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Comments on spin operators and spin-polarization states of 2 + 1 fermions

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Abstract

In this brief article, we discuss spin polarization operators and spin polarization states of $2+1$ massive Dirac fermions and represent a convenient representation by the help of 4-spinor for their descriprion. We stress that namely the use of such a representation allows us to introduce the conserved covariant spin operator in the $2+1$ field theory. Another advantage of this representation is related to the pseudoclassical limit of the theory. As one can see, quantizing the corresponding pseudoclassical model of spinning particle in $2+1$ dimensions, we obtain namely the 4-spinor representation as the adaquate realization of the operator algebra. In this realization, the operator of a first-class constraint that cannot be gauged out by imposing gauge conditions is just the operator represented in this article.

I. The $2+1$ spinor field theory [1] attracts in recent years great attention due to various reasons: e.g. because of nontrivial topological properties, and due to a possibility of the existence of particles with fractional spins and exotic statistics (anyons), having probably applications to fractional Hall effect, high-T_c superconductivity and so on [2]. In many practical situations the quantum behavior of spin 1/2 fermions (further simply fermions) in $2+1$ dim. can be described by the corresponding Dirac equation with external electromagnetic field. The main difference between the relativistic quantum mechanics of fermions in $3+1$ and in $2+1$ dim. is related to the different description of spin polarization states. It is well know that in $3+1$ dim. there exist two massive spin 1/2 fermions, the electron and the corresponding to it antiparticle positron. Both electron and positron have two spin polarization states. In $2+1$ dim. there exist

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four massive fermions, two different types of electrons and two corresponding positrons. In contrast to the situation in $3+1$ dim., each particle in $2+1$ dim. has only one polarization state. We recall that constructing covariant and conserved spin operators for the $3+1$ Dirac equation with external field is an important problem for finding exact solution of this equation and specification spin polarization states [3]. Here does not exist universal covariant conserved spin operators which serve for any external field, for each specific configuration of the external field one has to determine such operators [4]. On the first glance, the problem does not exist in $2+1$ dim. since each fermion there has only one spin polarization state. Nevertheless, the spin (or spin magnetic momentum) as a physical quantity in $2+1$ dim. does exist, and therefore, the corresponding operators do exist. One can see, solving the Dirac equation in $2+1$ dim., that knowledge of such spin operators is very useful for finding physically meaningful solutions. Moreover, it turns out that in $2+1$ dim. the appropriate spin operator serves at the same time as particle species operator, and its explicit expression is useful for interpretation of the theoretical constructions. In this brief article, we discuss spin polarization operators and spin polarization states of $2+1$ massive Dirac fermions and some convenient representations for their description.

II. It is well known that in $2+1$ dim. (as well as in any odd dimensions) there exist two inequivalent sets (representations) of gamma-matrices. For example, denoting these matrices via Γ_s^μ, $\mu = 0, 1, 2$, where the subscript $s = \pm 1$ labels the different representations, we can chose

$$\Gamma_s^0 = \sigma^3,\ \Gamma_s^1 = i\sigma^2,\ \Gamma_s^2 = -si\sigma^1,\ s = \pm 1\,, \tag{1}$$

where σ^i are the Pauli matrices. Respectively, there exist two different Dirac equations and two different Lagrangian of the corresponding spinor field. If an external electromagnetic field is present, then the particle ($\zeta = 1$) and antiparticle ($\zeta = -1$) with the charges ζe, $e > 0$ respectively obey the Dirac equations in which the operator $i\partial_\mu$ has to be replaced by $P_\mu = i\partial_\mu - \zeta e A_\mu(x)$, where $A_\mu(x)$ are electromagnetic potentials. Thus, in fact, in $2+1$ dim. we have four massive fermions (let us call further the two different type of fermions up and down particles) and respectively four types of solutions of the $2+1$ Dirac equation (2-spinors $\Psi^{(\zeta,s)}(x)$):

$$\begin{aligned}
&\left(\Gamma_s^\mu P_\mu - m\right)\Psi^{(\zeta,s)}(x) = 0\,,\ x = (x^\mu)\ ,\ \mu = 0,1,2\,,\\
&P_\mu = i\partial_\mu - \zeta e A_\mu(x)\ ,\ s,\zeta = \pm 1\,.
\end{aligned} \tag{2}$$

In such a picture (and in stationary external fields that do not violate the vacuum stability), the only states from the upper energy branch are physical, and only such states can be used for secondary quantization [?].

III. To define a spin magnetic momentum of the $2+1$ massive fermions let us set the external field to be uniform constant magnetic field. In $2+1$ dim. the magnetic field has only one component $F_{21} = -F_{12} = B = \text{const}$. The sign of B defines the "direction" of the field, the positive B corresponds the "up" direction whereas the negative B corresponds to the "down" direction. In such a background the equation (2) can be reduced to the stationary form

$$H^{(\zeta,s)}\Psi_n^{(\zeta,s)}(\mathbf{x}) = \varepsilon_n^{(\zeta,s)}\Psi_n^{(\zeta,s)}(\mathbf{x}) \,, \quad H^{(\zeta,s)} = -\Gamma_s^0\Gamma_s^k P_k + \Gamma_s^0 m \,,$$
$$\Psi^{(\zeta,s)}(x) = \exp\left(-i\varepsilon^{(\zeta,s)}x^0\right)\Psi^{(\zeta,s)}(\mathbf{x}) \,, \; \varepsilon_n^{(\zeta,s)} > 0 \,, \; \mathbf{x} = \left(x^1, x^2\right) . \quad (3)$$

As usual we pass to the squared equation by the ansatz

$$\Psi^{(\zeta,s)}(\mathbf{x}) = \left[\Gamma_s^0\varepsilon + \Gamma_s^k P_k + m\right]\Phi^{(\zeta,s)}(\mathbf{x}) \quad (4)$$

to obtain the following equation

$$\left[\varepsilon_n^2 - D^{(\zeta,s)}\right]\Phi_n^{(\zeta,s)}(\mathbf{x}) = 0 \,,$$
$$D^{(\zeta,s)} = m^2 + \mathbf{P}^2 + \frac{i}{4}\zeta e F_{\mu\nu}\left[\Gamma_s^\mu, \Gamma_s^\nu\right] = m^2 + \mathbf{P}^2 - s\zeta e B\sigma^3 \,, \; \mathbf{P} = \left(P^1, P^2\right) . \quad (5)$$

The 2-component spinors $\Phi_n^{(\zeta,s)}(\mathbf{x})$ may be chosen in the form $\Phi^{(\zeta,s)}(\mathbf{x}) = f_n^{(\zeta,s)}(\mathbf{x})\,\upsilon$, where $f_n^{(\zeta,s)}(\mathbf{x})$ are some functions and υ some constant 2-component spinors that classify spin polarization states. We select υ to obey the equation $\sigma^3\upsilon = \upsilon$. One can see that selecting υ to be the eigenvector of σ^3 with the eigenvalue -1, we do not obtain new linearly independent spinors $\Psi_n^{(\zeta,s)}(\mathbf{x})$. This is a reflection of the well known fact (see e.g. [?]) that massive $2+1$ Dirac fermions have only one spin polarization state. In a weak magnetic field it follows from (5):

$$\varepsilon_n^{(\zeta,s)} = \varepsilon_n^{(\zeta,s)}\Big|_{B=0} - \mu^{(\zeta,s)}B \,, \quad \mu^{(\zeta,s)} = \frac{s\zeta e}{2\sqrt{m^2 + \left(f_n^{(\zeta,s)}\right)^{-1}\mathbf{P}^2 f_n^{(\zeta,s)}}} \,. \quad (6)$$

We have to interpret $\mu^{(\zeta,s)}$ as the spin magnetic momentum of $2+1$ fermions. Thus in $2+1$ dim., we have

$$\operatorname{sign}\mu^{(\zeta,s)} = s\zeta \,. \quad (7)$$

One ought to remark that this result matches with the conventional description of spin polarization in $2+1$ dimensions. Considering the total angular momentum in the rest frame (see, for example, [6, 8]), one can define the operators $S_0^{(s)}$ of spin projection on the x^0-axis,

$$S_0^{(s)} = \frac{i}{4}\left[\Gamma_s^1, \Gamma_s^2\right] = \frac{s}{2}\sigma^3 \,. \quad (8)$$

In the nonrelativistic limit we obtain from (4) and (6),

$$\mu^{(\zeta,s)} = \frac{s\zeta e}{2m} \,, \; \Psi_n^{(\zeta,s)}(\mathbf{x}) = 2m\Phi_n^{(\zeta,s)}.$$

In such a limit the Dirac spinors $\Psi_n^{(\zeta,s)}(\mathbf{x})$ are eigenfunctions of the operators (8),

$$S_0^{(s)}\Psi_n^{(\zeta,s)}(\mathbf{x}) = \frac{s}{2}\Psi_n^{(\zeta,s)}(\mathbf{x}) .$$

Thus, one can consider

$$M^{(\zeta,s)} = \frac{\zeta e}{m} S_0^{(s)} \tag{9}$$

as spin magnetic momentum operator. However, the operators $S_0^{(s)}$ are not covariant and are not conserved in the external field. Below we represent a conserved and covariant spin operator for $2+1$ massive fermions.

IV. Let us use a 4-component spinor representation for the wave functions to describe particles in $2+1$ dimensions. Namely, let us introduce 4-component component spinors of the form

$$\psi^{(\zeta,+1)}(x) = \begin{pmatrix} \Psi^{(\zeta,+1)}(x) \\ 0 \end{pmatrix},\ \psi^{(\zeta,-1)}(x) = \begin{pmatrix} 0 \\ \sigma^1\Psi^{(\zeta,-1)}(x) \end{pmatrix}. \tag{10}$$

These 4-component spinors are representatives of 2-component spinors $\Psi^{(\zeta,+1)}(x)$ and $\Psi^{(\zeta,-1)}(x)$. At the same time it is convenient to use three 4×4 matrices γ^0, γ^1, and γ^2 taken from the following representation [7] of $3+1$ gamma-matrices

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} \Gamma^0_{+1} & 0 \\ 0 & -\Gamma^0_{-1} \end{pmatrix},\ \gamma^1 = \begin{pmatrix} \Gamma^1_{+1} & 0 \\ 0 & -\Gamma^1_{-1} \end{pmatrix}, \\ \gamma^2 &= \begin{pmatrix} \Gamma^2_{+1} & 0 \\ 0 & \Gamma^2_{-1} \end{pmatrix},\ \gamma^3 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \end{aligned} \tag{11}$$

In the new representation, the 4-component spinors (10) obey the Dirac equation of the following form

$$(\gamma^\mu P_\mu - m)\,\psi(x) = 0\,,\ P_\mu = i\partial_\mu - \zeta e A_\mu(x)\ ,\ x = (x^\mu)\ ,\ \mu = 0,1,2\,. \tag{12}$$

In fact, this equation can be considered as a result of a partial dimensional reduction of the $3+1$ Dirac equation. Stationary solutions of the equation (12) can be expressed via solutions $\Phi_n^{(\zeta,s)}(\mathbf{x})$ of the equation (5) as follows

$$\begin{aligned} &\psi_n^{(\zeta,s)}(x) = \exp\left(-i\varepsilon_n^{(\zeta,s)}x^0\right)\left[\gamma^0\varepsilon_n^{(\zeta,s)} + \gamma^k P_k + m\right]\varphi^{(\zeta,s)}(\mathbf{x})\ ,\ \mathbf{x} = \left(x^1, x^2\right), \\ &\varphi^{(\zeta,+1)} = \begin{pmatrix} \Phi^{(\zeta,+1)} \\ 0 \end{pmatrix},\ \varphi^{(\zeta,-1)} = \begin{pmatrix} 0 \\ \sigma^1\Phi^{(\zeta,-1)} \end{pmatrix},\ \varepsilon_n^{(\zeta,s)} > 0\,, \end{aligned} \tag{13}$$

whereas the energy spectrum is the same as for the equation (5). One can easily see that the 4-spinors $\varphi^{(\zeta,s)}$ are eigenvectors of the operator Σ^3 with the eigenvalues s being the particle species

$$\Sigma^3\varphi^{(\zeta,s)} = s\varphi^{(\zeta,s)}\,,\ \Sigma^3 = i\gamma^1\gamma^2 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}. \tag{14}$$

The operator Σ^3 commutes with the squared Dirac equation. This fact allows us to find a spin integral of motion for the Dirac equation (12). Such an integral of motion reads

$$\Lambda = \frac{\mathcal{H}\Sigma^3 + \Sigma^3\mathcal{H}}{4m}\,,\ \mathcal{H} = -\gamma^0\gamma^k P_k + \gamma^0 m\,. \tag{15}$$

In the case under consideration, we obtain

$$\Lambda\psi^{(\zeta,s)} = \frac{s}{2}\psi^{(\zeta,s)}\,,\ \ \Lambda = \frac{1}{2}\gamma^0\Sigma^3 = \frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{16}$$

V. Now we can consider $2+1$ QFT of the spinor field that obeys the equation (12). Such a QFT can be obtained by a standard quantization of the corresponding Lagrangian. Here the field operators have the form

$$\hat{\psi}\left(x\right) = \begin{pmatrix} \hat{\Psi}_{+1}\left(x\right) \\ \sigma^1\hat{\Psi}_{-1}\left(x\right) \end{pmatrix}, \tag{17}$$

where the 2-component operators $\hat{\Psi}_s\left(x\right)$ describe particles of the s-species. Decomposing the field (17) into the solutions (13), we obtain four type creation and annihilation operators: $a_{s,n}$ and $a^+_{s,n}$ which are operators of particles ($\zeta = +1$) and $b_{s,n}$ and $b^+_{s,n}$ which are operators of antiparticles ($\zeta = -1$). Thus, in the QFT under consideration all the types of $2+1$ fermions appear at the same footing.

In the QFT one can define the second-quantized operator $\hat{\Lambda}$ that corresponds to the operator Λ of the field theory,

$$\hat{\Lambda} = \frac{e}{m}\int \hat{\psi}^\dagger \Lambda \hat{\psi} d\mathbf{x}\,. \tag{18}$$

It is easily to verify that such an operator is a scalar under $2+1$ Lorentz transformations and is conserved in any external field. We call the operator $\hat{\Lambda}$ the spin magnetic polarization operator. One can easily see that this operator is expressed via charge operators $\hat{Q}_s$ of $2+1$ fermions as follows:

$$\hat{\Lambda} = \frac{1}{2m}\left(\hat{Q}_{+1} - \hat{Q}_{-1}\right), \tag{19}$$

where

$$\hat{Q}_s = \frac{e}{2}\int \left[\hat{\Psi}^\dagger_s, \hat{\Psi}_s\right] d\mathbf{x} = e\sum_n \left(a^+_{s,n}a_{s,n} - b^+_{s,n}b_{s,n}\right)\,,\ s = \pm 1\,. \tag{20}$$

Remark that the eigenvalues of the operator $\hat{\Lambda}$ in the one-particle sector coincide with the spin magnetic momenta $\mu^{(\zeta,s)} = s\zeta e/2m$ of the $2+1$ fermions in the rest frame.

We stress that namely the use of a spinor representation with more than 2-components allows us to introduce the conserved covariant spin operator in the $2+1$ field theory. There is another argument (which is related to the first quantization procedure) in favour of such representations is discussed below.

VI. It was demonstrated in [9] that relativistic quantum mechanics of all the massive $2+1$ fermions can be obtained in course of the first quantization of a corresponding pseudoclassical action where the particle species s is not fixed. General state vectors are 16-component columns. The states with a definite charge sign ζ can be described by 8-component columns ϕ_ζ. The operators of space coordinates $\hat{X}^k$ and momenta $\hat{\mathcal{P}}_k$ act on these columns as

$$\hat{X}^k = x^k \mathbf{I}\,,\ \hat{\mathcal{P}}_k = \hat{p}_k \mathbf{I}\,,\ \hat{p}_k = -i\partial_k\,.$$

Here, $\mathbf{I}$ is the 8×8 unit matrix. Besides the spin degrees of freedom are related to the operators

$$\hat{\xi}^1 = \frac{i}{2}\text{antidiag}\left(\gamma^1,\gamma^1\right)\ ,\ \hat{\xi}^2 = \frac{i}{2}\text{diag}\left(\gamma^2,\gamma^2\right)\ .$$

The operator of a conserved first-class (ungauged) constraint has a form

$$\hat{t} = \hat{\theta} - \hat{S}\,,\ \ \hat{\theta} = \text{diag}\left(\Lambda,\Lambda\right)\ ,\ \hat{S} = 2i\hat{\xi}^2\hat{\xi}^1\,.$$

To fix the gauge at the quantum level, one imposes according to Dirac the condition $\hat{t}\phi_\zeta = 0$ on physical state vectors. At the same time we chose ϕ_ζ to be eigenvectors of the matrix $\hat{\theta}$,

$$\hat{\theta}\phi_{\zeta,s} = \frac{s}{2}\phi_{\zeta,s}\,.$$

We see that in the first quantized theory under consideration the operator $\hat{S}$ acts as the operator Λ in the quantum mechanics of item IV,

$$\hat{S}\phi_{\zeta,s} = \frac{s}{2}\phi_{\zeta,s}\,.$$

Thus, we can interpret the operator $\hat{S}$ as spin operator.

Finally, there exists a relation between the representations of one-particle quantum states in terms of $\phi_{\zeta,s}$ and $\psi^{(\zeta,s)}$. Such a relation reads:

$$\phi_{\zeta,+1}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi^{(\zeta,+1)}(x)\\ \gamma^0\psi^{(\zeta,+1)}(x)\end{pmatrix},\ \ \phi_{\zeta,-1}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi^{(\zeta,-1)}(x)\\ \gamma^0\psi^{(\zeta,-1)}(x)\end{pmatrix}.$$

One can easily demonstrate that these two representations are physically equivalent.

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