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Publicação IF - 1445/2000

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ABSTRACT

The dynamics of Josephson-like oscillations between two coupled Bose-Einstein condensates is studied using the time-dependent variational method. We suppose that the quantum state of the condensates is a gaussian wave-packet which can translate and perform breathing shape oscillations. Under this hypotheses we study the influence of these degrees of freedom on the tunneling dynamics by comparing the full-model with one where these degrees of freedom are "frozen" at its equilibrium values. The result of our calculation shows that when the traps are not displaced the two models agree, whereas when they are, the models differ considerably, the former being now closer to its linear approximation.

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The investigation of collective excitations of Josephson-coupled two-component Bose-Einstein condensates has been the subject of many papers [1–3]. The main interest in those papers is the study of the dynamics associated to the exchange of atoms between the two condensates. The basic hypotheses in most of these studies is that this exchange is “coherent” [4], that is, without changing the quantum state of each condensate, which leads to a dynamics involving only the degree of freedom associated to the relative population and phase of the condensates. Based on this “frozen” model, a host of interesting phenomena is predicted, for example, the existence of a “phase-transition” to a non-symmetrical equilibrium composition in the limit of strong repulsion between the two condensates and “weak” Josephson coupling [1,2].

One exception, however, is the work of reference [3] where the effect of the change of the quantum state is taken into account. They use standard mean-field theory to derive coupled time-dependent Gross-Pitaevskii equations for the time-evolution of the order parameter of each condensate. They simplify the problem by treating the system in only one spatial dimension and they observe the effect of the mean-field on the non-linear oscillations of the relative population between the two condensates.

In this paper we study the effect of the change of the quantum state in the tunneling dynamics. Our approach is in the spirit of reference [3], however we take a more qualitative point of view along the line of references [5–8]. We suppose that the quantum state of the system is a gaussian wave-packet which can translate and perform breathing shape oscillations. Under this hypotheses, we study the influence of these degrees of freedom on the tunneling dynamics by comparing the full-model with the limit where these degrees of freedom are “frozen” at its equilibrium values. In our study, we also investigate, in the context of the full-model, under what conditions the linear dynamics is valid.

Our starting point is the action

$$S \ = \ \int d^3\vec{r}dt \, \Big[\ \sum_j i\hbar\psi_j^*(\vec{r},t)\dot{\psi}_j(\vec{r},t) - \mathcal{E}(\vec{r},t) \ \Big] \tag{1}$$

where $\psi_j(\vec{r},t)$, $j=1,2$ are the order parameters of each condensate and the energy density is given by

$$\mathcal{E}(\vec{r},t) = \sum_j \psi_j(\vec{r},t)^* \Big[-\frac{\hbar^2\nabla^2}{2m} + V^j_{trap} + \frac{\delta}{2}\gamma_j \Big] \psi_j + \frac{1}{2}\sum_{k,j} \lambda_{kj} |\psi_j(\vec{r},t)|^2 |\psi_k(\vec{r},t)|^2 + \Omega \Big[\psi_1^*\psi_2 + \psi_2^*\psi_1 \Big] \ . \tag{2}$$

In the above expression $V^j_{trap}(\vec{r})$ is the trapping potential of each component which we take as

$$V^j_{trap} = \frac{m}{2}\Big[\omega_\perp^2(x^2+y^2) + \omega_z^2(z+\gamma_j z_0)^2\Big] \tag{3}$$

where $\gamma_1 = 1$ and $\gamma_2 = -1$, λ_{jk} are proportional to the scattering lengths a_{jk} for the collisions of atoms j and k of equal mass, $\lambda_{jk} = \frac{4\pi}{m}a_{jk}$, the term that depends on δ takes into account the effects of detuning, and the last term is the Josephson interaction term where Ω is the tunneling intensity [3].

The condition that the action is stationary with respect to variations of the order parameters leads to the coupled mean-field equations for $\psi_j(\vec{r},t)$ [3]. In this paper we follow references [5–8] and instead we parametrize the time dependence of the order parameter through the variation of a set of $2n$ parameters [9] which we denote by $\mathbf{X} = \{X_1, X_2, ..., X_{2n}\}$,

$$\psi_j(\vec{r},t) = \psi_j(\vec{r},\mathbf{X}(t)) \tag{4}$$

Given this parametrization the action reduces to a "classical action"in terms X_i(t) [9]

$$S \ = \ \int dt \, \Big[\sum_k \Gamma_k^{(1)}(\mathbf{X})\dot{X}_k - E(\mathbf{X}) \Big] \tag{5}$$

where

$$\Gamma_k^{(1)} \ = \ \frac{i}{2}\sum_j \left(\langle\psi_j|\frac{\partial\psi_j}{\partial X_k}\rangle - \langle\frac{\partial\psi_j}{\partial X_k}|\psi_j\rangle \right) \tag{6}$$

In (6) we introduced the state vector $|\psi_j(X(t))\rangle$ whose norm is equal to the population of the component j of the condensate and whose wave function is given by equation (4), and $E(\mathbf{X})$ is the spatial integral of the energy density

$$E(\mathbf{X}) = \int d^3\vec{r}\mathcal{E}(\vec{r}, t) \tag{7}$$

with the order parameter parametrized as in eq. (4).

Requiring that the action is stationary with respect to variations of the parameters leads to Hamiltonians-type equations of motion for $X_i(t)$:

$$\sum_k \Gamma^{(2)}_{jk}\ \dot{X}_k = \frac{\partial E}{\partial X_j}\ , \tag{8}$$

where the antisymmetric matrix $\Gamma^{(2)}_{jk}$ is given by

$$\Gamma^{(2)}_{jk} = \frac{\partial \Gamma^{(1)}_k}{\partial X_j} - \frac{\partial \Gamma^{(1)}_j}{\partial X_k} \tag{9}$$

To proceed we should specify our choice of parameters. Previous work [6–8] have shown that a parametrization which reproduces very well the equilibrium properties and collective excitations of a single-component condensate is one where the condensate density is a gaussian wave-packet which can translate and perform quadrupole and monopole shape oscillations and its corresponding motions. The study of the two-component condensate dynamics with the above parametrization is feasible and is under investigation. Here, in the spirit of reference [3], we simplify the problem by treating the system in only one spatial dimension.

Thus we write the order parameter as:

$$|\psi_j\rangle = \sqrt{N_j(t)}e^{-i\theta_j}|\phi_j\rangle \tag{10}$$

where we put explicitly the degree of freedom associated to the population and phase of each condensate. The condensate orbitals $|\phi_i\rangle$ [10] are taken as gaussian wave-packets which can translate and perform breathing shape oscillations,

$$\phi_j(z,X)\rangle = \Big(\frac{1}{\sqrt{\pi}q_j}\Big)^{\frac{1}{2}} e^{iP_j(t)\left[z-Q_j(t)\right]-\left[\frac{1}{2q_j^2(t)}-i\frac{p_j(t)}{2q_j(t)}\right]\left[z-Q_j(t)\right]^2} \tag{11}$$

where the parameters (Q_j, P_j) are related to the translational degrees of freedom and (q_j, p_j) to the "breathing" oscillation. Notice that in this parametrization the quantum state of each condensate (called here the condensate orbital) $|\phi_j\rangle$ does not depend explicitly on the phase and population of the condensates.

Thus, we have twelve parameters which leads to a twelve dimensional "phase space". Since $E(\mathbf{X})$ depends only on the difference of phase between the two condensates, the equations of motion (8) give that the total number of particles is conserved. This reduces the number of degrees of freedom to ten, which we take as besides the $q_i, Q_i, p_i, P_i, i = 1, 2$, the relative phase and relative population fraction. Thus, in the full-dynamics, we solve ten coupled first order equations, given an initial configuration.

Since $\det\Gamma^{(2)}(\mathbf{X}) \neq 0$, the equilibrium configuration of the system is determined by the condition

$$\frac{\partial E}{\partial X_j}(X_0) = 0. \tag{12}$$

Linearizing the equation (8) in the neighborhood of the equilibrium configuration, one has

$$\sum_k \Gamma^{(2)}_{jk}(\mathbf{0})\dot{x}_k(t) = \sum_k \frac{\partial^2 E}{\partial X_j \partial X_k}(\mathbf{0}) x_k \tag{13}$$

where x is the displacement from equilibrium

$$x_k = X_k - X_{k0} \tag{14}$$

and $\mathbf{0}$ in these equations denote that the quantities are evaluated at the equilibrium configuration. The solution of the coupled linear equations (13) define the small oscillation dynamics around this point.

Finally, the frozen model is obtained by constraining all the degrees of freedom, except the relative population fraction and the relative phase at its equilibrium values. When this is done, the equations of motion (7) reduces to [1,2]

$$\dot{\theta} = \Delta + \frac{\Lambda\eta}{2} - \omega_R \, \frac{\eta}{(1-\eta^2)^{1/2}} \cos\theta \tag{15}$$

$$\dot{\eta} = \omega_R \, (1-\eta^2)^{1/2} \sin\theta \tag{16}$$

where $\theta = \theta_2 - \theta_1$ and $\eta = \eta_2 - \eta_1$, and in equations (15) the time is measured in unit of ω_0, with ω_0 the trap frequency. The parameters defining the frozen model are :

$$\Delta = \varepsilon^2 - \varepsilon^1 - \delta \tag{17}$$

$$\Lambda = \bar{\lambda}_{11} + \bar{\lambda}_{22} - 2\bar{\lambda}_{12} \tag{18}$$

$$\omega_R = \frac{2\Omega}{\hbar\omega_0} \int d^3\vec{r}\phi_1^*(\vec{r}, \mathbf{0})\phi_2(\vec{r}, \mathbf{0}) \tag{19}$$

$$\tag{20}$$

where ε^j is essentially the energy per-particle of each condensate (in units of ω_0)

$$\varepsilon^j = \left[\langle\phi_j|t + V^j_{trap}|\phi_j\rangle + \frac{N}{2}\lambda_{jj} \int d^3\vec{r}|\phi_j(\vec{r}, \mathbf{0})|^4\right]/\hbar\omega_0 \tag{21}$$

and the $\bar{\lambda}_{jk}$ are

$$\bar{\lambda}_{jk} = N\frac{\lambda_{jk}}{\hbar\omega_0} \int d^3\vec{r}|\phi_j(\vec{r}, \mathbf{0})|^2|\phi_k(\vec{r}, \mathbf{0})|^2 \tag{22}$$

We apply the formalism presented above to the system considered in reference [3], trapped ^{87}Rb atoms in hyperfine states $|f = 1, m_f = -1\rangle$ and $|f = 2, m_f = 1\rangle$. Following this reference, the initial configuration is such that the only degree of freedom which is not at equilibrium is the phase difference and the detuning is chosen in such a way that the system is initially driven resonantly [3], which implies that $\Lambda = 0$.

As a first example we will discuss the case where the traps are not displaced ($z_0 = 0$) and $a_{11} = a_{22} = a_{12} = a_{\mathrm{Rb}}$. In this case we have a complete agreement between the "frozen" and "full model", indicating that the tunneling dynamics is such that the quantum state of the condensates do not change. This result is a consequence of the fact that, in this symmetrical case, $q_i = q_{i0}$, $Q_i = Q_{i0}$, $p_i = P_i = 0$, $i = 1, 2$ is an invariant subspace (also called maximally decoupled subspace) of the system, that is, if the system is in this surface, it remains there at all time [11].

We can also examine the importance of non-linear effects in this symmetrical case. The linearization of the frozen model gives a frequency of oscillation equal to the Rabi frequency ω_R, that in our case is equal to $\omega_R = 0.100$, which coincides with the frequency of the lowest energy normal mode of equations (13), $\omega_{nm}^{(1)} = 0.100$. When compared to the full model, the linear approximation basically differs only in the amplitude of the oscillation, being larger for the latter.

As a more realistic application, we consider the case where the trap is not displaced ($z_0 = 0$) and the scattering lengths are equal to the experimental values as in Table 1 of reference [3]. In fig. 1 we show the results of the calculation corresponding to the frozen model, full-model and its linearization, when $\phi(0) = 0.7\pi$ as explained in the caption. As shown in these figures, there is again complete agreement between the "full" and "frozen"model. The translational degrees of freedom remain at its equilibrium values, whereas the displacements of the ones associated to the "breathing" oscillation are negligible (notice the scale of the figs. 1d and 1e). Therefore, we can conclude that the invariant subspace found in the symmetrical case survives and, as a consequence, we have a complete agreement between the two models.

In the case under investigation, the parameters of the frozen model are $\omega_R = 0.100$ and $\Lambda = -0.017$, showing that we are in the "strong Josephson" coupling regime $\frac{2\omega_R}{|\Lambda|} > 1$ [4,12]. The linearization of the frozen model gives a frequency equal to $\omega_F = \sqrt{\omega_R(\omega_R + \frac{\Delta}{2})}$ which coincides with the lowest energy normal mode of equation (13), whose frequency is $\omega_{nm}^{(1)} = 0.096$. As in the symmetrical case, the linear approximation of the full model differs from the other two models in the amplitude of the oscillation, with the former having a larger amplitude. Of course, as $\theta(0)$ approaches the equilibrium value, $\theta(0) = \pi$, this difference is less pronounced.

Now we will turn to the investigation of the effect of the displacement of the traps on the system. To this end, we repeat the previous calculation with $z_0 = 0.15 z_{sho}$ [3]. In fig. 2 we show the curves for the relative population fraction, translation and "breathing"oscillation of each condensate component in the three models, with $\phi(0) = 0.7\pi$ as explained in the captions.

From these figures we see that there is a good agreement between the full-model and its linear approximation whereas they both differ from the "frozen" model. From figure (2a) we observe that, for the relative population, the "frozen" model gives a much smaller amplitude and a much higher frequency, compared to the other two models. One the other hand, the results of the calculation according to the full model and its linear approximation are very close, the latter having a slighter higher amplitude and frequency. We observe the same qualitative pattern for the translational degrees of freedom, which oscillate in phase and the "breathing" ones, which oscillate out of phase.

In the present case, the parameters of the frozen model are $\Delta = 128.4$ and $\omega_R = 0.0523$, showing that we are in the regime of weak Josephson coupling $\frac{2\omega_R}{|\Delta|} < 1$. The decrease of ω_R and the increase of Δ can be understood as basically an effect of the displacement of the traps. This displacement diminishes the overlap between the condensate orbitals $\phi_j(\vec{r}, \mathbf{0})$, which decreases ω_R and $\bar{\lambda}_{12}$ and increases Δ.

The linear approximation of the frozen model gives a frequency of $\omega_F \approx 1.832$ much higher than the observed oscillation of the relative population. On the other hand, the lowest energy normal mode of the linear equations (13) has a frequency equal to 0.4466, which compares well with the frequency of oscillation for the relative population and the translation degree of freedom calculated in the linear approximation to the full-model. This result indicates that these degree of freedom has components basically in this normal mode. In turn, the time-evolution of the "breathing"variables, figs. (2d) and (2e), although dominated by this mode, show the presence of higher frequency modes.

Comparing the two calculations at different displacements, we see that the full model predicts that, when we displace the trap, the amplitude diminishes and the frequency increases, in agreement with reference [3]. The frozen model gives the same qualitative prediction, however it overestimates the size of the effect.

To summarize, we have investigated the influence of the change of the quantum state in the tunneling dynamics of Josephson coupled two-component BEC, for different trap geometries. From our calculations, we have observed that when the trap is not displaced the tunneling dynamics is "coherent", that is, the orbitals of each condensate do not change. As a consequence, we have agreement between the frozen model [1,2] and the full model. On the other hand, when we displace the traps, the change in the condensate orbitals has a strong influence in the tunneling dynamics. Compared to the frozen model, which neglects this change, the full model gives a smaller frequency and higher amplitude of oscillation. We also found that in this latter case the full-model is much closer to its linear approximation than to the frozen model.

As a final remark, it is clear that, by changing the initial condition, we can get a large variety of dynamical behaviors for the system. Our aim in this paper was to establish a framework where all these questions could be investigated in a scheme numerically feasible and which can be easily extended to a three dimensional calculation. The application of the formalism presented in this paper to this more realistic case is currently under investigation.

ACKNOWLEDGMENT: This work was partially supported by Fundacção de Amparo *à* Pesquisa do Estado de S*ã*o Paulo (FAPESP). E. J. V. de Passos was supported in part by CNPq. The work of Da-Shin Lee was supported in part by the National Science Council, ROC under the Grant NSC-89-2112-M-259-008-Y.

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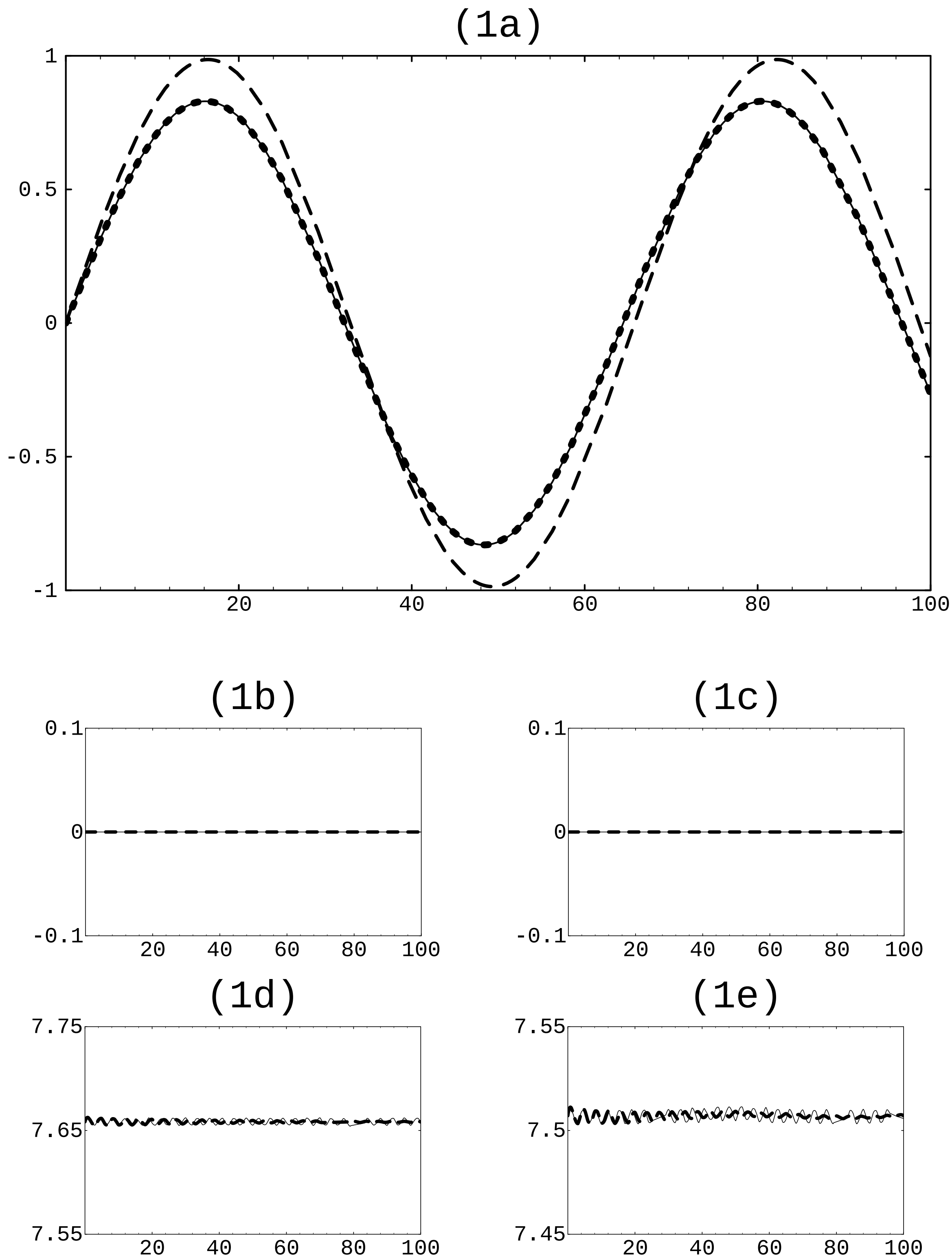
Figure Caption

Figure 1. Plot showing time evolution of relative population (1a), translation (1b-1c) and breathing (1d-1e). The solid curve corresponds to the full model, the doted curve to the "frozen" model and dashed to the linear approximation. The initial condition is the relative phase at $\theta(0) = 0.7\pi$ and the other variables are at its equilibrium values. The displacement of the trap z_0 is set equal to 0. The time is expressed in unit of $\frac{1}{\omega_0}$ and the lengths in unit of z_{sho}, which is the size parameter of the trap. See text for more details.

Figure 2. The conventions are the same as in figure 1. The displacement of the trap z_0 is set equal to $0.15 z_{sho}$. The initial phase is $\theta(0) = 0.7\pi$ and the other variables are at its equilibrium values. See text for more details.

Figure 1



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-1
20
40
60
80
100
(1b)
0.1
0
-0.1
20
40
60
80
100
(1c)
0.1
0
-0.1
20
40
60
80
100
(1d)
7.75
7.65
7.55
20
40
60
80
100
(1e)
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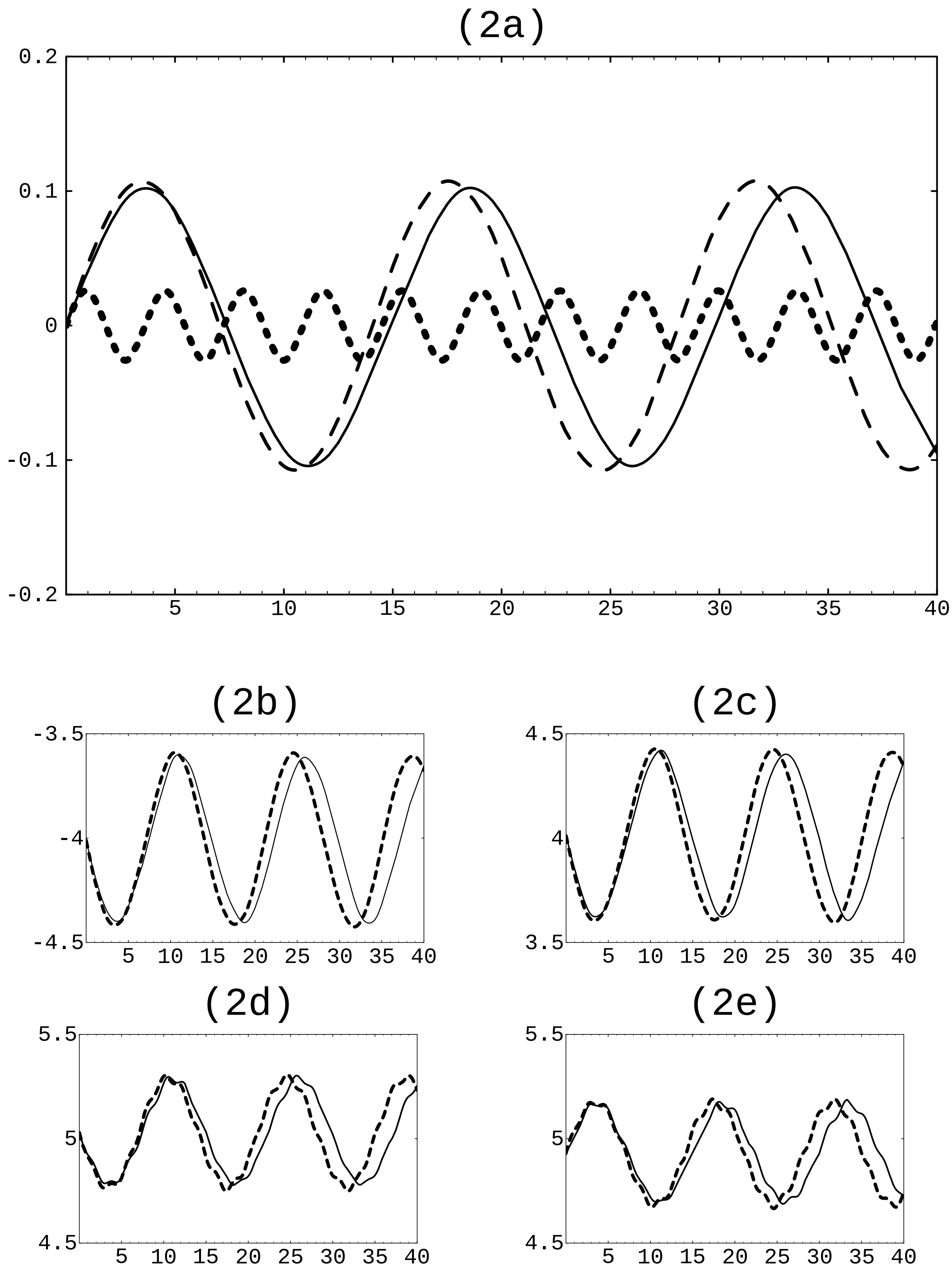
40
60
80
100

Figure 2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15
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(2b)
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(2c)
4.5
4
3.5
(2d)
5.5
5
4.5
(2e)
5.5
5
4.5