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Hamiltonization of Lagrangian theories with degenerate coordinates

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Hamiltonization of Lagrangian theories with degenerate coordinates.

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Abstract

We consider a special class of Lagrangian theories where part of the coordinates does not have any time derivatives in the Lagrange function (we call such coordinates degenerate). We advocate it is reasonable to reconsider the conventional definition of singularity based on the usual Hessian and, moreover, to simplify the conventional Hamiltonization procedure. In particular, in such a procedure, it is not necessary to complete the degenerate coordinates with the corresponding conjugate momenta.

1 Introduction

The Hamiltonization of Lagrangian theories is an important preliminary step towards their canonical quantization [1, 2, 3]. The procedure is quite different for nonsingular and singular theories. Whereas for nonsingular theories such a procedure is, in fact, the well-known Legendre transformation, the Hamiltonization of singular theories is sometimes a difficult task. The singularity property of a theory is usually defined by the corresponding Hessian, which is zero in the singular case. The Hamiltonization procedure also depends essentially on theory structure. In particular, it depends on the highest orders of time derivatives in the Lagrange function. In principle, the Hamiltonization procedure is quite well developed for theories of arbitrary orders $N \geq 1$ of time derivatives [5]. However, as we are going to demonstrate, for a special class of theories where part of the coordinates does not have any time derivatives in the Lagrange function (we call such coordinates degenerate) it is reasonable to reconsider the conventional definition of singularity and, moreover, to simplify the conventional Hamiltonization procedure. In particular, in such a procedure (we call it the generalized Hamiltonization procedure) we do not complete the degenerate coordinates with the corresponding conjugate momenta. Indeed, it seems exaggerate to introduce a momentum for the variable p in a theory whose Lagrange function is $L = p\dot{q} - V(q,p)$ (the corresponding action already has Hamiltonian form) and then to struggle with irrelevant constraints, see relevant remarks in [4]. We show that the degenerate coordinates may be treated on the same footing as usual velocities (or highest order time derivatives in the Lagrangian function). In fact, some observations about the possibility of a special treatment of the degenerate coordinates were already implicitly presented in literature. In this regard one can recall that sometimes, in the course of the Hamiltonization of the Maxwell theory, A_0 is considered a Lagrange multiplier to a constraint and no conjugate momentum to A_0 is introduced, see for example [1]. For theories with degenerate coordinates, the generalized Hamiltonization procedure contains less stages than the usual Hamiltonization procedure and needs less suppositions about the theory structure. There exist some models to which only the generalized Hamiltonization procedure is applicable. In this relation one ought to say that almost all modern physical gauge models are theories with degenerate coordinates.

The present paper is organized as follows: in Sect.II, using a simple but instructive example, we advocate a new definition of singularity and consider the possibility of simplifying Hamiltonization for theories with degenerate coordinates. Then in Sect.III we formulate the generalized Hamiltonization procedure and new criteria for singularity in the general case of a Lagrangian theory with degenerate coordinates. In Sect.IV we consider some relevant examples. In the Appendix we discuss an useful notion and properties of auxiliary variables, the latter being often used in the main text.

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2 Theories with degenerate coordinates

2.1 Which theories are conventionally called singular?

We first recall the conventional definition of singularity of a theory through the example of a theory without higher order time derivatives for which the action reads $S=\int Ldt$, and the Lagrange function has the form $L=L(q,\dot{q})$, where $q=(q^a;a=1,2,...,n)$ is the set of generalized coordinates and $\dot{q}=(\dot{q}^a\equiv dq^a/dt)$. In such a case, the Hessian M is used for the classification. Namely:

$$M=\det\left\|\frac{\partial^2 L}{\partial\dot{q}^a\partial\dot{q}^b}\right\|=\begin{cases}\neq 0, \text{ nonsingular theory}\\ =0, \text{ singular theory}\end{cases}. \tag{1}$$

Whenever a theory is nonsingular according to the above definition, the corresponding Euler-Lagrange equations of motion (EM) can be solved with respect to the highest time derivatives (here with respect to second-order derivatives) of all the coordinates. Indeed,

$$\begin{aligned}&\frac{\delta S}{\delta q^a}=\frac{\partial L}{\partial q^a}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}^a}=0\Longrightarrow M_{ab}\ddot{q}^b=K_a=\frac{\partial L}{\partial q^a}-\frac{\partial^2 L}{\partial\dot{q}^a\partial q^b}\dot{q}^b;\\ &\text{thus, }\frac{\delta S}{\delta q^a}\Longrightarrow\ddot{q}^a=M^{ab}K_b\,,\;M_{ab}=\frac{\partial^2 L}{\partial\dot{q}^a\partial\dot{q}^b}\,,\;M^{ab}M_{bc}=\delta^a_c\,.\end{aligned} \tag{2}$$

Which in turn means that the EM of a nonsingular theory of the above type always have a unique solution whenever $2n$ initial data are given. Hamiltonization of nonsingular theories leads to the Hamilton EM without any constraints on the phase-space variables q,p. Recall that the conventional Hamiltonization procedure may be formulated as follows: we pass to the first-order formalism introducing additional variables v, called velocities, and imposing the relation $v=\dot{q}$. Then an equivalent first-order action S^v reads:

$$\begin{aligned}&S^v=\int\left[L^v+p_a\left(\dot{q}^a-v^a\right)\right]dt=\int\left[p_a\dot{q}^a-H^v\right]dt\,,\\ &L^v=L|_{\dot{q}=v}=L(q,v)\,,\quad H^v=p_a v^a-L^v\,.\end{aligned} \tag{3}$$

The Lagrange multipliers p to the EM $v=\dot{q}$ are considered conjugate momenta to the coordinates q. The pairs q,p form the phase-space and all the variables $q\,,\,p\,,v$ form the extended phase-space. The first-order EM have the form:

$$\frac{\delta S^v}{\delta p_a}=\dot{q}-v=0,\;\Longrightarrow\dot{q}^a=\{q^a,H^v\}\,, \tag{4}$$

$$\frac{\delta S^v}{\delta q^a}=\frac{\partial L^v}{\partial q^a}-\dot{p}_a\;\Longrightarrow\dot{p}_a=\{p_a,H^v\}\,. \tag{5}$$

$$\frac{\delta S^v}{\delta v^a}=-\frac{\partial H^v}{\partial v^a}=0\;\Longrightarrow p_a=\frac{\partial L^v}{\partial v^a}\,. \tag{6}$$

Here $\{,\}$ are usual Poisson brackets in the phase space. Due to Eqs. (6), there exist two possibilities of treating the velocities v inside the Poisson brackets: as quantities independent of the q,p variables or as quantities dependent on the q,p variables. The set v,p can be expressed with the help of Eqs. (6,4) via the set $q,\dot{q}$ (this is possible both in nonsingular and in singular theories). Thus, the variables v,p can be treated as auxiliary variables for the action S^v (see Appendix). Substituting v,p as functions of $q,\dot{q}$ in the action S^v and in the EM (5), we reproduce both the initial action S and the Lagrangian EM. The theory with action S^v is equivalent to the theory with initial action S. Performing the Hamiltonization, we try to eliminate the velocities v from the action S^v and from the first-order EM. For nonsingular theories, due to the condition $M\neq 0$, we can solve the equations (6) with respect to all the velocities, $v=\bar{v}(q,p)$. Thus, all the velocities v are auxiliary variables in the first-order formulation of nonsingular theories and can be excluded from the action. One can easily verify that after such an exclusion, we get the usual Hamilton action $S_{\rm H}$, and usual Hamilton EM for the unconstrained phase-space variables:

$$\begin{aligned}&S_{\rm H}=S^v|_{v=\bar{v}}=\int_{t_1}^{t_2}(p\dot{q}-H)\,dt,\;H=H^v|_{v=\bar{v}}\;,\\ &\dot{q}^a=\{q^a,H\}\;,\;\dot{p}_a=\{p_a,H\}\,.\end{aligned}$$

Hamiltonization of singular theories is more complicated, see for example, [1, 2, 3]. In particular, there appear constraints both in the Lagrangian and the Hamiltonian formulations, and sometimes not all the velocities can be excluded from the first-order action. They remain in the Hamiltonian formulation as Lagrange multipliers to primary constraints.

In the general case (theories with higher derivatives), the action reads:

$$
\begin{aligned}
& S=\int L dt,\ L=L\left(t, q^{(l)}\right), \\
& q^{(l)}=\left(q^{a(l_a)} \equiv d^{l_a} q^a / dt^{l_a}\right),\ a=1, \ldots, n,\ l_a=0,1, \ldots, \bar{N}_a .
\end{aligned} \tag{7}
$$

Here L depends on the coordinates $q^a=q^{a(0)}$ and their time derivatives $q^{a(l_a)}$ up to some finite orders $\bar{N}_a$. Bearing in mind the same Lagrange function (7), it is sometimes convenient to assume it to be a function of the coordinates and their time derivatives up to some finite order $N_a \geq \bar{N}_a$, where $\bar{N}_a$ are the above mentioned orders of the time derivatives that actually enter in L. Thus, we introduce a set of theories with the same Lagrange function L but with different orders $\{N_a\}$. From the point of view of the Lagrangian formulation it is obvious that all theories with the same L and different orders $\{N_a\}$ are equivalent. Even though their Hamiltonization involves different extended phase spaces, we end up with equivalent formulations [5, 2].

A generalization of the definition (1) for theories with Lagrange function L and orders $\{N_a\}$ was proposed in [5, 2]. Such a definition is based on a simple generalization of the Hessian,

$$
M=\det\left\|\frac{\partial^2 L}{\partial q^{a(N_a)} \partial q^{b(N_b)}}\right\|=\begin{cases}\neq 0, & \text{nonsingular theory} \\ =0, & \text{singular theory}\end{cases},\ N_a \geq 1 . \tag{8}
$$

When we effected the conventional Hamiltonization, we proceeded from a system of first-order equations. To this end, we introduce new variables $q_{s_a}^a$, v^a and impose the relations $q_{s_a}^a \equiv q^{a(s_a-1)}$, $v^a \equiv q^{a(N_a)}$, $s_a=1, \ldots, N_a$. The variables v^a are called velocities. The variational principle for the initial action $S=\int L dt$ is equivalent to the one for the first-order action S^v,

$$
\begin{aligned}
S^v & =\int\left[L^v+\sum_{s_a=1}^{N_a-1} p_a^{s_a}\left(\dot{q}_{s_a}^a-q_{s_a+1}^a\right)+p_a^{N_a}\left(\dot{q}_{N_a}^a-v^a\right)\right] dt \\
& =\int\left[\sum_{s_a=1}^{N_a} p_a^{s_a} \dot{q}_{s_a}^a-H^v\right] dt, \\
L^v & =\left.L\right|_{q^{a(s_a-1)}=q_{s_a}^a,\ q^{a(N_a)}=v^a},\ H^v=\sum_{s_a=1}^{N_a-1} p_a^{s_a} q_{s_a+1}^a+p_a^{N_a} v^a-L^v .
\end{aligned} \tag{9}
$$

As before, the momenta p appear as Lagrange multipliers to the new imposed equations. The pairs q, p form the phase space and all the variables q, p, v form the extended phase space. The corresponding Euler-Lagrange EM read:

$$
\begin{aligned}
& \left.\begin{array}{c}
\frac{\delta S^v}{\delta p_a^{s_a}}=\dot{q}_{s_a}^a-q_{s_a+1}^a=0,\ s_a=1, \ldots, N_a-1 \\
\frac{\delta S^v}{\delta p_a^{N_a}}=\dot{q}_{N_a}^a-v^a=0
\end{array}\right\} \Longrightarrow\left\{\begin{array}{c}
\dot{q}_{s_a}^a=\{q_{s_a}^a, H^v\} \\
s_a=1, \ldots, N_a
\end{array}\right. ; \\
& \left.\begin{array}{c}
\frac{\delta S^v}{\delta q_1^a}=\frac{\partial L^v}{\partial q_1^a}-\dot{p}_1=0 \\
\frac{\delta S^v}{\delta q_{s_a}^a}=\frac{\partial L^v}{\partial q_{s_a}^a}-p_a^{s_a-1}-\dot{p}_a^{s_a}=0,\ s_a=2, \ldots, N_a
\end{array}\right\} \Longrightarrow\left\{\begin{array}{c}
\dot{p}_a^{s_a}=\{p_a^{s_a}, H^v\} \\
s_a=1, \ldots, N_a
\end{array}\right. ; \\
& \frac{\delta S^v}{\delta v^a}=\frac{\partial L^v}{\partial v^a}-p_a^{N_a}=0 \Longrightarrow \frac{\partial H^v}{\partial v^a}=0 .
\end{aligned} \tag{10}
$$

When performing the Hamiltonization, we try to eliminate the velocities v from the set (10). In nonsingular theories, according to the definition (8), it is possible to express all the velocities by means of the last set of Eq. (10) as $v=\bar{v}\left(q, p_a^{N_a}\right)$. Thus, in such a case all the velocities are auxiliary variables (see Appendix). They can be excluded from the action (9). Thus, we arrive at the Hamilton action S_{H} and at the Hamilton EM for unconstrained phase-space variables q_s^a, p_a^s:

$$S_{\mathrm{H}} = \int \left(\sum_{s=1}^{N_a} p_a^s \dot{q}_s^a - H \right) dt\,,\ H = H^v|_{v=\bar{v}}\,,$$
$$\dot{q}_s^a = \{q_s^a\,, H\}\,,\ \dot{p}_a^s = \{p_a^s\,, H\}\,.$$

First, the Hamiltonization of nonsingular theories with higher-order time derivatives was presented in [6]. Hamiltonization of singular theories with higher-order time derivatives, on the base of the equations (10), is completely analogous to the case of theories without higher time derivatives, see [5, 2].

2.2 Degenerate coordinates. An instructive example

Let us now suppose that some of the generalized coordinates do not have any time derivatives in the Lagrange function. In the general case (7), that means that $\bar{N}_a$ are zero for some of that coordinates. We shall call the coordinates with $\bar{N}_a = 0$ degenerate. According to the conventional definitions (1) or (8) any theory with degenerate coordinates is singular. However, here we are going to discuss the following question: is it always reasonable to treat theories with degenerate coordinates as singular and to follow the above described conventional Hamiltonization scheme? To answer this question it is instructive to first consider a class of theories with two coordinates x, u and with Lagrange functions of the form

$$L = L(x, \dot{x}, u)\,. \tag{11}$$

Here the Hessian (1) is zero, $M = 0$, therefore we are formally dealing with the singular case. Nevertheless, we can demonstrate that the corresponding Euler-Lagrange EM

$$\frac{\delta S}{\delta x} = \frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial \dot{x} \partial x}\dot{x} - \frac{\partial^2 L}{\partial \dot{x} \partial u}\dot{u} - \frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}}\ddot{x} = 0\,, \tag{12}$$
$$\frac{\delta S}{\delta u} = \frac{\partial L}{\partial u} = 0 \tag{13}$$

have a unique solution (thus the theory is not a gauge theory) whenever the determinant $\tilde{M}$ (we call it further the generalized Hessian)

$$\tilde{M} = \det \begin{Vmatrix} \frac{\partial^2 L}{\partial \dot{x}^2} & \frac{\partial^2 L}{\partial \dot{x} \partial u} \\ \frac{\partial^2 L}{\partial u \partial \dot{x}} & \frac{\partial^2 L}{\partial u^2} \end{Vmatrix} = \frac{\partial^2 L}{\partial \dot{x}^2}\frac{\partial^2 L}{\partial u^2} - \left(\frac{\partial^2 L}{\partial \dot{x} \partial u} \right)^2 = \tilde{M}\,(x, \dot{x}, u)\,, \tag{14}$$

is not zero and two initial data are given. Indeed, the condition $\tilde{M} \neq 0$ necessarily implies either case (a) or case (b):

$$a)\ \frac{\partial^2 L}{\partial u^2} \neq 0\,, \tag{15}$$
$$b)\ \frac{\partial^2 L}{\partial \dot{x} \partial u} \neq 0\,. \tag{16}$$

First consider case (a). In this case the equation (13) can be solved with respect to u ,

$$\frac{\partial L}{\partial u} = 0 \Longrightarrow u = \bar{u}\,(x, \dot{x})\,, \tag{17}$$

and the equation

$$\frac{d}{dt}\frac{\partial L}{\partial u} = \frac{\partial^2 L}{\partial u \partial x}\dot{x} + \frac{\partial^2 L}{\partial u \partial \dot{x}}\ddot{x} + \frac{\partial^2 L}{\partial u^2}\dot{u} = 0 \tag{18}$$

can be solved with respect to $\dot{u}$,

$$\dot{u} = -\left(\frac{\partial^2 L}{\partial u^2} \right)^{-1} \left[\frac{\partial^2 L}{\partial u \partial x}\dot{x} + \frac{\partial^2 L}{\partial u \partial \dot{x}}\ddot{x} \right]\,. \tag{19}$$

Substituting (19) into (12), we arrive at the following equation

$$\tilde{M}(x,\dot{x},u)\ddot{x} = F_1(x,\dot{x},u) ,$$
$$F_1(x,\dot{x},u) = \frac{\partial L}{\partial x}\frac{\partial^2 L}{\partial u^2} + \left[\frac{\partial^2 L}{\partial u \partial \dot{x}}\frac{\partial^2 L}{\partial u \partial x} - \frac{\partial^2 L}{\partial x \partial \dot{x}}\frac{\partial^2 L}{\partial u^2}\right]\dot{x} .$$

Since $\tilde{M} \neq 0$, the Euler-Lagrange EM can be reduced to the form

$$\ddot{x} = F_1(x,\dot{x},\bar{u})/\tilde{M}(x,\dot{x},\bar{u}) \ , \ u = \bar{u}(x,\dot{x}) \ . \tag{20}$$

They have a unique solution whenever two initial data are given, for example, x and $\dot{x}$ at the initial time instant.
Let us turn to the case (b). Here, due to (16), the equation (13) can be solved with respect to $\dot{x}$,

$$\dot{x} = \bar{v}(x,u) \ , \tag{21}$$

and the equation

$$\frac{d}{dt}\frac{\partial L}{\partial u} = \frac{\partial^2 L}{\partial u \partial x}\dot{x} + \frac{\partial^2 L}{\partial u \partial \dot{x}}\ddot{x} + \frac{\partial^2 L}{\partial u^2}\dot{u} = 0 \tag{22}$$

can be solved with respect to $\ddot{x}$,

$$\ddot{x} = -\left(\frac{\partial^2 L}{\partial u \partial \dot{x}}\right)^{-1}\left[\frac{\partial^2 L}{\partial u \partial x}\dot{x} + \frac{\partial^2 L}{\partial u^2}\dot{u}\right] . \tag{23}$$

We may substitute (23) into (5a) to get

$$\tilde{M}(x,\dot{x},u)\dot{u} = F_2(x,\dot{x},u) ,$$
$$F_2(x,\dot{x},u) = \frac{\partial L}{\partial x}\frac{\partial^2 L}{\partial u \partial \dot{x}} + \left[\frac{\partial^2 L}{\partial x \partial \dot{x}}\frac{\partial^2 L}{\partial u \partial \dot{x}} - \frac{\partial^2 L}{\partial u \partial x}\frac{\partial^2 L}{\partial \dot{x}^2}\right]\dot{x} .$$

Since $\tilde{M} \neq 0$, the Euler-Lagrange EM can be reduced to the form

$$\dot{u} = F_2(x,\bar{v},u)/\tilde{M}(x,\bar{v},u) \ , \ \dot{x} = \bar{v}(x,u) \tag{24}$$

They again have a unique solution whenever two initial data are given, for instance, x and u at the initial time instant. One ought to remark that provided both conditions a) and b) are satisfied, the EM can be written in both forms (20) and (24).

Let us turn to the Hamiltonization of theories under consideration. First we consider the conventional Hamiltonization procedure [1, 2, 3], that is, we choose $N_x = N_u = 1$. In the first-order formalism, the phase space is formed by the pairs x, p; u, p', and the extended phase space is formed by the variables x, p; $u, p'; v, v'$. The first-order formalism action reads:

$$S^{vv'} = \int \left[L^{vv'} + p(\dot{x} - v) + p'(\dot{u} - v')\right] dt = \int \left[p\dot{x} + p'\dot{u} - H^{vv'}\right] dt ,$$
$$L^{vv'} = L(x,v,u) , \quad H^{vv'} = pv + p'v' - L^{vv'} . \tag{25}$$

When performing the Hamiltonization, we have to try to eliminate the velocities v, v' from the action $S^{vv'}$ and from the Euler-Lagrange EM

$$\frac{\delta S^{vv'}}{\delta p} = 0 \Longrightarrow \dot{x} = \left\{x, H^{vv'}\right\}, \quad \frac{\delta S^{vv'}}{\delta x} = 0 \Longrightarrow \dot{p} = \left\{p, H^{vv'}\right\} ,$$
$$\frac{\delta S^{vv'}}{\delta p'} = 0 \Longrightarrow \dot{u} = \left\{u, H^{vv'}\right\}, \quad \frac{\delta S^{vv'}}{\delta u} = 0 \Longrightarrow \dot{p}' = \left\{p', H^{vv'}\right\} , \tag{26}$$
$$\frac{\delta S^{vv'}}{\delta v} = -\frac{\partial H^{vv'}}{\partial v} = \frac{\partial L^{vv'}}{\partial v} - p = 0 , \quad \frac{\delta S^{vv'}}{\delta v'} = -\frac{\partial H^{vv'}}{\partial v'} = -p' = 0 , \tag{27}$$

generated by the action $S^{vv'}$. The Hessian is zero, the theory is singular and we cannot exclude both velocities v, v' using Eqs. (27). [As is known, in this case there appear primary constraints and further Hamiltonization is related to the Dirac procedure we are going to demonstrate, for a special class of theories where part of the coordinates does not have any time derivatives in the Lagrange function (we call such variables degenerate) it is reasonable to reconsider the conventional definition of singularity and, moreover, to simplify the conventional Hamiltonization procedure (one ought to say that almost all modern physical gauge models are theories with degenerate variables). In particular, in such a procedure (we call it the generalized Hamiltonization scheme) we do not complete the degenerate coordinates with the corresponding conjugate momenta.[1, 2, 3].

Let us suppose, however, that the generalized Hessian (14) is not zero ($\tilde{M}(x, v, u) \neq 0$). Consider the following two possible cases:

a) $\partial^2 L^{vv'}/\partial v^2 = 0$. Then $L^{vv'} = v f_1(x, u) - f_2(x, u)$, and $\tilde{M} = -(\partial f_1/\partial u)^2 \neq 0 \Longrightarrow \partial f_1/\partial u \neq 0$. Therefore, the equation $\partial L^{vv'}/\partial v - p = 0 \Longrightarrow f_1(x, u) - p = 0$ can be solved with respect to u as $u = \bar{u}(x, p)$. Thus, we have two primary second-class constraints,

$$\Phi_1^{(1)} = p' = 0, \;\; \Phi_2^{(1)} = u - \bar{u}(x, p) = 0, \tag{28}$$

and both velocities v', v appear to be Lagrangian multipliers in the total Hamiltonian $H^{(1)} = f_2 + \lambda^1 \Phi_1^{(1)} + \lambda^2 \Phi_2^{(1)}$ which defines now the Hamilton dynamics of the phase-space variables. No more constraints appear. The constraints (28) have a special form [2] and can be used to exclude variables p' and u from the action and from the EM. Namely, we can substitute $p' = 0$ and $u = \bar{u}(x, p)$ directly into $H^{(1)}$ to get the Hamiltonian $H = f_2(x, \bar{u})$, which defines the Hamilton dynamics of the remaining phase-space variables x, p as: $\dot{x} = \{x, H\}$, $\dot{p} = \{p, H\}$.

b) $\partial^2 L^{vv'}/\partial v^2 \neq 0$ (we should suppose that this condition holds in a vicinity of the point $x = v = u = 0$). In this case the equation $\delta S^{vv'}/\delta v = \partial L^{vv'}/\partial v - p = 0$ can be solved with respect to v as $v = \bar{v}(x, u, p)$ and one primary constraint appears $\Phi^{(1)} = p' = 0$. The total Hamiltonian that defines now the Hamilton dynamics of the phase-space variables reads: $H^{(1)} = p\bar{v} - L(x, \bar{v}, u) + \lambda \Phi^{(1)}$. The consistency condition for the primary constraint gives a secondary constraint $\{p', H^{(1)}\} = \partial L^{vv'}/\partial u \big|_{v=\bar{v}} = 0$, which can be solved with respect to u as $\Phi^{(2)} = u - \bar{u}(x, p) = 0$. The variables p' and u can be excluded as in the previous case, and so we get a similar result[to the previous case result]. Thus, after the conventional Hamiltonization we are left with the set of equations

$$\begin{aligned} &\dot{x} = \{x, H\} \, , \; \dot{p} = \{p, H\} \, , \; u = \bar{u}(x, p) \, , \\ &v = \bar{v}(x, p) \, , \; H = p\bar{v} - L(x, \bar{v}, \bar{u}) \, . \end{aligned} \tag{29}$$

We see that whenever the determinant (14) is not zero, the sector x, p of the theory is not singular (no constraints on x, p), and the coordinate u can be treated as an auxiliary variable. The number of initial data for the EM is two. This fact matches the aforementioned Lagrangian treatment.

Moreover, in the present case, the conventional Hamiltonization scheme can be simplified (we call new Hamiltonization scheme the generalized one). Indeed, we may choose $N_u = \bar{N}_u = 0$. Since the derivative $\dot{u}$ is not present in the Lagrange function, we only introduce the x-velocity v and do not introduce the corresponding Lagrange multiplier. Then the equivalent first-order action reads:

$$\begin{aligned} &S^v = \int [L^v + p(\dot{x} - v)] \, dt = \int [p\dot{x} - H^v] \, dt \, , \\ &L^v = L(x, v, u) \, , \quad H^v = pv - L^v \, . \end{aligned} \tag{30}$$

Thus, the only conjugate momentum introduced is the x-momentum p. The phase space is formed by x, p and the extended phase space can be thought of as $x, p; v, u$. In the course of Hamiltonization, it is natural to treat both v and u on equal footing and try to exclude them from the corresponding action and from the Euler-Lagrange EM

$$\frac{\delta S^v}{\delta p} = 0 \Longrightarrow \dot{x} = \{x, H^v\} \, , \; \frac{\delta S^v}{\delta x} = 0 \Longrightarrow \dot{p} = \{p, H^v\} \, , \tag{31}$$

$$\frac{\delta S^v}{\delta v} = -\frac{\partial H^v}{\partial v} = \frac{\partial L^v}{\partial v} - p = 0 \, , \; \frac{\delta S^v}{\delta u} = -\frac{\partial H^v}{\partial u} = \frac{\partial L^v}{\partial u} = 0 \, . \tag{32}$$

Consider the case $\tilde{M} \neq 0$. In this case the equations (32) may be used to express both v and u via the canonical pair x, p as $v = \bar{v}(x,p)$, $u = \bar{u}(x,p)$. We see that the variables v, u are auxiliary and can be eliminated from the action S^v (see the Appendix) to obtain the action in the Hamiltonian form. The corresponding Hamiltonian H is obtained by substituting $v = \bar{v}(x,p)$, $u = \bar{u}(x,p)$ directly into H^v to get $H = p\bar{v} - L(x, \bar{v}, \bar{u}) = H(x,p)$ which determines the Hamilton dynamics of the pair x, p. Thus, after such generalized Hamiltonization we arrive at the same set of equations (29). Notice that in the generalized Hamiltonization scheme we did not use the condition: $\partial^2 L^{vv'}/\partial v^2 \neq 0$ in a vicinity of the point $x = v = u = 0$.

What can we learn from the above considerations? First of all, it is not necessary to introduce the momenta associated to the degenerate coordinates in the course of Hamiltonization. The scheme of Hamiltonization can be simplified. Moreover, the new generalized Hamiltonization scheme motivates us to change the definition of singularity (1) in the presence of degenerate coordinates. In this case, it is more reasonable to classify theories according to the generalized Hessian (14) and to consider them nonsingular whenever $\tilde{M} \neq 0$. Indeed, besides the natural consistency with the generalized Hamiltonization scheme, the generalized-Hessian criterion allows one to conclude immediately that a theory is not a gauge theory whenever $\tilde{M} \neq 0$. Looking upon the conventional Hessian, we cannot come to such a conclusion without additional analysis of the constraint structure. Below we present a generalization of the conventional Hamiltonization scheme and a generalized singularity criterion for a general Lagrangian theory.

3 Hamiltonization of a general Lagrangian theory

As was already said, in the general case the action reads $S = \int L dt$, the Lagrange function has the form (7), and let the orders of the highest derivatives be $\{N_a\}$. The corresponding Euler-Lagrange EM are

$$\frac{\delta S}{\delta q^a} = \sum_{l_a=0}^{N_a} (-1)^{l_a} \frac{d^{l_a}}{dt^{l_a}} \left[\frac{\partial L}{\partial q^{a(l_a)}} \right] = 0 . \tag{33}$$

We propose to classify Lagrangian theories as singular or nonsingular using the generalized Hessian $\tilde{M}$ in the following way:

$$\tilde{M} = \det \left\| \frac{\partial^2 L}{\partial q^{a(N_a)} \partial q^{b(N_b)}} \right\| = \begin{cases} \neq 0, \text{ nonsingular theory} \\ = 0, \text{ singular theory} \end{cases} , \; N_a \geq 0 . \tag{34}$$

We stress that the orders N_a can be zero in the presence of degenerate coordinates. The difference between definition (34) and definition (8) is related namely to the possibility of N_a to be zero. If we restrict all $N_a \geq 1$ even in the presence of degenerate coordinates, the generalized Hessian (34) and the Hessian (8) coincide. In what follows, we consider the Hamiltonization of general theories according to the generalized Hamiltonization scheme and present arguments in favour of the definition (34). In particular, we will demonstrate that in the nonsingular case (according to (34)) the Euler-Lagrange EM always have a unique solution under an appropriate choice of initial data and the Hamiltonization leads to usual Hamilton EM without any constraints in the appropriate phase space. Let us turn to the generalized Hamiltonization scheme supposing for simplicity that $N_a = \bar{N}_a$. In the beginning we pass to the first-order formulation, which differs from the one considered above whenever some of N_a are zero. To this end, we divide all the indices a, numbering the coordinates, into two groups,

$$a = (i, \mu) \; , \; N_i = 0 \, , \; N_\mu \geq 1 \, , \tag{35}$$

introduce new variables $x^\mu_{s_\mu}$, v^a and impose the relations

$$x^\mu_{s_\mu} = q^{\mu(s_\mu - 1)} \, , \; s_\mu = 1, ..., N_\mu \, , \; v^\mu = q^{\mu(N_\mu)} \, ,$$

$$\dot{x}^\mu_{s_\mu} = x^\mu_{s_\mu + 1} \, , \; s_\mu = 1, ..., N_\mu - 1 ; \; \dot{x}^\mu_{N_\mu} = v^\mu \, , \tag{36}$$

$$q^i = q^{i(0)} = q^{i(N_i)} = v^i \, . \tag{37}$$

The variables v^a will be called velocities. Thus, the degenerate coordinates have the status of velocities in the first-order formulation. The variational principle for the initial action S is equivalent to the one for the first-order action S^v,

$$S^v = \int \left[L^v + \sum_{s_\mu=1}^{N_\mu - 1} p_\mu^{s_\mu} \left(\dot{x}_{s_\mu}^\mu - x_{s_\mu+1}^\mu \right) + p_\mu^{N_\mu} \left(\dot{x}_{N_\mu}^\mu - v^\mu \right) \right] dt$$
$$= \int \left[\sum_{s_\mu=1}^{N_\mu} p_\mu^{s_\mu} \dot{x}_{s_\mu}^\mu - H^v \right] dt \,,$$
$$L^v = L|_{q^{\mu(s_\mu-1)} = x_{s_\mu}^\mu ,\, q^{a(N_a)} = v^a} \,, \quad H^v = \sum_{s_\mu=1}^{N_\mu - 1} p_\mu^{s_\mu} x_{s_\mu+1}^\mu + p_\mu^{N_\mu} v^\mu - L^v \,. \tag{38}$$

The variables $p_\mu^{s_\mu}$ should be treated as conjugate momenta to the coordinates $x_{s_\mu}^\mu$. The corresponding Euler-Lagrange EM read:

$$\left. \begin{array}{c} \frac{\delta S^v}{\delta p_\mu^{s_\mu}} = \dot{x}_{s_\mu}^\mu - x_{s_\mu+1}^\mu = 0, \; s_\mu = 1, ..., N_\mu - 1, \\ \frac{\delta S^v}{\delta p_\mu^{N_\mu}} = \dot{x}_{N_\mu}^\mu - v^\mu = 0 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \dot{x}_{s_\mu}^\mu = \left\{ x_{s_\mu}^\mu , H^v \right\} , \\ s_\mu = 1, ..., N_\mu \end{array} \right. ; \tag{39}$$

$$\left. \begin{array}{c} \frac{\delta S^v}{\delta x_1^\mu} = \frac{\partial L^v}{\partial x_1^\mu} - \dot{p}_1^\mu = 0, \\ \frac{\delta S^v}{\delta x_{s_\mu}^\mu} = \frac{\partial L^v}{\partial x_{s_\mu}^\mu} - p_\mu^{s_\mu - 1} - \dot{p}_\mu^{s_\mu} = 0, \; s_\mu = 2, ..., N_\mu, \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \dot{p}_\mu^{s_\mu} = \{ p_\mu^{s_\mu} , H^v \} , \\ s_\mu = 1, ..., N_\mu \end{array} \right. ; \tag{40}$$

$$\left. \begin{array}{c} \frac{\delta S^v}{\delta v^i} = \frac{\partial L^v}{\partial v^i} = 0 , \\ \frac{\delta S^v}{\delta v^\mu} = \frac{\partial L^v}{\partial v^\mu} - p_\mu^{N_\mu} = 0 \end{array} \right\} \Longrightarrow \frac{\partial H^v}{\partial v^a} = 0 \,. \tag{41}$$

It is easy to verify that after eliminating $p_\mu^{s_\mu}$, $s_\mu = 1, ..., N_\mu$, $x_{s_\mu}^\mu$, $s_\mu = 2, ..., N_\mu$, v^μ (these variables are auxiliary) from the action S^v and from Eqs. (39)-(41) we arrive at the action S and at the Euler-Lagrange EM (33). The phase space is formed by the variables $x_{s_\mu}^\mu$, $p_\mu^{s_\mu}$ only, and the extended phase space is formed by the variables $x_{s_\mu}^\mu$, $p_\mu^{s_\mu}$; v^a . By effecting the Hamiltonization, we must try to eliminate the velocities v^a from the action S^v.

Consider the nonsingular case according to the definition (34):

$$\tilde{M} = \det \left\| \frac{\partial^2 L^v}{\partial v^a \partial v^b} \right\| = \det \left\| \frac{\partial^2 H^v}{\partial v^a \partial v^b} \right\| \neq 0 \,.$$

Thus, Eqs. (41) can be solved with respect to all the velocities v, such that these velocities can be expressed in the form $v = \bar{v} \left(x, p_\mu^{N_\mu} \right)$. Now we can eliminate the variables v from the action S^v, since they are auxiliary variables (see Appendix). Therefore, the action S^v and Eqs. (39)-(40) are transformed into the ordinary Hamilton action $S_{\rm H}$ and Hamilton EM for the unconstrained phase-space variables x_s^a, p_a^s:

$$S_{\rm H} = \int \left(\sum_{s_\mu=1}^{N_\mu} p_\mu^{s_\mu} \dot{x}_{s_\mu}^\mu - H \right) dt \,, \; H = H^v|_{v=\bar{v}} \,,$$
$$\dot{x}_{s_\mu}^\mu = \left\{ x_{s_\mu}^\mu , H \right\} \,, \; \dot{p}_\mu^{s_\mu} = \{ p_\mu^{s_\mu} , H \} , \; s_\mu = 1, ..., N_\mu \tag{42}$$

One can see that the Hamiltonian H is the energy written in the phase-space variables. Eqs. (42) are solved with respect to the highest (here first order) time derivatives and therefore have a unique solution whenever $2 \sum_\mu N_\mu = 2 \sum_a N_a$ initial data are given. Since these Hamilton EM are equivalent to the Euler-Lagrange EM (33), we can say that for nonsingular (according to the new definition (34)) theories, the EM have a unique solution whenever $2 \sum_a N_a$ initial data are given!

In the singular case, the equations (41) do not allow expressing all the velocities through $x_{s_\mu}^a, p_a^{s_\mu}$, so part of the velocities appear as Lagrange multipliers to primary constraints. Then the Hamiltonization follows literally the way described in [5, 2].

We see that by performing the Hamiltonization procedure one does not need, in principle, to introduce the momenta conjugate to the degenerate coordinates. The singularity definition (34) differs from (8) for theories with degenerate coordinates and seems more reasonable in such cases. As was already remarked, considering the simple example, besides the natural consistency of the generalized Hamiltonization scheme, the generalized-Hessian criterion allows one to conclude immediately that the theory is not a gauge theory when $\tilde{M} \neq 0$. Examining the Hessian, we cannot come to such a conclusion without additional analysis of the constraint structure.

If we select some $N_a > \bar{N}_a$, a wider extended phase space is needed for the Hamiltonization. However, one can demonstrate, similarly to [5, 2], that the final Hamiltonized theory will be equivalent to the theory with orders $\{N_a = \bar{N}_a\}$.

4 Examples and discussion

1. As an example we consider the theory of a massive vector field A^μ . The theory is described by the Proca action

$$
\begin{aligned}
S &= \int \mathcal{L} dx\,,\ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A^2 \\
&= \frac{1}{2}\left(\dot{A}^i + \partial_i A^0\right)^2 - \frac{1}{4} F_{ik} F^{ik} + \frac{m^2}{2} A^2,\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \qquad (43)
\end{aligned}
$$

In this case the velocity $\dot{A}^0$ does not enter the Lagrangian. Thus, A^0 is a degenerate variable. Below we compare the conventional and the generalized Hamiltonization schemes.

a) The Hessian is zero, since the matrix $\partial^2 \mathcal{L} / \partial \dot{A}^\mu \partial \dot{A}^\nu$ is not invertible, and the theory is singular in the conventional definition. Consider the conventional Hamiltonization. Here we have to introduce momenta p_μ to all the coordinates A^μ . The action S^v in the first-order formalism reads

$$
\begin{aligned}
S^v &= \int \left[p_\mu \dot{A}^\mu - \mathcal{H}^v\right] dx\,, \\
\mathcal{H}^v &= p_\mu v^\mu - \frac{1}{2}\left[\left(v^i + \partial_i A_0\right)^2 - \frac{1}{2} F_{ik} F^{ik} + \frac{m^2}{2} A^2\right]. \qquad (44)
\end{aligned}
$$

Thus, A^μ, p_μ form the phase space and A^μ, $p_\mu; v^\mu$ form the extended phase space. The equations

$$
\delta S^v / \delta v^\mu = 0 \Leftrightarrow \left\{ \begin{array}{l} p_0 = \frac{\partial \mathcal{L}}{\partial v^0} = 0 \\ p_i = \frac{\partial \mathcal{L}}{\partial v^i} = v^i + \partial_i A^0\,, \end{array} \right. \qquad (45)
$$

do not allow one to express the velocity v^0 via the other variables. At the same time the velocities v^i are auxiliary variables (see Appendix), so they can be expressed via the other variables by aid of equations (45) and substituted into (44) to get the reduced equivalent action

$$
\begin{aligned}
S^{(1)} &= \int \left[p_\mu \dot{A}^\mu - \mathcal{H}^{(1)}\right] dx\,, \\
\mathcal{H}^{(1)} &= \frac{1}{2} p_i^2 - p_i \partial_i A^0 + \frac{1}{4} F_{ik} F^{ik} - \frac{m^2}{2} A^2 + \lambda p_0\,, \qquad (46)
\end{aligned}
$$

where $\lambda = v^0$. It follows from (46) that in the phase space A^μ, p_μ there is a primary constraint $\Phi^{(1)} = p_0 = 0$ and the dynamics is governed according to the Hamilton EM with Hamiltonian $H^{(1)} = \int \mathcal{H}^{(1)} dx$. Applying the consistency condition to the primary constraint, we find the secondary constraint $\Phi^{(2)} = \partial_i p_i - m^2 A^0 = 0$. There are no further secondary constraints and $\Phi = (\Phi^{(1)}, \Phi^{(2)})$ is the complete set of second-class constraints. Moreover, this set of constraints Φ is of the special form [2], such that one can use these constraints to eliminate the variables A^0 and p_0 from the action (46) to get still one more reduced equivalent Hamilton action

$$
S_{\mathrm{H}} = \int \left[p_i \dot{A}^i - \mathcal{H}\right] dx\,,\ \mathcal{H} = \frac{1}{2} p_i^2 + \frac{1}{2m^2}\left(\partial_i p_i\right)^2 + \frac{1}{4} F_{ik} F^{ik} + \frac{m^2}{2} A_i^2\,. \qquad (47)
$$

It follows from (47) that in the phase space A^i, p_i the dynamics is governed according to ordinary Hamilton EM with the Hamiltonian density $H = \int \mathcal{H} dx$ and without any constraints.

b) In the generalized Hamiltonization scheme we introduce the velocities according to the general prescription (see Sect.III) as $v^0 = A^0$, $v^i = \dot{A}^i$. The theory is not singular with respect to the new definition (34), since the matrix $\partial^2\mathcal{L}/\partial v^\mu \partial v^\nu$ is invertible. We introduce the momenta p_i conjugate to the coordinates A^i only. Thus, A^i, p_i form the phase space and A^i, $p_i; v^\mu$ form the extended phase space. The action S^v in the first-order formalism reads

$$S^v = \int \left[p_i \dot{A}^i - \mathcal{H}^v\right] dx \,,$$
$$\mathcal{H}^v = p_i v^i - \frac{1}{2}\left[\left(v^i + \partial_i v_0\right)^2 - \frac{1}{2}F_{ik}F^{ik} + m^2\left(v_0^2 - A_i^2\right)\right] . \tag{48}$$

Here the equations $\delta S^v/\delta v^\mu = 0$ allow one to express all the velocities via the momenta (all the velocities are auxiliary variables)

$$v_0 = m^{-2}\partial_i p_i \,,\; v^i = \left(\delta_{ij} - m^{-2}\partial_i\partial_j\right) p \,. \tag{49}$$

Substituting the expressions (49) into (48) we arrive immediately at the reduced equivalent Hamilton action (47).

2. Consider electrodynamics. The theory is described by the Maxwell action, which follows from the Proca action (43) at $m = 0$,

$$S = \int \mathcal{L} dx \,,\; \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\dot{A}^i + \partial_i A^0\right)^2 - \frac{1}{4}F_{ik}F^{ik} . \tag{50}$$

Here both the Hessian and the generalized Hessian are zero, thus the theory is singular in both definitions. As before A^0 is a degenerate variable. Let us compare the conventional and the generalized Hamiltonization schemes.

a) First consider the conventional Hamiltonization. Here we have to introduce momenta p_μ conjugate to all the coordinates A^μ. The action S^v in the first-order formalism reads

$$S^v = \int \left[p_\mu \dot{A}^\mu - \mathcal{H}^v\right] dx \,,\; \mathcal{H}^v = p_\mu v^\mu - \frac{1}{2}\left[\left(v^i + \partial_i A_0\right)^2 - \frac{1}{2}F_{ik}F^{ik}\right] . \tag{51}$$

Thus, A^μ, p_μ form the phase space and A^μ, $p_\mu; v^\mu$ form the extended phase space. As in the Proca case, we can get the reduced equivalent action

$$S^{(1)} = \int \left[p_\mu \dot{A}^\mu - \mathcal{H}^{(1)}\right] dx \,,\; \mathcal{H}^{(1)} = \frac{1}{2}p_i^2 - p_i\partial_i A^0 + \frac{1}{4}F_{ik}F^{ik} + \lambda p_0 \,, \tag{52}$$

where $\lambda = v^0$. It follows from (52) that in the phase space A^μ, p_μ there is a primary constraint $\Phi^{(1)} = p_0 = 0$ and the dynamics is governed according to the Hamilton EM with the Hamiltonian $H^{(1)} = \int \mathcal{H}^{(1)} d\mathbf{x}$. Applying the consistency condition to the primary constraint, we find the secondary constraint $\Phi^{(2)} = \partial_i p_i = 0$. One can see that $\Phi = (\Phi^{(1)}, \Phi^{(2)}) = 0$ is a set of first-class constraints, so that in this case λ is undetermined and new constraints do not arise. It is a gauge theory. Any rigid gauge needs two additional gauge conditions (in particular, to fix λ), and one can see that A^0 is not a physical variable [2].

b) In the generalized Hamiltonization scheme we introduce the velocities according to the general prescription (see Sect.III) as $v^0 = A^0$, $v^i = \dot{A}^i$. The theory is singular with respect to the new definition (34), since the matrix $\partial^2\mathcal{L}/\partial v^\mu \partial v^\nu$ is not invertible. We do not introduce momenta conjugate to A^0, and then the procedure becomes completely analogous to the generalized Hamiltonization of the Proca case. The action in the first-order formalism, S^v, reads

$$S^v = \int \left[p_i \dot{A}^i - \mathcal{H}^v\right] dx \,,\; \mathcal{H}^v = p_i v^i - \frac{1}{2}\left(v^i + \partial_i v_0\right)^2 + \frac{1}{4}F_{ik}F^{ik} . \tag{53}$$

Here the equations $\delta S^v/\delta v^\mu = 0$ result in $\delta S^v/\delta v^i = 0 \Longrightarrow v^i = \partial_i v_0 - p_i$ and in the primary constraint $\Phi^{(1)} = \partial_i p_i = 0$. Only the [The only] velocities v^i are auxiliary variables. They can be eliminated from the action using the corresponding equations. Therefore, we are left with the reduced equivalent Hamilton action

$$S_{\mathrm{H}} = \int \left[p_i \dot{A}^i - \mathcal{H}\right] dx \,,\; \mathcal{H} = \frac{1}{2}p_i^2 + \frac{1}{4}F_{ik}F^{ik} + A^0\partial_i p_i \,. \tag{54}$$

We see that in the generalized Hamiltonization scheme, the variable A^0 has naturally the status of a Lagrange multiplier to the primary constraint. The action (54) is physically equivalent to the one (52), the former corresponds to a partial gauge, i.e., fixing $\lambda = 0$ in the latter.

3. Consider the theory with degenerate coordinate u and Lagrange function of the form

$$L = \dot{x}u + \dot{x}^3.$$

The Euler-Lagrange EM give $\dot{u} = \dot{x} = 0$. Here not only the Hessian is zero, but also the Hessian matrix

$$\left\| \begin{array}{cc} \frac{\partial^2 L}{\partial \dot{x}\partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{x}\partial \dot{u}} \\ \frac{\partial^2 L}{\partial \dot{u}\partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{u}\partial \dot{u}} \end{array} \right\| = \left\| \begin{array}{cc} 6\dot{x} & 0 \\ 0 & 0 \end{array} \right\|$$

does not have a constant rank in the vicinity of the zero point $x = \dot{x} = u = \dot{u} = 0$, such that we may have additional difficulties when using the usual Hamiltonization procedure. In contrast to this, the generalized Hessian $\tilde{M}$ equals -1. Thus, the generalized Hamiltonization procedure can be completed without such difficulties. This example gives additional arguments in favour of the generalized Hamiltonization procedure for theories with degenerate coordinates.

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Appendix

Consider a classical system described by a set of generalized coordinates $q \equiv \{q^a; a = 1, 2, ..., n\}$ and by the action $S[q] = \int L dt$. Sometimes the EM allow one to uniquely express a part of the variables, which we denote by y, via the rest of the variables, which we denote by x, such that $q^a = (y^i, x^\mu)$. In such a case, one can try to eliminate the variables y from the initial action and ask whether the initial and the reduced theories are equivalent. In what follows, we consider the case in which a positive answer is possible, see in this regard [7, 8].

Suppose an action $S[q] = S[y, x]$ is given such that the EM $\delta S/\delta y = 0$ allow us to express uniquely the variables y as local functions of the variables x, namely:

$$\frac{\delta S[y,x]}{\delta y} = 0 \Longleftrightarrow y = \bar{y}(t, x^{(l)}). \tag{55}$$

Consider the action $\bar{S}[x] \equiv S[\bar{y}, x]$, which we call the reduced action and compare the EM corresponding to both actions. Consider the variation $\delta\bar{S}$ under arbitrary inner variations δx such that any surface terms vanish [1],

$$\delta\bar{S}[x] = \int \left(\left.\frac{\delta S[y,x]}{\delta y}\right|_{y=\bar{y}} \delta\bar{y} + \left.\frac{\delta S[y,x]}{\delta x}\right|_{y=\bar{y}} \delta x \right) dt = \int \frac{\delta\bar{S}[x]}{\delta x}\delta x dt. \tag{56}$$

In virtue of (55), the EM for the reduced theory read:

$$\frac{\delta\bar{S}[x]}{\delta x} = \left.\frac{\delta S[y,x]}{\delta x}\right|_{y=\bar{y}} = 0. \tag{57}$$

On the other hand, the EM for the initial theory are:

$$\frac{\delta S[y,x]}{\delta y} = 0 \Longleftrightarrow y = \bar{y}(x, \dot{x}, ...); \frac{\delta S[y,x]}{\delta x} = 0.$$

They are reduced to (57) in the sector of the x-variables . Thus, the initial action S and the reduced action $\bar{S}$ lead to the same EM for the variables x. For this reason, we can treat the y-variables as dependent and call them auxiliary variables. Thus, the auxiliary variables can be eliminated with the help of the EM derived from the action. The initial theory and the reduced one are equivalent. One ought to stress that this equivalence is a consequence of supposition (55), that is, it is important that the y-variables be expressed as functionals of x by means of the equations $\delta S/\delta y = 0$ only. If for this purpose some of the equations $\delta S/\delta x = 0$ are used as well, then the above equivalence may be untrue. Of course, solutions of the reduced theory, together with the

[1] To derive the equations of motion it is enough to only consider such inner variations.

relation $y = \bar{y}$, contain all the solutions of the initial theory as it is easily seen from Eq. (56). However, the reduced theory may have additional solutions. To illustrate this fact, we consider a Lagrange function of the form

$$L = \dot{x}^2/2 + xy\,. \tag{58}$$

The corresponding EM are $\delta S/\delta x = y - \ddot{x} = 0, \quad \delta S/\delta y = x = 0$. They have the unique solution $x = y = 0$. Now let us express the variable y via x using the equation $\delta S/\delta x = 0$. The reduced Lagrange function takes the form

$$\bar{L} = \dot{x}^2/2 + x\ddot{x}\,. \tag{59}$$

We see that the EM $\delta \bar{S}/\delta x = \ddot{x} = 0$ of the reduced theory together with $y = \ddot{x}$ have additional solutions in comparison with the initial theory. If we use the equation $\delta S/\delta y = x = 0$ for eliminating x from the Lagrange function, then the reduced theory for the variable y with $\bar{L} = 0$ even becomes a gauge theory, whereas the initial theory was not a gauge theory.

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