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Condensed vortex ground states of rotating Bose-Einstein condensate in harmonic atomic trap

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Abstract

We study a system of N Bose atoms trapped by a symmetric harmonic potential, interacting via weak central forces. Considering the ground state of the rotating system as a function of the two conserved quantities, the total angular momentum and its collective component, we develop a supersymmetric approach to derive exact wave functions and energies of these ground states. We describe a broad class of the interactions for which these results are valid. This universality class is defined by simple integral condition on the potential. The most the potentials of practical interest which have pronounced repulsive component belong to this universality class.

I. INTRODUCTION

Recent progress in creation of magnetic traps for cold atoms made possible the study of the effects of quantum degeneracy in systems of finite numbers of interacting identical particles [1]. One of the most interesting phenomena is the Bose-Einstein condensation, which can be studied in different coupling regimes in such dilute atomic gas systems. Theoretical studies of such systems of weakly interacting bosons, confined by parabolic potential, have become very active [2–6].

Response of such system to rotation and the onset of vorticity in the condensate are among the most interseting questions attracting attention of both experimentalists and theorists [2–8]. In this connection, it is very important to study the structure and the spectrum of the ground states of rotating condensates at a given angular momentum, called the *yrast states* [2–5], see Fig. 1. The notion of yrast states is borrowed from nuclear physics, where the states of many-body system having highest spins at a given excitation energy play also a special role [9].

A particular problem of wide recent interest [3–5] arises in the so-called weak coupling limit, when the repulsive interactions between bosonic atoms can be considered weak as compared to the spacing between different oscillator levels in the harmonic trap. This limit is expected [2–4] to be reached in future experiments. In particular, an interesting possibility to study different coupling regimes is related to using the Feshbach resonance [10] to vary the strength of the effective interatomic interactions. From the theoretical viewpoint, the advantage of the weak coupling limit is that it allows analytical treatment of the problem.

We will consider here N spinless Bose atoms in a spherically symmetric harmonic trap, assuming the interactions weak. It is sufficient to consider the two-dimensional case, because the three-dimensional case can be reduced to the former in the weak coupling limit, as was shown in Refs. [4,5]. It is expedient to start with the case of noninteracting particles. The noninteracting system has equidistant spectrum $\hbar\omega n$ where ω is the trapping oscillator frequency and n an integer. Each level is degenerate, and the degeneracy grows exponentially

with n at $n - N \gg 1$ [4]. This degeneracy is related to the number of ways to distribute the total energy $\hbar\omega n$ among the Bose particles. The short-range interactions $V(r)$ between the atoms are assumed weak in the sense that hoppings between different $\hbar\omega n$ levels can be neglected,

$$N\langle V\rangle \ll \hbar\omega, \tag{1}$$

where $\langle V\rangle$ is the typical matrix element of the interaction [4]. The problem is therefore to find nonperturbative solution for the highly degenerate states at a single level $\hbar\omega n$, which is similar in spirit to the problem of the lowest Landau level for the electrons in high magnetic field, which arises in the theory of the fractional quantum Hall effect [11,12] or to the problem of compound states in an atomic nucleus. The yrast states are those with minimal energy at given angular momentum, L. This is illustrated in Fig. 2, where the spectrum patterns are shown for the cases of zero and finite interaction.

As is usually the case for interacting many-body systems, the evaluation of the exact ground state is a prohibitive task, even with the simplification introduced by the weak coupling limit [3–5], [13–23]. The yrast states in the case of *attractive* δ-forces have been found analytically by Wilkin *et al.* [3]. Later, these results have been shown valid for a broad class of attractive interactions in Ref. [23]. The case of repulsive interaction is more difficult to analyze [3–5,13–19]. One of the first important results obtained for the repulsive forces was that of Bertsch and Papenbrock [5]: these authors diagonalized the repulsive δ-interaction numerically and suggested analytical formulae for the wave functions and energies of the yrast states. Later, it was shown analytically by several authors using various methods [16,14,15,13,17], that the states of the form of Bertsch and Papenbrock are indeed eigenstates of the Hamiltonian. Only recently, it was shown that these states indeed correspond to minimum energy [20], [21].

In this work, we provide rigorous analytical solution for the yrast states. In fact, we consider a more general problem [20], of finding the ground state as a function of two quantum numbers, the total angular momentum, L, and the angular momentum of internal

excitations (we discuss this quantity in the following sections). Such "generalized yrast states" include the usual yrast states as a limiting case. These solutions are valid in the region $L \leq N$. Physical interpretation of the results is quite transparent. In fact, the form of the yrast wave functions which was drawn from numerics [5] for the case of δ-interaction, turns out to be valid for a broad class of repulsive interactions. The universality class of such interactions [which includes, but is not limited to, delta-forces, Gaussian forces with arbitrary range, Coulomb $(1/r)$ and log-Coulomb $[log(r)]$ forces] is described by an explicit sufficiency condition.

The structure of the paper is the following. In section II, we present detailed formulation of the problem in the weak coupling limit, introducing the relevant "partition subspace" and the basic ingredients of the following consideration. In section III, we derive the operator expansion for the Hamiltonian projected onto the partition space. In section IV, we discuss the additional conserved quantum number, "seniority", which can be interpreted as a collective contribution to the angular momentum, and introduce the notion of generalized yrast states. The method of supersymmetric decomposition of the Hamiltonian is introduced and discussed in section V. Its application to the present problem is developed in section VI. Section VII is devoted to study the property of non-negative definiteness of the "supersymmetric perturbation". The results for the generalized yrast states are presented, in general form, in section VIII. In section IV, we consider various examples of the interaction potentials and discuss the results. Section X is devoted to detailed discussion of the applicability condition and of the corresponding universality class of attractive potentials. Section XI summarizes the work.

II. THE PARTITION SUBSPACE

In the weak coupling limit discussed in the previous section, the problem splits onto series of independent problems for each value of the total angular momentum L. At given angular momentum L, the Hamiltonian ($\hbar = m = 1$) is the sum

$$\tilde{H} = \omega \tilde{H}_0 + \tilde{V}, \tag{2}$$

where the first term (the c-number equal to the energy of degenerate level) comes from the noninteracting Hamiltonian in the parabolic trap,

$$\omega H_0 = \sum_i^N \left(\frac{\vec{p}_i^2}{2} + \frac{\omega^2}{2} \vec{r}_i^2 \right) \qquad \rightarrow \qquad \omega \tilde{H}_0, \tag{3}$$

with $\vec{p}_i$ and $\vec{r}_i$ denoting the momentum and coordinate of the i-th particle. Hereafter, we set $\omega \equiv 1$ for brevity. The second, nontrivial, term $\tilde{V}$ stems from the two-body interaction

$$V = \sum_{i>j} V(r_{ij}) \qquad \rightarrow \qquad \tilde{V}, \tag{4}$$

projected onto the single-level problem, which is worked out below.

In two dimensional problems involving harmonic potential, it is convenient to use the complex variable $z = x + iy$ and the conjugated $z^* = x - iy$ instead of vector $\vec{r} = (x, y)$. Introducing the notations $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\frac{\partial}{\partial z^*} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$, it is convenient to employ the tetrad of ladder operators a^+, a, b^+ and b

$$\begin{aligned} a^+ = \frac{z_i}{2} - \frac{\partial}{\partial z^*}, \quad b^+ = \frac{z^*}{2} - \frac{\partial}{\partial z}, \\ a = (a^+)^\dagger = \frac{z^*}{2} + \frac{\partial}{\partial z}, \quad b = (b^+)^\dagger = \frac{z}{2} + \frac{\partial}{\partial z^*}, \end{aligned} \tag{5}$$

for each particle. For brevity, the particle markers are suppressed in Eqs.(5), dagger denotes Hermitean conjugation. The role of these operator is to raise (lower) the powers of z_i and z_j^* in the preexponentials of many-body wave functions, which are all polynomials times the Gaussian factor $|0\rangle = exp(-1/2 \sum |z_k|^2)$. Thus, we have

$$\begin{aligned} z_i|0\rangle = a_i^+|0\rangle, \quad z_j^*|0\rangle = b_j^+|0\rangle, \\ a_i|0\rangle = 0, \quad b_i|0\rangle = 0, \qquad etc. \end{aligned} \tag{6}$$

The only nonzero commutators between (5) are given by

$$[a_i, a_j^+] = [b_i, b_j^+] = \delta_{ij}. \tag{7}$$

In two dimensions, the total angular momentum is the difference

$$L = L_+ - L_- \tag{8}$$

of numbers of "up" and "down" quanta, $L_+ = \sum_1^N a_k^+ a_k$ and $L_- = \sum_1^N b_k^+ b_k$, respectively. At the same time, the noninteracting Hamiltonian in the left hand side of (3) is given by

$$H_0 = L_+ + L_- + N. \tag{9}$$

By definition, the yrast states have minimum energy at given L. They must therefore belong to the subspace with

$$L_- = 0, \qquad L_+ = L. \tag{10}$$

The first term in (3) is therefore reduced to a constant given by

$$\tilde{H}_0 = N + L. \tag{11}$$

The subspace defined by (10 is spanned by the homogeneous symmetric polynomials of degree L, $poly_S^L(z_i)$ which do not involve conjugated z^*'s at all,

$$\Psi(L) = poly_S^L(a_i^+)|0\rangle \tag{12}$$

The orthogonal basis of these symmetric polynomials can be constructed as follows [24]. Consider the monomials

$$m \equiv \{l_1, l_2, ..., l_N\} \equiv z_1^{l_1} z_2^{l_2} ... z_N^{l_N} \tag{13}$$

corresponding to partition of integer L, $\sum\limits_n l_n = L$. The states $m|0\rangle$ with different sets $\{l_1, l_2, ..., l_N\}$ are mutually orthogonal. The basis symmetric polynomials [24] are given by symmetrized linear combinations of (13)

$$[l_1, l_2, ..., l_N] \equiv P_S m, \tag{14}$$

denoted by corresponding ordered partition $[l_1, l_2, ..., l_N]$ with $l_1 \geq l_2 \geq ... \geq l_N$. Here, the operator of symmetrization P_S is the standard sum over all the permutations. In the case $L \leq N$ which we are interested here, the number of the basis states (14) is given by "the

number of unrestricted partitions" $[l_1, l_2, ..., l_N]$ of positive integer L [25]. At $L \leq N$, this number does not depend on N; hereafter, it is denoted by $p(L)$. For example, at $N = L = 3$ the complete basis (14) of the partition space (12) is spanned by the following states

$$\begin{aligned}
&[1,1,1] = z_1 z_2 z_3 = |0,3,0,0...\rangle, \\
&[2,1,0] = z_1^2 z_2 + z_1 z_2^2 + z_2^2 z_3 + z_2 z_3^2 + z_3^2 z_1 + z_3 z_1^2 = |1,1,1,0...\rangle, \\
&[3,0,0] = z_1^3 + z_2^3 + z_3^3 = |0,0,3,0...\rangle,
\end{aligned} \tag{15}$$

which correspond to the number of partitions $p(3) = 3$ (normalization factors are suppressed for brevity). In Eqs.(15), the right hand side of each line gives interpretation of many-body state in terms of the single-particle occupation numbers of the oscillator states $|n_0, n_1, n_2, ...\rangle$ with n_i the number of Bose particles in the state with i oscillator quanta.

The number of partitions can be found from the generating function

$$\sum_{L=0}^{\infty} p(L) t^L = \prod_{k=1}^{\infty} \frac{1}{1-t^k}.$$

An explicit (however involved) expression for $p(L)$ is given in [25]. For small L, the values of $p(L)$ are

$$\begin{aligned}
&p(2) = 2, \quad p(3) = 3, \quad p(4) = 5, \quad p(5) = 7, \\
&p(6) = 11, \quad p(7) = 15, \quad p(8) = 22, \quad ...
\end{aligned} \tag{16}$$

At L large, the number of partitions grows exponentially [25]

$$p(L \gg 1) \propto e^{L^{\frac{1}{2}}}, \tag{17}$$

as is typical for many-body systems [9].

The problem of finding the spectrum consists of diagonalization of the interaction, the second term in (2), in the "partition subspace" (12,14); the eigenvalues of (2) have the form

$$\mathcal{E} = N + L + E, \tag{18}$$

where E is the interaction contribution.

In this paper, we restrict ourselves to the case $L \leq N$ only. At $L > N$, the whole situation is changed drastically. First, the number of basis states is not given by $p(L)$ anymore and it depends on both L and N. For example, the simplest states with partition $[1,1,1...1]$, which play the role of generating functions for the ground states at $L \leq N$ (see section V), can not be constructed at $L > N$. It is therefore reasonably to expect something similar to "phase transition" at $L = N$. In the earlier studies [5], the signatures of this phase transition have been observed in numerical simulations.

III. THE INTERACTION WITHIN THE PARTITION SUBSPACE. "OPERATOR EXPANSION"

In this section, we work out a convenient representation for the $\tilde{V}$, Cf. (4), the interaction V projected onto the "partition space" which was described in the previous section. Using condition Eq.(10), the projected interaction $\tilde{V}$ in Eq.(2) can be written [13,20] as

$$\tilde{V} = \sum_{L=-\infty}^{\infty} P_0^- P_L^+ V P_L^+ P_0^-, \tag{19}$$

where we use the standard number-of-quanta projector

$$P_L^+ = \frac{1}{2\pi} \int_0^{2\pi} d\alpha e^{i\alpha\left(L - \sum_i a_i^+ a_i\right)}, \tag{20}$$

and the similar expression for P_L^- with substitutions $a_i^+ \to b_i^+$ and $a_i \to b_i$. Due to construction, the transitions between different L-sectors are suppressed in (19), and the resulting effective interaction (19) applies equally to every L-sector.

With the help of (20), Eq.(19) can be cast in the form

$$\tilde{V} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha P_0^- e^{i\alpha \sum_i a_i^+ a_i} V e^{-i\alpha \sum_i a_i^+ a_i} P_0^- \tag{21}$$

In order to evaluate (21), we use the Fourier representation of the interaction in the right hand side of (21),

$$V = \sum_{i>j} \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y e^{i\left[q_-(a_{ij}^+ + b_{ij}) + q_+(b_{ij}^+ + a_{ij})\right]} V_q \tag{22}$$

where

$$V_q = \frac{1}{2\pi}\int_0^\infty r dr J_0\left(r\sqrt{q_x^2+q_y^2}\right)V(r)$$

is the two-dimensional Fourier transform of the central potential $V(r)$. Here, J_0 is the Bessel function [25] and notation $q_\pm{=}(q_x \pm i q_y)/\sqrt{2}$ is introduced. The two-particle operator combinations

$$a_{ij}^+ \equiv \frac{1}{\sqrt{2}}(a_i^+ - a_j^+), \quad b_{ij}^+ \equiv \frac{1}{\sqrt{2}}(b_i^+ - b_j^+),$$
$$a_{ij}^+ \equiv (a_{ij}^+)^\dagger, \quad b_{ij}^+ \equiv (b_{ij}^+)^\dagger \tag{23}$$

came from resolving x_i, y_i and x_j, y_j from (5). Substituting (22) in (19), we make use of Baker-Hausdorf relation

$$e^{a^+ - a} = e^{\frac{[a^+,a]}{2}} e^{a^+} e^{-a},$$

which is valid for any pair of boson operators a^+ and a. Expanding (22) to powers of q^2 and evaluating integrals term by term, we use the relation

$$\int_0^\infty dq e^{-q^2/2} q^{2n+1} J_0(qr) = 2^n n! M(n+1, 1, -r^2/2)$$

where $M(\mu, \nu, x)$ is the Kummer confluent hypergeometric function [25]. Proceeding in this manner, we obtain expansion of $\tilde{V}$

$$\tilde{V} = \sum_{k=0}^{2[L/2]} (-1)^k s_k B^k = \frac{N(N-1)}{2} - s_1 B^1 + ... + \tag{24}$$

in terms of the two-particle normal-ordered operators defined as

$$B^k = \sum_{j>i}^{N} B_{ij}^k, \qquad B_{ij}^k = a_{ij}^{\dagger k} a_{ij}^k. \tag{25}$$

In Eq.(24), $[q]$ denotes integer part of real number q. The "strength parameters", s_k, are related to the potential, $V(r)$, via the integrals

$$s_k = \frac{1}{k!}\int_0^\infty dt M(k+1, 1, -t) V(\sqrt{2t}). \tag{26}$$

whith M the Kummer function [25]. For k integer, the Kummer function can be expressed in terms of the Laguerre polynomials L_n^0

$$M(k+1,1,-t) = e^{-t} L_n^0(t).$$

In the expansion (24), the highest possible order k of the operators B^k is L for L even, and $L-1$ for L odd. In the lowest order term, no operators are involved, and we have $B^0 \equiv N(N-1)/2$.

Eq.(24) can be written in another convenient form

$$\tilde{V} = \sum_{k=0}^{[L/2]} V_k,$$
$$V_k \equiv s_{2k}(B^{2k} - B^{2k-1}) + (s_{2k+2} - s_{2k+1})B^{2k+1}. \qquad (27)$$

Here, $B^{-1} \equiv 0$. This second form of the expansion contains operator structures collected in the way convenient to apply the method of "supersymmetric decomposition" described in the following sections.

The operator expansion (24,27) is quite universal, while the particular shape of the potential $V(r)$ is described by the integrals with Kummer function M [25] We should stress that expansion (27) is exact for any interaction $V(r)$ whose moments s_k are finite [13,22].

IV. THE QUANTUM NUMBER "SENIORITY", GENERALIZED YRAST STAES AND CORRESPONDENCE RULE

In this section, we discuss an extra conserved quantum number the system enjoys, in addition to the energy and the total angular momentum. [3,4]. This quantity, which is very important for classification of states, is reminiscent to the "center-of mass" mode in nuclear problems [9]. Here, it can be interpreted as collective contribution to the total angular momentum L. Indeed, the pair of the mutually conjugated collective ladder operators

$$\mathcal{A}^+ = \sum_{i=1}^{N} \frac{a_i^+}{\sqrt{N}}, \qquad \mathcal{A} = \sum_{i=1}^{N} \frac{a_i}{\sqrt{N}}, \qquad [\mathcal{A}, \mathcal{A}^+] = 1, \qquad (28)$$

commute with any two-body combinations in (27) as well as with the angular momentum,

$$[\mathcal{A}, a_{ij}^{+}] = 0, \qquad [\mathcal{A}, L_{+}] = 0, \qquad [\mathcal{A}, L_{-}] = 0, \tag{29}$$

and they therefore commute with the both terms in the Hamiltonian $\tilde{H}$ (2), thus we have $[\mathcal{A}, \tilde{H}] = 0$. The number of collective quanta, $\mathcal{A}^{+}\mathcal{A}$, is therefore a conserved quantity. The mutual eigenfunctions of the triad of operators $\tilde{H}$, L and $\mathcal{A}^{+}\mathcal{A} = v$ can be found in following factorized form

$$\Psi_k(L, v) = \qquad Z^v \quad poly_S^{L-v}(\tilde{z}_i) \quad e^{-\sum \frac{|z_k|^2}{2}}, \tag{30}$$

where $\tilde{z}_i = z_i - Z$ and

$$Z \equiv \sum_{1}^{N} z_k/N \tag{31}$$

is the collective variable. The additional index k stands to distingush between the different states in the same (L, v)-sector.

In the state (30), the degree of the pre-exponential polynomial, L, which is the total angular momentum, is redistributed between internal excitations, J=L−v, and the contribution due to the collective motion, v, which we call *seniority* for brevity.

The energies of the states (30), i.e., the eigenavlues of (2) have the form

$$\mathcal{E}_k(L, v) = N + L + E_k(L, v) \tag{32}$$

where $E_k(L, v)$, the interaction energy, comes from diagonalization of $\tilde{V}$. It is therefore meaningful to consider the ground state as a function of both L and v, as illustracted Fig. 3. We call these states,

$$\Psi_0(L, v), \qquad \mathcal{E}_0(L, v) = min_k\{\mathcal{E}_k(L, v)\} \tag{33}$$

with minimum energy at fixed values of L and v the "generalized yrast states" [20], [21]. The usual yrast states are those among the (33), which minimize the energy $\mathcal{E}_0(L, v)$ with respect to the seniority,

$$\mathcal{E}_{yrast}(L) = min_v\{\mathcal{E}_0(L,v)\}. \tag{34}$$

At given L, one has exactly L allowed values of the seniority, they are

$$v = 0, 1, 2, ..., L-2, L. \tag{35}$$

The value $v = L-1$ is excluded because the symmetric degree $J = 1$ polynomial in variables $z_i - Z$ is evidently reduced to zero,

$$poly_S^1(z_i - Z) = \sum_{i=1}^{N}(z_i - Z) \equiv 0.$$

The states with definite seniority of type (30), can be obtained by applying the seniority projector [Cf. Eq.(20)]

$$\mathcal{P}_v = \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{i\phi(v-\mathcal{A}^+\mathcal{A})} \tag{36}$$

to the states of type (14).

The quantum number seniority helps to establish very useful "corresponding rules" between the states in different (L,v)-sectors and to relate their spectra. Indeed, by virtue of (19),(36) and 29), we can write

$$\tilde{V}\Psi_k(L+1, v+1) = \tilde{V}\mathcal{A}^+\Psi_k(L,v) =$$
$$= \mathcal{A}^+\tilde{V}\Psi_k(L,v) = E_k(L,v)\mathcal{A}^+\Psi_k(L,v). \tag{37}$$

Eq.(37) means the following: let $\{\Psi_k(L,v)\}$ be the $p(L)$ normalized eigenstates the Hamiltonian (2) in the sector L. Then exactly $p(L)$ eigenstates in the sector $L+1$ (having nonzero v!) can be obtained simply applying the collective ladder operator to the states in L sector,

$$\Psi_k(L+1, v+1) = \left(\frac{v}{v+1}\right)^{1/2}\mathcal{A}^+\Psi_k(L,v), \tag{38}$$

while their interaction energies will be the same,

$$E_k(L+1, v+1) = E_k(L,v). \tag{39}$$

Similar relationships appear in the fermion problem of the fractional quantum Hall effect [12].

As the total angular momentum grows from $L-1$ to L, the only new structures in the wave functions uppear in the **seniority zero sector**, where we have new states of total number equal to

$$g_0(L) = p(L) - p(L-1), \tag{40}$$

They must be obtained by diagonalization of $\tilde{V}$ in the sector $v = 0$.

Using (40) recursively from v to $v+1$, we calculate the total numbers of states, $g_v(L)$, in the sectors,

$$g_v(L) = p(L-v) - p(L-v-1) \qquad for \qquad v \leq L-2$$
$$and \qquad g_{v=L}(L) = 1. \tag{41}$$

These important results will be very usefull in the analysis given in the following sections.

V. SUPERSYMMETRIC DECOMPOSITION OF THE HAMILTONIAN

Regular methods to obtain exact ground state without solving the whole spectrum are not available. We use the approach [22] which we loosely nicknamed "supersymmetry" [26]. Suppose that the Hamiltonian can be written as a sum,

$$\tilde{V} = V_0 + V_S, \tag{42}$$

such that

(a) the first term, V_0, is simple, and one can find out its ground state $|0)$ with eigenvalue $\mathcal{E}_{min}$, possibly degenerate. If the second term, V_S, has the two "supersymmetric" properties:

(b) V_S annihilates the state $|0)$, so $V_S|0) = 0$,

(c) V_S is *non-negative definite*, $V_S \geq 0$, (it does not have negative eigenvalues),

then the state $|0)$ will still be the ground state for the full Hamiltonian $V_0 + V_S$, with the same eigenvalue $\mathcal{E}_{min}$. Indeed, **(b)** implies that $|0)$ is an eigenstate for the sum $V_0 + V_S$ with its eigenvalue intact. As one knows form linear algebra, if an Hermitean operator V_0

is perturbed by *a non-negative definite* Hermitean operator V_S, the eigenvalues can only increase (see, e.g. [27]). This means that the states other than $|0)$ can gain energy [Cf. **(b)**]. As $|0)$ is already the ground state for V_0, it will be the same for $V_0 + V_S$ [28].

A trivial but helpful example is the one-dimensional linear oscillator with the Hamiltonian $V = 1/2 + a^+a$. The state $|0)$, which obeys $a|0) = 0$, is "the ground state" for $V_0 \equiv 1/2$ because any other state $|n)$ has the same eigenvalue $1/2$. The non-negative definite operator $V_S = a^+a$, $V_S \geq 0$ can be considered as the "supersymmetric perturbation". It removes this degeneracy and leaves $|0)$ with lowest eigenvalue of the total Hamiltonian V [28].

The same arguments apply to the case with additional conserved quantum number v, such as $[v, V_0] = [v, V_S] = 0$. In the above scheme, the single state $|0)$ is replaced by the set of states $|L, v)$, each having the property **(a)** and **(b)** in their v-sectors. This is illustrated in Fig. 4, where the spectrum of V_0 is taken degenerate in each v-sector.

VI. SUPERSYMMETRIC TRIAD

The starting strategy is to look for states annihilated by a part of V which can be then shown to be non-negative definite operator. In our case, the supersymmetric triad V_0, V_S and $|0)$ can be established by inspecting action of terms V_k in (27) on selected states of partition basis (14). We observe first, that using the algebraic identity

$$N\sum_i^N a_i^+ a_i = \frac{1}{2}\sum_{i,j}^N (a_i^+ - a_j^+)(a_i - a_j) + \sum_i^N a_i^+ \sum_i^N a_j,$$

one can cast the first term, V_0, of the operator expansion (27) in the following form,

$$V_0 = \frac{N}{2}[(N-1)s_0 - (L - \mathcal{A}^+\mathcal{A})(s_1 - s_2)]. \tag{43}$$

We recall now that the operator $\mathcal{A}^+\mathcal{A}$ is diagonal in the seniority basis (30). In this basis, the operator V_0 is therefore reduced to a combination of quantum numbers, (43), with $\mathcal{A}^+\mathcal{A} = v$. Next, one can see that the simplest basis state, $|L)$, [Cf. Eq. (14)] with partition $[1, 1, ...1]$ [1]

[1] An example of such states is given by the first line of (15).

$$|L\rangle = P_S a_1^+ a_2^+ ... a_L^+ |0\rangle, \tag{44}$$

is annihilated by the remainder of the Hamiltonian (27),

$$(\tilde{V} - V_0)|L\rangle = 0. \tag{45}$$

In order to prove (45), we use Eqs.(7,23) to evaluate the commutator

$$[a_{12}^2, a_1^+ a_2^+] = -2(a_{12}^+ a_{12} + 1). \tag{46}$$

From this equation, it follows that

$$a_{12}^k a_1^+ a_2^+ a_3^+ ... a_L^+ |0\rangle = 0$$

for any $k > 2$, by virtue of (6). Therefore, we have

$$B^k |L\rangle = 0 \tag{47}$$

for any $k > 2$. From the same commutator (46), we have

$$[a_{12}^{+2} a_{12}^2, a_1^+ a_2^+]|0\rangle = [a_{12}^+ a_{12}, a_1^+ a_2^+]|0\rangle = -a_{12}^{+2}|0\rangle,$$

and thus $(B^2 - B^1)|L\rangle = 0$. This relation together with (47) results in (45).

Equations (43) and (45) hint that the state $|L\rangle$ and the operator V_0 may constitute important ingredients of the supersymmetric decomposition, Cf. Eq.(42) described in the previous section. However, the state $|L\rangle$ does not have definite seniority and it can not be eigenstate of the Hamiltonian. Nevertheless, as we will see, the $|L\rangle$ turns to be a generating function for states with definite seniority. Indeed, substitution $z_i = \tilde{z}_i + Z$ [see (30)] transforms $|L\rangle$ to a sum

$$|L\rangle = \left[P_S \tilde{z}_1 \tilde{z}_2 ... \tilde{z}_L + ... + Z^{L-2} P_S \tilde{z}_1 \tilde{z}_2 + Z^L\right]|0\rangle = \\ = \sum_{v=0}^{v=L(v\neq L-1)} \mathcal{P}_v |L\rangle \tag{48}$$

of exactly L states of the form (30), each being the eigenvector of $\mathcal{A}^+\mathcal{A}$ and therefore of V_0, with the eigenvalue (43). We notice that each v-sector is represented by a single term in (48), identified with $\mathcal{P}_v|L\rangle$.

The spectrum of V_0 is very simple. A schematic example is shown in Fig.4, left hand side. It consists of L equidistant (except $v \neq L-1$), $g_v(L)$-fold degenerate levels with energies given by (43). The degeneracies are given by Eq.(41). Each v-level contains one and only one state $|L,v) = \mathcal{P}_v|L\rangle$ from the sum (48). Therefore, the set of states $|L,v)$ obey the criterion **(a)** with the operator V_0. The property **(b)** with $V_S \equiv \tilde{V} - V_0$ holds by virtue of (45). In particular, **(a)** together with **(b)** mean that $|L,v)$ are the eigenvectors of $\tilde{V} = V_0 + V_S$ with eigenvalues (43). This holds for any interaction $V(r)$.

The supersymmetric representation for the triad V_0, V_S and $|L,v) = \mathcal{P}_v|L\rangle$ would be complete if we succeeded to prove non-negative definiteness $V_S \geq 0$ of the remainder of the Hamiltonian $\tilde{V} - V_0$, criterion **(c)**. So far, we did not specify form of the interaction $V(r)$ in (27). We will study now general case and specify the class of potentials which have $V_S \geq 0$.

VII. NON-NEGATIVE DEFINITENESS OF THE SUPERSYMMETRIC PERTURBATION

We have to check signs of all the eigenvalues of V_S in the partition space (14). To avoid solving the whole spectrum, in the space of symmetrized states, we use the following trick. By definition, the nonzero eigenvalues of V_S in the space (14) coincide with the nonzero eigenvalues of $P_S V_S P_S$ in the **full space of monomials** m in (14). This latter space has dimensionality much higher than $p(L)$ and it includes wave functions of all possible symmetries, including boson sector (fully symmetric), fermion sector (fully antisymmetric) *etc.* In this extended space, the analysis of signs of eigenvalues is however crucially simplified, while the contributions from the symmetric sector can be accurately separated. Using the symmetry of V_S under permutations of particles, we can write

$$P_S V_S P_S = P_S \sum_{i>j} V_{S,ij} P_S = \frac{N(N-1)}{2} P_S V_{S,12} P_S \tag{49}$$

where $V_{S,ij}$ is the contribution from pair of particles i,j to V_S [Cf.(27,26,42)]. In order to see that

$$P_S V_S P_S \geq 0 \tag{50}$$

for a given interaction potential $V(r)$, it is *sufficient* to show that $V_{S,12} \geq 0$, because application of any projector P_S from both sides in (49) can add new zero eigenvalues, but can not add negative eigenvalues. This follows from the known "inertia theorem" of linear algebra [27]. Now, we study the eigenvalues of $V_{S,12}$. Let π_{12} be the operator of permutation of variables 1 and 2. The triad

$$T = \{\pi_{12}, \quad a_{12}^{+} a_{12}, \quad V_{S,12}\}$$

forms a set of mutually commuting operators. Indeed, $V_{S,12}$ is expressed in terms of $B_{12}^{k} = a_{12}^{+k} a_{12}^{k}$ [see (27)]. For fixed pair of particle indices, the operators a_{12} and a_{12}^{+} commute like Bose operators, $[a_{12}, a_{12}^{+}] = 1$. Consequently, any operator B_{12}^{k} can be expressed in terms of $a_{12}^{+} a_{12}$, using the standard boson calculus formula

$$a_{12}^{+k} a_{12}^{k} = a_{12}^{+} a_{12} (a_{12}^{+} a_{12} - 1) ... (a_{12}^{+} a_{12} - k + 1). \tag{51}$$

The triad T is diagonalized simultaneously in the basis of monomials $m \equiv z_1^{l_1} z_2^{l_2} z_3^{l_3} .... z_N^{l_N}$, (14) with the only substitutions $z_1 \rightarrow \frac{1}{\sqrt{2}}(z_1 - z_2)$, $z_2 \rightarrow \frac{1}{\sqrt{2}}(z_1 + z_2)$,

$$\left[\frac{1}{\sqrt{2}}(z_1 - z_2)\right]^{l_1} \left[\frac{1}{\sqrt{2}}(z_1 + z_2)\right]^{l_2} z_3^{l_3} ... z_N^{l_N} \tag{52}$$

In this basis, the eigenvalues of the triad T depend only on l_1 through the subfactor $\left[\frac{1}{\sqrt{2}}(z_1 - z_2)\right]^{l_1}$ of the eigenvector (52). These eigenvalues are given by

$$T = \left\{(-1)^{l_1}, \quad l_1, \quad \lambda_{l_1}\right\}, \tag{53}$$

respectively, where λ_{l_1} is the eigenvalue of $V_{S,12}$ which can be readily calculated. Using Eqs.(27) and (26) and the summation formula

$$\sum_{k=0}^{N} \frac{(-1)^k N!}{k!(N-k)!} M(k+1, 1, -t) = \frac{e^{-t} t^N}{N!}$$

we obtain the expression for the eigenvalue of $V_{S,12}$ in the form of the integral

$$\lambda_{l_1} = \int_0^\infty r dr V(r) f_{l_1}(r) \tag{54}$$

with f the functions defined as

$$f_{l_1}(r) = e^{-r^2/2}\left[\frac{r^{2l_1}}{2^{l_1} l_1!} - 1 + \frac{l_1}{4}\left(2 - \frac{r^4}{4}\right)\right]. \tag{55}$$

The eigenvalue $(-1)^{l_1}$ of π_{12} helps now to separate out the states with wrong symmetry: the eigenvectors with l_1 odd are antisymmetric in z_1, z_2, and the projector P_S in (49) eliminates their contributions. All even values of l_1 ($\leq L$) can contribute to the bosonic sector, and the corresponding λ's must be checked. Therefore, the set of inequalities for the *control eigenvalues*

$$\lambda_{2n} \geq 0 \tag{56}$$

for all values of integer n obeying $2n \leq L$ forms the sufficient condition for (50), i.e., for the supersymmetric operator V_S to be non-negative definite, criterion **(c)** [2]. Under these conditions, the triad V_0, $V_S \equiv \tilde{V} - V_0$ and $\{\mathcal{P}_v|L\rangle\}$ obeys the supersymmetric representation with properties **(a)**, **(b)** and **(c)**, with $\mathcal{P}_v|L\rangle$ being the ground state in its sector L, v with the energy $\mathcal{E}_{min}$ equal to eigenvalue of V_0.

The conditions (56) can be used to describe the universality class of the interactions $V(r)$. They are analyzed explicitly in the following sections.

VIII. GENERALIZED YRAST STATES AND THEIR SPECTRA

With the criterion (56) and Eqs.(48) and (43) at hand, we can formulate very general result: For any *bona fide* two-body potential $V(r)$ which satisfies the integral condition

$$\int_0^\infty V(\sqrt{2t}) e^{-t}\left[\frac{t^{2n}}{(2n)!} - 1 + n\left(1 - \frac{t^2}{2}\right)\right] \geq 0$$
$$for \quad any \quad n \leq L'/2, \tag{57}$$

---

[2]Note that $f_0(r) \equiv 0$ and $f_2(r) \equiv 0$, therefore $\lambda_0 = \lambda_2 = 0$ anyway.

the eigenstates of the Hamiltonian $\tilde{H}$ **with minimal energies at given pair of** v **and** $L(\leq \min\{L', N\})$ have universal form, as follows from (48),

$$|L, v) = e^{-\frac{1}{2}\sum |z_i|^2} Z^v \left(\frac{\partial}{\partial Z}\right)^{N-L+v} \prod_{k=1}^{N} (z_k - Z),$$

$$Z \to \frac{1}{N}\sum_{i=1}^{N} z_i. \tag{58}$$

Their energies are given by (43) and they can be expressed through the simple moments of $V(r)$,

$$\mathcal{E}_{min}(L, v) = L + N + \frac{N(N-1)}{2} s_0 + \frac{s_0 - s_1}{2} N(v - L) \tag{59}$$

$$s_0 = \int_0^\infty dt e^{-t} V(\sqrt{2t}), \quad s_1 = \int_0^\infty dt e^{-t} \left(\frac{1}{2} + \frac{t^2}{4}\right) V(\sqrt{2t})$$

which are equal to expectation values of interaction between two bosons both in the ground $s_0 = \langle \psi_0^\dagger \psi_0^\dagger | V | \psi_0 \psi_0 \rangle$ and the first excited state of oscillator $s_1 = \langle \psi_1^\dagger \psi_1^\dagger | V | \psi_1 \psi_1 \rangle$, respectively. Here, $\langle \psi^\dagger \psi^\dagger | V | \psi \psi \rangle$ denote the usual two-body matrix element and $\psi_0 = |0\rangle$ and $\psi_1 = z|0\rangle$ It is worth to note that this relation does not mean that the eigenvalues (59) can be obtained as an expectation value over a simple state, say, with the two lowest oscillator levels occupied; in fact, the eigenstates (58) are much more complicated when written in the second quantization representation.

At fixed L, we have exactly L such equidistant *generalized yrast* states, marked by $v = 0, 1, 2, ..., L$ $(v \neq L - 1)$, Cf. Fig. 3., right hand part. Each such state is the "ground state" in the sector L, v (of course, there are other states in each sector with higher energies).

Example of the spectrum of a real system for $N = L = 6$ and the δ-interaction is shown in Fig. 5.

The usual yrast states are those of (58) that minimize $\mathcal{E}_{min}(L, v)$ with respect to v, Cf. Eq.(34). As $\mathcal{E}_{min}(L, v)$ in (59) depend linearly on v, it is immediately seen that there is only two options [3] :

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[3] The rare case of exact eqality in (61) corresponds to complete degeneracy of the generalized yrast states with different values of v, having the same N and L, see, e.g. Ref. [22]).

$$\textbf{(I)} \qquad v = 0, \qquad if \qquad D = s_0 - s_1 > 0, \tag{60}$$

$$\textbf{(II)} \qquad v = L \qquad if \qquad D = s_0 - s_1 \leq 0. \tag{61}$$

where the "spectral discriminant", D, is given by

$$D = \int_0^\infty dt e^{-t} \left(\frac{1}{2} - \frac{t^2}{4} \right) V(\sqrt{2t}). \tag{62}$$

For the *bona fide* potentials obeying (57), the first option is usually realized, as is assumed in Figs. 3 and 4, see also Figs. 5 and 7 for particular interaction potentials. For example, integrating by parts in (62), the spectral discriminant D is seen positive for any decreasing potential

$$\frac{dV(r)}{dr} < 0 \tag{63}$$

which allows the representation (27). The condition (63) can be replaced by

$$\frac{dV(r)}{dr} \leq 0, \tag{64}$$

if strict inequality in (63) holds for at least single value of r.

The inequality $(\mathcal{V}_0 - \mathcal{V}_1)(v - L) < 0$ corresponding to **(I)** means that internal rotational excitations with higher $J = L - v$ are energetically favorable, once the interaction energy between two bosons in the state $z|0\rangle$ is smaller than that in the state $|0\rangle$.

Physically, the *yrast* wave functions (58) with $v = 0$ correspond to condensation to a vortex, rotating around the "center-of-mass", as discussed, for example, in [3].

In contrast, the maximum seniority states with $v = L$, which correspond to purely collective rotation [4] with no internal excitations, were shown [3] to be energetically favorable for attractive δ-forces. It is curios that there exists a broad class of predominantly attractive interactions, for which the yrast states have the same form as for the attractive δ-function interaction [23].

The results desribed in this section are very general [20]. The condition (57) defines the class of the potentials for which the results (58,59) are valid. In the next section, we show that (57) holds for many potentials of physical interest and consider few explicit examples. Discussion of general properties of (57) will be given in section 10.

IX. THE RESULTS FOR PARTICULAR INTERACTIONS

We consider now few applications of the results obtained in general form in the previous section to particular cases of various potentials of interaction between the bosonic atoms.

Gaussian forces with variable range and δ-interaction.

We start with considering the case of repulsive Gaussian interaction,

$$V(r) = \frac{U_0}{\pi R^2} e^{-r^2/R^2}, \qquad U_0 \geq 0, \tag{65}$$

with U_0 a nonnegative strength and R the radius which can be varied from zero to infinity. We obtain from Eq. (57) the control eigenvalues

$$\lambda_{2n} = \frac{U_0}{\pi(2+R^2)} \left[\left(\frac{R^2}{2+R^2} \right)^{2n} - 1 + 4n \frac{1+R^2}{(2+R^2)^2} \right]. \tag{66}$$

From this equation, it can be easily seen that the non-negative definiteness condition (57) $\lambda_{2n} \geq 0$ is fulfilled for any n irrespectively to the value of R, see Fig. 6. This can be easily proved by induction in n. Indeed, we have from (66) taht $\lambda_0 = \lambda_2 = 0$ and

$$\lambda_{2(n+1)} - \lambda_{2n} = \frac{4U_0(1+R^2)}{\pi(2+R^2)} \left[1 - \left(\frac{R^2}{2+R^2} \right)^{2n} \right] > 0,$$

therefore $\lambda_{2n} > 0$ for any $n \geq 2$. The results given by Eqs.(58,59) are therefore valid for any $L \leq N$, and the spectrum of the generalized yrast states for the case of the Gaussian potential is given by

$$\mathcal{E}_{min}(L,v) = L + N + \frac{U_0}{\pi(2+R^2)} \left[N(N-1)/2 - \frac{(1+R^2)}{(2+R^2)^2} N(L-v) \right] \tag{67}$$

One can see that the usual yrast states correspond to $v = 0$.

Taking the zero range limit, $R \to 0$, in Eqs. (65,66,67), we pass to the case of the δ-function repulsive interaction

$$V = U_0 \delta(\vec{r}) = U_0 \frac{\delta(r)}{2\pi r}. \tag{68}$$

In this case, we obtain instead of (66)

$$\lambda_{l_1} = \frac{U_0}{2\pi}\left[\delta_{l_1,0} + \frac{l_1}{2} - 1\right]. \tag{69}$$

It is seen that while $\lambda_1 = -1/2$, for any $l_1 = 2n$ even the condition $\lambda_{2n} \geq 0$ (57) holds, Cf. Fig. 6. The energies of the generalized yrast states (59) are now given by

$$\mathcal{E}_{min}(L, v) = L + N + \frac{U_0}{8\pi} N(2N - L + v - 2). \tag{70}$$

Yrast states have $v = 0$, and $\mathcal{E}_{min}(L, 0)$ agrees with that obtained numerically [5], see also [14], [15], [16].

It is expedient to look at the full spectrum of the system. At low N and L, the problem can be diagonalized analytically. Below, we present the results for the total angular momentum $L = 6$ in the system of six particles interacting via repulsive δ-function interaction. The total number of states (dimensionality of the partition basis) is $p(6) = 11$. The interaction energies $E_k(L, v)$ of the states are given by

$$\begin{aligned}
E_0(6,6) = 15\frac{U_0}{2\pi}, \qquad E_0(6,4) = 12\frac{U_0}{2\pi}, \qquad E_0(6,3) = \frac{21}{2}\frac{U_0}{2\pi}, \\
E_0(6,2) = 9\frac{U_0}{2\pi}, \qquad E_1(6,2) = \frac{21}{2}\frac{U_0}{2\pi}, \\
E_0(6,1) = \frac{15}{2}\frac{U_0}{2\pi}, \qquad E_1(6,1) = \frac{81}{8}\frac{U_0}{2\pi}, \\
E_0(6,0) = \frac{U_0}{2\pi}6, \qquad E_1(6,0) = \frac{U_0}{2\pi}\left[-\eta + \frac{69}{8} + \zeta\right], \\
E_2(6,0) = \frac{U_0}{2\pi}\left[-\eta + \frac{69}{8} - \zeta\right], \qquad E_3(6,0) = \frac{U_0}{2\pi}\left(2\eta + \frac{69}{8}\right)
\end{aligned} \tag{71}$$

with

$$\eta = \frac{\xi}{32} + \frac{21}{4\xi}, \quad \zeta = i\frac{3\sqrt{3}}{2}\left(\frac{\xi}{48} - \frac{7}{2\xi}\right),$$

$$\xi = (324 + 12i\sqrt{32199})^{1/3}$$

Here, the index k marks degree of excitation of the state within the same L, v-sector: so $\mathcal{E}_0(L, v) = \mathcal{E}_{min}(L, v)$ correspond to the generalized yrast states (59,70). There are no excited

states in the sectors $v = 6, 4$ and 3, the sectors $v = 4$ and 5 have one excited state each. The sector $v = 0$ has three excited states, their energies are found by solving a cubic equation. The spectrum is shown in Fig. 5.

Two- and three dimensional Coulomb forces

Of special physical interest are the long-range forces of Coulomb type. For the conventional Coulomb interaction

$$V = \frac{U_0}{r}, \qquad U_0 \geq 0 \tag{72}$$

we obtain in (57)

$$\lambda_{2n} = U_0\sqrt{2\pi}\left(\frac{5n}{16} - \frac{1}{2} + \frac{\Gamma(2n+1/2)}{2\sqrt{\pi}(2n)!}\right). \tag{73}$$

It is easily seen, using induction in n, that the inequalities $\lambda_{2n} \geq 0$ (57) hold (Cf. Fig. 6), and the wave functions of the generalized yrast states are therefore given by (58). Their energies are given by

$$\mathcal{E}_{min}(L, v) = L + N + \frac{U_0\sqrt{2\pi}}{64}N[16(N-1) - 5(L-v)].. \tag{74}$$

The yrast states correspond to $v = 0$.

Similar formulas can be obtained for the two-dimensional Coulomb log-interaction

$$V = U_0 log\left(\frac{1}{r}\right), \qquad U_0 \geq 0, \tag{75}$$

which corresponds to repulsion at small distances. We have

$$\lambda_{2n} = \frac{U_0}{4}\left[3n - \frac{1}{n} + 2\psi(1) - 2\psi(2n)\right] \geq 0 \tag{76}$$

where ψ is digamma function [25], the inequality in $\lambda_{2n} \geq 0$ for (76) can again be easily proven by induction, Cf. Fig. 6, so condition (57) holds. The energies of the generalized yrast states are given by

$$\mathcal{E}_{min}(L, v) =$$
$$L + N - \frac{U_0 N}{4}\left[(log(2) - \gamma)(N-1) - \frac{3}{4}(v - L)\right] \tag{77}$$

with $\gamma = -\psi(1) = 0.57721...$ the Euler constant. The yrast states correspond to $v = 0$.

The results very similar to the above can be obtained for the screened Coulomb (Yukawa) forces $V \propto exp(r/r_0)/r$.

The results for the genearlized yrast states in the cases considered in this section are illustrated in Fig. 7 for $N = L = 6$.

X. THE SUFFICIENCY CONDITION AND THE UNIVERSALITY CLASS OF BONA FIDE REPULSIVE POTENTIALS

The simple sufficiency condition (57) can be checked straightforwardly for any interaction potential $V(r)$ as was done in the previous section. In this section, we show that the applicability condition (57) imposes in fact **only weak** restrictions on the class of repulsive forces $V(r)$ to which the results (58,59) apply. We will discuss the condition (57) in more details and define some subclasses of the bona fide potentials of interest.

Short-range interactions

Indeed, at small $r \simeq 0$, the factor-function $f_{2n}(r)$ in $\int_0^\infty r dr f_{2n}(r) U(r)$ in (57) approach positive constant values with zero derivatives,

$$f_n|_{r=0} = n - 1, \qquad \frac{df_n}{dr}\Big|_{r=0} = 0, \tag{78}$$

while the first node of the functions $f_{2n}(r)$ occurs at

$$r_0 = \sqrt{2\sqrt{12 - 2\sqrt{30}}} \simeq 1.43 \tag{79}$$

which is of order of the oscillator length, in our units ($\hbar = m = \omega = 1$). The factor functions are plotted in Fig. 8. From Eqs.(78), it follows therefore that for any short range (as compared to the characteristic length of the trap) interaction, the condition (57) reduces to

$$\int d^2\vec{r} V(r) \geq 0, \tag{80}$$

Thus, the results (58,59) hold for short-range interactions, which are repulsive on average. I is seen also that inequality (80) gives v=0 for the usual yrast states, according to (60).

It is interesting that the sort-range potentials do not have to be purely repulsive to match the condition (57), (80). Instead, the condition (57) implies that $V(r)$ has strong repulsive component. For example, consider the two-parameter family of the potentials in the form

$$V(r) = -\frac{|V_0|}{R^2}(1-(r/R)^2)e^{-\frac{r^2}{2R^2}} \tag{81}$$

These potentials can be regarded as attractive in the usual sense, having the potential well art short distance. Examples are shown on Fig. 9. In fact, if the well is deep enough, it can even support bound state. Indeed, considering the Schroedinger equation of relative motion (s-wave) of the pair of particle with unit mass each, interaction via (81),

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\phi_0 + V(r)\phi_0 = \varepsilon_0\phi_0,$$

for the wave function of the ground state, ϕ_0, we obtain rigorous variational upper bound on the ground state energy, ε_0, using the trial wave functon $\phi_0 = exp(-\sqrt{3}r^2/R^2)$:

$$\varepsilon_0 < \frac{\sqrt{6V_0}\left[720V_0^2 + 1 - 8(6V_0)^{5/2}/3 - 46V_0 - 2(6V_0)^{3/2}\right]}{R^2(24V_0-1)^2}$$

It is seen that we have negative energy of the ground state, $\varepsilon_0 < 0$, if

$$V_0 > \frac{1}{216}\left[(3763+18\sqrt{502})^{1/3} + \frac{241}{(3763+18\sqrt{502})^{1/3}} + 13\right]^2 \simeq 8.99 \tag{82}$$

in such cases the potential well indeed supports the bound state(s). On the other hand, the potential (81) satisfies the basic condition (57) if

$$R \le \frac{1}{10}\sqrt{5+5\sqrt{41}} \simeq 0.608, \tag{83}$$

with all its control λ_{2n} nonnegative, irrespectively of the magnitude of V_0, Cf. Fig. 9, lower panel. Therefore, the family of the potentials (81) satifying (82) and (83) will be

"predominantly repulsive" in the sense of the condition (57), having the solutions for the generalized yrast states in the form (58) and (59), while having strong attractive component sufficient for the bound state. In the sense of three-dimensional scattering theory, the scattering length can be negative.

Long-range forces

The condition (57) holds even for many long-range interactions. This is seen from behavior of function $f_4(r)$ ($n = 0$ and $n = 1$ give $f \equiv 0$) which is positive at $r < r_0$ and

$$r > r_1 = \sqrt{2\sqrt{12 + 2\sqrt{30}}} \simeq 3.10, \tag{84}$$

and f_4 is negative only in the interval $r_0 \leq r \leq r_1$ ($f_n(r)$ for higher n behave similarly). By direct calculation, it is easy to see that the integrals of the functions rf_{2n} [Cf. Fig. 8, lower panel] over r vanish,

$$\int_0^\infty r dr f_{2n}(r) = S_1 + S_2 + S_3 = 0,$$

$$S_1 = \int_0^{r_0} r dr f_{2n}(r), \quad S_2 = \int_{r_0}^{r_1} r dr f_{2n}(r), \quad S_3 = \int_{r_1}^{\infty} r dr f_{2n}(r), \tag{85}$$

so the net areas coming form the regions where $f > 0$ and $f < 0$ coincide,

$$S_1 + S_3 = |S_2|,$$

as is shown in Fig. 10, upper panel, for the case f_4. From this geometry, which is also illustrated in Fig. 10, lower panel, it is clear that (57) holds, if $V(r)$ decreases monotonically and fast enough, as is the case for the long range Gaussian and Coulomb forces. Example of the potentials of such type are shown in Fig. 10, lower panel.

Integrating by parts, one sees that (57) holds for any monotonically decreasing and convex, interaction $V(r)$,

$$\frac{dV(r)}{dr} \leq 0, \qquad \frac{d^2V(r)}{dr^2} \geq 0,$$

not necessarily repulsive everywhere.

One can therefore summarize that the condition holds for any physically meaningful repulsive interaction.

XI. CONCLUSION

To conclude, we considered the problem of weakly interacting Bose atoms in symmetric harmonic trap. Our main purpose was to study the yrast states of the system. In order to treat the problem of the ground state of the system at a given angular momentum, we developed an expansion of the interaction in powers of ladder operators. This universal operator expansion is exact and it provides a very convenient way to study the eigenvalue problem in the coordinate representation.

Taking into account the additional conserved quantity, associated with the collective contribution to the total angular momentum, we considered a more general problem, namely, the ground state of the system as a function of two conserved quantum numbers: the total angular momentum and the angular momentum of internal excitations. We called this series of states "generalized yrast states".

A supersymmetric representation for the interaction has been developed to derive analytically the states with minimum energy at a given angular momentum and seniority (generalized yrast states). The results apply, in particular, to the usual yrast states. The wave functions of the generalized yrast state have the form of "condensed vortex states". Their energies are expressed through their quantum numbers and simple integrals of the interaction potential.

We studied the condition on the interaction potential which allows the use of our solution. Analysis shows that there exists a broad *universality* class of the repulsive interactions for which these results are valid. We described this universality class by simple integral condition on the interaction potential.

The results of the work allow further generalizations and developments. The three-dimensional case can be done using the same method. It is also interesting to study region of higher angular momenta $L > N$, where the structure of the basis polynomials will be changed [24], while the numerical studies indicate signs of phase transition [5]. The method of "supersymmetric decomposition" developed here is not restricted to this particular problem

and can be applied to fermions and even to the particles with parastatistics.

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Figure Captions

Fig. 1. Definition of the yrast states. The spectrum of a system (circles) is shown versus the angular momentum, L. The lowest energy states at each value of L, connected by the curve, compose the "yrast line" of the system.

Fig. 2. Fragment of the spectrum of the system (schematic plot) within approximation of the weak coupling limit. The left hand side is the spectrum of the system without interactions. The levels are equidistant with the spacing $\hbar\omega$, each level is p-fold degenerate. Central part is the spectrum of interacting system. Each L-level splits onto p levels. The sequence of the states with minimum energy at given L is the yrast-line (right hand part).

Fig. 3. Illustration to the definition of the generalized yrast states. The left-hand side is the part of the spectrum of the interacting system with definite angular momentum L (corresponding to a bunch of levels in the central part of Fig. 1.). These states can be grouped onto "bands" with definite values of the seniority, v, as is shown on the right hand side. The sequence of lowest energy states in their v-sector composes the generalized yrast states (they are marked with asterisks). One of those, with absolute minimum of energy, is the usual ground state.

Fig. 4. Illustration of supersymmetric decomposition. The spectrum of V_0 (left) is sequence of degenerate levels, labeled by the conserved quantum number v. Supersymmetric perturbation $V_S \geq 0$ splits each level, pushing the states up and leaving the lowest energy in each v-sector intact. The resulting spectrum of $V_0 + V_S$ is shown on the right.

Fig. 5. The spectrum of the system with repulsive δ-interaction calculated analytically for the case of six particles in the sector with total angular momentum $L = 6$. The total number of states is $p(6) = 11$. The energy of the levels $e(L, v) = \frac{2\pi}{U_0}[\mathcal{E}(L, v) - L - N]$, Eq.(70), are plotted against the internal angular momentum, $J = L - v$. The generalized yrast states drop on the straight line described by Eq.(70), they are connected by dashed

line.

Fig. 6. "Redge trajectories" for λ_{l_1} (54). Upper panel: the values of $a = \frac{2\pi}{U_0}\lambda_l$ are plotted versus l (circles connected by curves) for two different interaction potentials. The curves correspond to l considered as contionuos variable. Solid curve - repulsive Gaussian potential with $R = 1$ and dashed curve - repulsive δ-function interaction. The double circles denote the control eigenvalues with even $l = 2n$, Eqs.(57,66).

Lower panel: the same but for the Coulomb interaction (solid curve) and *log*-Coulomb interaction (dashed curve), Cf. Eqs.(73,76).

Fig. 7. The specta of the generalized yrast states $e = \frac{2\pi}{U_0}\mathcal{E}_{min}(L, v)$ are plotted against $J = L - v$ ($N = L = 6$) for the Gaussian interaction with $R = 1$ (small circles connected by solid line), δ-interaction (small circles, dashed line), Coulomb interaction (big circles, solid line) and the *log*-Coulomb interaction (big circles, dashed line).

Fig. 8. Upper panel: Factor-functions $f_{2n}(r)$ for $n = 2, 3, 4, 5$ are plotted against r: f_4 (curve 1), f_6 (curve 2), f_8 (curve 3) and f_{10} (curve 4).

Lower panel: the same as above but for the $rf_{2n}(r)$.

Fig. 9. Upper panel: the potentials [Eq.(81)] supporting the bound state with $V_0 = 10$ obeing Eq.(81) are plotted against r for three values of R: $R = 0.3$ - solid curve, $R = 0.6$ - dashed curve, and $R = 1.0$ - dashed-dotted curve.

Lower panel: the "Redge trajectories" for λ_{l_1} (54). The eigenvalues λ_l in combinations $a = \frac{1}{U_0}\lambda_l$ are plotted as function of l (circles) for the potentials shown on the upper panel. The curves connecting the symbols correspond to eigenvalues as continuous function of l: $R = 0.3$ - solid curve, $R = 0.6$ - dashed curve, and $R = 1.0$ - dashed-dotted curve. The double circles denote the control eigenvalues with even $l = 2n$.

Fig. 10. Upper panel: areas under curve $rf_4(r)$ (illustration to Eq. (85).

Lower panel: the function $rf_4(r)$ (thin curve) is plotted against r together with potentials

matching condition (57). The latter are shown by thick curves: 1 - Gaussian potential $V(r) = \frac{U_0}{\pi R^2} exp(-r^2/R^2)$ with $R = U_0 = 1.5$, 2 - Coulomb potential $V(r) = \frac{U_0}{r}$ with $U_0 = 1/5$ and 3 - log-Coulomb potential $V(r) = U_0 log(1/r)$ with $U_0 = 1/3$.

Fig. 1. Hussein & Vorov "Condensed..."

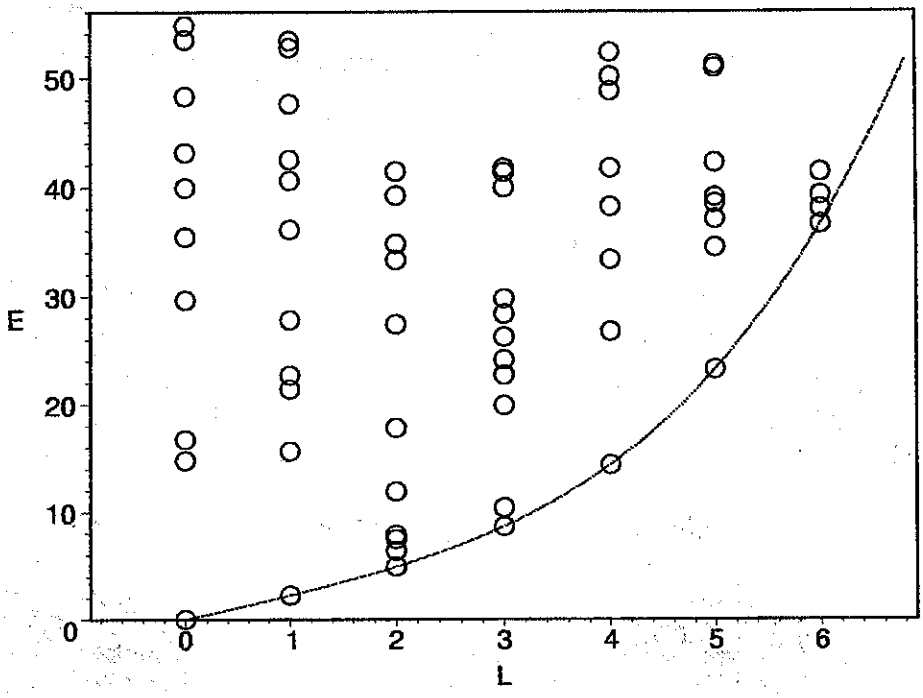
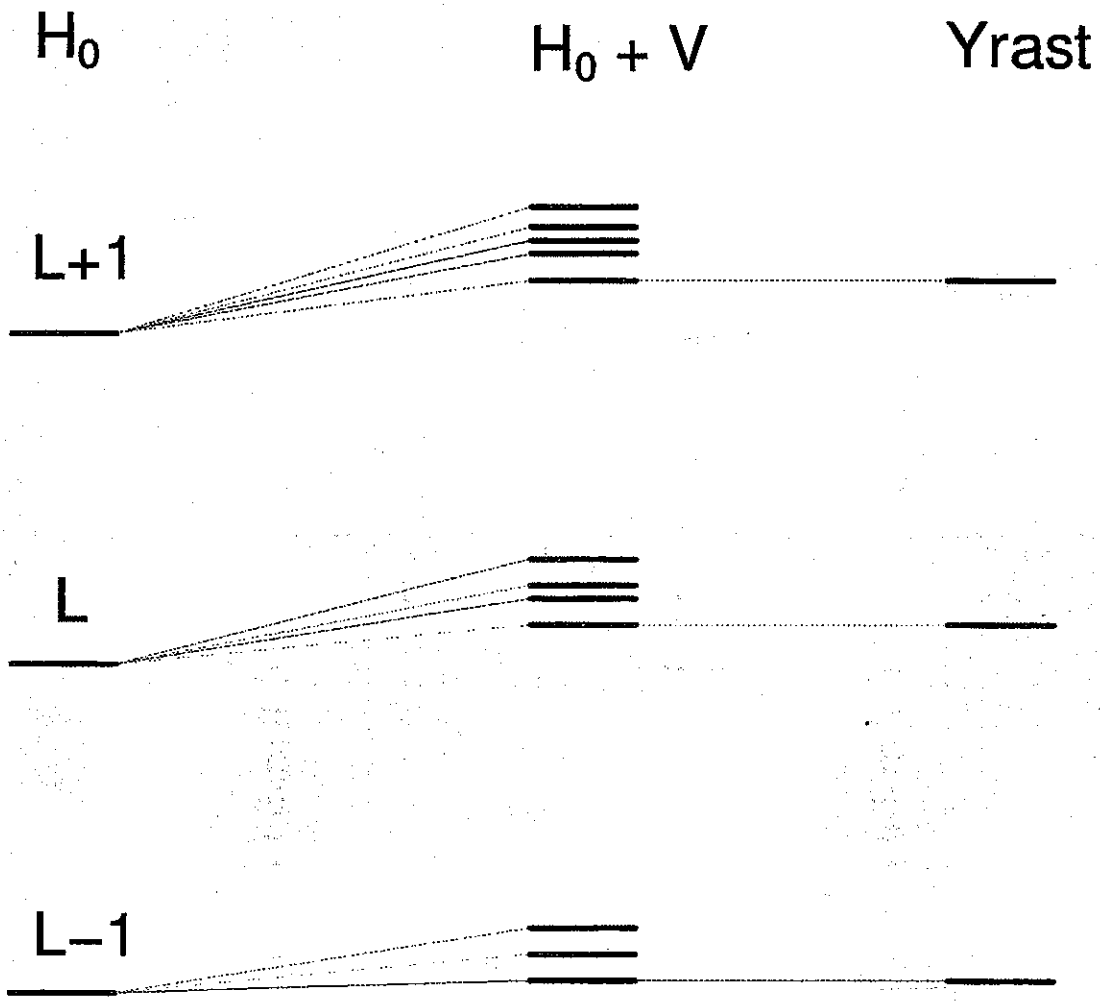


Fig. 2. Hussein & Vorov "Condensed..."



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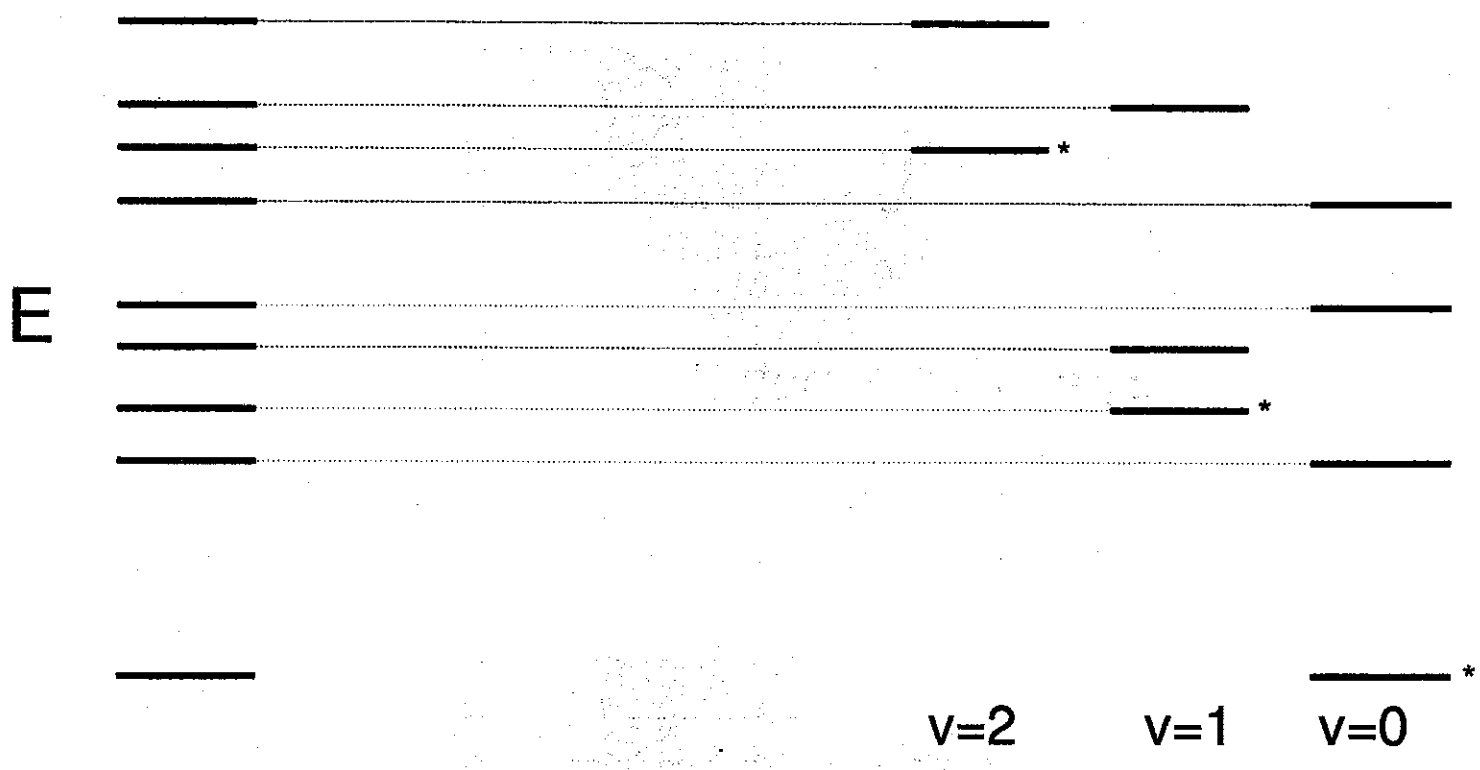

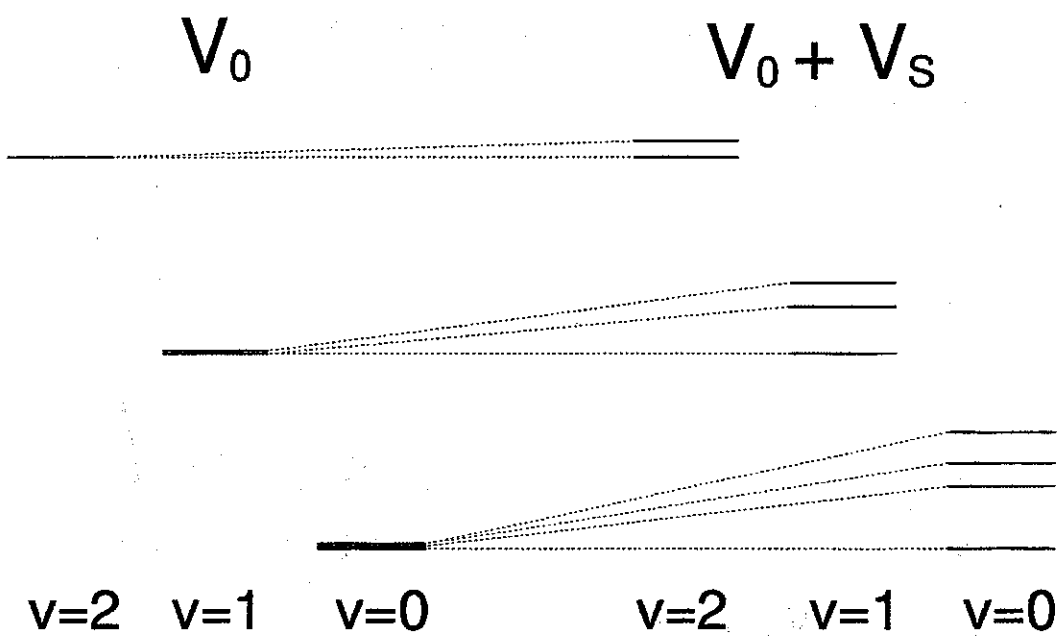

Fig. 7. Hussein & Vorov "Condensed..."

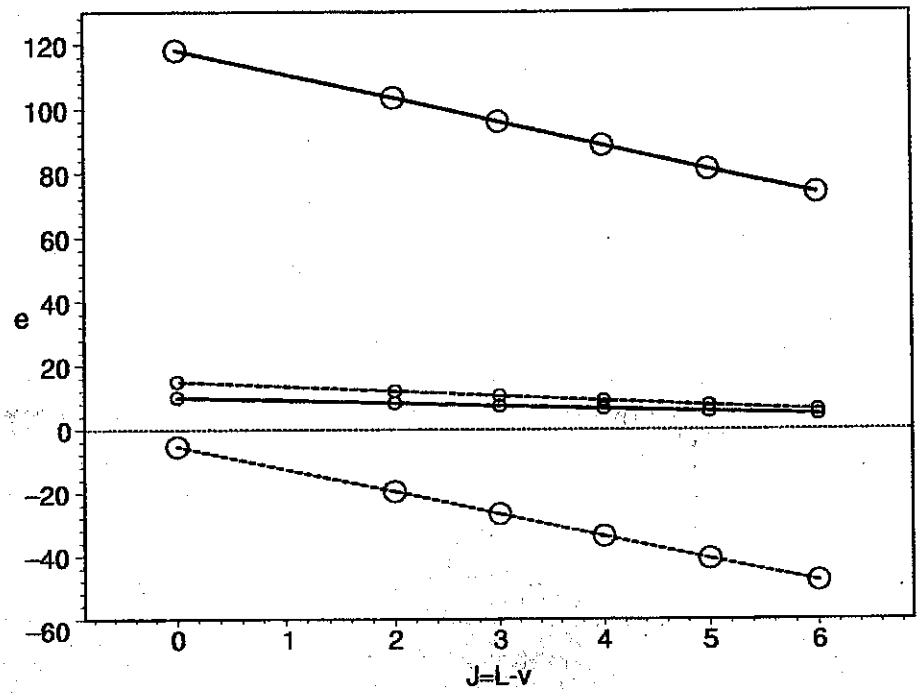

Fig. 8. Hussein & Vorov "Condensed..."