Instituto de Física
Universidade de São Paulo

Dirac equation in the magnetic-solenoid field

Gavrilov, SP
Tomsk State Pedagogical University, 634041 Russia
Gitman, DM; Smirnov, AA
Instituto de Física, Universidade de São Paulo, SP, Brasil

Publicação IF - 1555/2002

Dirac equation in the magnetic-solenoid field

S.P. Gavrilov[a], D.M. Gitman[b], and A.A. Smirnov[c]
Instituto de Fisica, Universidade de São Paulo,
C.P 66318, 05315-970 São Paulo, SP, Brasil

Abstract

In the present article we consider the Dirac equation in the magnetic-solenoid field and in the regularized magnetic-solenoid field. First we construct self-adjoint extensions of the Dirac Hamiltonian using von Neumann's theory of deficiency indices. We find a one-parameter family of self-adjoint extensions of the Dirac Hamiltonian and define a one-parameter family of the allowed boundary condition at the AB solenoid. In the regularized magnetic-solenoid field, we find for the first time solutions of the Dirac equation. We study the structure of these solutions and their dependence on the behavior of the magnetic field inside the solenoid. Then we exploit the latter solutions to specify boundary conditions for the singular magnetic-solenoid field.

03.65.Pm, 03.65.Ge

1 Introduction

The present article is a natural continuation of the works [1, 2, 3, 4] where solutions of the Schrödinger, Klein-Gordon, and Dirac equations in the superposition of the Aharonov-Bohm (AB) field (the field of an infinitely long and infinitesimally thin solenoid) and a collinear uniform magnetic field were studied. In what follows, we call the latter superposition the magnetic-solenoid field. In particular, in the paper [4] solutions of the Dirac equation in the magnetic-solenoid field in $2+1$ and $3+1$ dimensions were studied in detail. Then, in [5], these solutions were used to calculate various characteristics of the particle radiation in such a field. In fact, the AB effect in synchrotron radiation was investigated. However, a number of important and interesting aspects related to the rigorous treatment of the solutions of the Dirac equation in the magnetic-solenoid field were not considered. One ought to say that in the work [4] it was pointed out that a critical subspace exists where the Hamiltonian of the problem is not self-adjoint. But the corresponding self-adjoint extensions of the Hamiltonian were not studied. The completeness of the solutions was not considered from this point of view as well.

One has to remark that even for the pure AB field it was not simple to solve the two aforementioned problems. First, the construction of self-adjoint extensions of the nonrelativistic Hamiltonian in the AB field was studied in detail in [6]. In the work [6] solutions in the regularized AB field were thoroughly considered as well. The need to consider self-adjoint extensions of the Dirac Hamiltonian in the pure AB field in $2+1$ dimensions was recognized in [7, 8]. The interaction between the magnetic momentum of a charged particle and the AB field essentially changes the behavior of the wave functions at the magnetic string [8, 9, 10]. It was shown that a one-parameter family of boundary

[a]Tomsk State Pedagogical University, 634041 Russia; e-mail: gavrilovsp@hotmail.com
[b]e-mail: gitman@dfn.if.usp.br
[c]e-mail: saa@dfn.if.usp.br

conditions at the origin arises. Self-adjoint extensions of the Dirac Hamiltonian in $3+1$ dimensions were found in [11]. The works [12, 13] present an alternative method of treating the Hamiltonian extension problem in $2+1$ and in $3+1$ dimensions. It was shown in [14] that in $2+1$ dimensions only two values of the extension parameter correspond to the presence of the point-like magnetic field at the origin. Thus, other values of the parameter correspond to additional contact interactions. One possible boundary condition was obtained in [9, 15, 16] by a specific regularization of the Dirac delta function, starting from a model in which the continuity of both components of the Dirac spinor is imposed at a finite radius, and then this radius is shrunk to zero. Other extensions in $2+1$ and $3+1$ dimensions were constructed in the works [17, 18, 19] by imposing spectral boundary conditions of the Atiyah-Patodi-Singer type [20] (MIT boundary conditions) at a finite radius, and then the zero-radius limit is taken. In the works [21, 22] it was shown that, given certain relations between the extension parameters, it is possible to find the most general domain where the Hamiltonian and the helicity operator are self-adjoint. The bound state problem for particles with magnetic moment in the AB potential was considered in detail in the works [23, 24, 25]. The physically motivated boundary conditions for the particle scattering on the AB field and a Coulomb center was studied in [26].

The study of similar problems in the magnetic-solenoid field is a nontrivial task. Indeed, the presence of the uniform magnetic field changes the energy spectrum of the spinning particle from continuous to discrete. Thus, the boundary conditions that were obtained for a continuous spectrum cannot be automatically used for the discrete spectrum. By analogy with the pure AB field it is important to consider the regularized magnetic-solenoid field (we call the regularized magnetic-solenoid field the superposition of a uniform magnetic field and the regularized AB field). Here one has to study solutions of the Dirac equation in such a field. The latter problem was not solved before, and is of particular interest regardless of the extension problem in the AB field. One ought to say that the Pauli equation in the magnetic-solenoid field was recently studied in [27, 28, 29]. The Klein-Gordon equation in this field was considered in [29] together with boundary conditions.

In the present article we consider the Dirac equation in the magnetic-solenoid field and in the regularized magnetic-solenoid field. First we construct self-adjoint extensions of the Dirac Hamiltonian using von Neumann's theory of deficiency indices. We demonstrate how to reduce the $(3+1)$-dimensional problem to the $(2+1)$-dimensional one by a proper choice of the spin operator. We find a one-parameter family of self-adjoint extensions of the Dirac Hamiltonian, and define a one-parameter family of the allowed boundary conditions at the AB solenoid. Then, we study properties of the corresponding solutions and energy spectra. We discuss the spectrum dependence upon the extension parameter. In the regularized magnetic-solenoid field, we find for the first time solutions of the Dirac equation. We study the structure of these solutions and their dependence on the behavior of the magnetic field inside the solenoid. Then we use these solutions to specify boundary conditions for the singular magnetic-solenoid field. To this end, we consider the zero-radius limit of the solenoid.

2 Exact solutions

Consider the Dirac equation ($c = \hbar = 1$) in $(3+1)$- and $(2+1)$-dimensions,

$$i\partial_0 \Psi = H\Psi, \; H = \gamma^0 (\gamma \mathbf{P} + M) . \tag{1}$$

Here $\gamma^\nu = (\gamma^0, \gamma)$, $\gamma = \left(\gamma^k\right)$, $P_k = i\partial_k - qA_k$, $k = 1, 2$, for $2+1$, and $k = 1, 2, 3$, for $3+1$, $\nu = (0, k)$; q is an algebraic charge, for electrons $q = -e < 0$. As an external electromagnetic field we take the magnetic-solenoid field. The magnetic-solenoid field is a collinear superposition of a constant uniform magnetic field B and the Aharonov-Bohm field B^{AB} (the AB field is a field of an infinitely long and infinitesimally thin solenoid). The complete Maxwell tensor has the form:

$$F_{\lambda\nu} = \overline{B}\left(\delta_\lambda^2 \delta_\nu^1 - \delta_\lambda^1 \delta_\nu^2\right), \; \overline{B} = B^{AB} + B .$$

The AB field is singular at $r = 0$,

$$B^{AB} = \Phi \delta(x^1) \delta(x^2) .$$

The AB field creates the magnetic flux Φ. It is convenient to present this flux as:

$$\Phi = (l_0 + \mu)\, \Phi_0, \Phi_0 = 2\pi/e , \tag{2}$$

where l_0 is integer, and $0 \leq \mu < 1$. We suppose Φ and B can take positive and negative values. The case when the quantities are positive means the axis z and the direction of the corresponding fields coincide. One ought to remark that the case $\Phi < 0$ does not require a special study, but can be obtained from the case $\Phi > 0$ by changing $\mu \to 1 - \mu$.

If we use the cylindric coordinates $\varphi, r :\; x^1 = r\cos\varphi$, $x^2 = r\sin\varphi$, then the potentials have the form:

$$\begin{aligned} &A_0 = 0, \; eA_1 = [l_0 + \mu + A(r)] \frac{\sin\varphi}{r}, \; eA_2 = -[l_0 + \mu + A(r)] \frac{\cos\varphi}{r} , \\ &(A_3 = 0 \text{ in } 3+1), \; A(r) = eBr^2/2 . \end{aligned} \tag{3}$$

2.1 Solutions in 2+1 dimensions

First, we consider the problem in $2+1$ dimensions. In $2+1$ dimensions there are two inequivalent representations for γ-matrices:

$$\gamma^0 = \sigma^3, \; \gamma^1 = i\sigma^2, \; \gamma^2 = -i\sigma^1 \zeta, \;\; \zeta = \pm 1 ,$$

where the "polarizations" $\zeta = \pm 1$ correspond to "spin up" and "spin down" respectively, $\boldsymbol{\sigma} = (\sigma^i)$ are Pauli matrices. In our stationary case, we may select the following form for the spinors $\psi(x)$:

$$\Psi(x) = \exp\left\{-i\varepsilon x^0\right\} \psi_\varepsilon^{(\zeta)}(x_\perp) \;, \; \zeta = \pm 1, \; x_\perp = \left(x^1, x^2\right) . \tag{4}$$

Then the Dirac equation in both representations implies:

$$\left(\boldsymbol{\sigma} \mathbf{P}_\perp + M\sigma^3\right) \psi_\varepsilon^{(1)}(x_\perp) = \varepsilon \psi_\varepsilon^{(1)}(x_\perp) , \tag{5}$$

$$\left(\sigma^1\boldsymbol{\sigma}\mathbf{P}_\perp\sigma^1 + M\sigma^3\right)\psi_\varepsilon^{(-1)}(x_\perp) = \varepsilon\psi_\varepsilon^{(-1)}(x_\perp)\,. \tag{6}$$

One can see that

$$\psi_\varepsilon^{(-1)}(x_\perp) = \sigma^2\psi_{-\varepsilon}^{(1)}(x_\perp)\,. \tag{7}$$

Further, we are going to use the representation defined by $\zeta = 1$.

As the total angular momentum operator we select $J = -i\partial_\varphi + \sigma^3/2$. It commutes with the Hamiltonian H and is the dimensional reduction of the operator J^3 in $3+1$. Then the spinors $\psi_\varepsilon^{(1)}$ have to satisfy Eq. (5) and the equation

$$J\psi_\varepsilon^{(1)}(x_\perp) = \left(l - l_0 - \frac{1}{2}\right)\psi_\varepsilon^{(1)}(x_\perp)\,. \tag{8}$$

Presenting the spinors $\psi_\varepsilon^{(1)}$ in the form

$$\psi_\varepsilon^{(1)}(x_\perp) = g_l(\varphi)\psi_l(r)\,,\;\; g_l(\varphi) = \frac{1}{\sqrt{2\pi}}\exp\left\{i\varphi\left(l - l_0 - \frac{1}{2}\left(1+\sigma^3\right)\right)\right\}, \tag{9}$$

we find that the radial spinor $\psi_l(r)$ obeys the equation

$$h\psi_l(r) = \varepsilon\psi_l(r),\;\; h = \Pi + \sigma^3 M\,. \tag{10}$$

Here

$$\Pi = -i\left[\partial_r + \sigma^3\frac{1}{r}\left(l - \frac{1}{2}\left(1-\sigma^3\right) + \mu + A(r)\right)\right]\sigma^1\,. \tag{11}$$

defines the spin projection operator action on the radial spinor in the subspace with a given value l. Namely,

$$\boldsymbol{\sigma}\mathbf{P}_\perp g_l(\varphi)\psi_l(r) = g_l(\varphi)\Pi\psi_l(r)\,.$$

It is convenient to present the radial spinor in the following form

$$\psi_l(r) = \left[\sigma^3(\varepsilon - \Pi) + M\right]u_l(r), \tag{12}$$

where

$$u_l(r) = \sum_{\sigma=\pm1} c_\sigma u_{l,\sigma}(r)\,,\; u_{l,\sigma}(r) = \phi_{l,\sigma}(r)v_\sigma\,,$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\;\; v_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{13}$$

and c_σ are some constants. The radial functions $\phi_{l,\sigma}(r)$ satisfy the following equation:

$$\left\{\rho\frac{d^2}{d\rho^2} + \frac{d}{d\rho} - \frac{\rho}{4} + \frac{1}{2}\left[\frac{\omega}{\gamma} - \xi\left(\mu + l - \frac{1}{2}(1-\sigma)\right)\right] - \frac{\nu^2}{4\rho}\right\}\phi_{l,\sigma}(r) = 0\,, \tag{14}$$

where

$$\rho = \gamma r^2/2,\;\; \gamma = e\,|B|\,,\;\; \xi = sgnB,\;\; \nu = \mu + l - 1/2\,(1+\sigma)\,.$$

Solutions of the equation (14) were studied in [4]. They can be expressed via the Laguerre functions $I_{n,m}(\rho)$ which are defined in the Appendix. Taking into account these results, we get:

a) For any l, there exist a set of solutions $\phi_{l,\sigma} = (\phi_{m;l,\sigma},\; m = 0,1,2,\dots)$ where

$$\phi_{m;l,\sigma}(r) = I_{m+|\nu|,m}(\rho)\,. \tag{15}$$

The functions $\phi_{m,l,\sigma}(r)$ are regular at $r=0$.

b) For $l=0$ and $\mu \neq 0$, there exist additional solutions irregular[1] at $r=0$:

$$\begin{aligned} \phi^{ir}_{m;1}(r) &= I_{m+\mu-1,m}(\rho) \ , \ \sigma=1, \ m=0,1,2,\ldots, \\ \phi^{ir}_{m;-1}(r) &= I_{m-\mu,m}(\rho) \ , \ \sigma=-1, \ m=0,1,2,\ldots . \end{aligned} \tag{16}$$

All the corresponding solutions $\psi_l(r)$ of Eq. (10) are square integrable on the half-line with the measure rdr . The Laguerre functions in Eqs. (15), (16) are expressed via the Laguerre polynomials.

The eigenvalues ω depend on $sgnB$. For $B>0$ the spectrum of ω corresponding to the functions $\phi_{m,l,\sigma}(r)$ reads

$$\omega = \begin{cases} 2\gamma(m+l+\mu), \ l-1/2(1+\sigma) \geq 0 \\ 2\gamma(m+1/2(1+\sigma)), \ l-1/2(1+\sigma) < 0 \end{cases}, \tag{17}$$

and the spectrum of ω corresponding to the functions $\phi^{ir}_{m,\sigma}(r)$ reads

$$\omega = \begin{cases} 2\gamma(m+\mu), \ \sigma=1 \\ 2\gamma m, \ \sigma=-1 \end{cases}. \tag{18}$$

For $B<0$ the spectrum of ω corresponding to the functions $\phi_{m,l,\sigma}(r)$ is

$$\omega = \begin{cases} 2\gamma(m-l+1-\mu), \ l-1/2(1+\sigma) < 0 \\ 2\gamma(m+1/2(1-\sigma)), \ l-1/2(1+\sigma) \geq 0 \end{cases}, \tag{19}$$

and the spectrum of ω corresponding to the functions $\phi^{ir}_{m,\sigma}(r)$ is

$$\omega = \begin{cases} 2\gamma(m+1-\mu), \ \sigma=-1 \\ 2\gamma m, \ \sigma=1 \end{cases}. \tag{20}$$

Besides, for $l=0$ and $\mu \neq 0$, a general solution of the equation (14) has the form:

$$\begin{aligned} &\phi_{\omega,\sigma}(r) = \psi_{\lambda,\alpha}(\rho) = \rho^{-1/2} W_{\lambda,\alpha/2}(\rho), \\ &\alpha = \mu - 1/2(1+\sigma), \ 2\lambda = \omega/\gamma - \xi(\mu - 1/2(1-\sigma)), \end{aligned} \tag{21}$$

where $W_{\lambda,\alpha/2}$ are the Whittaker functions (see [30] , 9.220.4). The spinors in (10) constructed by the help of the latter functions are square integrable for arbitrary complex λ. Therefore, eigenvalues ω are not defined when the latter functions are used. The functions $\psi_{\lambda,\alpha}$ were studied in detail in [4], some important relations for these functions are presented in the Appendix. The functions $\phi_{\omega,\sigma}(r)$ are irregular at $r=0$.

We demand the spinors $u_l(r)$ to be eigenvector for Π, such that the functions $u_{m,l,\pm}$ have to obey the equation

$$\Pi u_{m,l,\pm}(r) = \pm\sqrt{\omega} u_{m,l,\pm}(r) . \tag{22}$$

[1]Here we use the terms "regular", "irregular" at $r=0$ in the following sense. We call a function to be regular if it behaves as r^c at $r=0$ with $c \geq 0$, and irregular if $c<0$. We call a spinor to be regular when all its components are regular, and irregular when at least one of its components is irregular.

In the case $\omega=0,\ B>0$,

$$u_{0,l}(r)=\left(\begin{array}{l}0\\ \phi_{0,l,-1}(r)\end{array}\right),\ l\leq-1;\quad u_0^I(r)=\left(\begin{array}{l}0\\ \phi_{0,-1}^{ir}(r)\end{array}\right),\ l=0\,. \tag{23}$$

That can be easily seen from the relations (92) - (95) for the Laguerre functions $I_{n,m}(\rho)$.

In the case $\omega\neq0,\ B>0$,

$$\begin{aligned}
u_{m,l,\pm}(r) &= \left(\begin{array}{c}\phi_{m,l,1}(r)\\ \pm i\phi_{m,l,-1}(r)\end{array}\right),\ l\geq1\,,\ \omega=2\gamma\,(m+l+\mu)\,,\\
u_{m+1,l,\pm}(r) &= \left(\begin{array}{c}\phi_{m,l,1}(r)\\ \mp i\phi_{m+1,l,-1}(r)\end{array}\right),\ l\leq-1\,,\ \omega=2\gamma\,(m+1)\,,\\
u_{m+1,\pm}^I(r) &= \left(\begin{array}{c}\phi_{m,0,1}(r)\\ \mp i\phi_{m+1,-1}^{ir}(r)\end{array}\right),\ l=0\,,\ \omega=2\gamma\,(m+1)\,,\\
u_{m,\pm}^{II}(r) &= \left(\begin{array}{c}\phi_{m,1}^{ir}(r)\\ \pm i\phi_{m,0,-1}(r)\end{array}\right),\ l=0\,,\ \omega=2\gamma\,(m+\mu)\,.
\end{aligned} \tag{24}$$

In the case $\omega=0,\ B<0$,

$$u_{0,l}(r)=\left(\begin{array}{l}\phi_{0,l,1}(r)\\ 0\end{array}\right),\ l\geq1;\quad u_0^{II}(r)=\left(\begin{array}{l}\phi_{0,1}^{ir}(r)\\ 0\end{array}\right),\ l=0. \tag{25}$$

In the case $\omega\neq0,\ B<0$,

$$\begin{aligned}
u_{m,l,\pm}(r) &= \left(\begin{array}{c}\phi_{m,l,1}(r)\\ \mp i\phi_{m,l,-1}(r)\end{array}\right),\ l\leq-1\,,\ \omega=2\gamma\,(m-l+1-\mu)\,,\\
u_{m+1,l,\pm}(r) &= \left(\begin{array}{c}\phi_{m+1,l,1}(r)\\ \pm i\phi_{m,l,-1}(r)\end{array}\right),\ l\geq1\,,\ \omega=2\gamma\,(m+1)\,,\\
u_{m+1,\pm}^{II}(r) &= \left(\begin{array}{c}\phi_{m+1,1}^{ir}(r)\\ \pm i\phi_{m,0,-1}(r)\end{array}\right),\ l=0\,,\ \omega=2\gamma\,(m+1)\,,\\
u_{m,\pm}^{I}(r) &= \left(\begin{array}{c}\phi_{m,0,1}(r)\\ \mp i\phi_{m,-1}^{ir}(r)\end{array}\right),\ l=0\,,\ \omega=2\gamma\,(m+1-\mu)\,.
\end{aligned} \tag{26}$$

For $\omega\neq0$, we will construct solutions of the Dirac equation using the spinors corresponding to the positive eigenvalues of the operator II. These solutions have the form:

$$\begin{aligned}
\psi_{m,l}(r) &= N\left[\sigma^3\left(\varepsilon-\sqrt{\omega}\right)+M\right]u_{m,l,+}(r),\ l\neq0,\\
\psi_m^{I,II}(r) &= N\left[\sigma^3\left(\varepsilon-\sqrt{\omega}\right)+M\right]u_{m,+}^{I,II}(r),\ l=0,
\end{aligned} \tag{27}$$

where N is the normalization constant. Substituting (27) into Eq. (10), we obtain two types of states corresponding to particles ${}_+\psi$ and antiparticles ${}_-\psi$ with $\varepsilon={}_\pm\varepsilon=\pm\sqrt{M^2+\omega}$ respectively. The particle and antiparticle spectra are symmetric, that is $|{}_+\varepsilon|=|{}_-\varepsilon|$ for the given quantum numbers $m,\ l$.

The case $\omega=0$ is special. Consider first $B>0$. Then $\omega=0$ at $m=0,\ l\leq-1$ (we note the sign of l for the states $\omega=0$ is opposite to the sign of B), and at $m=0,\ l=0$ for u_0^I spinor. Taking all that into account, one can see from the equation (10) that there exist only antiparticle nontrivial solutions with $\varepsilon = {}_{-}\varepsilon = -M$. They coincide with the corresponding spinors u up to a normalization constant

$$ {}_{-}\psi_{0,l}(r) = N u_{0,l}(r),\ l \le -1;\quad {}_{-}\psi_0^I(r) = N u_0^I(r),\ l = 0\ . \tag{28}$$

Thus, in this case only the antiparticle has the rest energy level. The particle lowest energy level for $l \le 0$ is ${}_{+}\varepsilon = \sqrt{M^2 + 2\gamma}$.

When the AB potential is present, if $B < 0$ then $\omega = 0$ at $m = 0$, $l \ge 1$, and at $m = 0$, $l = 0$ for the spinor u_0^{II}. One can see from the equation (10) that there exist only particle nontrivial solutions with $\varepsilon = {}_{+}\varepsilon = M$. They coincide with the corresponding spinors u up to a normalization constant,

$$ {}_{+}\psi_{0,l}(r) = N u_{0,l}(r),\ l \ge 1;\quad {}_{+}\psi_0^{II}(r) = N u_0^{II}(r),\ l = 0. \tag{29}$$

Thus, in the case $B < 0$ only the particle has the rest energy level, and the antiparticle states spectrum begins from ${}_{-}\varepsilon = -\sqrt{M^2 + 2\gamma}$.

Thus, we observe the spectrum asymmetry. There is a relation between the three-dimensional chiral anomaly and fermion zero modes in a uniform magnetic field [31] (for review see [32, 33]). One can see the effect also takes place in the AB potential presence (see discussion in Sect. IV). The spectrum asymmetry is known in 2+1 QED for the uniform magnetic field when the AB potential is absent. Here the spectrum changes mirror-like with the change of the magnetic field sign.

One can see that for $l \neq 0$ the spectrum is similar to the spectrum of the uniform external field. The presence of the AB potential is especially essential for the states with $l = 0$, when the particle penetrates the solenoid. The spectrum peculiarities for the states with $l = 0$ will be discussed in Sect. III.

All the radial spinors $\psi_{m,l}(r)$ are orthogonal for different m. The same is true both for the spinors ψ_m^I and ψ^{II}. In the general case, the spinors of the different types are not orthogonal. By the help of Eq. (98) from the Appendix, one can prove this fact and at the same time calculate the normalization factor, which has the same form for all types of the spinors,

$$ N = \frac{\sqrt{\gamma}}{\sqrt{2\left[\left(\varepsilon - \sqrt{\omega}\right)^2 + M^2\right]}}\ . \tag{30}$$

On the subspace $l \neq 0$, the radial spinors $\psi_{m,l}(r)$ are square integrable on the half-line with the measure rdr. By the help of the completeness relation for the Laguerre functions (99), one can verify that the spinors $\psi_{m,l}(r)$ form a complete set in all the subspaces with $l \neq 0$. In the case $l = 0$, $\mu = 0$ the spinors $\psi_{m,l}(r)$ are square integrable and form a complete set as well. In what follows, considering the subspace $l = 0$, we will always assume $\mu \neq 0$.

Besides, there are special solutions of (10) that exist only on the subspace $l = 0$. They are expressed via the functions $\psi_{\lambda,\alpha}(\rho)$. We present these solutions as follows:

$$\begin{aligned} \psi_\omega(r) &= \left[\sigma^3\left(\varepsilon - \Pi\right) + M\right] u_\omega(r), \\ u_\omega(r) &= c_1 u_{\omega,1}(r) + c_{-1} u_{\omega,-1}(r),\ \ u_{\omega,\sigma}(r) = \phi_{\omega,\sigma}(r) v_\sigma\ , \end{aligned} \tag{31}$$

where c_σ are some coefficients. Using the relations (104) for the functions $\psi_{\lambda,\alpha}(\rho)$, we obtain the useful expressions

$$ B > 0,\ \ \Pi u_{\omega,1}(r) = i\sqrt{2\gamma} u_{\omega,-1}(r),\ \ \Pi u_{\omega,-1}(r) = -i\frac{\omega}{\sqrt{2\gamma}} u_{\omega,1}(r), \tag{32}$$

$$B<0,\;\; \Pi u_{\omega,-1}(r)=i\sqrt{2\gamma}u_{\omega,1}(r),\;\; \Pi u_{\omega,1}(r)=-i\frac{\omega}{\sqrt{2\gamma}}u_{\omega,-1}(r). \tag{33}$$

By the help of Eq. (106) from the Appendix, one can see that the spinors $\psi_\omega(r)$ and $\psi_{\omega'}(r)$, $\omega\neq\omega'$ are not orthogonal in the general case.

Thus, on the subspace $l\neq 0$ the complete set of orthonormal eigenfunctions of the radial Hamiltonian h has been obtained. Therefore, the Hamiltonian h is self-adjoint on this subspace. The situation with the subspace $l=0$, which we call the critical subspace, is more delicate. We have found that for $l=0$ the solutions are not orthogonal and irregular at $r=0$. Thus, one has to study the Hamiltonian on the critical subspace in order to construct self-adjoint extensions for it.

One can see that the radial Hamiltonian is symmetric if

$$\int_0^\infty r\chi^\dagger(r)\,h\psi(r)\,dr=\int_0^\infty r\,[h\chi(r)]^\dagger\,\psi(r)\,dr \tag{34}$$

for arbitrary spinors $\chi(r)$, $\psi(r)$. It follows as the end-point condition

$$\lim_{r\to 0} r\chi^\dagger(r)\,\sigma^1\psi(r)=0 \tag{35}$$

holds true. A symmetric operator is self-adjoint whenever its domain coincides with the domain of the corresponding adjoint operator. It is well known that in the pure AB field, the regularity of the spinor wave functions at $r=0$ is too strong a requirement. The same situation takes place in the background under consideration. The domain of the adjoint Hamiltonian contains functions which are singular at $r=0$ and therefore the adjoint operator has a larger domain. Hence one has to impose a boundary condition such that the requirement (35) entails the same boundary condition in the dual space, i.e. a self-adjoint extension of the Hamiltonian is required. This will be done in the next Section.

2.2 Solutions in 3+1 dimensions

To exploit the symmetry of the problem under z translations, we use the following representations for γ-matrices (see, e.g. [15]):

$$\gamma^0=\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix},\ \gamma^1=\begin{pmatrix}i\sigma^2 & 0\\ 0 & -i\sigma^2\end{pmatrix},\ \gamma^2=\begin{pmatrix}-i\sigma^1 & 0\\ 0 & i\sigma^1\end{pmatrix},\ \gamma^3=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}.$$

In $3+1$ dimensions a complete set of commuting operators can be chosen as follows ($\gamma^5=-i\gamma^0\gamma^1\gamma^2\gamma^3$):

$$H,\ P^3=-i\partial_3,\ J^3=-i\partial_\varphi+\Sigma^3/2,\ S^3=\gamma^5\gamma^3\left(M+\gamma^3P^3\right)/M. \tag{36}$$

$$\begin{aligned} H\Psi &= \varepsilon\Psi, && (37)\\ P^3\Psi &= p^3\Psi, && (38)\\ J^3\Psi &= j^3\Psi, && (39)\\ S^3\Psi &= s\widetilde{M}/M\Psi\,, && (40)\end{aligned}$$

where $\widetilde{M}=\sqrt{M^2+(p_3)^2}$, p^3 is z-component of the momentum and j^3 is z-component of the total angular momentum. Remark that the energy eigenvalues can be positive,

$\varepsilon = {}_{+}\varepsilon > 0$, or negative, $\varepsilon = {}_{-}\varepsilon < 0$. The eigenvalues j^3 are half-integer, for us it is convenient to use the representation: $j^3 = \left(l - l_0 - \frac{1}{2}\right)$, where $l = 0, \pm 1, \pm 2, \ldots$. As the spin operator, we choose the operator S^3 which is the z-component of the polarization pseudovector [34]

$$S^0 = -\frac{1}{2M}\left(H\gamma^5 + \gamma^5 H\right),\ S^i = \frac{1}{2M}\left(H\Sigma^i + \Sigma^i H\right)\ ,$$

the corresponding eigenvalues are $s = \pm 1$.

Usually, the helicity operator $S_h = \mathbf{\Sigma P}/|\mathbf{P}|$ is used as the spin operator. In this case it is necessary to find a common domain for two operators: H and S_h. That is not a trivial problem even in the special case $p^3 = 0$ [21, 22]. Moreover, not for all the extension parameter values of the Hamiltonian there exists a self-adjoint extension of the operator S_h. We suggest to select the operator S^3 to specify the spin degree of freedom. Then, in $3+1$ dimensions, one can separate the spin and coordinate variables and get the following representation for the spinors Ψ:

$$\begin{aligned} \Psi(x) &= \exp\left\{-i\varepsilon x^0 + ip^3x^3\right\}\Psi_s(x_\perp)\,, \\ \Psi_s(x_\perp) &= N\begin{pmatrix} \left[1 + \left(p^3 + s\widetilde{M}\right)/M\right]\psi_{\varepsilon,s}(x_\perp) \\ \left[-1 + \left(p^3 + s\widetilde{M}\right)/M\right]\psi_{\varepsilon,s}(x_\perp) \end{pmatrix}. \end{aligned} \tag{41}$$

Here $\psi_{\varepsilon,s}(x_\perp)$ are two-component spinors, $x_\perp = (0, x^1, x^2, 0)$, and N is a normalization factor.

As a result, the equation (37) is reduced to the following form

$$\left(\boldsymbol{\sigma}\mathbf{P}_\perp + s\widetilde{M}\sigma^3\right)\psi_{\varepsilon,s}(x_\perp) = \varepsilon\psi_{\varepsilon,s}(x_\perp),\ P_\perp = (0, P_1, P_2, 0)\ . \tag{42}$$

At fixed s and p^3 , the equation is similar to the equation in $2+1$ dimensions (5). Thus, the spinor $\psi_{\varepsilon,1}(x_\perp)$ in $3+1$ dimensions can be obtained from the spinor $\psi_\varepsilon^{(1)}(x_\perp)$ in $2+1$ dimensions (4) with the substitution M by $\widetilde{M}$. Note that

$$\psi_{\varepsilon,-1}(x_\perp) = \sigma^3\psi_{-\varepsilon,1}(x_\perp)\ . \tag{43}$$

Using the results in $2+1$ dimensions we conclude that for $l \neq 0$ solutions form complete orthonormal set. And in the critical subspace the Hamiltonian needs a self-adjoint extension.

3 Self-adjoint extension

First, we study the $(2+1)$-dimensional case. As well-known [7, 8, 15], the radial Hamiltonian in the pure AB field requires a self-adjoint extension for the critical subspace $l = 0$. To this end one may use the standard theory of von Neumann deficiency indices [35]. As a result [8] one gets a one-parameter family of acceptable boundary conditions. In the case of our interest, the external background is more complicated, it includes besides the AB field an uniform magnetic field. The wave functions and the spectrum in such a background differ in a nontrivial manner from ones in the pure AB field. Thus, the problem of self-adjoint extension of the Dirac Hamiltonian in such a background, which is considered below, is not trivial.

We recall that for the extension problem the subspace $l = 0$ is only important. To establish the boundary condition we shall analyze the von Neumann deficiency indices. To this aim we have to solve the following auxiliary problem: Let h be the radial Hamiltonian with the domain $\mathcal{D} = \{\Psi(r)\}$, where Ψ are absolutely continuous, square integrable on the half-line with measure rdr and regular at the origin. Then one has to construct the eigenspaces $\mathcal{D}^{\pm}$ of $h^{\dagger}$ with the corresponding eigenvalues $\pm iM$,

$$h^{\dagger}\psi^{\pm}(r) = \pm iM\psi^{\pm}(r),\ h^{\dagger} = \Pi^{\dagger} + \sigma^3 M\,, \tag{44}$$

where

$$\Pi^{\dagger} = -i\left\{\partial_r + \frac{\sigma^3}{r}\left[\mu - \frac{1}{2}\left(1-\sigma^3\right) + A\left(r\right)\right]\right\}\sigma^1 . \tag{45}$$

Using the functions (21), we find for $B > 0$:

$$\psi^{\pm}(r) = N\begin{pmatrix} \phi_1(r) \\ \pm e^{\pm i\pi/4}\frac{\sqrt{\gamma}}{M}\phi_{-1}(r) \end{pmatrix}, \tag{46}$$

and for $B < 0$:

$$\psi^{\pm}(r) = N\begin{pmatrix} \phi_1(r) \\ \pm e^{\pm i\pi/4}\frac{M}{\sqrt{\gamma}}\phi_{-1}(r) \end{pmatrix}, \tag{47}$$

where

$$\phi_\sigma(r) = \psi_{\lambda,\alpha}(\rho),\ \ 2\lambda = -2M^2/\gamma - \xi\left(\mu - 1/2\left(1-\sigma\right)\right)\,,\ \sigma = \pm 1\,.$$

Thus, the deficiency indices are $(1,1)$. Then there exist the following isometry from a subspace $\mathcal{D}^{+}$ into $\mathcal{D}^{-}$:

$$\psi^{+}(r) \to e^{i\Omega}\psi^{-}(r)\,,$$

where Ω is a real number. By von Neumann's theorem, the extensions of a symmetric operator[2] are in one-to-one correspondence with a set of isometries. The extensions are self-adjoint if the deficiency indices have the same values. Thus, there exists a one-parameter family h^{Ω} of self-adjoint extensions of the original operator h . The domain $\mathcal{D}^{\Omega}$ of h^{Ω} reads:

$$\mathcal{D}^{\Omega} = \left\{\chi\left(r\right) = \Psi\left(r\right) + C\left(\psi^{+}\left(r\right) + e^{i\Omega}\psi^{-}\left(r\right)\right) :\ \Psi\left(r\right) \in \mathcal{D}\right\}\,,$$

where χ is a two component spinor, $\chi = (\chi^1, \chi^2)$, and C is an arbitrary complex constant. Using the behavior (105) of the function $\psi_{\lambda,\alpha}(\rho)$ at small ρ, we find:

$$\lim_{r\to 0}\frac{\chi^1\left(r\right)\left(Mr\right)^{1-\mu}}{\chi^2\left(r\right)\left(Mr\right)^{\mu}} = \begin{cases} \frac{i2^{1-\mu}\Gamma(1-\mu)\Gamma(\mu+M^2/\gamma)}{\left(\tan\frac{\Omega}{2}-1\right)\Gamma(\mu)\Gamma(1+M^2/\gamma)}\left(\frac{M^2}{\gamma}\right)^{1-\mu}, & B > 0\,, \\ \frac{i2^{1-\mu}\Gamma(1-\mu)\Gamma(1+M^2/\gamma)}{\left(\tan\frac{\Omega}{2}-1\right)\Gamma(\mu)\Gamma(1-\mu+M^2/\gamma)}\left(\frac{M^2}{\gamma}\right)^{-\mu}, & B < 0 \end{cases} \tag{48}$$

For our purposes it is convenient to pass from the parametrization by Ω to the parametrization by the angle Θ such that at any B :

$$\lim_{r\to 0}\frac{\chi^1\left(r\right)\left(Mr\right)^{1-\mu}}{\chi^2\left(r\right)\left(Mr\right)^{\mu}} = i\tan\left(\frac{\pi}{4} + \frac{\Theta}{2}\right), \tag{49}$$

[2]We remark, that von Neumann's theorem stated for a closed symmetric operator, but as known [35] every symmetric operator has a closure, and the operator and its closure have the same closed extensions.

Then one can see that:

$$\chi(r) \sim \left(\begin{array}{c} i\,(Mr)^{\mu-1} \sin(\pi/4+\Theta/2) \\ (Mr)^{-\mu} \cos(\pi/4+\Theta/2) \end{array} \right) \text{ as } r \to 0\,. \tag{50}$$

The boundary condition (50) is the same as in the case of pure AB field, see [8]. One can verify that the limit $\gamma \to 0$ of the right hand sides of (48) coincides with the corresponding expression obtained in [8] in the case of pure AB field.

Thus, solving the above auxiliary problem, we have obtained needed boundary conditions for our real problem (10). Solutions for the latter problem with an arbitrary boundary condition at $r = 0$ we found in Sec. II. These solutions are two-spinors $\psi_\omega(r) = (\psi^1(r), \psi^2(r))$ defined by (31). We subject them to the boundary condition (50). Then, with the help of (32), (33) and (105), we find

$$\lim_{r\to 0} \frac{\psi^1(r)\,(Mr)^{1-\mu}}{\psi^2(r)\,(Mr)^{\mu}} = i\tan\left(\frac{\pi}{4}+\frac{\Theta}{2}\right)$$
$$= \left\{ \begin{array}{ll} -\frac{i(\varepsilon+M)}{M}\frac{\Gamma(1-\mu)\Gamma(\mu-\omega/2\gamma)}{2^{\mu}\Gamma(\mu)\Gamma(1-\omega/2\gamma)}\left(\frac{M^2}{\gamma}\right)^{1-\mu}, & B>0 \\ \frac{iM}{(\varepsilon-M)}\frac{\Gamma(1-\mu)\Gamma(1-\omega/2\gamma)}{2^{\mu-1}\Gamma(\mu)\Gamma(1-\mu-\omega/2\gamma)}\left(\frac{M^2}{\gamma}\right)^{-\mu}, & B<0 \end{array} \right. . \tag{51}$$

Let us study spectra of the self-adjoint extensions h^{Ω}. To this end we have to solve the transcendental equations (51) for ω considering two branches of ε, one for particles and another one for antiparticles, ${}_{\pm}\varepsilon = \pm\sqrt{M^2+\omega}$. Introducing the notations

$$\begin{aligned} \omega &= 2\gamma x,\;\; x = {}_{\varsigma}x = \left({}_{\varsigma}\varepsilon^2 - M^2\right)/2\gamma\,,\;\; Q(x) = \frac{\varepsilon}{M}+1\,,\; \varsigma = \pm\;, \\ \eta &= \frac{2^{\mu}\Gamma(\mu)}{\Gamma(1-\mu)}\tilde{\eta}(\mu)\,,\;\; \tilde{\eta}(\mu) = -\tan\left(\frac{\pi}{4}+\frac{\Theta}{2}\right)\left(\frac{\gamma}{M^2}\right)^{1-\mu}\,, \end{aligned} \tag{52}$$

we may rewrite Eq. (51) for $B > 0$ as follows:

$$Q\left({}_{\varsigma}x\right)\frac{\Gamma(\mu - {}_{\varsigma}x)}{\Gamma(1-{}_{\varsigma}x)} = \eta\,. \tag{53}$$

Having ω for $B > 0$, one can obtain ω for $B < 0$ making the transformation

$$\varsigma \to -\varsigma,\quad \tilde{\eta}(\mu) \to 1/\tilde{\eta}(\mu)\,,\;\; \mu \to 1-\mu\,.$$

Therefore, below we consider the case $B > 0$ only.

Possible solutions $x = x(\eta)$ of the equation (53) are functions of the parameter η (of μ, γ/M^2, Θ) and are labelled by $m = 0, 1, \ldots$. One can find the following asymptotic representations for these solutions at $|\eta| \to 0$:

$$\begin{aligned} x_m(\eta) &= m + \Delta x_m,\;\; \Delta x_m = \frac{\sin(\pi\mu)\,\Gamma(m+1-\mu)}{\pi\Gamma(m)Q(m)}\eta\,,\;\; m = 1,2,3,\ldots, \\ {}_{-}x_0(\eta) &= -\frac{\eta M^2}{\gamma\Gamma(\mu)}\,. \end{aligned} \tag{54}$$

All $x_m(0)$, $m = 1, 2, \ldots$ are positive and integer. The asymptotic representation of ${}_{+}x_0(\eta)$ at $|\eta| \to 0$ is discussed below. The function ${}_{+}x_0(\eta)$ vanishes at the point $\eta = 2\Gamma(\mu)$ and, in the neighborhood of the latter point, has the form

$${}_{+}x_0(\eta) = \frac{\Gamma(\mu) - \eta/2}{\Gamma(\mu)\left(\psi(\mu) - \psi(1)\right)}\,. \tag{55}$$

Here $\psi(x)$ is the logarithmic derivative of the gamma function $\Gamma(x)$, and $-\psi(1) \simeq 0.577$ is the Euler-Mascheroni constant [36]. At $|\eta| \to \infty$ we found the following asymptotic representations:

$$
\begin{aligned}
{}_{\varsigma}x_m(\varsigma\eta) &= m + \mu + \Delta x_m, \quad m = 0, 1, 2, \ldots, \ \eta \to \infty, \\
{}_{\varsigma}x_m(\varsigma\eta) &= m - 1 + \mu + \Delta x_m, \quad m = 1, 2, 3, \ldots, \ \eta \to -\infty, \\
\Delta x_m &= -\frac{\sin(\pi\mu)\,\Gamma(m+\mu)Q(m+\mu)}{\pi\Gamma(m+1)\eta}.
\end{aligned} \tag{56}
$$

These approximations hold true only for $|\Delta x_m| \ll \mu$, $|x_0(\eta)| \ll \mu$.

According to [37] (see there Theorem 8.19, Corollary 1) if T_1 and T_2 are two self-adjoint extensions of the same symmetric operator with the equal finite defect indices (d, d) then any interval $(a, b) \subset \mathcal{R}$ that does not contain T_1-spectrum points, may contain only isolated points of T_2-spectrum with total multiplicity $\leq d$. Let us select the extension h^{Ω} at $\Theta = \pi/2$ with the eigenvalues ${}_{+}\varepsilon = M\sqrt{1 + 2\gamma\, {}_{+}x_0(\infty)/M^2}$ and ${}_{\pm}\varepsilon = \pm M\sqrt{1 + 2\gamma\, {}_{\pm}x_m(\pm\infty)/M^2}$, $m \geq 1$. Then the above theorem implies that if (a, b) is an open interval where a, b are two subsequent eigenvalues of h^{Ω} at $\Theta = \pi/2$, then any self-adjoint extension h^{Ω} at $\Theta \neq \pi/2$ has at most one eigenvalue in (a, b). According to [38] (see there Chapter VIII Sect. 105 Theorem 3) for any $\varepsilon \in (a, b)$, there exist a self-adjoint extension h^{Ω} with the eigenvalue ε. As it follows from (53,56), on the ranges $(m - 1 + \mu \leq {}_{\pm}x_m(\eta) \leq m + \mu, m \geq 1)$, $(-\infty \leq {}_{+}x_0(\eta) \leq \mu)$ the functions ${}_{\pm}x(\eta) = ({}_{\pm}\varepsilon^2 - M^2)/2\gamma$ are one-valued and continuous. This observation is in complete agreement with the above general Theorems. The functions ${}_{\pm}x_m(\eta)$ were found numerically in the weak field, $\gamma/M^2 \ll 1$, for some first m's. The plots of these functions (for $\mu = 0.8$) see on Figs. 1 and 2.

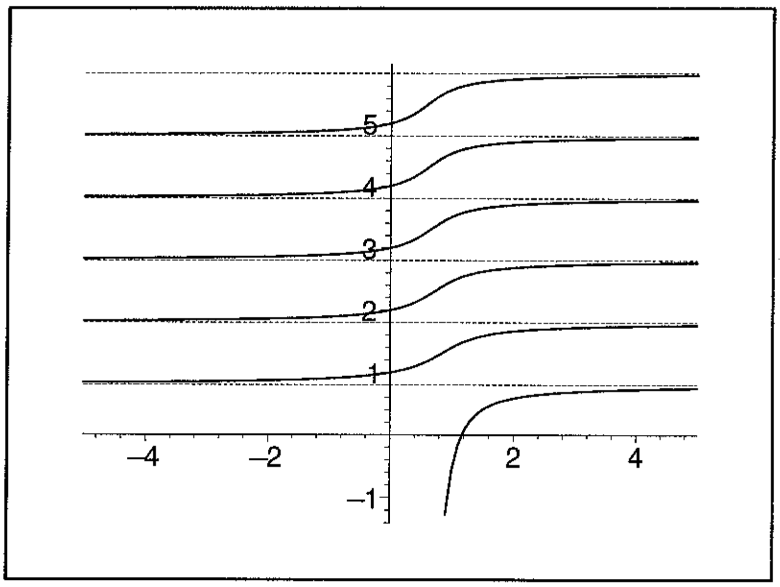

Figure 1: Particle lowest energy levels in dependence on the parameter $\eta_+ = \frac{\Gamma(\mu)}{\Gamma(1-\mu)}\left(\frac{\gamma}{2M^2}\right)^{1-\mu}\tan\left(\frac{\pi}{4} + \frac{\Theta}{2}\right)$

One can see that $\delta x_m = x_{m+1}(\eta) - x_m(\eta) \to 1$ with increasing m. It follows from

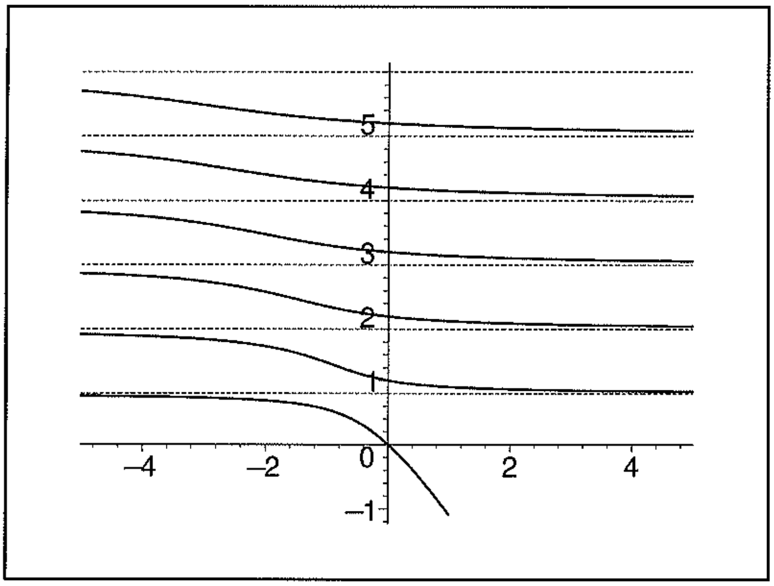

Figure 2: Antiparticle lowest energy levels in dependence on the parameter $\eta_- = \frac{\Gamma(\mu)}{\Gamma(1-\mu)}\left(\frac{\gamma}{2M^2}\right)^{-\mu}\tan\left(\frac{\pi}{4}+\frac{\Theta}{2}\right)$

the equation (53) that

$$\delta x_m - 1 = \pi^{-1}\left\{\cot\left(\pi x_m\right) - \cot\left[\pi\left(x_m-\mu\right)\right]\right\}^{-1}\left(\frac{1-\mu}{x_m} - \delta Q\right),\ m \gg 1\,, \tag{57}$$

where $\delta Q = \frac{d}{dx}\ln Q(x)\Big|_{x=x_m} \leq 1/x_m$. The curve $x_5\left(\eta\right)$ may give an idea how the functions $x_m\left(\eta\right)$ behave at big m.

Below we discuss some limiting cases.

Consider weak fields B, for which $\gamma/M^2 \ll 1$, and nonrelativistic electron energies, $x_m\left(\eta\right)\gamma/M^2 \ll 1$. Here the functions ${}_{\pm}x\left(\eta\right)$ change significantly in the neighborhood of $\eta = 0$ only. The asymptotic behavior at $|\eta| \to \infty$ reads:

$$
{}_{\varsigma}x_m\left(\varsigma\eta\right) \to m - 1 + \mu\,,\ \varsigma\eta < 0\,,
$$
$$
{}_{\varsigma}x_m\left(\varsigma\eta\right) \to m + \mu\,,\ \varsigma\eta > 0\,.
$$

In the ultrarelativistic case, $x_m\left(\eta\right)\gamma/M^2 \gg 1$, the behavior of $x_m\left(\eta\right)$ qualitatively depends on μ. One can distinguish three cases: $\mu < 1/2$, $\mu > 1/2$, $\mu = 1/2$. If $\mu < 1/2$ then the interval near $\eta = 0$ on which the functions change significantly diminishes with m increasing. If $\mu > 1/2$ then this interval grows with m increasing. For $\mu = 1/2$ and $-\frac{1}{2} < \left(\frac{1}{4}+\frac{\Theta}{2\pi}\right) < \frac{1}{2}$, we get the asymptotic representation:

$${}_{\varsigma}x_m\left(\eta\right) = m + \varsigma\left(\frac{1}{4}+\frac{\Theta}{2\pi}\right),\ m \gg 1\,. \tag{58}$$

One can see that negative ${}_{\pm}x_0\left(\eta\right)$ exist only for $\eta > 0$. The minimal admissible negative $x_0\left(\eta\right)$ is defined by the condition $\varepsilon = 0$. In strong fields B, for which $\gamma/M^2 \sim 1$, the quantity $x_0\left(\eta\right)$ is close to zero.

Let Θ_0 correspond to such an extension that admits $\varepsilon = 0$. The value of Θ_0 is defined by the expression

$$\tan\left(\frac{\pi}{4}+\frac{\Theta_0}{2}\right) = -\frac{\Gamma(1-\mu)\Gamma(\mu+M^2/2\gamma)}{2^{\mu}\Gamma(\mu)\Gamma(1+M^2/2\gamma)}\left(\frac{M^2}{\gamma}\right)^{1-\mu}. \tag{59}$$

In weak fields, $\gamma/M^2 \ll 1$, the angle Θ_0 is defined by the expression

$$\tan\left(\frac{\pi}{4}+\frac{\Theta_0}{2}\right) = -\frac{\Gamma(1-\mu)}{2^{2\mu-1}\Gamma(\mu)}, \tag{60}$$

and does not depend on the magnetic field. It follows from (59) that in the superstrong fields B, for which $\gamma/M^2 \gg 1$, the angle Θ_0 does not depend on the magnetic field as well.

One can see that in weak magnetic fields, $\gamma/M^2 \ll 1$, and for nonrelativistic energy values, $x_0\gamma/M^2 \ll 1$, there exist negative $x_0\,(\eta)$ with big absolute values,

$$_{+}x_0\,(\eta) = -\,(2/\eta)^{1/(1-\mu)}\,, \tag{61}$$

$$_{-}x_0\,(\eta) = -\left(\eta M^2/\gamma\right)^{1/\mu}. \tag{62}$$

Let us consider the particular case $\Theta = -\pi/2$. It follows from (51) that for $B > 0$, there exists $_{-}\varepsilon = -M$. The energies $|\varepsilon| > M$ are defined by poles of $\Gamma(1-x)$ or of $\Gamma(1-\mu-x)$ for $B > 0$ or $B < 0$ respectively. The spectrum ε coincides with one defined by Eqs. (26), (27) for ψ^I. Moreover, using the relation (102), we can see that the spinors $\psi_\omega(r)$ coincide with ψ^I,

$$\psi_\omega\,(r) \equiv \psi^I\,(r)\ \text{ for } \Theta = -\pi/2\,. \tag{63}$$

In the case $\Theta = \pi/2$ we have the following picture: It follows from (51) that for $B < 0$ there exists $_{+}\varepsilon = M$. The energies $|\varepsilon| > M$ are defined by poles of $\Gamma(\mu - x)$ or of $\Gamma(1-x)$ for $B > 0$ or $B < 0$ respectively. The spectrum ε coincides with one found by Eqs. (26), (27) for ψ^{II}. From (102) it follows that the spinor $\psi_\omega(r)$ coincides with ψ^{II},

$$\psi_\omega(r) \equiv \psi^{II}\,(r)\ \text{ for } \Theta = \pi/2\,. \tag{64}$$

Thus, that in the problem under consideration for $\pi/2 < \Theta < 3\pi/2$ there exist the only one particle state and the only one antiparticle state with energies $|\varepsilon| < M$. The same situation was observed in the pure AB field case [8]. However, in contrast to the latter case, in the presence of the uniform magnetic field, there exists an angle Θ_0 which admits zero value for ε.

The form of solutions in 3 + 1 dimensions (41) allows us to divide all Hilbert space of solutions into two orthogonal subspaces with respect to the value of the spin quantum number:

$$D\,(H) = \{\Psi_{+1}\} \oplus \{\Psi_{-1}\}\,, \tag{65}$$

and apply von Neumann's theory of deficiency indices to each of the subspaces. Thus, we have to solve the problem (66)

$$\begin{gathered}\mathcal{D}_s^{\pm} \equiv Ker\left(-H^{\dagger} \pm i\widetilde{M}\right)_s\,,\\ H^{\dagger}\Psi_s^{\pm}\,(x_\perp) = \pm i\widetilde{M}\Psi_s^{\pm}\,(x_\perp)\,,\ H^{\dagger} = H,\ s = \pm 1\,,\end{gathered} \tag{66}$$

and define deficiency indices in each subspace. Solutions of this problem is given in (67)

$$\Psi_s^{\pm}(x_\perp) = N \begin{pmatrix} \left[1 + \left(p^3 + s\widetilde{M}\right)/M\right] g_0(\varphi)\, \psi^{\pm}(r) \\ \left[-1 + \left(p^3 + s\widetilde{M}\right)/M\right] g_0(\varphi)\, \psi^{\pm}(r) \end{pmatrix}, \tag{67}$$

where $g_0(\varphi)$ is defined in (9), and $\psi^{\pm}(r)$ are functions from (46) or (47). It follows from (67) that deficiency indices in both subspaces $\{\Psi_{+1}\}$ and $\{\Psi_{-1}\}$ are equal to $(1,1)$. Therefore, in each subspace $\{\Psi_s\}$, $s = \pm 1$, solutions must be subjected to one-parameter boundary conditions. Using the parametrization similar to (51) we obtain the boundary conditions in the form (68)

$$\lim_{r\to 0} \frac{\psi^1_{\omega,s}(r)\,(Mr)^{1-\mu}}{\psi^2_{\omega,s}(r)\,(Mr)^{\mu}} = \frac{\psi^3_{\omega,s}(r)\,(Mr)^{1-\mu}}{\psi^4_{\omega,s}(r)\,(Mr)^{\mu}} = i\tan\left(\frac{\pi}{4} + \frac{\Theta_s}{2}\right), \; s = \pm 1 \, . \tag{68}$$

Thus, in $3+1$ dimensions there exist the two-parameter family of self-adjoint extensions of the Hamiltonian. Spectra in $3+1$ dimensions can be obtained from the results in $2+1$ dimensions with the substitution M by $\widetilde{M}$, and the relation (43). In particular, Fig.1 presents energy lowest levels for particles with spin $s = 1$, and Fig. 2 presents energy lowest levels for particles with spin $s = -1$.

4 Solenoid regularization

One can introduce the AB field as a limiting case of a finite radius solenoid field (the regularized AB field). In this way, one can fix the extension parameter Θ. First, the manner of doing that in the pure AB field was presented by Hagen [9]. Below, we consider the problem in the presence of the uniform magnetic field. To this end we have to study solutions of the Dirac equation (1) in the combination of the regularized AB field and the uniform magnetic field.

Let the solenoid have a radius R. We assume that inside the solenoid there is an axial-symmetrical magnetic field $B^{in}(r)$ that creates the flux $\Phi = (l_0 + \mu)\,\Phi_0$, $\Phi_0 = 2\pi/e$. Outside the solenoid ($r > R$) the field $B^{in}(r)$ vanishes. Thus,

$$e\int_0^R B^{in}(r)\, r dr = l_0 + \mu \, .$$

The function $B^{in}(r)$ is arbitrary but such that integrals in the functions $\vartheta(x)$, $b(x)$ in (73) are not divergent. The potentials of the field $B^{in}(r)$, we select in the form

$$eA_1^{in} = \vartheta(x)\frac{\sin\varphi}{Rx}, \; eA_2 = -\vartheta(x)\frac{\cos\varphi}{Rx}, \tag{69}$$

where

$$\vartheta(x) = \int_0^x f(x')\, x' dx', \; f(x) = R^2 e B^{in}(xR), \; x = r/R \, .$$

The potentials of the uniform magnetic field are

$$A_0 = 0, \; A_1 = A(r)\frac{\sin\varphi}{r}, \; A_2 = -A(r)\frac{\cos\varphi}{r}, \;\; A(r) = Br^2/2 \, . \tag{70}$$

Outside the solenoid the potentials have the form (3).

Let us analyze solutions of the Dirac equation in the above defined field. To this end we have to solve the equation inside and outside the solenoid and continuously join the corresponding solutions. The former Dirac spinors we are going to call the inside spinors, whereas the latter ones the outside spinors.

First, we study the problem in $2+1$ dimensions. By the same manner as in the Sect. II , we can find that the inside radial spinors $\psi^{in}_{\omega,l}(r)$ $(r \le R)$ obey the equation:

$$h^{in}\psi^{in}_{\omega,l}(r) = \varepsilon\psi^{in}_{\omega,l}(r)\,,\ \ h^{in} = \Pi^{in} + \sigma^3 M\,,$$

where

$$\Pi^{in} = -\frac{i}{R}\left\{\partial_x + \frac{\sigma^3}{x}\left[l - l_0 - \frac{1}{2}\left(1-\sigma^3\right) + \vartheta(x) + \xi\rho_R x^2\right]\right\}\sigma^1,\ \rho_R = \gamma R^2/2\,. \quad (71)$$

For $\omega \neq 0$ we present the spinors in the form

$$\psi^{in}_{\omega,l}(r) = \begin{pmatrix} \psi^{in}_1(r) \\ \psi^{in}_2(r) \end{pmatrix} = \left[\sigma^3\left(\varepsilon - \Pi^{in}\right) + M\right]\left[c_1\phi^{in}_{l,1}(x)\upsilon_1 + ic_{-1}\phi^{in}_{l,-1}(x)\upsilon_{-1}\right],$$

where c_σ are arbitrary constants. The functions $\phi^{in}_{l,\sigma}(x)$ satisfy the equation

$$\left[\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x} - \frac{1}{x^2}\left(\alpha + \vartheta(x) + \xi\rho_R x^2\right)^2 + \omega R^2 - \sigma\left(f(x) + 2\xi\rho_R\right)\right]\phi^{in}_{l,\sigma}(x) = 0\,. \quad (72)$$

We demand the functions $\phi^{in}_{l,\sigma}(x)$ to be square integrable at $r=0$. We are interested in the limiting case $R \to 0$. For our purposes it is enough to use the approximation $\rho_R \ll 1$, $\omega R^2 \ll 1$. Dropping terms proportional to R^2 in (71) and (72), we find that solutions of Eq. (72) have the form

$$\phi^{in}_{l,\sigma}(x) = \begin{cases} cx^{|\eta|}e^{\sigma b(x)}\,, & \sigma\eta \ge 0 \\ cx^{-|\eta|}e^{\sigma b(x)}\int_0^x d\tilde{x}\tilde{x}^{2|\eta|-1}e^{-2\sigma b(\tilde{x})}\,, & \sigma\eta < 0 \end{cases}, \quad (73)$$

$$b(x) = \int_0^x d\tilde{x}\tilde{x}^{-1}\vartheta(\tilde{x}),\ \vartheta(x) = \int_0^x f(x')\,x'dx',\ f(x) = R^2 eB^{in}(xR)\,,$$
$$\eta = l - l_0 - (1+\sigma)/2,\ x = r/R\,.$$

For $\omega = 0$ $(|\varepsilon| = M)$ the solutions read:

$$\begin{aligned} {}_+\psi^{in}_{0,l}(r) &= \phi^{in}_{0,l,1}(x)\upsilon_1\,,\ l - l_0 \ge 1\,, \\ {}_-\psi^{in}_{0,l}(r) &= \phi^{in}_{0,l,-1}(x)\upsilon_{-1}\,,\ l - l_0 \le 0\,, \\ \phi^{in}_{0,l,\sigma}(x) &= cx^{|\alpha|}\exp\left\{\sigma\int_0^x d\tilde{x}\tilde{x}^{-1}\left(\vartheta(\tilde{x}) + \xi\rho_R\tilde{x}^2\right)\right\},\ \alpha = l - l_0 - (1+\sigma)/2\,, \end{aligned} \quad (74)$$

where c is an arbitrary constant.

The outside solutions $(r \ge R)$ obey the equation

$$h\psi^{out}_\omega(r) = \varepsilon\psi^{out}_\omega(r)\,. \quad (75)$$

and the condition $\psi^{out}(\infty) \to 0$. Here h is defined by Eqs. (10), (11). The general form of the outside solutions reads:

$$\psi^{out}_\omega(r) = \left[\sigma^3(\varepsilon - \Pi) + M\right]\left(c_1\phi^{out}_{\omega,1}(r)\upsilon_1 + ic_{-1}\phi^{out}_{\omega,-1}(r)\upsilon_{-1}\right),$$
$$\phi^{out}_{\omega,\sigma}(r) = \psi_{\lambda,\alpha}(\rho)\ ,\alpha = l + \mu - 1/2(1+\sigma)\,,\ 2\lambda = \omega/\gamma - \xi\left(l + \mu - 1/2(1-\sigma)\right) \quad (76)$$

The solutions $\psi_{\omega}^{out}(r)$ and $\psi_{\omega,l}^{in}(r)$ must be joined continuously at $r=R$,

$$\psi^{out}(R)=\psi^{in}(R) \ . \tag{77}$$

The joining condition (77) leads to the following conditions for the functions $\phi_{\omega,\sigma}^{in}(r)$ and $\phi_{\omega,\sigma}^{out}(r)$ at $r=R$:

$$\phi(R-\epsilon)=\phi(R+\epsilon) \ , \ \frac{d}{dr}\phi(R-\epsilon)=\frac{d}{dr}\phi(R+\epsilon) \ . \tag{78}$$

It is convenient to use in (76) the representation (102) for $\psi_{\lambda,\alpha}(\rho)$. Then, the functions $\phi_{\omega,\sigma}^{out}(r)$ read:

$$\begin{aligned} &\phi_{\omega,\sigma}^{out}(r)=a_\sigma I_{n_\sigma,m_\sigma}(\rho)+b_\sigma I_{m_\sigma,n_\sigma}(\rho) \ , \\ &a_\sigma=\frac{\sqrt{\Gamma(1+n_\sigma)\Gamma(1+m_\sigma)}}{\sin(n_\sigma-m_\sigma)\pi}\sin n_\sigma\pi \ , \\ &b_\sigma=-\frac{\sqrt{\Gamma(1+n_\sigma)\Gamma(1+m_\sigma)}}{\sin(n_\sigma-m_\sigma)\pi}\sin m_\sigma\pi \ , \\ &n_\sigma=\lambda-\frac{1-\alpha}{2}, \ m_\sigma=\lambda-\frac{1+\alpha}{2} \ . \end{aligned} \tag{79}$$

where n_σ, m_σ are real numbers.

For the case $l-l_0\le 0$, the coefficients a_σ, b_σ can be found using (78),

$$\begin{aligned} &a_1=\rho_R^{-(l+\mu-1)/2}\left\{\frac{1}{2}\left[1+\frac{-|l-l_0-1|+l_0+\mu}{l+\mu-1}\right]\phi_{l,1}^{in}(1)+\frac{ce^{-b(1)}}{l+\mu-1}\right\} \ , \\ &b_1=\rho_R^{(l+\mu-1)/2}\left\{\frac{1}{2}\left[1-\frac{-|l-l_0-1|+l_0+\mu}{l+\mu-1}\right]\phi_{l,1}^{in}(1)-\frac{ce^{-b(1)}}{l+\mu-1}\right\} \ ; \end{aligned} \tag{80}$$

$$\begin{aligned} &a_{-1}=\rho_R^{(l+\mu)/2+1}g_1 \ , \\ &b_{-1}=\rho_R^{(l+\mu)/2}\frac{1}{2}\left[1-\frac{|l-l_0|-l_0-\mu}{l+\mu}\right]ce^{-b(1)} \ ; \end{aligned} \tag{81}$$

where g_1 is some coefficient not depending on R. Here in (81) we had to compute the next higher power of ρ_R in the coefficient of $I_{n,m}(\rho)$. At $R\to 0$ we obtain from the above expressions

$$\frac{a_\sigma}{b_\sigma}R\to 0\to\left\{\begin{array}{l} \infty, \ l\ge 1 \\ 0, \ l\le 0 \end{array}\right. .$$

Besides, at $R\to 0$ one finds

$$m_\sigma=0,1,2,..., \ l\ge 1; \ \ n_\sigma=0,1,2,..., \ l\le 0$$

from (79). Thus, for $l-l_0\le 0$ the functions read:

$$\begin{aligned} &\phi_{\omega,1}^{out}(r)=\phi_{l,1}^{out}(r)=CI_{m+|\alpha|,m}(\rho) \ , \ \alpha=l+\mu-(1+\sigma)/2 \ , \\ &\phi_{\omega,-1}^{out}(r)=\left\{\begin{array}{l} \phi_{l,-1}^{out}(r)=CI_{m+|\alpha|,m}(\rho) \ , \ l\ne 0 \\ \phi_{0,-1}^{out}(r)=CI_{m-\mu,m}(\rho) \ , \ l=0 \end{array}\right. , \\ &\omega/2\gamma=\left\{\begin{array}{l} m+l+\mu, \ l\ge 1 \\ m+1, \ l\le 0 \end{array}\right. , \ m=0,1,2,\dots , \end{aligned} \tag{82}$$

where C is a constant.

For the case $l-l_0>0$, the coefficients a_σ, b_σ read:

$$a_1=\rho_R^{-(l+\mu-1)/2}\left\{\frac{1}{2}\left[1+\frac{(|l-l_0-1|+l_0+\mu)}{l+\mu-1}\right]\phi_{l,1}^{in}(1)\right\},$$
$$b_1=\rho_R^{(l+\mu-1)/2+1}g_2\,; \tag{83}$$
$$a_{-1}=\rho_R^{-(l+\mu)/2}\frac{1}{2}\left\{\left[1-\frac{|l-l_0|+l_0+\mu}{l+\mu}\right]\phi_{l,-1}^{in}(1)+\frac{ce^{-b(1)}}{l+\mu}\right\},$$
$$b_{-1}=\rho_R^{(l+\mu)/2}\frac{1}{2}\left\{\left[1+\frac{|l-l_0|+l_0+\mu}{l+\mu}\right]\phi_{l,-1}^{in}(1)-\frac{ce^{-b(1)}}{l+\mu}\right\}; \tag{84}$$

where g_2 is some coefficient not depending on R. In (83) we had to calculate the next higher power of ρ_R in the coefficient of $I_{m,n}(\rho)$. At $R\to 0$ we obtain from the above expressions:

$$\frac{a_\sigma}{b_\sigma}R\to 0\to\left\{\begin{array}{l}0,\ l\leq -1\,,\\ \infty,\ l\geq 0\,,\end{array}\right.,$$

Besides, at $R\to 0$ one finds:

$$n_\sigma=0,1,2,...,\ l\leq -1\ ;\ m_\sigma=0,1,2,...,\ l\geq 0$$

from (79). Thus, for $l-l_0>0$ the functions read:

$$\phi_{\omega,-1}^{out}(r)=\phi_{l,-1}^{out}(r)=CI_{m+|\alpha|,m}(\rho)\,,\ \alpha=l+\mu-(1+\sigma)/2\,,$$
$$\phi_{\omega,1}^{out}(r)=\left\{\begin{array}{l}\phi_{l,1}^{out}(r)=CI_{m+|\alpha|,m}(\rho)\,,\ l\neq 0\\ \phi_{0,1}^{out}(r)=CI_{m+\mu-1,m}(\rho)\,,\ l=0\end{array}\right.,$$
$$\omega/2\gamma=\left\{\begin{array}{l}m+1,\ l\leq -1\\ m+l+\mu,\ l\geq 0\end{array}\right.,\ m=0,1,2,...\,, \tag{85}$$

where C is a constant.

Substituting the functions (82), (85) into (73), one can see that the solutions $\psi_\omega^{out}(r)$ coincide with the solutions $\psi_{m,l}(r)$ (27) for $l\neq 0$. For $l=0$ the form of solutions $\psi_\omega^{out}(r)$ depends on $sgn\,l_0$. Namely,

$$\psi_\omega^{out}(r) = N\left[\sigma^3\left(\varepsilon-\sqrt{\omega}\right)+M\right]\left(c_1I_{m-\mu+1,m}(\rho)\,v_1+ic_{-1}I_{m-\mu,m}(\rho)\,v_{-1}\right),\ l_0\geq 0\,,$$
$$\psi_\omega^{out}(r) = N\left[\sigma^3\left(\varepsilon-\sqrt{\omega}\right)+M\right]\left(c_1I_{m+\mu-1,m}(\rho)\,v_1+ic_{-1}I_{m+\mu,m}(\rho)\,v_{-1}\right),\ l_0<0 \tag{86}$$

We recall that $sgnl_0$ defines the sign of the solenoid flux Φ. Substituting (86) into (73), we find that the solutions $\psi_\omega^{out}(r)$ coincide with either $\psi_m^I(r)$ or $\psi_m^{II}(r)$ (27) for $l=0$:

$$\psi_\omega^{out}(r)=\left\{\begin{array}{l}\psi_m^I(r)\,,\ sgn\,(q\Phi)=+1\\ \psi_m^{II}(r)\,,\ sgn\,(q\Phi)=-1\end{array}\right..$$

In Sect. III we have found the relation between the extension parameter values and solution types in the critical subspace $l=0$ (63), (64). Now we are in position to refine this relation. Namely, if one introduces the AB field as a field of a finite radius solenoid for a zero-radius limit, then the extension parameter Θ is fixed to be $\Theta=sgn\,(q\Phi)\,\pi/2$.

Besides, this way of the AB field introduction explicitly implies no additional interaction in the solenoid core.

To solve the problem in $3+1$ dimensions we use the results in $2+1$ dimensions presented above. In the limit $R \to 0$ solutions in the critical subspace have the form (87)

$$\Psi_s^{out}(x_\perp) = N \begin{pmatrix} \left[1 + \left(p^3 + s\widetilde{M}\right)/M\right] g_0(\varphi)\, \psi_\omega^{out}(r) \\ \left[-1 + \left(p^3 + s\widetilde{M}\right)/M\right] g_0(\varphi)\, \psi_\omega^{out}(r) \end{pmatrix}, \tag{87}$$

where N is a normalization constant, and the functions $g_0(\varphi)$, $\psi_\omega^{out}(r)$ are defined in (9) and (86) respectively. We specify the values of the extension parameters in $3+1$ dimensions as follows:

$$\Theta_{+1} = \Theta_{-1} = sgn(q\Phi)\,\pi/2\,. \tag{88}$$

5 Summary

We have studied in detail solutions of the Dirac equation in the magnetic-solenoid field in $2+1$ and $3+1$ dimensions. In the general case, solutions in $2+1$ and $3+1$ dimensions are not related in a simple manner. However, it has been demonstrated that solutions in $3+1$ dimensions with special spin quantum numbers can be constructed directly on the base of solutions in $2+1$ dimensions. To this end, one has to choose the z-component of the polarization pseudovector S^3 as the spin operator in $3+1$ dimensions. This is a new result not only for the magnetic-solenoid field background, but for the pure AB field as well. The choice S^3 as the spin operator was convenient from different points of view. For example, solutions with arbitrary momentum p^3 are eigenvectors of the operator S^3. This allows us to separate explicitly spin and coordinate variables in $3+1$ dimensions. Thus, in $3+1$ dimensions one has to study self-adjoint extensions of the radial Hamiltonian only. Moreover, boundary conditions in such a representation do not violate translation invariance along the natural direction which is the magnetic-solenoid field direction. The self-adjoint extensions of the Dirac Hamiltonian in the magnetic-solenoid field have been constructed using von Neumann's theory of deficiency indices. A one-parameter family of allowed boundary conditions in $2+1$ dimensions and a two-parameter family in $3+1$ dimensions have been constructed. By that the complete orthonormal sets of solutions have been found. The energy spectra dependent on the extension parameter Θ have been defined for the different self-adjoint extensions. Besides, for the first time solutions of the Dirac equation in the regularized magnetic-solenoid field have been described in detail. We considered an arbitrary magnetic field distribution inside a finite-radius solenoid. It was shown that similarly to the pure AB field, the extension parameters $\Theta = sgn(q\Phi)\pi/2$ in $2+1$ dimensions and $\Theta_{+1} = \Theta_{-1} = sgn(q\Phi)\pi/2$ in $3+1$ dimensions correspond to the limiting case $R \to 0$ of the regularized magnetic-solenoid field.

6 Acknowledgments

D.M.G. thanks CNPq and FAPESP for permanent support. A.A.S. thanks FAPESP for support. S.P.G. thanks Brazilian foundation CAPES for support. S.P.G. and A.A.S. thank the Department of Physics of Universidade Federal de Sergipe (Brazil) for hospitality.

A Useful relations

1.The Laguerre function $I_{n,m}(x)$ is defined by the relation

$$I_{n,m}(x) = \sqrt{\frac{\Gamma(1+n)}{\Gamma(1+m)}\frac{\exp(-x/2)}{\Gamma(1+n-m)}}x^{(n-m)/2}\Phi(-m, n-m+1; x)\,. \tag{89}$$

Here $\Phi(a, b; x)$ is the confluent hypergeometric function in a standard definition (see [30], 9.210). Let m be a non-negative integer number; then the Laguerre function is related to Laguerre polynomials $L_m^\alpha(x)$ ([30], 8.970, 8.972.1) by the equation

$$I_{\alpha+m,m}(x) = \sqrt{\frac{m!}{\Gamma(m+\alpha+1)}}e^{-x/2}x^{\alpha/2}L_m^\alpha(x)\,, \tag{90}$$

$$L_m^\alpha(x) = \frac{1}{m!}e^x x^{-\alpha}\frac{d^m}{dx^m}e^{-x}x^{m+\alpha}\,. \tag{91}$$

Using well-known properties of the confluent hypergeometric function ([30], 9.212; 9.213; 9.216), one can easily get the following relations for the Laguerre functions

$$2\sqrt{x(n+1)}I_{n+1,m}(x) = (n-m+x)I_{n,m}(x) - 2xI'_{n,m}(x)\,, \tag{92}$$

$$2\sqrt{x(m+1)}I_{n,m+1}(x) = (n-m-x)I_{n,m}(x) + 2xI'_{n,m}(x)\,, \tag{93}$$

$$2\sqrt{xn}I_{n-1,m}(x) = (n-m+x)I_{n,m}(x) + 2xI'_{n,m}(x)\,, \tag{94}$$

$$2\sqrt{xm}I_{n,m-1}(x) = (n-m-x)I_{n,m}(x) - 2xI'_{n,m}(x). \tag{95}$$

Using properties of the confluent hypergeometric function, one can get a representation

$$I_{n,m}(x) = \sqrt{\frac{\Gamma(1+n)}{\Gamma(1+m)}\frac{\exp(x/2)}{\Gamma(1+n-m)}}x^{\frac{n-m}{2}}\Phi(1+n, 1+n-m; -x)\,, \tag{96}$$

and a relation ([30], 9.214)

$$I_{n,m}(x) = (-1)^{n-m}I_{m,n}(x),\ n-m \text{ integer}\,. \tag{97}$$

The functions $I_{\alpha+m,m}(x)$ obey the orthonormality relation

$$\int_0^\infty I_{\alpha+n,n}(x)\,I_{\alpha+m,m}(x)\,dx = \delta_{m,n}\,, \tag{98}$$

which follows from the corresponding properties of the Laguerre polynomials ([30], 7.414.3). The set of the Laguerre functions

$$I_{\alpha+m,m}(x),\ m = 0, 1, 2...\,,\ \alpha > -1$$

is complete in the space of square integrable functions on the half-line ($x \geq 0$),

$$\sum_{m=0}^{\infty} I_{\alpha+m,m}(x)I_{\alpha+m,m}(y) = \delta(x-y)\,. \tag{99}$$

2. The function $\psi_{\lambda,\alpha}(x)$ is even with respect to index α,

$$\psi_{\lambda,\alpha}(x) = \psi_{\lambda,-\alpha}(x). \tag{100}$$

It can be expressed via the confluent hypergeometric functions

$$\psi_{\lambda,\alpha}(x) = e^{-\frac{x}{2}}\left[\frac{\Gamma(-\alpha)\, x^{\frac{\alpha}{2}}}{\Gamma\left(\frac{1-\alpha}{2}-\lambda\right)}\Phi\left(\frac{1+\alpha}{2}-\lambda, 1+\alpha; x\right)\right.$$
$$\left.+\frac{\Gamma(\alpha)\, x^{-\frac{\alpha}{2}}}{\Gamma\left(\frac{1+\alpha}{2}-\lambda\right)}\Phi\left(\frac{1-\alpha}{2}-\lambda, 1-\alpha; x\right)\right], \tag{101}$$

or, using (89), via the Laguerre functions

$$\psi_{\lambda,\alpha}(x) = \frac{\sqrt{\Gamma(1+n)\Gamma(1+m)}}{\sin(n-m)\pi}\left(\sin n\pi I_{n,m}(x) - \sin m\pi I_{m,n}(x)\right),$$
$$\alpha = n-m,\ 2\lambda = 1+n+m,\ n = \lambda - \frac{1-\alpha}{2},\ m = \lambda - \frac{1+\alpha}{2}. \tag{102}$$

There are the following relations of the functions $\psi_{\lambda,\alpha}(x)$,

$$\begin{aligned}
\psi_{\lambda,\alpha}(x) &= \sqrt{x}\psi_{\lambda-\frac{1}{2},\alpha-1}(x) + \frac{1+\alpha-2\lambda}{2}\psi_{\lambda-1,\alpha}(x), \\
\psi_{\lambda,\alpha}(x) &= \sqrt{x}\psi_{\lambda-\frac{1}{2},\alpha+1}(x) + \frac{1-\alpha-2\lambda}{2}\psi_{\lambda-1,\alpha}(x), \\
2x\psi'_{\lambda,\alpha}(x) &= (2\lambda-1-x)\psi_{\lambda,\alpha}(x) + \frac{1}{2}(2\lambda-1-\alpha)(2\lambda-1+\alpha)\psi_{\lambda-1,\alpha}(x), \\
2x\psi'_{\lambda,\alpha}(x) &= (\alpha-x)\psi_{\lambda,\alpha}(x) + (2\lambda-1-\alpha)\sqrt{x}\psi_{\lambda-\frac{1}{2},\alpha+1}(x) \\
&= (x-2\lambda-1)\psi_{\lambda,\alpha} - 2\psi_{\lambda+1,\alpha}.
\end{aligned} \tag{103}$$

As a consequence of these properties we get

$$A_\alpha\psi_{\lambda,\alpha}(x) = \frac{2\lambda-1+\alpha}{2}\psi_{\lambda-\frac{1}{2},\alpha-1}(x),\ A_\alpha^+\psi_{\lambda-\frac{1}{2},\alpha-1}(x) = \psi_{\lambda,\alpha}(x),$$
$$A_\alpha = \frac{x+\alpha}{2\sqrt{x}} + \sqrt{x}\frac{d}{dx},\quad A_\alpha^+ = \frac{x+\alpha-1}{2\sqrt{x}} - \sqrt{x}\frac{d}{dx}. \tag{104}$$

Using well-known asymptotics of the Whittaker function ([30], 9.227), we have

$$\psi_{\lambda,\alpha}(x) \sim x^{\lambda-\frac{1}{2}}e^{-\frac{x}{2}},\quad x\to\infty;\ \psi_{\lambda,\alpha}(x) \sim \frac{\Gamma(|\alpha|)}{\Gamma\left(\frac{1+|\alpha|}{2}-\lambda\right)}x^{-\frac{|\alpha|}{2}},\ \alpha\neq 0,\quad x\sim 0. \tag{105}$$

The function $\psi_{\lambda,\alpha}(x)$ is correctly defined and infinitely differentiable for $0<x<\infty$ and for any complex λ,α. In this respect one can mention that the Laguerre function are not defined for negative integer n,m. In particular cases, when one of the numbers n,m is non-negative and integer, the function $\psi_{\lambda,\alpha}(x)$ coincides (up to a constant factor) with the Laguerre function.

According to (105), the functions $\psi_{\lambda,\alpha}(x)$ are square integrable on the interval $0 \leq x < \infty$ whenever $|\alpha| < 1$. It is not true for $|\alpha| \geq 1$. The corresponding integrals at $\alpha \neq 0$ can be calculated as following ([30], 7.611),

$$\int_0^\infty \psi_{\lambda,\alpha}(x)\,\psi_{\lambda',\alpha}(x)\;dx = \frac{\pi}{(\lambda'-\lambda)\sin\alpha\pi}\left\{\left[\Gamma\left(\frac{1+\alpha-2\lambda'}{2}\right)\Gamma\left(\frac{1-\alpha-2\lambda}{2}\right)\right]^{-1}\right.$$
$$\left.-\left[\Gamma\left(\frac{1-\alpha-2\lambda'}{2}\right)\Gamma\left(\frac{1+\alpha-2\lambda}{2}\right)\right]^{-1}\right\}, \quad |\alpha| < 1, \tag{106}$$
$$\int_0^\infty |\psi_{\lambda,\alpha}(x)|^2\,dx = \frac{\pi}{\sin\alpha\pi}\,\frac{\psi\left(\frac{1+\alpha-2\lambda}{2}\right)-\psi\left(\frac{1-\alpha-2\lambda}{2}\right)}{\Gamma\left(\frac{1+\alpha-2\lambda}{2}\right)\Gamma\left(\frac{1-\alpha-2\lambda}{2}\right)}, \quad |\alpha| < 1, \tag{107}$$

Here $\psi(x)$ is the logarithmic derivative of the Γ-function ([30], 8.360). In the general case, the functions $\psi_{\lambda,\alpha}(x)$ and $\psi_{\lambda',\alpha}(x)$, $\lambda' \neq \lambda$, are not orthogonal, as it follows from (106).

References

[1] R.R. Lewis, *Aharonov-Bohm effect for trapped ions*, Phys. Rev. A**28** (1983) 1228-1236.

[2] V.G. Bagrov, D.M. Gitman,V.D. Skarzhinsky, *Aharonov-Bohm effect for stationary and coherent states of an electron in homogeneous magnetic field*, Trudy FIAN (Proceedings of Lebedev Institite), **176** (1986) 151-166.

[3] E.M. Serebrianyi,V.D. Skarzhinsky, Sov. Phys.-Lebedev Inst.Rep. (Kratk. Soobshch. Fiz.) **6** (1988) 56; Trudy FIAN (Proceedings of Lebedev Institute), **197** (1989) 181.

[4] V.G. Bagrov, D.M. Gitman, and V.B. Tlyachev, *Solutions of relativistic wave equations in superpositions of Aharonov-Bohm, magnetic, and electric fields*, J. Math. Phys. **42** (2001) 1933-1959.

[5] V.G. Bagrov, D.M. Gitman, A. Levin and V.B. Tlyachev, *Aharonov-Bohm effect in cyclotron and synchrotron radiations*, Nucl. Phys. **B 605** (2001) 425-454.

[6] I.V. Tyutin, *Electron scattering on a solenoid,* Lebedev Physical Institute, Preprint N 27 (1974) pp.1-46 (unpublished).

[7] Ph. Gerbert and R. Jackiw, *Classical and quantum scattering on a spinning cone*, Commun. Math. Phys. **124** (1989) 229-260.

[8] Ph. Gerbert, *Fermions in an Aharonov-Bohm field and cosmic strings*, Phys. Rev. D**40** (1989) 1346-1349.

[9] C.R. Hagen, *Aharonov-Bohm scattering of particles with spin*, Phys. Rev. Lett., **64** (1990) 503-506.

[10] C.R. Hagen, *Spin dependence of the Aharonov-Bohm effect*, Int. J. Mod. Phys. **A6** (1991) 3119-3150.

[11] S.A. Voropaev, D.V. Galtsov and D.A. Spasov, *Bound states for fermions in the gauge Aharonov-Bohm field*, Phys. Lett. **B 267** (1991) 91-94.

[12] J.A. Audretsch, U. Jasper, and V.D. Skarzhinsky, *A pragmatic approach to the problem of the self-adjoint extension of Hamiltonian operators with the Aharonov-Bohm potential*, J. Phys. **A 28** (1995) 2359-2367.

[13] J.A. Audretsch, U. Jasper, and V.D. Skarzhinsky, *Bremsstrahlung of relativistic electrons in the Aharonov-Bohm potential*, Phys. Rev. **D 53** (1996) 2178-2189.

[14] C. Manuel and R. Tarrach, *Contact interactions and Dirac anyons*, Phys. Lett. **B 301** (1993) 72-76.

[15] M.G. Alford, J. March-Russel and F. Wilczek, *Enhanced baryon number violation due to cosmic strings*, Nucl. Phys., **B 328** (1989) 140-158.

[16] E.G. Flekkøy and J.M. Leinaas, Int. J. Mod. Phys. *Vacuum currents around a Magnetic Flux String*, **A 6** (1991) 5327-5347.