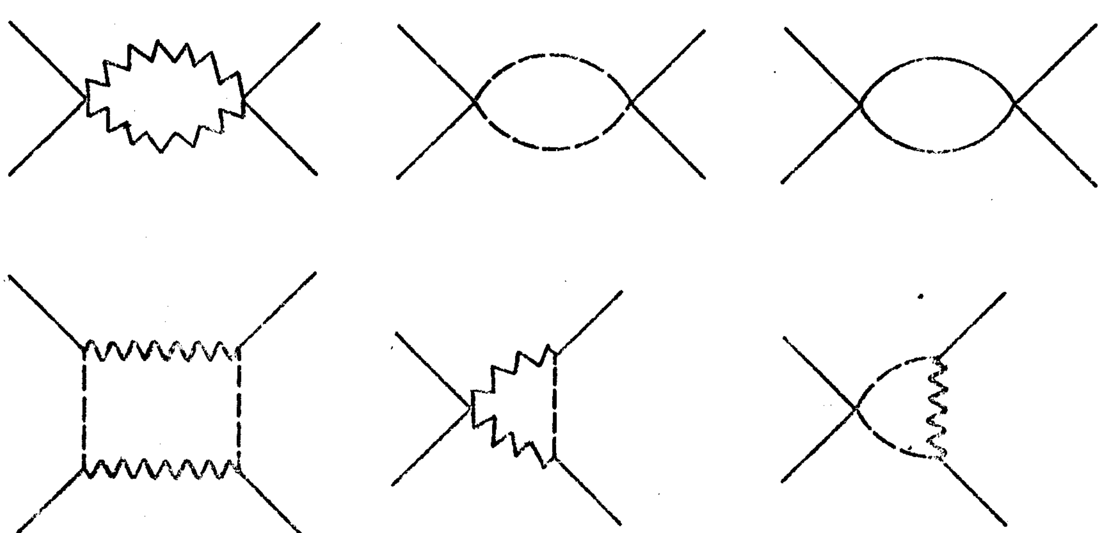

Fig. 4

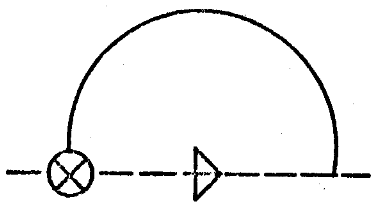

Fig. 5

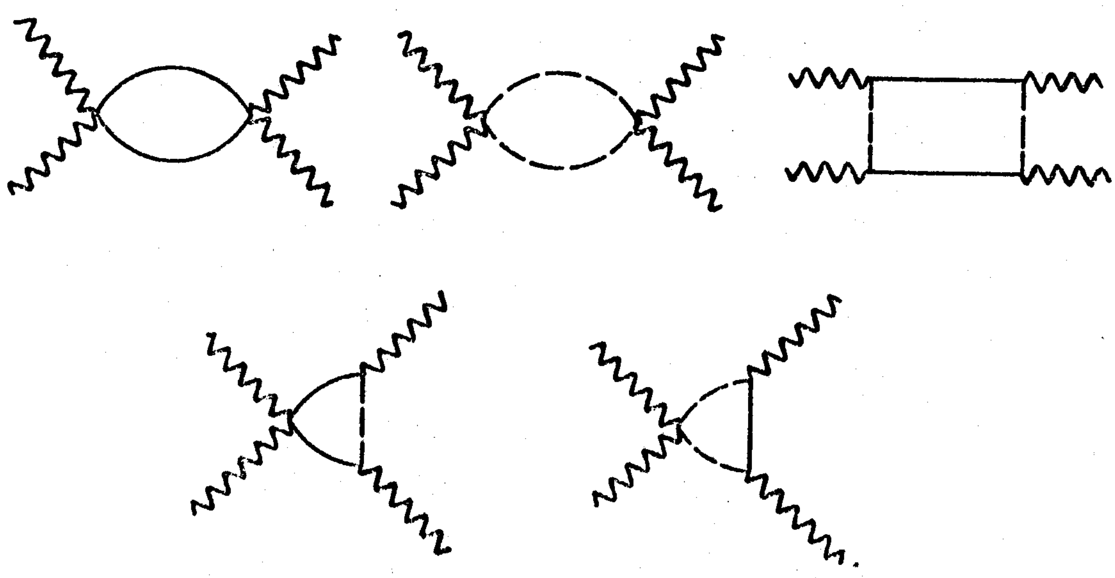

Fig. 6