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Cohomologies of the Poisson superalgebra of the Grassmann-valued functions on 2-dimensional space.

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Abstract

Cohomology spaces of the Poisson superalgebra realized on smooth Grassmann-valued functions with compact support on $\mathbb{R}^2$ are investigated under suitable continuity restrictions on cochains. The first and second cohomology spaces with trivial coefficients and the zeroth, first and second cohomology spaces with coefficients in the adjoint representation of the Poisson superalgebra are found for the case of a nondegenerate constant Poisson superbracket.

The hope to construct the quantum mechanics on nontrivial manifolds is connected with geometrical or deformation quantization [1] - [4]. The functions on the phase space are associated with the operators, and the product and the commutator of the operators are described by associative *-product and *-commutator of the functions. These *-product and *-commutator are the deformations of usual product and of usual Poisson bracket.

To find the deformations of Poisson superalgebra, one should calculate the second cohomology of the Poisson superalgebra.

In [6], the lower cohomologies (up to second) were calculated for the Poisson algebra consisting of smooth complex-valued functions on $\mathbb{R}^n$. The pure Grassmanian case $n = 0$ is considered in [7] and [8].

In [9], the lower cohomologies of the Poisson superalgebra in the trivial (up to third cohomology) and adjoint representation (up to second cohomology) for the case $n > 2$ are calculated. It occurred that the case $n = 2$ needs a separate consideration and an additional cohomologies arise in this case. The results of this consideration are presented in this publication. In [10] the central extensions of the algebras considered in this paper is considered.

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Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We denote by $\mathcal{D}(\mathbb{R}^n)$ the space of smooth $\mathbb{K}$-valued functions with compact support on $\mathbb{R}^n$. This space is endowed by its standard topology: by definition, a sequence $\varphi_k \in \mathcal{D}(\mathbb{R}^n)$ converges to $\varphi \in \mathcal{D}(\mathbb{R}^n)$ if $\partial^\lambda \varphi_k$ converge uniformly to $\partial^\lambda \varphi$ for every multi-index λ, and the supports of all φ_k are contained in a fixed compact set. We set

$$\mathbf{D}^{n_-}_{n_+} = \mathcal{D}(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}, \quad \mathbf{E}^{n_-}_{n_+} = C^\infty(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}, \quad \mathbf{D}'^{n_-}_{n_+} = \mathcal{D}'(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-},$$

where $\mathbb{G}^{n_-}$ is the Grassmann algebra with n_- generators and $\mathcal{D}'(\mathbb{R}^{n_+})$ is the space of continuous linear functionals on $\mathcal{D}(\mathbb{R}^{n_+})$. The generators of the Grassmann algebra (resp., the coordinates of the space $\mathbb{R}^{n_+}$) are denoted by ξ^α, $\alpha = 1, \ldots, n_-$ (resp., x^i, $i = 1, \ldots, n_+$). We shall also use collective variables z^A which are equal to x^A for $A = 1, \ldots, n_+$ and are equal to ξ^{A-n_+} for $A = n_+ + 1, \ldots, n_+ + n_-$. The spaces $\mathbf{D}^{n_-}_{n_+}$, $\mathbf{E}^{n_-}_{n_+}$, and $\mathbf{D}'^{n_-}_{n_+}$ possess a natural grading which is determined by that of the Grassmann algebra. The parity of an element f of these spaces is denoted by $\varepsilon(f)$. We also set $\varepsilon_A = 0$ for $A = 1, \ldots, n_+$ and $\varepsilon_A = 1$ for $A = n_+ + 1, \ldots, n_+ + n_-$.

Let $\partial/\partial z^A$ and $\overleftarrow{\partial}/\partial z^A$ be the operators of the left and right differentiation. The Poisson bracket is defined by the relation

$$\{f,g\}(z) = f(z)\frac{\overleftarrow{\partial}}{\partial z^A}\omega^{AB}\frac{\partial}{\partial z^B}g(z) = -\sigma(f,g)\{g,f\}(z), \tag{1}$$

where $\sigma(f,g) = (-1)^{\varepsilon(f)\varepsilon(g)}$ and the symplectic metric $\omega^{AB} = (-1)^{\varepsilon_A\varepsilon_B}\omega^{BA}$ is a constant invertible matrix. For definiteness, we choose it in the form

$$\omega^{AB} = \begin{pmatrix} \omega^{ij} & 0 \\ 0 & \lambda_\alpha\delta^{\alpha\beta} \end{pmatrix}, \quad \lambda_\alpha = \pm 1, \ i,j = 1, ..., n_+, \ \alpha, \beta = 1 + n_+, ..., n_- + n_+$$

where ω^{ij} is the canonical symplectic form (if $\mathbb{K} = \mathbb{C}$, then one can choose $\lambda_\alpha = 1$). The nondegeneracy of the matrix ω^{AB} implies, in particular, that n_+ is even. The Poisson superbracket satisfies the Jacobi identity

$$\sigma(f,h)\{f,\{g,h\}\}(z) + \mathrm{cycle}(f,g,h) = 0, \quad f,g,h \in \mathbf{E}^{n_-}_{n_+}. \tag{2}$$

By Poisson superalgebra, we mean the space $\mathbf{D}^{n_-}_{n_+}$ with the Poisson bracket (1) on it. The relations (1) and (2) show that this bracket indeed determines a Lie superalgebra structure on $\mathbf{D}^{n_-}_{n_+}$.

The integral on $\mathbf{D}^{n_-}_{n_+}$ is defined by the relation $\int \mathrm{d}z\, f(z) = \int_{\mathbb{R}^{n_+}} \mathrm{d}x \int \mathrm{d}\xi\, f(z)$, where the integral on the Grassmann algebra is normed by the condition $\int \mathrm{d}\xi\, \xi^1 \ldots \xi^{n_-} = 1$. We identify $\mathbb{G}^{n_-}$ with its dual space $\mathbb{G}'^{n_-}$ setting $f(g) = \int \mathrm{d}\xi\, f(\xi)g(\xi)$, $f, g \in \mathbb{G}^{n_-}$. Correspondingly, $\mathbf{D}'^{n_-}_{n_+}$, i.e., the space of continuous linear functionals on $\mathbf{D}^{n_-}_{n_+}$ is identified with the space $\mathcal{D}'(\mathbb{R}^{n_+}) \otimes \mathbb{G}^{n_-}$. As a rule, the value $m(f)$ of a functional $m \in \mathbf{D}'^{n_-}_{n_+}$ on a test function $f \in \mathbf{D}^{n_-}_{n_+}$ will be written in the "integral" form: $m(f) = \int \mathrm{d}z\, m(z)f(z)$.

Let L be a Lie superalgebra acting in a $\mathbb{Z}_2$-graded space V (the action of $f \in L$ on $v \in V$ will be denoted by $f \cdot v$). The space $C_p(L,V)$ of p-cochains consists of all multilinear super anti-symmetric mappings from L^p to V. The space $C_p(L,V)$ possesses a natural $\mathbb{Z}_2$-grading: by definition, $M_p \in C_p(L,V)$ has the definite parity $\varepsilon(M_p)$ if

$$\varepsilon(M_p(f_1, \ldots, f_p)) = \varepsilon(M_p) + \varepsilon(f_1) + \ldots + \varepsilon(f_1)$$

for any $f_j \in L$ with parities $\epsilon(f_j)$. The differential d_p^V is defined to be the linear operator from $C_p(L,V)$ to $C_{p+1}(L,V)$ such that

$$d_p^V M_p(f_1,...,f_{p+1}) = -\sum_{j=1}^{p+1}(-1)^{j+\varepsilon(f_j)|\varepsilon(f)|_{1,j-1}+\varepsilon(f_j)\varepsilon_{M_p}} f_j \cdot M_p(f_1,...,\check{f}_j,...,f_{p+1}) -$$
$$-\sum_{i<j}(-1)^{j+\varepsilon(f_j)|\varepsilon(f)|_{i+1,j-1}} M_p(f_1,...f_{i-1},\{f_i,f_j\},f_{i+1},...,\check{f}_j,...,f_{p+1}), \quad (3)$$

for any $M_p \in C_p(L,V)$ and $f_1,\dots f_{p+1} \in L$ having definite parities. Here the sign $\check{}$ means that the argument is omitted and the notation

$$|\varepsilon(f)|_{i,j} = \sum_{l=i}^{j}\varepsilon(f_l)$$

has been used. The differential d^V is nilpotent (see [5]), i.e., $\mathrm{d}_{p+1}^V\mathrm{d}_p^V = 0$ for any $p = 0,1,\dots$. The p-th cohomology space of the differential d_p^V will be denoted by H_V^p. The second cohomology space H_{ad}^2 in the adjoint representation is closely related to the problem of finding formal deformations of the Lie bracket $\{\cdot,\cdot\}$ of the form $\{f,g\}_* = \{f,g\} + \hbar^2\{f,g\}_1 + \dots$. The condition that $\{\cdot,\cdot\}$ is a 2-cocycle is equivalent to the Jacobi identity for $\{\cdot,\cdot\}_*$ modulo the $\hbar^4$-order terms.

In the case of Poisson algebra, the problem of finding deformations can hardly be solved in such a general setting, and additional restrictions on cochains (apart from linearity and anti-symmetry) are usually imposed. In some papers on deformation quantization it is supposed that the kernels of n-th order deformations $\{\cdot,\cdot\}_n$ are bidifferential operators. In the present paper, this requirement is replaced by the much weaker condition that cochains should be separately continuous multilinear mappings. It will be shown that this gives rise to additional cohomologies. We study the cohomologies of the Poisson algebra $\mathbf{D}_{n+}^{n-}$ in the following representations:

1. The trivial representation: $V = \mathbb{K}$, $f \cdot a = 0$ for any $f \in \mathbf{D}_{n+}^{n-}$ and $a \in \mathbb{K}$. The space $C_p(\mathbf{D}_{n+}^{n-},\mathbb{K})$ consists of separately continuous anti-symmetric multilinear forms on $(\mathbf{D}_{n+}^{n-})^p$. The cohomology spaces and the differentials will be denoted by H_{tr}^p and $\mathrm{d}_p^{\mathrm{tr}}$ respectively.

2. $V = \mathbf{D}_{n+}^{\prime n-}$ and $f \cdot m = \{f,m\}$ for any $f \in \mathbf{D}_{n+}^{n-}$ and $m \in \mathbf{D}_{n+}^{\prime n-}$. The space $C_p(\mathbf{D}_{n+}^{n-},\mathbf{D}_{n+}^{\prime n-})$ consists of separately continuous anti-symmetric multilinear mappings from $(\mathbf{D}_{n+}^{n-})^p$ to $\mathbf{D}_{n+}^{\prime n-}$. The continuity of $M \in C_p(\mathbf{D}_{n+}^{n-},\mathbf{D}_{n+}^{\prime n-})$ means that the $(p+1)$-form

$$(f_1,\dots,f_{p+1}) \to \int \mathrm{d}z\, M(f_1,\dots,f_p)(z) f_{p+1}(z)$$

on $(\mathbf{D}_{n+}^{n-})^{p+1}$ is separately continuous. The cohomology spaces will be denoted by $H_{\mathbf{D}'}^p$.

3. $V = \mathbf{E}_{n+}^{n-}$ and $f \cdot m = \{f,m\}$ for every $f \in \mathbf{D}_{n+}^{n-}$ and $m \in \mathbf{E}_{n+}^{n-}$. The space $C_p(\mathbf{D}_{n+}^{n-},\mathbf{E}_{n+}^{n-})$ is the subspace of $C_p(\mathbf{D}_{n+}^{n-},\mathbf{D}_{n+}^{\prime n-})$ consisting of mappings taking values in $\mathbf{E}_{n+}^{n-}$. The cohomology spaces will be denoted by $H_{\mathbf{E}}^p$.

4. The adjoint representation: $V = \mathbf{D}^{n_-}_{n_+}$ and $f \cdot g = \{f, g\}$ for any $f, g \in \mathbf{D}^{n_-}_{n_+}$. The space $C_p(\mathbf{D}^{n_-}_{n_+}, \mathbf{D}^{n_-}_{n_+})$ is the subspace of $C_p(\mathbf{D}^{n_-}_{n_+}, \mathbf{D}'^{n_-}_{n_+})$ consisting of mappings taking values in $\mathbf{D}^{n_-}_{n_+}$. The cohomology spaces and the differentials will be denoted by H^p_{ad} and $\mathrm{d}^{\mathrm{ad}}_p$ respectively.

For the representations 2 and 3, we shall denote the differentials by the same symbol $\mathrm{d}^{\mathrm{ad}}_p$ as in the adjoint representation. Note that in the case of the trivial representation, $\mathbb{K}$ is considered as a graded space whose odd subspace is trivial. We shall call p-cocycles $M^1_p, \dots M^k_p$ independent if they give rise to linearly independent elements in H^p. For a multilinear form M_p taking values in $\mathbf{D}^{n_-}_{n_+}$, $\mathbf{E}^{n_-}_{n_+}$, or $\mathbf{D}'^{n_-}_{n_+}$, we write $M_p(x|f_1, \dots, f_p)$ instead of more cumbersome $[M_p(f_1, \dots, f_p)](x)$.

Below we assume that $n_+ = 2$.

The results of the work are given by the following three theorems.

Theorem 1.

1. $H^1_{\mathrm{tr}} \simeq \mathbb{K}$; the linear form $M_1(f) = \bar{f} \stackrel{\mathrm{def}}{=} \int \mathrm{d}z\, f(z)$ is the nontrivial cocycle.
2. Let bilinear forms M^1_2 and M^2_2 be defined by the relations

$$M^1_2(f,g) = \bar{f}\bar{g}, \quad M^2_2(f,g) = \int dz z^A \left(\frac{\partial f(z)}{\partial z^A} g(z) - \sigma(f,g) \frac{\partial g(z)}{\partial z^A} f(z) \right), \quad f, g \in \mathbf{D}^{n_-}_2.$$

If n_- is even and $n_- \neq 6$, then $H^2_{\mathrm{tr}} = 0$;

if $n_- = 6$, then $H^2_{\mathrm{tr}} \simeq \mathbb{K}$ and the form M^2_2 is a nontrivial cocycle;

if n_- is odd, then $H^2_{\mathrm{tr}} \simeq \mathbb{K}$ and the form M^1_2 is a nontrivial cocycle.

Theorem 2.

1. $H^0_{\mathbf{D}'} \simeq H^0_{\mathbf{E}} \simeq \mathbb{K}$; the function $M_0(z) \equiv 1$ is a nontrivial cocycle.
2. $H^1_{\mathbf{D}'} \simeq H^1_{\mathbf{E}} \simeq \mathbb{K}^2$; independent nontrivial cocycles are given by

$$M^1_1(z|f) = \bar{f}, \quad M^2_1(z|f) = \left(1 - \frac{1}{2} z^A \frac{\partial}{\partial z^A}\right) f(z).$$

3. Let the bilinear mappings M^1_2, M^2_2, M^3_2, M^4_2 and M^5_2 from $(\mathbf{D}^{n_-}_2)^2$ to $\mathbf{E}^{n_-}_2$ be defined by the relations

$$\begin{aligned}
M^1_2(z|f,g) &= f(z) \left(\frac{\overleftarrow{\partial}}{\partial z^A} \omega^{AB} \frac{\partial}{\partial z^B} \right)^3 g(z), \\
M^2_2(z|f,g) &= \bar{f}\bar{g}, \\
M^3_2(z|f,g) &= \bar{g} M^2_1(z|f) - \sigma(f,g) \bar{f} M^2_1(z|g), \\
M^4_2(z|f,g) &= \int du u^A \left(\frac{\partial f(u)}{\partial u^A} g(u) - \sigma(f,g) \frac{\partial g(u)}{\partial u^A} f(u) \right), \\
M^5_2(z|f,g) &= \Delta(x|\partial_2 f g) - \Delta(x|f \partial_2 g) - 2(-1)^{\varepsilon(f)} \partial_2 f(z) \Delta(x|g) + 2\Delta(x|f)\partial_2 g(z) \\
&\quad + \frac{1}{2} \mathring{f}(z) \left(z^A \partial_A g(z)\right) - \frac{1}{2} (-1)^{\varepsilon(f)} \left(z^A \partial_A f(z)\right) \mathring{g}(z),
\end{aligned}$$

where $\Delta(x|f) \stackrel{def}{=} \int du\delta(x_1 - u_1)\theta(x_2 - u_2)f(u)$ and $\overset{\circ}{f}(z) \stackrel{def}{=} \int d\xi_1 ... d\xi_{n_-} f(z)$.

If n_- is even and $n_- \neq 6$ and $n_- \neq 0$, then $H^2_{\mathbf{D}'} \simeq H^2_{\mathbf{E}} \simeq \mathbb{K}^2$ and the cochains M_2^1 and M_2^3 are independent nontrivial cocycles;

if $n_- = 0$, then $H^2_{\mathbf{D}'} \simeq H^2_{\mathbf{E}} \simeq \mathbb{K}^3$ and the cochains M_2^1, M_2^3 and M_2^5 are independent nontrivial cocycles;

if $n_- = 6$, then $H^2_{\mathbf{D}'} \simeq H^2_{\mathbf{E}} \simeq \mathbb{K}^3$ and the cochains M_2^1, M_2^3, and M_2^4 are independent nontrivial cocycles;

if n_- is odd and $n_- \neq 1$, then $H^2_{\mathbf{D}'} \simeq H^2_{\mathbf{E}} \simeq \mathbb{K}^3$ and the cochains M_2^1, M_2^2 and M_2^3 are independent nontrivial cocycles;

if $n_- = 1$, then $H^2_{\mathbf{D}'} \simeq H^2_{\mathbf{E}} \simeq \mathbb{K}^4$ and the cochains M_2^1, M_2^2, M_2^3 and M_2^5 are independent nontrivial cocycles.

Theorem 3.

1. $H^0_{\mathrm{ad}} = 0$.
2. Let V_1 be the one-dimensional subspace of $C_1(\mathbf{D}_2^{n_-}, \mathbf{D}_2^{n_-})$ generated by the cocycle M_1^2 defined in Theorem 2. Then there is a natural isomorphism $V_1 \oplus (\mathbf{E}_2^{n_-}/\mathbf{D}_2^{n_-}) \simeq H^1_{\mathrm{ad}}$ taking $(M_1, T) \in V_1 \oplus (\mathbf{E}_2^{n_-}/\mathbf{D}_2^{n_-})$ to the cohomology class determined by the cocycle $M_1(z|f) + \{t(z), f(z)\}$, where $t \in \mathbf{E}_2^{n_-}$ belongs to the equivalence class T.
3. Let V_2 be the two-dimensional subspace of $C_2(\mathbf{D}_2^{n_-}, \mathbf{D}_2^{n_-})$ generated by the cocycles M_2^1 and M_2^3 defined in Theorem 2. Then there is a natural isomorphism $V_2 \oplus (\mathbf{E}_2^{n_-}/\mathbf{D}_2^{n_-}) \simeq H^2_{\mathrm{ad}}$ taking $(M_2, T) \in V_2 \oplus (\mathbf{E}_2^{n_-}/\mathbf{D}_2^{n_-})$ to the cohomology class determined by the cocycle
$$M_2(z|f,g) - \{t(z), f(z)\}\bar{g} + \sigma(f,g)\{t(z), g(z)\}\bar{f},$$
where $t \in \mathbf{E}_2^{n_-}$ belongs to the equivalence class T.

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