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Publicação IF – 1592/2004

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Density operators for particles created by strong backgrounds

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May 27, 2004

Abstract

In this work the Fock operator which represents the density matrix for charged bosons and fermions in the presence of an arbitrary external field is constructed in Furry´s representation. By introducing a unitary operator which contains the complete information about the physical processes occurring in the many-particle system described by a quadratic Hamiltonian, the mean number of particles and antiparticles in the final state is explicitly calculated. The one-particle distribution functions are found and the developed formalism is applied to a number of physical situations involving slowly varying external fields. Time and temperature dependences of the integral mean numbers of created particles are also discussed.

1 Introduction

The effect of particles creation from vacuum by an external field (vacuum instability in an external field) ranks among the most intriguing nonlinear phenomena in quantum theory. Its consideration is theoretically important, since it requires one to go beyond the scope of the perturbation theory, and its experimental observation would verify the validity of the theory in the superstrong field domain. The study of the effect began, in fact, in connection with the so-called Klein [1] paradox, which revealed the possibility of electron penetration through an arbitrary high barrier formed by an external field. Then Schwinger [2] had calculated the vacuum-to-vacuum transition probability for the quantum spinor field in an external constant electric field. It became clear that the effect can

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actually be observed as soon as the external field strength approaches the characteristic value (critical field) $E_c = m^2c^3/|e|\hbar \simeq 1,3 \cdot 10^{16}\ V/cm$. Although a real possibility of creating such fields under laboratory conditions does not exist at present, these fields can play a role in astrophysics, where the characteristic values of electromagnetic fields and gravitational fields near black holes are enormous. One can also mention that Coulomb fields of superheavy nuclei can create electron-positron pairs [3]. The particle creation effect for the quantum spinor field in macroscopic electric-like external fields were considered in detail by Narozhnyi and Nikishov (see the survey [4] and references therein) and by other authors as well (see e.g. [5, 6] and references therein) and the general formulation of QED in backgrounds violating the vacuum stability was elaborated in [7]. Discussions on particle creation in various nonsingular gravitational backgrounds can be found in [8] and references therein. The problem of the e^+e^--pair production in strong laser beams is now discussed intensively [9], and there exist recent experimental evidence of such an effect [10]. There exist a number of problems that are closely related physically or technically to the above vacuum instability problem, for example, phase transitions in field theories, the problem of boundary conditions or topology influence on the vacuum, the problem of consistently constructing the vacuum in QCD and string theories, multiple particles creation, and so on.

Calculation of the particle creation in the black hole gravitational field was first performed by Hawking [11] and then studied by numerous authors, see e.g. [12] and references therein (the effect is important, in particular, for understanding of Early Universe dynamics). In this particular case, due to the singularity of the black hole metric (for the existence of the horizon), one is enforced to consider the density matrix of a partial amount of created particles. It was discovered that such a density matrix describes a thermal distribution with a specific temperature (Hawking temperature). One can find many other examples when the particles creation in external gravitation fields (and due to a nontrivial topology) can be described by means of an effective temperature [12] (see also [13] and references therein). In the case of the black hole background (and in similar backgrounds) the particle creation problem cannot be formulated as a scattering problem by the analogy to the electric field background case due to specific technical (probably not only technical but in fact principal) difficulties related to the absence of the properly classified sets of solutions of the Dirac (or another corresponding) wave equation. In what follows, we call such background singular backgrounds. Namely, this circumstance does not allow one to clarify completely until the moment the reason of the intriguing appearance of the thermal distributions of particles created by singular backgrounds. Is the appearance of the thermal distributions related to the specific external backgrounds, or to the averaging over a part of the created particles, or to the combination of both factors? What is the role of the initial state in this case? In the works [15] the authors tried to answer these questions considering a quantized field (spinor or scalar) placed in a nonsingular background that admits the treatment of particle creation as a scattering problem. We have considered both general backgrounds of such a kind as well as specific backgrounds like an electric field that switches off at the initial and final time instants. Technically, the density matrices of particles created from the initial vacuum state were constructed averaging over all the created antiparticles. It was demonstrated that such a density matrix is not thermal. Nevertheless, the thermal distribution appears if one transforms such a matrix to that in the gravitationaL case using the equivalence principle. In such a way, the Hawking temperature can be reproduced with accuracy of a factor, or exactly, if one takes into account the vacuum polarization[6] effect. The attempt to study the dependence of the particle density matrix on the choice of the initial states was made in [16]. A formal expression for the density matrix of particles created by a nonsingular external background from an arbitrary initial state was obtained by a functional integration, but it was not sufficiently analyzed. Moreover, besides a few misprints in the final expression, essencial details in its derivation are completely omitted. For these reasons and due to the importance of the theme one finds the occasion to return to the problem of the density matrix construction in the present article, in order to clarify fundamental issues. The article is organized as follows:

In section 2, we consider a formal derivation of density matrices of particles created by external nonsingular backgrounds from an arbitrary initial state. To this end, we introduce and study a generating density operator for a quantized field (for a system of particles and antiparticles). In principal, such an operator allows one to derive density matrices of all the particles, or of a part of particles, created by external backgrounds from different initial states. However, it is not a simple technical task. In order to use the path integration methods to solve the above problems, we have obtained a normal form for such a generating density operator. Using the normal form and the possibility of functional integration, we have derived various density operators for created particles.

As an important technical result, in section 3 we find the generating functional for the ensemble average at the final time instant for any possible initial state and, by averaging over states of antiparticles, present a reduced generating operator which describes the subsystem of particles; in particular, the simplest one particle distribution functions are found in section 4. For this purpose, some necessary notions of the generalized Furry representation for a quadratic Hamiltonian system are presented in the Appendix, which contains all essential technical details and trace formulas.

In sections 5 and 6, we apply the formulated formalism to a number of the topical problems in slowly varying external fields: electric, chromoelectric and metric of the expanding Friedmann-Robertson-Walker universe; we consider the multiple particle creation through the distribution of the created particles and find the thermal-like density matrix description of particles created from the vacuum by a slowly varying electric-like field, then establishing a relation between the corresponding distribution and the Hawking's thermal distribution of particles created by the static gravitational field of a black hole. Finally, we discuss the time and temperature dependences of the integral mean numbers of particles created in an external background field.

We are using a convention of summation/integration over discrete/continuous repeated indices and a compact notation where all summations/integrations are suppresses, for example $\psi_n G_{nm} = (\psi G)_m$. In addition $\hbar = c = 1$ throughout this paper.

2 Generating density operator

2.1 General

In what follows, we consider a quantized field $\psi(x)$ placed in a nonsingular arbitrary external background. We do not specify the quantized field, namely, it can be either bosonic or fermionic. In our consideration, we are going to take the interaction with the background exactly, using the generalized Furry picture [7] (see more details in the Appendix). In the backgrounds under consideration, there always exist two complete sets of solutions (in-solutions ${}_\zeta\psi_n(x)$ and out-solutions ${}^\zeta\psi_n(x)$, $\zeta = \pm$) of the corresponding classical wave equations for the field $\psi(x)$. Solutions that are labeled by the upperscript $\zeta = +$ ($\zeta = -$) describe particles (antiparticles) at an initial time instant x^0_{in} , and solutions that are labeled by the subscript$\zeta = +$ ($\zeta = -$) describe particles (antiparticles) at a final time instant x^0_{out} . Here n stands for a set of all the corresponding quantum numbers. In fact, we consider the quantum theory of noninteracting particles and antiparticles among themselves, placed in an external background. In the general case the vacuum of such a system is unstable under particle creation. This effect is the main goal of our consideration.

Let us introduce the operator $\hat{\Omega}(J)$ which depends on the sources $J = \left(J_n^{(\zeta)}\right)$ as follows

$$\hat{\Omega}(J) = \mathcal{N}_{in} \exp\left\{a^\dagger(in)\left(\alpha^{(+)} - 1\right)a(in) + b^\dagger(in)\left(\alpha^{(-)} - 1\right)b(in)\right\} . \quad (1)$$

Here $\alpha^{(\zeta)}$ are diagonal matrices, $\alpha^{(\zeta)}_{mn} = \delta_{mn}J^{(\zeta)}_n$, and $\mathcal{N}_{in}$ is the sign of the normal ordering with respect to in-operators. One can see that $\hat{\Omega}(J)$ is a generating operator for density operators (density matrices) of the quantum system under consideration at the initial time instant x^0_{in}. In particular, specifying the sources, we can obtain different density matrices for the initial state . Let us consider the following examples:

Choosing all the sources to be zero ($J = 0$), we obtain the density operator $\hat{\rho}(0)$ which describes the initial vacuum state $|0, in\rangle$ $(a(in)|0, in\rangle = b(in)|0, in\rangle = 0)$ (a pure state). Namely,

$$\hat{\rho}(0) = \hat{\Omega}(0) = |0, in\rangle\langle 0, in| \quad (2)$$

is the projector on the initial vacuum state. The latter fact follows from the well-known relation (first represented by F.Beresin [37])

$$\mathcal{N}\exp\left\{-A^\dagger A\right\} = |0\rangle\langle 0| \, , \, (A|0\rangle = 0) \, ,$$

which is valid for any set of creation and annihilation operators $A^\dagger$ and A.

The density operator $\hat{\rho}\,(m_1,\ldots,m_M;n_1,\ldots,n_N)$ of a pure initial state with M paricles and N antiparicles (with the quantum numbers $m_1,\ldots,m_M$ and $n_1,\ldots,n_N$ respectively) can be obtained from the generating operator (1) as follows:

$$\hat{\rho}\,(m_1,\ldots,m_M;n_1,\ldots,n_N) = Z^{-1}\left.\frac{\partial^{M+N}\hat{\Omega}\,(J)}{\partial(J^{(+)}_{m_1}\ldots J^{(+)}_{m_M}J^{(-)}_{n_1}\ldots J^{(-)}_{n_N})}\right|_{J=0} . \tag{3}$$

The factor Z has to be calculated from the normalization condition $\mathrm{tr}\hat{\rho} = 1$.

The density operator $\hat{\rho}\,(\beta)$ of the grand canonical ensemble of particles and antiparticles in the initial state (further the temperature density operator) has the form

$$\hat{\rho}\,(\beta) = Z^{-1}\exp\left\{-\sum_{\zeta,n}E^{(\zeta)}_n\hat{N}^{(\zeta)}_n\,(in)\right\}\,,\; E^{(\zeta)}_n = \beta\left(\varepsilon^{(\zeta)}_n - \mu^{(\zeta)}\right), \tag{4}$$

where $\varepsilon^{(\zeta)}_n$ are energies of particles ($\zeta = +$) or antiparticles ($\zeta = -$), β^{-1} is the temperature Θ, by $\mu^{(\zeta)}$ we denote chemical potentials, and by $\hat{N}^{(\zeta)}_n\,(in)$ the operators of perticle numbers in the initial state,

$$N^{(+)}_n(in) = a^\dagger_n(in)a_n(in)\,,\; N^{(-)}_n(in) = b^\dagger_n(in)b_n(in)\,. \tag{5}$$

Such a density operator can follows from the generating density operator (1) by the following choice of the sourses J,

$$J^{(\zeta)}_n = \exp\left\{-E^{(\zeta)}_n\right\}. \tag{6}$$

Namely,

$$\hat{\rho}\,(\beta) = Z^{-1}\hat{\Omega}\,(J)\Big|_{J^{(\zeta)}_n=\exp\left\{-E^{(\zeta)}_n\right\}} . \tag{7}$$

The latter fact follows from the relation (see Eq. (81) proven in the Appendix)

$$\exp\left\{A^\dagger DA\right\} = \mathcal{N}\exp\left\{A^\dagger\left(e^D - 1\right)A\right\}, \tag{8}$$

which is valid for any set of creation and annihilation operators $A^\dagger$ and A and any matrix D.

2.2 Normal form of the generating density operator

Now, we are interested in obtaining the totally ordered form for $\hat{\Omega}$. It is a far from trivial problem and to do it we need the closed form (83) for Wick's theorem, given in Appendix. Let us adapt it to our problem, calculating an explicit normal form for the expression

$$Y(A,B) = e^{-bBa}e^{-a^\dagger Ab^\dagger}$$

where A and B are quadratic matrices and there are two species of creation and annihilation operators: $a^\dagger$ and a for particles and $b^\dagger$ and b for antiparticles. All of them can be either fermions or bosons. Applying (83) in Appendix to $Y(A,B)$ gives

$$Y(A,B) = \exp\left\{\frac{\partial_r}{\partial a_n}\frac{\partial_l}{\partial a_n^\dagger} + \frac{\partial_r}{\partial b_n}\frac{\partial_l}{\partial b_n^\dagger}\right\} : e^{-bBa}e^{-a^\dagger A b^\dagger} : .$$

Calculating the derivatives with respect to b and $b^\dagger$ gives

$$Y(A,B) = \exp\left\{\frac{\partial_r}{\partial a_n}\frac{\partial_l}{\partial a_n^\dagger}\right\} : \exp\left\{-bBa - a^\dagger A b^\dagger - a^\dagger ABa\right\} : . \tag{9}$$

In order to finish the calculations it is effective to use a path integral representation. For the fermion case we use a path integral over anticommuting (Grassmann) variables which is understood as Berezin's integral [37],

$$: e^{-a^\dagger K a} := \det K : \int \exp\left\{\lambda^* K^{-1}\lambda + a^\dagger\lambda + \lambda^* a\right\} \Pi d\lambda^* d\lambda : \tag{10}$$

For the boson case we use a path integral over commuting variables [39],

$$: e^{-a^\dagger K a} := \det K^{-1} : \int \exp\left\{-\varphi^* K^{-1}\varphi + a^\dagger\varphi + \varphi^* a\right\} \Pi d\varphi^* d\varphi : \tag{11}$$

Representing $: e^{-a^\dagger ABa} :$ in (9) with the help of either the integral (10) or (11), respectively, one can calculate the derivatives with respect to a and $a^\dagger$. After integrating one finally finds the normal form of $Y(A,B)$,

$$\begin{aligned} Y(A,B) &= \det(1+\kappa AB)^{\kappa} : \exp\left\{-a^\dagger A_{++}a - b^\dagger A_{--}b - a^\dagger A_{+-}b^\dagger - bA_{-+}a\right\} : , \\ A_{++} &= \kappa AB(1+\kappa AB)^{-1} \ , \ A_{--}^T = \kappa BA(1+\kappa BA)^{-1} \ , \\ A_{+-} &= (1+\kappa AB)^{-1}A \ , \ A_{-+} = B(1+\kappa AB)^{-1} \ , \end{aligned} \tag{12}$$

By using the formulas (81) in Appendix and (12) above, then applying successively Wick's theorem (83), we finally get

$$\begin{aligned} \hat{\Omega}\left(J^{(+)}, J^{(-)}\right) &= |c_v|^2 \det(1+\kappa AB)^{-\kappa} \mathcal{N}_{out}\left(\exp\left\{-a^\dagger(out)(1-D_+)a(out)\right.\right. \\ &\quad \left.\left. - b^\dagger(out)(1-D_-)b(out) - a^\dagger(out)\mathcal{B}^\dagger b^\dagger(out) - b(out)\mathcal{B}a(out)\right\}\right) , \\ D_+ &= w(+|+)(1+\kappa AB)^{-1}\alpha^{(+)}w(+|+)^\dagger , \ D_-^T = w(-|-)^\dagger\alpha^{(-)}(1+\kappa BA)^{-1}w(-|-) , \\ \mathcal{B} &= w(-|-)^\dagger\alpha^{(-)}B(1+\kappa AB)^{-1}\alpha^{(+)}w(+|+)^\dagger + w(+-|0)^\dagger , \\ A &= \alpha^{(+)}B^\dagger\alpha^{(-)}, \ B = w(0|-+) . \end{aligned} \tag{13}$$

It is the principal general result. Due to the normal form representation in (13), one can calculate the trace by using the effective path integral techniques presented in Appendix (formulas (85) and (86)). The above constructed density operator contains the complete information of all possible physical processes leading to a final configuration at a given initial temperature or at a given initial pure state. Its structure also suggests a possible connection with a consistent S-matrix operator in Fock space which describes the dynamical processes ocurring in a many-body system in the presence of an external field described by a quadratic Hamiltonian, in a real time formulation of finite temperature quantum field theory as, for example, thermo field dynamics [40]. As another possible application of the above formalism is the study of the behaviour of atomic bound state systems at finite temperature, through the analysis of the corresponding thermal field correlation functions and susceptibilities for the physical abservables [47].

To show an example of application of expression (13) note that the in-vacuum projection operator has a quite simple form in terms of the out-operators of creation and annihilation,

$$\begin{aligned} \hat{\Omega}(0) &= |c_v|^2 \mathcal{N}_{out} \left(\exp\left\{-a^\dagger(out)a(out) - b^\dagger(out)b(out)\right.\right. \\ &\left.\left. -a^\dagger(out)w\left(+-|0\right)b^\dagger(out) - b(out)w\left(+-|0\right)^\dagger a(out)\right\}\right). \quad (14) \end{aligned}$$

This expression is interesting in itself since it gives us an explicit example of a density matrix for a pure state which is not a state with a well-defined number of particles, this is a kind of a generalized coherent state (squeezed state) [41]. Using this formula and selecting the function $\hat{F}$, for example, as a projection operator on a quantum state with M pairs of particles and antiparticles at the final time instant one can write the probability of creating these pairs from vacuum.

3 Density matrices

Suppose the density matrix of an initial state is $\hat{\rho}$, and suppose the physical quantity F at the final time instant is given by the operator function

$$\hat{F} = F\left(a^\dagger(out), a(out), b^\dagger(out), b(out)\right).$$

Then its mean value at the final time instant is given by the formula

$$\langle F\rangle = \mathrm{tr}\,\hat{F}\hat{\rho}. \quad (15)$$

For any operator $\mathcal{A}$ the trace can be calculated in the Fock space as follows

$$\mathrm{tr}\mathcal{A} = \sum_{M,N=0}^{\infty}\sum_{\{m\}\{n\}} (M!N!)^{-1}\,\Psi^\dagger\left(\{m\}_M, \{n\}_N; in\right)\mathcal{A}\Psi\left(\{m\}_M, \{n\}_N; in\right),$$

$$\Psi\left(\{m\}_M, \{n\}_N; in\right) = a^\dagger_{m_1}(in)\dots a^\dagger_{m_M}(in)b^\dagger_{n_1}(in)\dots b^\dagger_{n_N}(in)|0, in> \quad (16)$$

To do it one needs to express all the out-operators of creation and annihilation in the function $\hat{F}$ via the in-operators of particle creation and annihilation, using Bogolyubov transformations (67), and then to find the normal form with respect to the in-operators of creation and annihilation for each given function $\hat{F}$ independently. It does not work if one is interested in a behavior of the functional $\langle F \rangle$ in general.

We can propose a better way to do the trace calculations. Let us consider another basis of the Fock space using the unitary transformation (73),

$$\begin{aligned} \Psi\left(\{m\}_M, \{n\}_N; in\right) &= V\Psi\left(\{m\}_M, \{n\}_N; out\right), \\ \Psi\left(\{m\}_M, \{n\}_N; out\right) &= a^\dagger_{m_1}(out)\ldots a^\dagger_{m_M}(out) b^\dagger_{n_1}(out)\ldots b^\dagger_{n_N}(out)|0, out> . \end{aligned} \tag{17}$$

Then the trace (16) can be transformed to the following form

$$\mathrm{tr}\mathcal{A} = \sum_{M,N=0}^{\infty} \sum_{\{m\}\{n\}} (M!N!)^{-1} \Psi^\dagger\left(\{m\}_M, \{n\}_N; out\right) \mathcal{A} \Psi\left(\{m\}_M, \{n\}_N; out\right). \tag{18}$$

We do not need to do anything special with the operator function $\hat{F}$ in the such a representation. Of course, all above mentioned problem in expressing one kind of creation and annihilation operators via the another ones and proceeding the re-ordering in the generating operator $\hat{\rho}$ still remains. We can solve this problem for the given expression of the operator (1). First, we find the necessary expression of $\hat{\rho}$ via the out-operators of creation and annihilation due to the unitary transformation (73) as follows,

$$\hat{\Omega}\left(J^{(+)}, J^{(-)}\right) = V\mathcal{N}_{out} \exp\left[a^\dagger(out)\left(\alpha^{(+)} - 1\right) a(out) + b^\dagger(out)\left(\alpha^{(-)} - 1\right) b(out)\right] V^\dagger, \tag{19}$$

where $\mathcal{N}_{out}$ () is the normal ordering operator with respect to the out-operators and V is defined by (74).

We see that the state of the system at the final time instant contains both particles and antiparticles due to pair creation by the external field independently of the initial state composition. On the other hand, a typical case occurs when the physical quantity F only is related to either the particle (+) or the antiparticle (−) subsystems in the final time instant. The corresponding operators $\hat{F}_\pm$ are functions of either operators $a^\dagger(out)$ and $a(out)$ or $b^\dagger(out)$ and $b(out)$,

$$\hat{F}_+ = F_+\left(a^\dagger(out), a(out)\right), \ \hat{F}_- = F_-\left(b^\dagger(out), b(out)\right). \tag{20}$$

In this case we can average over the states of one of the subsystems and obtain a description in terms of the density matrix defined for the remaining subsystem. Let us present the state vector $\Psi\left(\{m\}_M, \{n\}_N; out\right)$ as following

$$\begin{aligned} &\Psi\left(\{m\}_M, \{n\}_N; out\right) = \Psi\left(\{m\}_M; out\right) \otimes \Psi\left(\{n\}_N; out\right), \\ &\Psi_a\left(\{m\}_M; out\right) = a^\dagger_{m_1}(out)\ldots a^\dagger_{m_M}(out)|0, out>_a, \\ &\Psi_b\left(\{n\}_N; out\right) = b^\dagger_{n_1}(out)\ldots b^\dagger_{n_N}(out)|0, out>_b, \end{aligned} \tag{21}$$

where the relation $|0, out> = |0, out>_a \otimes |0, out>_b$ is used. Then

$$< F_\pm > = \mathrm{tr}_+ \mathrm{tr}_- \hat{F}_\pm \hat{\rho}, \tag{22}$$

where the traces over the subsystem states are

$$\begin{aligned}
\mathrm{tr}_{+}\mathcal{A} &= \sum_{M=0}^{\infty}\sum_{\{m\}}(M!)^{-1}\,\Psi_a^{\dagger}\left(\{m\}_M\,;out\right)\mathcal{A}\Psi_a\left(\{m\}_M\,;out\right),\\
\mathrm{tr}_{-}\mathcal{A} &= \sum_{M=0}^{\infty}\sum_{\{m\}}(M!)^{-1}\,\Psi_b^{\dagger}\left(\{m\}_M\,;out\right)\mathcal{A}\Psi_b\left(\{m\}_M\,;out\right).
\end{aligned} \tag{23}$$

Let us define the reduced generating operators of the corresponding density matrices $\hat{\rho}_{\pm}$ of the remaining subsystems as

$$\hat{\Omega}_{\pm} = \mathrm{tr}_{\mp}\hat{\Omega}. \tag{24}$$

Then one can rewrite (22) in the form

$$< F_{\pm} >= \mathrm{tr}_{\pm}\hat{F}_{\pm}\hat{\rho}_{\pm}, \tag{25}$$

Even though the initial state is the vacuum, the matrix $\hat{\rho}_{\pm}$ describes a mixed state. In this case $\hat{\rho}_{\pm}$ is called the density matrix of particles/antiparticles created in the external field. Sometimes, the use of the such a density matrix becomes a prime necessity. For example, in the problem of quantum particle creation in the strong gravitation field of a black hole [11, 12], the density matrix of particles created outside the black hole comes into play, due to the impossibilty of observing those part of the created particles that appear behind the event horizon. In an external electromagnetic field the necessity of the such a description may arise in the case where the particle detectors are located in a region which can only be reached by one kind of charged particles. The density matrice of particles created by an external electromagnetic field from the vacuum was studied in [15] (by using an explicit decomposition of the in-vacuum state $|0, in>$ via the complete set of the out-states $\Psi\left(\{m\}_M\,,\{n\}_N\,;out\right)$) and in [16] (by the functional method).

Due to the special path integral techniques presented in Appendix and the above obtained representation (13), we can find an explicit expression for the reduced generating operators $\hat{\Omega}_{\pm}$ in the general case described by the generating operator $\hat{\Omega}$. Calculating the traces in (24) by using formulas (85) for fermions and (86) for bosons we get

$$\begin{aligned}
\hat{\Omega}_{+} &= Z_{+}^{-1}\mathcal{N}_{out}\left(\exp\left\{-a^{\dagger}(out)\left(1-K_{+}\right)a(out)\right\}\right),\\
\hat{\Omega}_{-} &= Z_{-}^{-1}\mathcal{N}_{out}\left(\exp\left\{-b^{\dagger}(out)\left(1-K_{-}\right)b(out)\right\}\right),\\
Z_{\pm}^{-1} &= \left|c_v\right|^2\det\left(1+\kappa AB\right)^{\kappa}\det\left(1+\kappa D_{\mp}\right)^{\kappa},\\
K_{\pm} &= D_{\pm}+\mathcal{B}^{\dagger}\left(1+\kappa D_{\mp}^{T}\right)^{-\kappa}\mathcal{B}.
\end{aligned} \tag{26}$$

1. Selecting all $J_n^{(\zeta)} = 0$ in (26) we have

$$K_{\pm} = w\left(+-|0\right)w\left(+-|0\right)^{\dagger},\; Z_{\pm}^{-1} = \left|c_v\right|^2, \tag{27}$$

obtaining the density matrix of particles created by an external field from the vacuum, $\hat{\rho}^0_\pm = \hat{\Omega}_\pm\Big|_{J=0}$. The result agrees with that obtained in [16, 15].

2. Having a pure quantum state with either a particle or a antiparticle at the initial time instant in (26) we get

$$
\begin{aligned}
Z\hat{\rho}^{m;0}_+ &= \left.\frac{\partial\hat{\Omega}_+}{\partial J^{(+)}_m}\right|_{J=0} = \left[a^\dagger(out)w\,(+|+)\right]_m \hat{\rho}^0_+ \left[w\,(+|+)^\dagger a(out)\right]_m , \\
Z\hat{\rho}^{0;m}_- &= \left.\frac{\partial\hat{\Omega}_-}{\partial J^{(-)}_m}\right|_{J=0} = \left[w\,(-|-)\,b^\dagger(out)\right]_m \hat{\rho}^0_- \left[b(out)w\,(-|-)^\dagger\right]_m , \\
Z\hat{\rho}^{0;m}_+ &= \left.\frac{\partial\hat{\Omega}_+}{\partial J^{(-)}_m}\right|_{J=0} = \hat{\rho}^0_+ \left[w\,(-|-)\,w\,(-|-)^\dagger\right]_{mm} \\
&\quad - \left[a^\dagger(out)w\,(+-|0)\,w\,(-|-)^\dagger\right]_m \hat{\rho}^0_+ \left[w\,(-|-)\,w\,(+-|0)^\dagger a(out)\right]_m , \\
Z\hat{\rho}^{m;0}_- &= \left.\frac{\partial\hat{\Omega}_-}{\partial J^{(+)}_m}\right|_{J=0} = \hat{\rho}^0_- \left[w\,(+|+)^\dagger\, w\,(+|+)\right]_{mm} \\
&\quad - \left[b^\dagger(out)w\,(+-|0)\,w\,(+|+)^*\right]_m \hat{\rho}^0_- \left[w\,(+|+)^T w\,(+-|0)^\dagger\, b(out)\right]_m ,
\end{aligned}
$$

where the generating formula (2) is used. Here and only in this example, we are not using the convention of summation over repeated m.

3. Selecting $J^{(\zeta)}_n = e^{-E^{(\zeta)}_n}$ in (26) we get the particle/antiparticle density matrix describing the final state of the evolution for the initial thermodynamic equilibrium in an external field. First attempt [16] to find such a density matrix was not completely satisfactory.

4 Distribution functions

Note that matrices $K_\pm$ in (26) are not diagonal in general case. Then, it is useful to define the following partition functions,

$$
\begin{aligned}
\mathcal{Z}_+(\bar{\jmath}, \jmath) &= Z_+^{-1}\mathrm{tr}_+\mathcal{N}_{out}\left(\exp\left\{-a^\dagger(out)\,(1 - IK_+)\,a(out)\right\}\right), \\
\mathcal{Z}_-(\bar{\jmath}, \jmath) &= Z_-^{-1}\mathrm{tr}_-\mathcal{N}_{out}\left(\exp\left\{-b^\dagger(out)\,(1 - IK_-)\,b(out)\right\}\right),
\end{aligned} \tag{28}
$$

where the matrix elements $I_{mn} = \delta_{mn} + \bar{\jmath}_m \jmath_n$, $\bar{\jmath}_m$ and $\jmath_n$ represent some new sources, and the normalization condition $\mathcal{Z}_\zeta(0,0) = 1$ holds. One can easily calculate the traces in (28) by using the formulas (85) and (86) accordingly to statistics and get

$$
\mathcal{Z}_\zeta(\bar{\jmath}, \jmath) = Z_\zeta^{-1}\exp\left\{\kappa\left[\ln\left(1 + \kappa IK_\zeta\right)\right]_{nn}\right\}. \tag{29}
$$

Let us consider the one-particle distributions

$$
\begin{aligned}
R^{(+)}_{nm} &= \mathrm{tr}_+\left\{a^\dagger_n(out)a_m(out)\hat{\rho}_+\right\}, \\
R^{(-)}_{nm} &= \mathrm{tr}_-\left\{b^\dagger_n(out)b_m(out)\hat{\rho}_-\right\}.
\end{aligned} \tag{30}
$$

The diagonal matrix elements of $R^{(\zeta)}$ represent the mean number of particles and antiparticles in the mode m at the final time instant

$$N_m^{(\zeta)}(out) = R_{nm}^{(\zeta)},\ n = m. \tag{31}$$

Using Wick's theorem one can rewrite (30) as the traces of the normal ordered operators,

$$\begin{aligned} R_{nm}^{(+)} &= \mathrm{tr}_+ \left\{ a_n^\dagger(out)\hat{\rho}_+ \left[K_+ a(out)\right]_m \right\}, \\ R_{nm}^{(-)} &= \mathrm{tr}_- \left\{ b_n^\dagger(out)\hat{\rho}_- \left[K_- b(out)\right]_m \right\}. \end{aligned} \tag{32}$$

The expressions (32) can be found from the partition functions (28) by the formula

$$R_{nm}^{(\zeta)} = \frac{\partial}{\partial \bar{\jmath}_n}\frac{\partial}{\partial \jmath_m} \mathcal{Z}_\zeta\left(\bar{x}, x\right)\big|_{\bar{\jmath}=\jmath=0}.$$

Then, from (29), we get the one-particle distributions in terms of the K_ζ matrices given in (26), as follows

$$R_{nm}^{(\zeta)} = \left(\frac{K_\zeta}{1+\kappa K_\zeta}\right)_{mn}. \tag{33}$$

This form for the matrix $R^{(\zeta)}$ is preferred when one needs to have the explicit expression via the elementary probability amplitudes (70).

Frequently, it is preferred to express the matrix $R^{(\zeta)}$ via the one-particle distribution function at the initial time instant,

$$\begin{aligned} R_{nm}^{(+)}(in) &= \mathrm{tr}\left\{ a_n^\dagger(in) a_m(in)\hat{\rho} \right\}, \\ R_{nm}^{(-)}(in) &= \mathrm{tr}\left\{ b_n^\dagger(in) b_m(in)\hat{\rho} \right\}. \end{aligned} \tag{34}$$

It is clear that $R_{nm}^{(\zeta)}(in) = \delta_{nm} N_m^{(\zeta)}(in)$, where

$$N_m^{(\zeta)}(in) = \left(1/J_m^{(\zeta)} + \kappa\right)^{-1} \tag{35}$$

is the generating function for mean number of particles/antiparticles in the mode m at the initial time instant. To express the $R^{(\zeta)}$ via the $R^{(\zeta)}(in)$ one can represent the first ones as the following

$$\begin{aligned} R_{nm}^{(+)} &= \mathrm{tr}\left\{ a_n^\dagger(out) a_m(out)\hat{\rho} \right\}, \\ R_{nm}^{(-)} &= \mathrm{tr}\left\{ b_n^\dagger(out) b_m(out)\hat{\rho} \right\}. \end{aligned}$$

Then, using the canonical transformations (67) and calculating the traces one finds the desired expression,

$$\begin{aligned} R^{(+)T} &= G\left({}^+|_+\right) R^{(+)}(in) G\left({}_+|^+\right) + G\left({}^+|_-\right)\left[1-\kappa R^{(-)}(in)\right] G\left({}_-|^+\right), \\ R^{(-)} &= G\left({}^-|_-\right) R^{(-)}(in) G\left({}_-|^-\right) + G\left({}^-|_+\right)\left[1-\kappa R^{(+)}(in)\right] G\left({}_+|^-\right) \end{aligned} \tag{36}$$

As a special case of the vacuum initial state, one has from (36) the well-known expressions [7] for the distributions of the particles and antiparticles created by an external field,

$$R_m^{(+)cr} = \left[G\left({}^+|_-\right)G\left({}_-|^+\right)\right]_{nm}, \; R_m^{(-)cr} = \left[G\left({}^-|_+\right)G\left({}_+|^-\right)\right]_{nm}, \; n=m. \quad (37)$$

Note that solving relation (33) with respect to the matrix K_ζ we get

$$K_\zeta = \frac{R^{(\zeta)T}}{1-\kappa R^{(\zeta)T}}. \quad (38)$$

With relations (36) and (38) we have the expressions of the matrix densities (26) via the initial distribution $R^{(\zeta)}(in)$. For that case we can rewrite Z_ζ in terms of the K_ζ as

$$Z_\zeta = \exp\left\{\kappa\left[\ln\left(1+\kappa K_\zeta\right)\right]_{nn}\right\}.$$

5 Quasiconstant fields

Let us discuss some applications of the presented general formalism. Supposing that the states can be specified by eigenvalues of the integrals of motion (the same for particles and antiparticles) we have that all the matrices $G\left({}_\zeta|^{\zeta'}\right)$ are diagonal, and the mean numbers (37) of the particles and antiparticles created from the vacuum are the same, $R_m^{(+)cr} = R_m^{(-)cr} = R_m^{cr}$. In this case the distributions are considerably simplified. It is the case where we have, for example, an uniform external field. Then from the formulas (31), (36), (37) and the unitarity relations (65) the mean number of (anti)particles in the mode m at the final time instant is

$$N_m^{(\zeta)}(out) = \left(1-\kappa R_m^{cr}\right)N_m^{(\zeta)}(in) + R_m^{cr}\left(1-\kappa N_m^{(-\zeta)}(in)\right). \quad (39)$$

If the initial state is different from the vacuum, the distribution of the (anti)particles created is defined by the difference

$$N_m^{(\zeta)cr} = N_m^{(\zeta)}(out) - N_m^{(\zeta)}(in). \quad (40)$$

Thus we have $N_m^{(+)cr} = N_m^{(-)cr} = N_m^{cr}$ and

$$N_m^{cr} = R_m^{cr}\left[1-\kappa\left(N_m^{(+)}(in) + N_m^{(-)}(in)\right)\right]. \quad (41)$$

It is clear that only for fermions ($\kappa=+1$) N_m^{cr} can change sign as a function of the initial distributions. The mean number of (anti)particles in the mode m is conserved in an external field disturbing a vacuum stability ($R_m^{cr} \neq 0$) if $N_m^{(+)}(in) + N_m^{(-)}(in) = 1$. For bosons ($\kappa = -1$) the mean number of (anti)particles in the final state always grows in such a field since $N_m^{cr} > 0$. Note that the presence of a matter in the initial state increases the mean number of the bosons created, N_m^{cr}. At large $N_m^{(\zeta)}(in)$ the increment of bosons, N_m^{cr}, is many times more than the mean number of the bosons created from vacuum, R_m^{cr}. Nevertheless, the relative increment is a decreasing function of the initial distributions,

$$\frac{N_m^{cr}}{N_m^{(+)}(in)+N_m^{(-)}(in)} = R_m^{cr}\left[1+\frac{1}{N_m^{(+)}(in)+N_m^{(-)}(in)}\right].$$

Let us consider examples with quasiconstant uniform field creating pairs. For simplicity sake assume that any other external fields are absent. Then states are specified by continuous quantum numbers of momentum projections $\mathbf{p}$ and spin projections $r=\pm 1$ (for bosons it is formally assumed a value $r=0$). From now on we will assume that the standard volume regularization is used, so that $\delta(\mathbf{p}-\mathbf{p}')\to\delta_{\mathbf{p},\mathbf{p}'}$ for the condition of normalization. Thus we have the set of discrete quantum numbers $m=(\mathbf{p},r)$.

5.1 Electric field

The state of the quantum system in question is far-from-equilibrium due to the influence of the time dependent potential of an electric field. On the other hand it is well known that the distribution of pairs created from the vacuum by a constant electric field, R_m^{cr}, is parametrized by only one parameter and, in this sense, can be compared with a thermal distribution. Nevertheless, the problem of time dependence exists and must be discussed. Let us remind that, in a physically correct statement of the problem, we only refer to a quasiconstant electric field which is effectively acting during a finite time. In order to analyze the time dependence of particle creation effects let us consider the two quasi-constant examples of a uniform electric field. This field is nonstationary, but with a constant direction in space. Then one can always direct it along the x^3 axis. The charge of the particle is q and its mass is M. Let us first consider the case wihere the strength $E(x^0)$ has the form

$$E(x^0)=\begin{cases}0, & x^0\in I\\ E, & x^0\in II\\ 0, & x^0\in III\,,\end{cases} \tag{42}$$

where the time intervals are: $I=(-\infty,t_1)$, $II=[t_1,t_2]$, $III=(t_2,+\infty)$, $t_2-t_1=T$, $t_2=-t_1$, with $qE>0$. Thus, in fact, we consider a constant electric field E, which is acting during finite time T. Further we will call it a T-constant field. The corresponding potential $A_3(x^0)$ can be chosen in the form

$$A_3(x^0)=\begin{cases}Et_1, & x^0\in I\\ Ex^0, & x^0\in II\\ Et_2, & x^0\in III\,.\end{cases}$$

If the time T is sufficiently large: $\sqrt{qE}T/2\gg 1$, $\sqrt{qE}T/2\gg\lambda$, and

$qET/2 \gg |p_3|$, then the distribution function R_m^{cr} is [6]

$$R_m^{cr} = e^{-\pi\lambda}\left[1 + O\left(\left[\frac{1+\lambda}{\xi_1}\right]^3\right) + O\left(\left[\frac{1+\lambda}{\xi_{-1}}\right]^3\right)\right], \quad -\sqrt{qE}\frac{T}{2} \le \xi_1 \le -K. \tag{43}$$

where K is a given number $K >> 1 + \lambda$ and

$$\lambda = \frac{M^2 + \mathbf{p}_\perp^2}{qE}, \; \mathbf{p}_\perp = (p^1, p^2, 0), \; \xi_{\pm 1} = (\mp qET/2 - p_3)/\sqrt{qE}. \tag{44}$$

The distribution R_m^{cr} for large longitudinal momenta $|p_3|$ decreases, and for $|p_3| >> qET/2$ it rapidly decreases rapidly according to the power law $R_m^{cr} = O\left(\left[\lambda/\xi_1^{-2}\right]^3\right)$. The latter expression allows one to consider the limit $T \to \infty$ at any given $\mathbf{p}$. In this limit the distribution function takes the simple form

$$R_m^{cr} = e^{-\pi\lambda} \tag{45}$$

and coincides with the expressions obtained in the constant electric field [4]. One can see that the stabilization of the distribution function in the asymptotic form (45) for finite longitudinal momenta is reached at $T >> T_0$, where $T_0 = (1+\lambda)/\sqrt{qE}$. The characteristic time T_0 is called stabilization time. It is clear that the effect can actually be observed as soon as the external field strength approaches the characteristic value (critical field) $E_c = M^2/|q|$.

Let us consider another example of a quasiconstant electric field. This field switches on and off adiabatically at $x^0 \to \pm\infty$, and is quasiconstant at finite times. In this case, the function $E(x^0)$ has the following form

$$E(x^0) = E\cosh^{-2}\left(\frac{x^0}{\alpha}\right). \tag{46}$$

The corresponding nonzero potential is

$$A_3(x^0) = \alpha E \tanh\frac{x^0}{\alpha}.$$

We will call it adiabatic field. The distribution function was found in [4]. We only consider a large α parameter, $\alpha >> (1+\sqrt{\lambda})/\sqrt{qE}$. Then the distribution function R_m^{cr} for fermions and bosons have a form,

$$R_m^{cr} = \exp\left\{-\pi\alpha(\varpi_+ + \varpi_- - 2qE\alpha)\right\}, \tag{47}$$

where $\varpi_\pm = \sqrt{M^2 + \mathbf{p}_\perp^2 + (p_3 \mp qE\alpha)^2}$. Let us take small longitudinal momenta $|p_3| << qE\alpha$, so that

$$R_m^{cr} = \exp\left\{-\pi\lambda\left[1 + \left(\frac{p_3}{qE\alpha}\right)^2\right]\right\}. \tag{48}$$

Considering the limit $\alpha \to \infty$, one gets the formula (45), which means that the effects of switching on and off are not essential at large times and finite longitudinal momenta. For large longitudinal momenta $|p_3| >> qE\alpha$, the distribution function are exponentially small,

$$R_m^{cr} = \exp\left\{-2\pi\alpha(|p_3| - qE\alpha)\right\}. \tag{49}$$

The stabilization of the distribution function in the adiabatic field in the asymptotic form (45) comes for longitudinal momenta $|p_3| << qE\alpha$ at $\alpha >> \alpha_0$, $\alpha_0 = (1+\sqrt{\lambda})/\sqrt{qE}$. For large α the adiabatic field varies slowly and nearly coincides with the constant one in the time interval $|x^0| \leq \alpha$. Then α_0 is a characteristic time of the stabilization in this field. Thus, the stabilization time α_0 in the adiabatic field differs from the corresponding time T_0 in the T-constant field, and one can believe that the stabilization process depends on the switching effects. In the case $E/E_c < 1$, which corresponds to $\lambda > 1$, one can see the stabilization comes quicker for the adiabatic field than for the T-constant one ($\alpha_0 < T_0$), i.e. the quantum system is less affected by the adiabatic form of switching. If $E/E_c \geq 1$, there exists a domain of the transversal momenta $\mathbf{p}_\perp$ where $\lambda \leq 1$. In this case, the stabilization times in both cases are the same, $\alpha_0 \sim T_0 \sim 1/\sqrt{qE}$, so that for any E the relation $\alpha_0 \leq T_0$ holds. We see that for $T >> T_0$ the effects of switching on and off are negligible. By comparing the distribution function for the T-constant and adiabatic fields one can interpret $T_0^f = \sqrt{\lambda/qE}$ as the time of pair formation. A semiclassical consideration confirms this interpretation. Thus, a virtual particle with initial zero energy gets from the electric field by the time T_0^f the energy $\sqrt{M^2 + \mathbf{p}_\perp^2}$ necessary for the materialization. It is easy to see that the time T_0^f is always either less than the stabilization times T_0, α_0 or equal to them.

5.2 Effective temperature parameter

Using the obtained $\hat{\rho}_\pm^0$ density matrix description we can establish a relation between the distribution of pairs created from the vacuum by a slowly varying electric-like field and the Hawking's thermal distribution of particles created by the static gravitational field of a black hole. Due to the time dependence of the potential $A_3(x^0)$, which defines the quasiconstant (slowly varying for relatively large period of time T) electric field, the level of the vacuum energy changes with time. Taking into account such a shift of the vacuum energy in those states which remain vacuum states, the universal form of the distribution function was found [6],

$$R_m^{cr} = \exp\left\{-2\pi\frac{\omega}{g}\right\}, \tag{50}$$

where 2ω is the work which the external field accomplishes for the creation of a pair in a given state. The corresponding work with respect to a particle is ω,

$$\omega = \frac{1}{2}\left[p_0(t_f) + p_0(t_i) + \Delta\epsilon_{vac}\right],$$

where $p_0(t_f)$ and $p_0(t_i)$ are the particle energies at the final time instant t_f and at the initial time instant t_i, correspondingly, and $\Delta\epsilon_{vac}$ is a contribution due to an evolution shift of the vacuum energy. The quantity g is the classical acceleration of a particle in the final time instant. For the quasiconstant electric field one has

$$\begin{aligned} p_0\left(t_{f/i}\right) &= \sqrt{M^2+\mathbf{p}_\perp^2+(p_3-qA_3(t_{f/i}))^2}, \\ \Delta\epsilon_{vac} &= -\left|qA_3(t_i)-qA_3(t_f)\right|. \end{aligned}$$

In the case of the T-constant field $t_{f/i} = \pm T/2$ and $A_3(t_{f/i}) = \pm ET/2$. Then, due to the conditions of the stabilization $T >> T_0$ and $|p_3| \ll qET/2$, one finds

$$\omega = \frac{\lambda qE}{2p_0(t_f)} = \frac{\lambda}{T}, \quad g = \frac{qE}{p_0(t_f)} = \frac{2}{T}. \tag{51}$$

The adiabatic field for the finite period of time T from $t_i = -T/2$ to $t_f = T/2$ coincides with the T-constant field if condition $T/2\alpha \ll 1$ holds. For $T >> T_0$ the effects of switching on and off are negligible. Then it is natural we have the same result (51) for ω and g.

It is of interest to compare the particle creation in external electromagnetic fields and in external fields of a different nature, for example, in external gravitational fields. To this end one can use the results obtained in the quasiconstant electric fields and in the static gravitational fields. The latter problem was considered first by Hawking [11] who, in particular, calculated the distribution of particles created by the static gravitational field of a black hole with mass M in a specific thermal environment providing equilibrium,

$$R_m^{cr} = \left[\exp\left\{2\pi\frac{\omega}{g_{(H)}}\right\}+\kappa\right]^{-1}, \tag{52}$$

where ω is the energy of the created particle , which are supposed to be dependent on a complete set of quantum numbers m, and $g_{(H)} = \frac{GM}{r_g^2}$, r_g being the gravitational radius, so that $g_{(H)}$ is free falling acceleration at this radius. This spectrum was interpreted as a Planck distribution with the temperature $\theta_{(H)} = \frac{g_{(H)}}{2\pi k_B}$ (k_B is the Boltzmann constant). As before $\kappa = +1$ for fermions and $\kappa = -1$ for bosons. It is also known [45] that an observer, which is moving with a constant acceleration $g_{(R)}$ (with respect to its proper time), will probably register in the Minkowski vacuum some particles (Rindler particles). However, there is another opinion (see [46]) that the existence of the Unruh effect is an open problem. The mean numbers of Rindler bosons have the same Planck form (52) (with $\kappa = -1$), where one has to replace $g_{(H)}$ by $g_{(R)}$, so that the correspondent temperature is $\theta_{(R)} = \frac{g_{(R)}}{2\pi k_B}$. One can find many other examples where the particle creation in external quasistatic gravitation fields (and due to a nontrivial topology) can be described by means of an effective temperature [8].

In the case of a quasistatic gravitation field the evolution shift of the vacuum energy is $\Delta\epsilon_{vac} = 0$ so that one identifies the work ω, we have introduced, with the energy of a particle in formula (52). Thus we see that the distribution (50) is, in fact, the Boltzmann one with the temperature $\theta = \frac{g}{2\pi k_B}$ having literally the Hawking form. It is a direct consequence of the equivalence principle since we can compare equations of motion for a classical particle in a constant electric field $d\pi/dx^0 = q\mathbf{E}$ with the ones in the static gravitational field $d\pi/dx^0 = \omega\mathbf{g}$. In the latter, ω is the total energy of the test particle and $\mathbf{g}$ is the three-dimensional gravitational field strength vector. Let us discuss now the possible origin of the differences in the electrodynamical and gravitational formulas. First of all, formula (52) is derived in the formalism of the stationary scattering theory, where it is not necessary to take separately into account a shift of the vacuum level. In this case the energy of the created particle may coincide with the corresponding work of the field. Second, the different form of the thermal distributions (Boltzmann, Planck) can be stipulated by essentially different situations in both cases. We believe that in the gravitational problems the Planck distribution arises necessarily due to the horizon of events formation (there is a boundary of the domain of the Hamiltonian), that is, due to the condition in which the space domain of the particle vacuum and the antiparticle vacuum are not the same. On the other hand, the final state can be treated as an equilibrium state. In contrast, in the uniform electric field we deal in fact with both the particle vacuum and the antiparticle vacuum defined over all space, that is, these space domains coincide. In this case, the mixed state of particles (antiparticles) described by the $\hat{\rho}^0_+$ ($\hat{\rho}^0_-$) density matrix can be represented as a pure state in an extended phase space where the space domains for both the particle vacuum and the antiparticle vacuum are the same, and is the state of a far-from-equilibrium system. In the strict sense, the Boltzmann distribution of the relativistic particles created is not but thermal-like (the relativistic thermal distribution is, of course, the Planck one). However, at $\omega/g << 1$ the Boltzmann spectrum closely approximates the Planck's one. In this case, one can believe that the form (50) for the distribution function of created particles is universal and applicable to any theory with quasiconstant external fields.

5.3 Metric field

An important example of stabilization of the distribution function of created pairs can be presented in the context of a cosmological model which can be reformulated as a quasiconstant field model. Let us consider massive scalar and spinor fields in the expanding Friedmann-Robertson-Walker (FRW) universe with a scale factor $\Omega(\eta)$ obeying (in terms of the conformal time η) $\Omega^2(\eta) = b^2\eta^2 + a^2$. Such a scale factor corresponds to the expanding radiation-dominated FRW Universe. In terms of the physical time t the corresponding metric may be written as follows:

$$ds^2 = dt^2 - \Omega^2(t)(dx^2 + dy^2 + dz^2)\,, \tag{53}$$

where for small times $|t| \ll a^2/b$, $\Omega^2(t) \simeq a^2[1 + (bt/a^2)^2]$, and for large times $|t| \gg a^2/b$, $\Omega^2(t) \simeq 2b|t|$ (see [43]). Making a conformal transformation, we are led to quantum electrodynamics in a flat background with time coordinate $x^0 = \eta$, but with a time-dependent mass (QED-Ω theory), $M\,\Omega(\eta)$. That is, the term $M\,\Omega(\eta)$ can be treated as a time dependent potential well. On the other side the term bMx^0 can be treated as an electric-like field potential from extradimensions for a theory with mass aM, where bM plays the role of qE. The distribution function for fermions and bosons created at $x_2^0 = T/2$ from the in-vacuum at $x_1^0 = -T/2$ has the form [42, 43, 44]

$$R_m^{cr} = e^{-\pi\tilde{\lambda}},\ \tilde{\lambda} = \left(a^2M^2 + \mathbf{p}^2\right)/bM, \tag{54}$$

when the time $T = x_2^0 - x_1^0$ is large enough: $\sqrt{bM}\,T \gg 1$, $\sqrt{bM}T \gg \tilde{\lambda}$.

Another example of a thermal distribution of created particles can also be presented in the context of the FRW cosmological model. Following the analysis in [6] and using the above interpretation of bMx^0 (see details in [44]) we can rewrite (54) in the Boltzmann distribution form (50) with

$$\omega = \frac{\tilde{\lambda}}{T},\ g = \frac{2}{T}.$$

Here the quantity g is calculated as the extradimensional classical acceleration of a particle in the final time instant. In $3+1$ dimensions g is the specific power at the final time instant,

$$g = \frac{1}{M}\frac{dp_0}{dx^0} = \left.\frac{bM}{p_0}\right|_{x^0=T/2},$$

where $p_0 = \sqrt{M^2\Omega^2(x^0) + \mathbf{p}^2}$. Thus we see that the effective temperature of particles created due to expansion of the FRW Universe is $\theta = \frac{g}{2\pi k_B}$, i.e., the same as in the case of quantum electrodynamics in the T-constant electric field.

5.4 Multiple particle creation

As we noted in the introduction, the typical scenario with an application of the Schwinger mechanism in the modern QFT is the chromoelectric flux tube model [24]. In this model, if further interactions are absent, the back reaction of created pairs induces mean field and plasma oscillations [27]. Depending on details of the model, its stage and the field strength, both the pair production at finite temperature and at the zero temperature can be relevant (see e.g [28, 25]). The consideration of various time scales in the heavy-ion collisions shows that the time scale of the stabilization T_0 is far less than the period of the mean field and plasma oscillations. Thus, it may be reasonable to neglect the dynamic backreaction effects and thermalization, and to determinate the pair production distribution at zero temperature but in the presence of the pairs created at the previous stages. In other words, we can consider the pair creation from the initial generalized coherent state given by the distribution of particles previously created. The formula (41) is relevant in this analysis..

Starting from the initial vacuum state one has $N_m^{cr} = R_m^{cr}$. Then, at the end of the first stage, when the mean field is depleted for the first time, the distribution of particles (it is equal for antiparticles) is

$$N_m^{(1)} = R_m^{cr}.$$

During the second stage, the direction of the mean field is opposite to the field direction at the first stage. Due to the condition of stabilization it is of no importance since the R_m^{cr} is an even function of qE. Thus, when the mean field is depleted for the second time, using (41) one has the relation

$$N_m^{(2)} = R_m^{cr} + \left(1 - 2\kappa R_m^{cr}\right) N_m^{(1)},$$

and at the end of the n stage

$$N_m^{(n)} = R_m^{cr} + \left(1 - 2\kappa R_m^{cr}\right) N_m^{(n-1)}.$$

Consequently, the total number of the particles created at the end of the n stage is

$$N_m^{(n)} = R_m^{cr} \sum_{l=0}^{n-1} \left(1 - 2\kappa R_m^{cr}\right)^l .$$

We have this result if the created particles do not leave the area of the acting field. To take into account a possible loss of particles due to the interaction, movement etc. we assume that the total number of particles in the initial state of the n stage is less than the $N^{(n-1)}$ number of particles created at the end of the $n-1$ stage and is $\gamma N^{(n-1)}$, where $\gamma < 1$ is a loss factor. Then the modified relation is

$$N_m^{(n)} = R_m^{cr} + \left(1 - 2\kappa R_m^{cr}\right) \gamma N_m^{(n-1)}, \tag{55}$$

and we finally have

$$N_m^{(n)} = R_m^{cr} \sum_{l=0}^{n-1} \gamma^l \left(1 - 2\kappa R_m^{cr}\right)^l . \tag{56}$$

Supposing that γ is a constant, one can calculate the sum in (56):

$$N_m^{(n)} = R_m^{cr} \frac{1 - r^n}{1 - r}, \; r = \gamma \left(1 - 2\kappa R_m^{cr}\right). \tag{57}$$

For fermions $\kappa = +1$ then $N_m^{(n)} \leq 1$. The energy dissipation after a period of oscillation is estimated (for real parameters of heavy-ion collisions) not to being large so that the damping is small and the number of oscillation can be quite large; the damping decreases with an increasing field stength. If the number of cycles is sufficiently large we get the limiting Planck-like distribution (see the interpretation of λ in terms of characteristic temperature in subsection 6.2)

$$N_m^{\Sigma} = \frac{R_m^{cr}}{1 - \gamma \left(1 - 2\kappa R_m^{cr}\right)} = \frac{1}{(1-\gamma)} \cdot \frac{1}{e^{\pi\lambda} + \kappa 2\gamma / (1-\gamma)} \tag{58}$$

In other words, the system reaches a quasiequilibrium state. For bosons $\kappa = -1$ then $N_m^{(n)}$ grows. This is a resonance phenomenon and the increase can be either limited or unlimited depending on the factor γ. The increase is limited as long as $r < 1$. In this case formula (58) is valid for bosons, as well. We see that the back reaction induced plasma oscillations can reach a quasistationary form specified by the quasithermal distribution both for fermions and bosons.

6 Time-depending rates at finite temperature

We are ready now to present explicitly, for example, the mean number of (anti)particles in the mode m (with the finite longitudinal momenta, $|p_3| \leq \sqrt{qE}\left(\sqrt{qE}T/2 - K\right)$) for the final state of evolution in a quasiconstant field from the initial thermodynamic equilibrium, $N_m^{(\zeta)}(in) = \left(e^{E_m} + \kappa\right)^{-1}$, at equal chemical potentials $\mu^{(+)} = \mu^{(-)} = \mu$, ($\mu < M$ for bosons),

$$N_m^{(\zeta)}(out) = \left(e^{E_m} + \kappa\right)^{-1} + e^{-\pi\lambda}\left(\tanh\left(E_m/2\right)\right)^{\kappa} \tag{59}$$

where $E_m = \beta\left(\varepsilon_m - \mu\right)$, $\varepsilon_m = \sqrt{M^2 + \mathbf{p}_\perp^2 + \left(\pi_3\right)^2}$, $\pi_3 = p_3 + qET/2$, and we mean that λ is given by (44) for the electric field and by $\tilde{\lambda}$ from (54) for the FRW universe. This result for the electric field coincides with one obtained in [16]. Due to the effect of stabilization it seems that a time dependence of the final distributions in question is absent. However, the integral mean numbers vary as long as a quasiconstant field acts.

It is of interest to establish some general behaviour of the integral mean numbers of created particles when the effects of switching on and off are negligible. As shown above, we can satisfy this condition selecting the action time T of the T-constant field ($T >> T_0$) as an effective period of pair creation. We mean that, in general, the final time instant, t_f, and the initial time instant, t_i, are so selected that a quasiconstant field is closely approximated by the T-constant field for a period from t_i to t_f, and $t_f - t_i = T$.

Let us estimate the sum over the longitudinal momentum p_3 of N_m^{cr}, which is the mean number of particles created with all possible p_3 values. In order to treat the integral mean numbers we go over to the continuum, $\sum_{\mathbf{p}} \to \frac{V}{(2\pi)^3}\int d\mathbf{p}$, where V is the space volume. As above, at $T >> T_0$, R_m^{cr} is quasiconstant in the range $|p_3| \leq \sqrt{qE}\left(\sqrt{qE}T/2 - K\right)$ and decreases rapidly for $|p_3| > \sqrt{qE}\left(\sqrt{qE}T/2 + K\right)$. The contribution to the integral from the region $\sqrt{qE}\left(\sqrt{qE}T/2 - K\right) \leq |p_3| \leq \sqrt{qE}\left(\sqrt{qE}T/2 + K\right)$ is less then $2\sqrt{qE}K$. Then, one can conclude that the integral density distribution of particles created with all possible p_3 is finite and can be presented as

$$\begin{aligned} n_{\mathbf{p}_\perp,r}^{cr} &= \frac{V}{(2\pi)^3}\int_{-\infty}^{\infty} N_m^{cr} dp_3 = \frac{V}{(2\pi)^3}\left[e^{-\pi\lambda}\int_{-qET/2}^{qET/2} n_m(\beta) dp_3 + \sqrt{qE}O(K)\right], \\ n_m(\beta) &= \left(\tanh\left(E_m/2\right)\right)^{\kappa}. \end{aligned} \tag{60}$$

From (60) one can estimate the density distribution of the particle production rate,

$$\frac{dn^{cr}_{\mathbf{p}_\perp,r}}{dT} = \frac{qEV}{(2\pi)^3}\, n_m(\beta)|_{\pi_3=qET}\, e^{-\pi\lambda}. \tag{61}$$

We have $(qET)^2 \gg M^2 + \mathbf{p}_\perp^2$ according to the condition of the stabilization. Then, low temperature and high temperature limits for the production rate are only defined by the final longitudinal kinetic momentum qET and the temperature Θ relation: $\beta qET \gg 1$ and $\beta qET \ll 1$, respectively. For simplicity, assume that $qET \gg \mu$. Considering these limits one has for the temperature dependent term in (61) that

$$\begin{aligned} n_m(\beta)|_{\pi_3=qET} &= 1 - 2\kappa e^{-\beta qET},\ \beta qET \gg 1, \\ n_m(\beta)|_{\pi_3=qET} &= [\beta qET/2]^\kappa,\ \beta qET \ll 1. \end{aligned}$$

We see that at high temperatures the rate $\frac{dn^{cr}_{\mathbf{p}_\perp,r}}{dT}$ is time dependent: it is much lower than the zero temperature value but increasing for fermions and rather higher than the zero temperature value but decreasing for bosons. Consequently, the frequently used notion of a number of particles created per unit of time makes sense only at low temperatures and in this limit it coincides with the zero temperature value of the production rate. We consider two temperature limits for the integral distribution density (60): low temperatures at $\beta(\varepsilon_\perp - \mu) \gg 1$, $\varepsilon_\perp = \sqrt{M^2 + \mathbf{p}_\perp^2}$, when all the energies of the particles created in the modes with a given $\mathbf{p}_\perp$ are rather higher than the temperature Θ, and high temperatures at $\beta qET \ll 1$, when all the energies of the particles created are much lower than the temperature Θ,

$$\begin{aligned} n^{cr}_{\mathbf{p}_\perp,r} &= \frac{V\sqrt{qE}}{(2\pi)^3}\left[\sqrt{qE}Te^{-\pi\lambda} + O(K)\right],\ \kappa = \pm 1,\ \beta(\varepsilon_\perp - \mu) \gg 1, \\ n^{cr}_{\mathbf{p}_\perp,r} &= \frac{V\beta qE}{(2\pi)^3}\left[qET^2/2 + O\left(\sqrt{qE}T\right)\right]e^{-\pi\lambda},\ \kappa = +1,\ \beta qET \ll 1, \\ n^{cr}_{\mathbf{p}_\perp,r} &= \frac{V\sqrt{qE}}{(2\pi)^3}\left[\frac{4}{\beta\sqrt{qE}}\ln\left(\sqrt{qE}T/K\right)e^{-\pi\lambda} + O(K)\right],\ \kappa = -1,\ \beta qET \ll 1. \end{aligned} \tag{62}$$

The result at low temperatures is not different from the zero temperature result [6] within the accuracy of the analysis. Integrating expresions (62) over $\mathbf{p}_\perp$ one finds the total number of particles created at low temperature and high temperature limits, respectively:

$$\begin{aligned} N^{cr} &= 2^{(\kappa+1)/2}\frac{V(qE)^2T}{(2\pi)^3}e^{-\pi M^2/qE},\ \beta(M-\mu) \gg 1, \\ N^{cr} &= \frac{V\beta(qE)^3T^2}{(2\pi)^3}e^{-\pi M^2/qE},\ \kappa = +1,\ \beta qET \ll 1, \\ N^{cr} &= \frac{VqE\ln\left(\sqrt{qE}T\right)}{2\pi^3\beta}e^{-\pi M^2/qE},\ \kappa = -1,\ \beta qET \ll 1, \end{aligned} \tag{63}$$

where the summation over $r = \pm 1$ is carried out for the fermions, and only the leading T dependent terms are shown. From (62),(63) we can see that the values of the integral mean numbers for fermions at high temperatures are much lower than the corresponding values at low temperatures. For bosons, the integral mean numbers at high temperatures are rather higher than the corresponding values at low temperatures.

As mentioned in the introduction, thermally influenced pair production in a constant electric field has been searched via several approaches with extremely contrary results, varying from the absence of the creation to values of the fermion production rate higher than the rate at the zero temperature. Now we are ready to discuss such contradictions. As shown above, the initial thermal distribution affects the number of states in which pairs are created by the quasiconstant field. Hence, the pair production exists at any temperature and, in particular, the fermion production rate cannot be higher than the rate at the zero temperature by no way. Note that our calculations are based on the generalized Furry representation elaborated especially for the case of vacuum instabilty in accordance with basic principles of quantum field theory. On the other hand, all conclusions in [25, 31, 33, 34] about the pair production rate or/and the mean numbers of pairs created at non-zero temperatures are based on either the standard real-time or imaginary-time one-loop effective actions. However, such formalisms do not work in the presence of unstable modes. The real part of the standard effective action describes effects of a vacuum polarization and has nothing to say about the time dependent conduction current of created pairs. For example, it can be seen at the zero temperature. In this case the information about pair creation comes from the imaginary part of the standard effective action. The extension of real-time techniques for finite temperature quantum electrodynamics with unstable vacuum was presented in [36]. In this article one can see that the relevant Green functions in a constant electric field are quite different from the standard proper-time representation given by Schwinger. Then, the relevant real-time one-loop effective action must be different from the standard one[1]. The standard imaginary-time formalism was derived under the assumption of thermal equilibrium and the appearance of a contradiction with Pauli exclusion principle shows that the attempts of the generalization to a far-from-equilibrium system failed. The functional Schrödinger picture used in [32] to calculate the N^{cr} at high temperatures seems relevant. Its asymptotic expressions for N^{cr} at high temperatures agree with ours in (63).

7 Summary

For a quadratic Hamiltonian system in an time dependent external field pair creating we presented an effective calculation method and find the generating functional for the ensemble average at final time instant for any possible initial state, both pure and mixed. Averaging over the states of one of the subsystems (particles or antiparticles) we obtained a description in terms of the density

[1] We will present the relevant real-time one-loop effective action anywhere.

matrix defined for the remaining subsystem. We found the one-particle distribution functions and applied the formulated formalism to a number of the topical problems in slowly varying external fields: electric, chromoelectric, and metric of the expanding Friedmann-Robertson-Walker universe. We considered the multiple particle creation from the generalized coherent state presented by the distribution of particles previously created and found that the back reaction induced plasma oscillations can reach a quasistationary form specified by the quasithermal distribution; we also found the thermal-like density matrix description of particles created from the vacuum by a slowly varying electric-like field, establishing a relation between the corresponding distribution and Hawking's thermal distribution of particles created by the static gravittional field of a black hole. Finally, we discussed the time and temperature dependence of the integral mean numbers of created particles and found the correct expression at high temperatures.

Acknowledgement S.P.G. and J.L.T. are grateful to FAPESP. D.M.G. acknowledges the support of FAPESP, CNPq and DAAD.

Appendix

Canonical transformations between in- and out-operators

I. One can find decomposition coefficients $G\left({}_{\zeta}|{}^{\zeta'}\right)$ of the out-solutions in the in-solutions, and vice versa, for the one-particle equations of motion in the presence of external fields (from now on we will as a rule drop the suffices of quantum numbers),

$$\begin{aligned} {}^{\zeta}\psi(x) &= {}_{+}\psi(x)G\left({}_{+}|{}^{\zeta}\right) + \kappa\ {}_{-}\psi(x)G\left({}_{-}|{}^{\zeta}\right)\ , \\ {}_{\zeta}\psi(x) &= {}^{+}\psi(x)G\left({}^{+}|{}_{\zeta}\right) + \kappa\ {}^{-}\psi(x)G\left({}^{-}|{}_{\zeta}\right),\ \zeta = \pm \end{aligned} \tag{64}$$

The matrices $G\left({}_{\zeta}|{}^{\zeta'}\right) = G\left({}^{\zeta'}|{}_{\zeta}\right)^{\dagger}$ obey the following relations,

$$\begin{aligned} &G\left({}_{\zeta}|{}^{+}\right)G\left({}_{\zeta}|{}^{+}\right)^{\dagger} + \kappa G\left({}_{\zeta}|{}^{-}\right)G\left({}_{\zeta}|{}^{-}\right)^{\dagger} = (\zeta\mathbf{I})^{\frac{1-\kappa}{2}}, \\ &G\left({}_{+}|{}^{+}\right)G\left({}_{-}|{}^{+}\right)^{\dagger} + \kappa G\left({}_{+}|{}^{-}\right)G\left({}_{-}|{}^{-}\right)^{\dagger} = 0\ , \\ &G\left({}_{+}|{}^{+}\right)^{\dagger}G\left({}_{+}|{}^{-}\right) + \kappa G\left({}_{-}|{}^{+}\right)^{\dagger}G\left({}_{-}|{}^{-}\right) = 0 \end{aligned} \tag{65}$$

where $\mathbf{I}$ is the unit matrix and $\kappa = +1$ for fermions and $\kappa = -1$ for bosons, respectively. Relations (65) can be derived from the orthonormality relations for ${}^{\zeta}\psi$ and ${}_{\zeta}\psi$.

Let us define $a_n^{\dagger}(in)$, $b_n^{\dagger}(in)$, $a_n(in)$, $b_n(in)$ as creation and annihilation operators of in-particles and in-antiparticles respectively and $a_n^{\dagger}(out)$,$b_n^{\dagger}(out)$,

$a_n(out)$, $b_n(out)$ as the ones of out-particles and out-antiparticles,

$$\begin{aligned}
\hat{\psi}(x) &= a(in)\,{}_{+}\psi(x) + b^{\dagger}(in)\,{}_{-}\psi(x), \\
\hat{\psi}^{\dagger}(x) &= a^{\dagger}(in)\,{}_{+}\psi^{\dagger}(x) + b(in)\,{}_{-}\psi^{\dagger}(x), \\
\hat{\psi}(x) &= a(out)\,{}^{+}\psi(x) + b^{\dagger}(out)\,{}^{-}\psi(x), \\
\hat{\psi}^{\dagger}(x) &= a^{\dagger}(out)\,{}^{+}\psi^{\dagger}(x) + b(out)\,{}^{-}\psi^{\dagger}(x),
\end{aligned} \tag{66}$$

From (66) one has the canonical transformations between in- and out- creation and annihilation operators (called Bogolyubov transformations) as follows:

$$\begin{aligned}
&a(out) = G\left({}^{+}|{}_{+}\right) a(in) + G\left({}^{+}|{}_{-}\right) b^{\dagger}(in), \\
&b^{\dagger}(out) = G\left({}^{-}|{}_{+}\right) a(in) + G\left({}^{-}|{}_{-}\right) b^{\dagger}(in), \\
&a^{\dagger}(out) = a^{\dagger}(in) G\left({}_{+}|{}^{+}\right) + b(in) G\left({}_{-}|{}^{+}\right), \\
&b(out) = a^{\dagger}(in) G\left({}_{+}|{}^{-}\right) + b(in) G\left({}_{-}|{}^{-}\right),
\end{aligned} \tag{67}$$

and

$$\begin{aligned}
a(in) &= G\left({}_{+}|{}^{+}\right) a(out) + G\left({}_{+}|{}^{-}\right) b^{\dagger}(out), \\
b^{\dagger}(in) &= G\left({}_{-}|{}^{+}\right) a(out) + G\left({}_{-}|{}^{-}\right) b^{\dagger}(out), \\
a^{\dagger}(in) &= a^{\dagger}(out) G\left({}^{+}|{}_{+}\right) + b(out) G\left({}^{-}|{}_{+}\right), \\
b(in) &= a^{\dagger}(out) G\left({}^{+}|{}_{-}\right) + b(out) G\left({}^{-}|{}_{-}\right),
\end{aligned} \tag{68}$$

We suppose this canonical transformations are proper transformations, that is, the condition

$$\mathrm{tr} G\left({}_{-}|{}^{+}\right) G\left({}_{-}|{}^{+}\right)^{\dagger} < \infty \,. \tag{69}$$

holds.

All the information about the processes of particle creation, annihilation, and scattering in an external field (without radiative corrections) can be extracted from the matrices $G\left({}_{\zeta}|{}^{\zeta'}\right)$ [7], expressing the elementary probability amplitudes,

$$\begin{aligned}
w\left(+|+\right)_{mn} &= w\left(\overset{+}{m}|\overset{+}{n}\right) = c_v^{-1} < 0, out\left|a_m(out) a_n^{\dagger}(in)\right| 0, in >= G^{-1}\left({}_{+}|{}^{+}\right)_{mn}, \\
w\left(-|-\right)_{nm} &= c_v^{-1} < 0, out\left|b_m(out) b_n^{\dagger}(in)\right| 0, in >= G^{-1}\left({}^{-}|{}_{-}\right)_{nm}, \\
w\left(0|-+\right)_{nm} &= c_v^{-1} < 0, out\left|b_n^{\dagger}(in) a_m^{\dagger}(in)\right| 0, in > \\
&= -\left[G^{-1}\left({}^{-}|{}_{-}\right) G\left({}^{-}|{}_{+}\right)\right]_{nm} = \kappa\left[G\left({}_{-}|{}^{+}\right) G^{-1}\left({}_{+}|{}^{+}\right)\right]_{nm}, \\
w\left(+-|0\right)_{mn} &= c_v^{-1} < 0, out\left|a_m(out) b_n(out)\right| 0, in > \\
&= \kappa\left[G^{-1}\left({}_{+}|{}^{+}\right) G\left({}_{+}|{}^{-}\right)\right]_{mn} = -\left[G\left({}^{+}|{}_{-}\right) G^{-1}\left({}^{-}|{}_{-}\right)\right]_{mn}.
\end{aligned} \tag{70}$$

where the in-vacuum and out-vacuum are defined by

$$\begin{aligned}
&a(in)|0, in> = b(in)|0, in>= 0, \\
&a(out)|0, out> = b(out)|0, out>= 0,
\end{aligned} \tag{71}$$

and a vacuum-vacuum transition amplitude is

$$c_v = < 0, out|0, in > . \tag{72}$$

The proper linear canonical transformations (68) can be represented as a unitary transformation [7]

$$\begin{pmatrix} a(in) \\ b^\dagger(in) \\ a^\dagger(in) \\ b(in) \end{pmatrix} = V \begin{pmatrix} a(out) \\ b^\dagger(out) \\ a^\dagger(out) \\ b(out) \end{pmatrix} V^\dagger \tag{73}$$

where $V^\dagger = V^{-1}$,

$$\begin{aligned} V \;=\; & \exp\left(-\kappa a^\dagger(out) w\,(+-|0)\, b^\dagger(out)\right) \exp\left(-\kappa b(out) \ln w\,(-|-)\, b^\dagger(out)\right) \\ & \exp\left(a^\dagger(out) \ln w\,(+|+)\, a(out)\right) \exp\left(-b(out) w\,(0|-+)\, a(out)\right). \end{aligned} \tag{74}$$

Using this expression, one can easily find

$$c_v = < 0, out|V|0, out > = \exp\left(-\kappa \mathrm{tr} \ln w\,(-|-)\right). \tag{75}$$

Some operator relations

II. Let $A_n^\dagger$ and A_n be a set of creation and annihilation operators. Consider the operator function

$$F(s) = e^{sA^\dagger DA}, \tag{76}$$

where $D = D_{nm}$ is a matrix, and s is a real parameter. At the same time this operator function can be defined as a solution of the following differential equation

$$\frac{dF(s)}{ds} = A^\dagger DAF(s), \tag{77}$$

with the initial condition $F(0) = 1$. Using the well-known formula,

$$e^{-sA^\dagger DA} A e^{sA^\dagger DA} = e^{sD} A \Longleftrightarrow AF(s) = F(s) e^{sD} A, \tag{78}$$

which hold true both for bosonic and fermionic cases, we may rewrite the equation (77) as follows

$$\frac{dF(s)}{ds} = A^\dagger F(s) D e^{sD} A. \tag{79}$$

Now one can easily see that the equation (79) has a solution

$$F(s) = \mathcal{N}\left[\exp\left\{A^\dagger\left(e^{sD} - 1\right) A\right\}\right], \tag{80}$$

where $\mathcal{N}$ is the symbol of the normal ordering. Both operators (76) and (80) obey the same equation and the same initial condition $F(0) = 1$. Thus, we justify the formula

$$\exp\left\{A^\dagger DA\right\} = \mathcal{N} \exp\left\{A^\dagger\left(e^D - 1\right) A\right\}, \tag{81}$$

which hold true both for bosonic and fermionic cases.

Wick's Theorem

III. The vacuum projection operator $P_0 = |0><0|$ can be written in the normal form,

$$|0><0| =: e^{-a^\dagger a} : \tag{82}$$

represented first by F.Berezin [37]. To see this note that the equations

$$aP_0 = 0,\ P_0 a^\dagger = 0,\ P_0|0>= |0>$$

hold for this operator. By using Wick's theorem one can verify that the operator $: e^{-a^\dagger a} :$ satisfies these equations.

IV. It is very effective to use Wick's theorem in a closed form [38]. Let us consider a T-ordering product of the number of creation and annihilation operators $a_n^\dagger(t)$ and $a_m(t')$ depending on the formal ordering of the "time" parameter t, where $t \neq t'$. This parameter t is only needed for ordering and is omitted after realization of the procedure, $a_n^\dagger(t) = a_n^\dagger$ and $a_m(t') = a_m$ for all t and t'. The vaccuum $|0>$ is defined by the condition $a_n|0>= 0$ for all n. According to canonical commutation relations of the operators $a_n^\dagger$ and a_m, the vacuum mean value for the T-ordering pair of $a_n^\dagger(t)$ and $a_m(t')$ is

$$\Delta_{mn}(t',t) =< 0|Ta_m(t')a_n^\dagger(t)|0>= \delta_{mn}\Theta(t'-t)\,.$$

In this case, the T-product can be expressed in the normal form by using the following functional form (Wick's theorem),

$$\begin{aligned} & Ta_{n_1}^\dagger(t_1)\dots a_{n_N}^\dagger(t_N)a_{m_1}(t_1')\dots a_{m_M}(t_M') \\ = \ & \exp\left\{\int \frac{\delta_r}{\delta a_m(t')}\Delta_{mn}(t',t)\frac{\delta_l}{\delta a_n^\dagger(t)}dtdt'\right\} : a_{n_1}^\dagger(t_1)\dots a_{n_N}^\dagger(t_N)a_{m_1}(t_1')\dots a_{m_M}(t_M') \end{aligned} \tag{83}$$

Theorem (83) is valid both for fermions and bosons.

Trace formulas

V. The trace of a normal product of creation and annihilation operators can be calculated by using the following path integral representation. Let $: X\left(a^\dagger, a\right) :$ be a normal product of creation and annihilation operators a and $a^\dagger$. The trace in the Fock space,

$$\operatorname{tr}\left\{: X\left(a^\dagger,a\right) :\right\} = \sum_{M=0}^{\infty}\sum_{\{m\}}(M!)^{-1} < 0|a_{m_M}\dots a_{m_1} : X\left(a^\dagger,a\right) : a_{m_1}^\dagger\dots a_{m_M}^\dagger|0>,$$

can be represented as the following vacuum mean value,

$$\operatorname{tr}\left\{: X\left(a^\dagger,a\right) :\right\} =< 0|T : X\left(a^\dagger,a\right) : e^{a(t_f)a^\dagger(t_i)}|0> \tag{84}$$

where $a = a(t)$, $a^\dagger = a^\dagger(t)$, and $t_f > t > t_i$. Let us consider the fermion case. Using the path integral representation (10), one can rewrite (84) as

$$\mathrm{tr}\left\{: X\left(a^\dagger, a\right) :\right\} =< 0| \int \exp\left\{\lambda^*\lambda + \lambda^* a\left(t_f\right)\right\} : X\left(a^\dagger, a\right) : \exp\left\{a^\dagger\left(t_i\right)\lambda\right\} \Pi d\lambda^* d\lambda |0 >$$

Then one uses Wick's theorem where three kind of a pairing appear:

$$\begin{aligned} \mathrm{tr}\left\{: X\left(a^\dagger, a\right) :\right\} &= \exp\left\{\frac{\partial_r}{\partial a\left(t_f\right)}\frac{\partial_l}{\partial a^\dagger} + \frac{\partial_r}{\partial a}\frac{\partial_l}{\partial a^\dagger\left(t_i\right)} + \frac{\partial_r}{\partial a(t_f)}\frac{\partial_l}{\partial a^\dagger\left(t_i\right)}\right\} \\ &< 0| : \int \exp\left\{\lambda^*\lambda + \lambda^* a\left(t_f\right) + a^\dagger\left(t_i\right)\lambda\right\} X\left(a^\dagger, a\right) : \Pi d\lambda^* d\lambda |0 > \end{aligned}$$

Calculating the derivatives with respect to $a(t_f)$ and $a^\dagger(t_i)$, one finds the convenient representation for the trace of the operator $: X\left(a^\dagger, a\right) :$ in the fermion case,

$$\begin{aligned} \mathrm{tr}\left\{: X\left(a^\dagger, a\right) :\right\} &= \exp\left\{\frac{\partial_r}{\partial a\left(t_f\right)}\frac{\partial_l}{\partial a^\dagger} + \frac{\partial_r}{\partial a}\frac{\partial_l}{\partial a^\dagger\left(t_i\right)}\right\} \\ &< 0| : \int \exp\left\{2\lambda^*\lambda + \lambda^* a\left(t_f\right) + a^\dagger\left(t_i\right)\lambda\right\} X\left(a^\dagger, a\right) : \Pi d\lambda^* d\lambda |0 \end{aligned} \quad (85)$$

For the boson case, starting from the path integral representation (11) and following in a similar way what was used for fermions, one finds

$$\begin{aligned} \mathrm{tr}\left\{: X\left(a^\dagger, a\right) :\right\} &= \exp\left\{\frac{\partial}{\partial a\left(t_f\right)}\frac{\partial}{\partial a^\dagger} + \frac{\partial}{\partial a}\frac{\partial}{\partial a^\dagger\left(t_i\right)}\right\} \\ &< 0| : \int \exp\left\{\varphi^* a\left(t_f\right) + a^\dagger\left(t_i\right)\varphi\right\} X\left(a^\dagger, a\right) : \Pi d\varphi^* d\varphi |0 \end{aligned} \quad (86)$$

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