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Local Superfield Lagrangian BRST Quantization

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Abstract

A θ-local formulation of superfield Lagrangian quantization in non-Abelian hypergauges is proposed on the basis of an extension of general reducible gauge theories to special superfield models with a Grassmann parameter θ. We solve the problem of describing the quantum action and the gauge algebra of an L-stage-reducible superfield model in terms of a BRST charge for a formal dynamical system with first-class constraints of $(L+1)$-stage reducibility. Starting from θ-local functions of the quantum and gauge-fixing actions, with an essential use of Darboux coordinates on the antisymplectic manifold, we construct a superfield generating functionals of Green's functions, including the effective action. We present two superfield forms of BRST transformations, considered as θ-shifts along vector fields defined by Hamiltonian-like systems constructed in terms of the quantum and gauge-fixing actions and an arbitrary θ-local boson function, as well as via corresponding fermion functionals, in terms of Poisson brackets with opposite Grassmann parities. The gauge independence of the S-matrix is proved. The Ward identities are derived. Connection is established between the BV method [3], the multilevel Batalin–Tyutin formalism [21], and an extension of the superfield quantization rules [6,7] to the case of general coordinates.

1 Introduction

The construction of superfield versions of Hamiltonian [1,2] and Lagrangian [3] quantization methods for gauge theories on the basis of the BRST symmetry principle [4] has been covered in a number of papers [5–7]. These works are based on the use of nontrivial (represented by the operator $D = \partial_\theta + \theta\partial_t$, $[D,D]_+ = 2\partial_t$) and trivial relations between the even t and odd θ components of supertime $\chi = (t,\theta)$, introduced in [8]. In [5–7], the geometric interpretation of BRST transformations is realized in the form of special translations in superspace, which originally provided the basis for a superspace description of quantum Yang–Mills type theories [10].

The study of superfield quantization is closely related to generalized Poisson sigma-models [11], described from a superfield geometric viewpoint in [12], and then developed algorithmically by Batalin and Marnelius in [13]. The geometry of $D = 2$ supersymmetric sigma-models [14] with an arbitrary, $N \geq 1$, number of Grassmann coordinates has been adapted to the classical and quantum description of $D = 1$ sigma-models by Hull, and, independently, to the construction of the partition function for $N = 2$, by Gozzi et al [15]. Quantization with a single fermion supercharge, $Q(t,\theta)$, containing the BRST charge and the unitarizing Hamiltonian [5], was recently extended to $N = 2$ (non-spacetime) supersymmetries [16], and then, in [17], to the case of an arbitrary number of supercharges, $Q^k(t,\theta^1,...,\theta^N)$, $k = 1,...,N$, depending on Grassmann variables θ^k. The superfield modification [18] of the procedure [5] reveals a close interplay between the quantum action of the Batalin–Vilkovisky (BV) method [3] and the BRST charge of the Batalin–Fradkin–Vilkovisky (BFV) method [1]. Finally, note that the superfield approach is used in the description of second-class constrained systems as gauge models [19] as well as in the second quantization of gauge theories [20].

The superfield Lagrangian partition function of [5] is derived from a Hamiltonian partition function through functional integration over momenta. On the other hand, the quantization rules [6,7] present a superfield modification of the BV method by including non-Abelian hypergauges [21]. The corresponding hypergauge functions are introduced into a gauge-fixing action which obeys (following the ideas of [22]) the same generating equation that holds for the quantum action [6,7], except that the first-order operator V in this equation is replaced by the first-order operator U. The operators V, U are crucial ingredients of [6,7] from the viewpoint of a superspace interpretation of BRST transformations.

The formalism [6,7] achieves a comparatively detailed analysis of the properties of superfield quantization (BRST invariance, S-matix gauge-independence). This analysis [6,7] is based on the structure of solutions to the generating equations; however, a detailed correspondence between these solutions and a gauge model is not indicated. To achieve a better understanding of quantum properties based on solutions of the superfield generating equations, it is natural to equip the method [6,7] with an *explicit superfield description* of gauge algebra structure functions that determine a given model. So far, this problem has remained unsolved. For instance, the definition of a classical action of superfields, $\mathcal{A}^i(\theta) = A^i + \lambda^i\theta$, on a superspace with coordinates (x^μ,θ), $\mu = 0,...,D-1$, as an integral of a nontrivial odd density, $\mathcal{L}(\mathcal{A}(x,\theta)$, $\partial_\mu\mathcal{A}(x,\theta),...;x,\theta) \equiv \mathcal{L}(x,\theta)$, is a problem for every given model. Here, by trivial densities $\mathcal{L}(x,\theta)$ we understand those of the form

$$\int d^D x\, d\theta\, \mathcal{L}(x,\theta) = \int d\theta\, \theta\, S_0\left(\mathcal{A}(\theta)\right) = S_0(A),$$

where $S_0(A)$ is a usual classical action.

A peculiarity of the generating functional of Green's functions $Z[\Phi^*]$ in [6] and of the vacuum functional Z in [7] is the dependence of the gauge fermion $\Psi[\Phi]$, and of the quantum action $S[\Phi,\Phi^*]$, on the components λ^A of superfields $\Phi^A(\theta)$ in the multiplet $(\Phi^A,\Phi^*_A)(\theta)=(\phi^A+\lambda^A\theta,\phi^*_A-\theta J_A)$, where the variables $(\phi^A,\phi^*_A,\lambda^A,J_A)$ constitute the complete set of variables of the BV method [3]. Another feature of [6,7] is that the structure of superantifields $\Phi^*_A(\theta)$ and the explicit form of $Z[\Phi^*]$ allow one to introduce, in a non-contradictory manner, although violating the superfield content of the variables,[1] an effective action Γ, by using a Legendre transformation of $\ln Z[\Phi^*]$ with respect to $P_1(\theta)\Phi^*_A(\theta)$,[2]

$$\Gamma[P_0(\Phi,\Phi^*)]=\frac{\hbar}{i}\ln Z[\Phi^*]+\partial_\theta\left\{[P_1(\theta)\Phi^*_A(\theta)]\,\Phi^A(\theta)\right\},\ \Phi^A(\theta)=-\frac{\hbar}{i}\frac{\delta\ln Z[\Phi^*]}{\delta\left(P_1(\theta)\Phi^*_A(\theta)\right)},\tag{1}$$

with the standard Ward identity $(\Gamma,\Gamma)=0$ in terms of a superantibracket [6].

In this paper, we propose a local version of superfield Lagrangian quantization, in which we realize the quantities of an initial classical theory in the framework of a *θ-local superfield model* (LSM). The idea of LSM is to represent the objects of a gauge theory (classical action, generators of gauge transformations, etc.) in terms of θ-local functions, trivially related[3] to the spacetime coordinates. Using an analogy with classical mechanics (or classical field theory), we reproduce the dynamics and gauge invariance (in particular, BRST transformations) of the initial theory (the one with $\theta=0$) in terms of θ-local equations, called *Lagrangian* and *Hamiltonian systems* (LS, HS) with a dynamical θ.[4]

On the basis of the proposed formalism, we solve the following problems:

1. We develop a *dual description*[5] of an arbitrary reducible LSM of Ref. [18] in the case of irreducible gauge theories (with bosonic classical fields), in terms of a BRST charge related to a formal dynamical system with first-class constraints of a higher stage of reducibility.

2. An HS constructed from θ-local quantities, i.e., a quantum action, a gauge-fixing action, and an arbitrary bosonic function, encodes, through a θ-local antibracket, both anticanonical and BRST transformations in terms of a universal set of equations underlying the gauge-independence of the S-matrix. This set of equations is generated, in terms of an even superfield Poisson bracket, by a linear combination of fermionic functionals corresponding to the above θ-local quantities, e.g., the quantum and gauge-fixing actions and the bosonic function.

3. For the first time in the framework of superfield approach, we introduce a *superfield effective action* (also in the case of non-Abelian hypergauges).

4. We extend the superfield quantization of Refs. [6, 7] to the case of general coordinates on the manifold of super(anti)fields and establish a relation with the proposed local quantization.

The paper is organized as follows. In Section 2, a Lagrangian formulation of an LSM is proposed as an extension of a usual model of classical fields A^i, $i=1,\dots,n=n_++n_-$, to a θ-local theory, defined on the odd tangent bundle $T_{\mathrm{odd}}\mathcal{M}_{\mathrm{CL}}\equiv\Pi T\mathcal{M}_{\mathrm{CL}}=\left\{\mathcal{A}^I,\partial_\theta\mathcal{A}^I\right\}$, $I=1,\dots N=N_++N_-$[6], $(n_+,n_-)\leq(N_+,N_-)$. The superfields $(\mathcal{A}^I,\partial_\theta\mathcal{A}^I)(\theta)$ are defined on the superspace $\mathcal{M}=\widetilde{\mathcal{M}}\times\widetilde{P}$, parameterized by $\left(z^M,\theta\right)$, where the spacetime coordinates $z^M\subset i\subset I$ include Lorentz vectors and spinors of the superspace $\widetilde{\mathcal{M}}$. We investigate the superfield equations of motion, introduce the notions of reducible *general* and *special* superfield gauge theories and apply Noether's first theorem to θ-translations. Section 3 is devoted to the Hamiltonian formulation of an LSM, on the odd cotangent bundle $T^*_{\mathrm{odd}}\mathcal{M}_{\mathrm{CL}}\equiv\Pi T^*\mathcal{M}_{\mathrm{CL}}=\left\{\mathcal{A}^I,\mathcal{A}^*_I\right\}$. Here, we establish a connection to the Lagrangian formalism and investigate the

[1]By *violation* of the superfield content, we understand the fact that the derivative of $Z[\Phi^*]$, which defines the effective action in a Legendre transformation, is calculated with respect to only one superfield component, namely, the θ-component of $\Phi^*_A(\theta)$, so that the resulting effective action depends only on ϕ^A and ϕ^*_A, which can be formally expressed as $P_0(\theta)\left(\Phi^A,\Phi^*_A\right)(\theta)=\left(\phi^A,\phi^*_A\right)$.

[2]Here, $P_1(\theta)$ and the operator $\delta/\delta\left(P_1(\theta)\Phi^*_A(\theta)\right)$ in (1) are, respectively, the projector from the system $\{P_a(\theta)=\delta_{a0}(1-\theta\partial_\theta)+\delta_{a1}\theta\partial_\theta, a=0,1\}$ on the supermanifold with coordinates $(\Phi^A,\Phi^*_A)(\theta)$ and the superfield variational derivative with respect to $P_1(\theta)\Phi^*_A(\theta)$.

[3]By *trivial* relation to spacetime coordinates, we imply, in contrast to Hamiltonian formalism, that derivatives with respect to the even t and odd θ component of supertime are taken independently.

[4]By *dynamical* θ, we imply that this coordinate enters an LS or HS not as a parameter, but rather as part of a differential operator ∂_θ that describes the θ-evolution of a system.

[5]*Dual description* stands for the possibility of an interrelated description of a reducible gauge model, namely, a description that relates the Lagrangian and Hamiltonian formalism (the latter understood in the sense of a *formal* dynamical system).

[6]Π denotes the exchange operation of the coordinates of a tangent fiber bundle $T\mathcal{M}_{\mathrm{CL}}$ over a configuration $\mathcal{A}^I$ into the coordinates of the opposite Grassmann parity [23], and N_+, N_- are the numbers of bosonic and fermionic fields, among which there may be superfields corresponding to the ghosts of the minimal sector in the BV quantization scheme (in condensed notations [24] used in this paper).

existence of a Noether integral, related to θ-translations, that leads to the fulfillment of a θ-local master equation. The quantization rules are given in Section 4. As a first step, in Subsection 4.1, we transform the reducibility relations of a special *restricted* LSM into a sequence of new gauge transformations for the ghost superfields of the minimal sector. Together with the gauge transformations of the classical superfields $\mathcal{A}^i(\theta)$, extracted from $\mathcal{A}^I(\theta)$, the new gauge transformations are translated into a Hamiltonian system related to the restricted HS. A requirement of superfield integrability for the resulting HS produces a deformation of the θ-local Hamiltonian in powers of the ghosts and superantifields of the minimal sector, and leads to a quantum action, and, independently, to a gauge-fixing action (Subsection 4.3), subject to different θ-local master equations. In Subsection 4.2, we construct the dual description of an LSM. In Subsection 4.3, we define, in terms of the above-mentioned actions, a generating functional of Green's functions, $\mathcal{Z}(\theta)$, and an effective action, $\Gamma(\theta)$, using an invariant description of super(anti)fields on a general antisymplectic manifold. An essential feature in introducing $\mathcal{Z}(\theta)$ and $\Gamma(\theta)$ is the choice of Darboux coordinates $(\varphi,\varphi^*)(\theta)$ compatible with the properties of the quantum action. In Subsection 5.1, on the basis of a component formulation of the local superfield quantization, we establish its connection with the first-level formalism [21], with the BV scheme, and (in Subsection 5.2) with the proposed extension of the superfield method [6,7]. In Conclusion, we discuss the results of the present work.

In addition to DeWitt's condensed notation [24], we partially use the conventions of Refs. [6,7]. Besides, we distinguish between two types of superfield derivatives, namely, the right (left) variational derivative $\delta_{(l)}F/\delta\Phi^A(\theta)$ of a functional F with respect to superfields, $\Phi^A(\theta)$, and the right (left) derivative $\partial_{(l)}\mathcal{F}(\theta)/\partial\Phi^A(\theta)$ of a function $\mathcal{F}(\theta)$, for a fixed θ, with respect to $\Phi^A(\theta)$. Derivatives with respect to super(anti)fields and their components are understood as right (left), for instance, $\delta/\delta\lambda^A$, or $\delta/\delta\Phi^*_A(\theta)$, and the corresponding left (right) derivatives are labelled by the subscript "$l(r)$". For right-hand derivatives with respect to $\mathcal{A}^I(\theta)$, with a fixed θ, we use the notation $\mathcal{F}_{,I}(\theta) \equiv \partial\mathcal{F}(\theta)/\partial\mathcal{A}^I(\theta)$. The $\delta(\theta)$-function and integration over θ are given, respectively, by $\delta(\theta)=\theta$ and left-hand differentiation over θ.

The rank of an even supermatrix with Z_2-grading, ε, is characterized by a pair of numbers $\overline{m} = (m_+, m_-)$, where m_+ (m_-) is the rank of the Bose–Bose (Fermi–Fermi) block of the supermatrix. With respect to the same Grassmann parity ε, we understand the dimension of a smooth supersurface, also characterized by a pair of numbers, in the sense of the definition of a supermanifold in Ref. [25]. On these pairs, we consider the operations of component addition, $\overline{m}+\overline{n} = (m_+ + n_+, m_- + n_-)$, and comparison,

$$\overline{m} = \overline{n} \Leftrightarrow m_\pm = n_\pm,\ \overline{m} > \overline{n} \Leftrightarrow (m_+ > n_+,\ m_- \geq n_-) \text{ or } (m_+ \geq n_-,\ m_+ > n_-).$$

2 Lagrangian Formulation

The basic objects of the Lagrangian formulation of an LSM are a *Lagrangian action* S_L: $\Pi T\mathcal{M}_{\rm CL} \times \{\theta\} \to \Lambda_1(\theta;\mathbb{R})$, being a $C^\infty(\Pi T\mathcal{M}_{\rm CL})$-function with values in a real Grassmann algebra, $\Lambda_1(\theta;\mathbb{R})$, and a (nonequivalent) functional $Z[\mathcal{A}]$, whose θ-density is defined with accuracy up to an arbitrary function $f((\mathcal{A},\partial_\theta\mathcal{A})(\theta),\theta) \in \ker\{\partial_\theta\}$, $\vec{\varepsilon}(f)=\vec{0}$,

$$Z[\mathcal{A}] = \partial_\theta S_L(\theta),\ \vec{\varepsilon}(Z) = \vec{\varepsilon}(\theta) = (1,0,1),\ \vec{\varepsilon}(S_L) = \vec{0}. \tag{2}$$

The values $\vec{\varepsilon} = (\varepsilon_P, \varepsilon_{\bar{J}}, \varepsilon)$, $\varepsilon = \varepsilon_P + \varepsilon_{\bar{J}}$, of Z_2-grading, with the auxiliary components $\varepsilon_{\bar{J}}$, ε_P related to the respective coordinates (z^M,θ) of a superspace $\mathcal{M}$, are defined on superfields $\mathcal{A}^I(\theta)$, by the relation $\vec{\varepsilon}(\mathcal{A}^I) = ((\varepsilon_P)_I, (\varepsilon_{\bar{J}})_I, \varepsilon_I)$. Note that $\mathcal{M}$ may be realized as the quotient of a symmetry supergroup, $J = \bar{J} \times P$, $P = \exp\{i\mu p_\theta\}$, of the functional $Z[\mathcal{A}]$, where μ, p_θ are, respectively, a nilpotent parameter and a generator of θ-shifts, where $\bar{J}$ is chosen as the spacetime SUSY group. The quantities $\varepsilon_{\bar{J}}$, ε_P, introduced in [26], are the respective Grassmann parities of the coordinates of representation spaces of the supergroups $\bar{J}$, P. The introduced objects allow one to achieve a correct incorporation of the spin-statistic relation into operator quantization.

Among $S_L(\theta)$, $Z[\mathcal{A}]$, invariant under the action of a J-superfield representation T restricted to P, $T|_P$, only $S_L(\theta)$ is nontrivially transformed with respect to the total representation T under $\mathcal{A}^I(\theta) \to \mathcal{A}'^I(\theta) = (T|_{\bar{J}}\mathcal{A})^I(\theta-\mu)$,

$$\delta S_L(\theta) = S_L\left(\mathcal{A}'(\theta), \partial_\theta\mathcal{A}'(\theta), \theta\right) - S_L(\theta) = -\mu\left[\frac{\partial}{\partial\theta} + P_0(\theta)(\partial_\theta U_+)(\theta)\right] S_L(\theta). \tag{3}$$

Here, we have introduced the nilpotent operator $(\partial_\theta U_+)(\theta) = \partial_\theta\mathcal{A}^I(\theta)\partial_l/\partial\mathcal{A}^I(\theta) = [\partial_\theta, U_+(\theta)]_-$, $U_+(\theta) = P_1\mathcal{A}^I(\theta)\partial_l/\partial\mathcal{A}^I(\theta)$.

The dynamics of an LSM is encoded by the superfield Euler–Lagrange equations

$$\frac{\delta_l Z[\mathcal{A}]}{\delta\mathcal{A}^I(\theta)}=\left[\frac{\partial_l}{\partial\mathcal{A}^I(\theta)}-(-1)^{\varepsilon_I}\partial_\theta\frac{\partial_l}{\partial(\partial_\theta\mathcal{A}^I(\theta))}\right]S_L(\theta)\equiv\mathcal{L}^l_I(\theta)S_L(\theta)=0, \tag{4}$$

equivalent, in view of $\partial^2_\theta\mathcal{A}^I(\theta)\equiv 0$, to an LS, characterized by $2N$ formally second-order differential equations in θ,

$$\partial^2_\theta\mathcal{A}^J(\theta)\frac{\partial^2_l S_L(\theta)}{\partial(\partial_\theta\mathcal{A}^I(\theta))\partial(\partial_\theta\mathcal{A}^J(\theta))}\equiv\partial^2_\theta\mathcal{A}^J(\theta)(S''_L)_{IJ}(\theta)=0,$$
$$\Theta_I(\theta)\equiv\frac{\partial_l S_L(\theta)}{\partial\mathcal{A}^I(\theta)}-(-1)^{\varepsilon_I}\left[\frac{\partial}{\partial\theta}\frac{\partial_l S_L(\theta)}{\partial(\partial_\theta\mathcal{A}^I(\theta))}+(\partial_\theta U_+)(\theta)\frac{\partial_l S_L(\theta)}{\partial(\partial_\theta\mathcal{A}^I(\theta))}\right]=0. \tag{5}$$

The *Lagrangian constraints* $\Theta_I(\theta)=\Theta_I(\mathcal{A}(\theta),\partial_\theta\mathcal{A}(\theta),\theta)$ restrict the setting of the Cauchy problem for the LS and may be functionally dependent, as first-order equations in θ.

Provided that there exists (at least locally) a supersurface $\Sigma\subset\mathcal{M}_{\mathrm{CL}}$ such that

$$\Theta_I(\theta)|_\Sigma=0,\ \dim\Sigma=\overline{M},\ \operatorname{rank}\left\|\mathcal{L}^l_J(\theta_1)\left[\mathcal{L}^l_I(\theta_1)S_L(\theta_1)(-1)^{\varepsilon_I}\right]\right\|_\Sigma\delta(\theta_1-\theta)=\overline{N}-\overline{M}, \tag{6}$$

there exist $M=M_++M_-$ independent identities:

$$\int d\theta\frac{\delta Z[\mathcal{A}]}{\delta\mathcal{A}^I(\theta)}\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0)=0,\ \hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0)=\sum_{k\geq 0}\left((\partial_\theta)^k\delta(\theta-\theta_0)\right)\hat{\mathcal{R}}_k{}^I_{\mathcal{A}_0}(\mathcal{A}(\theta),\partial_\theta\mathcal{A}(\theta),\theta). \tag{7}$$

The generators $\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0)$ of *general gauge transformations*,

$$\delta_g\mathcal{A}^I(\theta)=\int d\theta_0\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0)\xi^{\mathcal{A}_0}(\theta_0),\ \vec{\varepsilon}(\xi^{\mathcal{A}_0})=\vec{\varepsilon}_{\mathcal{A}_0},\ \mathcal{A}_0=1,...,\ M_0=M_{0+}+M_{0-},$$

that leave $Z[\mathcal{A}]$ invariant, are functionally dependent, under the assumption of locality and $\vec{J}$-covariance, provided that

$$\operatorname{rank}\left\|\sum_{k\geq 0}\hat{\mathcal{R}}_k{}^I_{\mathcal{A}_0}(\theta)\,(\partial_\theta)^k\right\|_\Sigma\delta(\theta-\theta_0)=\overline{M}<\overline{M}_0.$$

The dependence of $\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0)$ implies the existence (on solutions of the LS) of proper zero-eigenvalue eigenvectors, $\hat{\mathcal{Z}}^{\mathcal{A}_0}_{\mathcal{A}_1}(\mathcal{A}(\theta_0),\partial_{\theta_0}\mathcal{A}(\theta_0),\theta_0;\theta_1)$, having a structure analogous to $\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta;\theta_0)$ in (7), which exhaust the zero-modes of the generators and are dependent in case

$$\operatorname{rank}\left\|\sum_k\hat{\mathcal{Z}}_k{}^{\mathcal{A}_0}_{\mathcal{A}_1}(\theta_0)\,(\partial_{\theta_0})^k\right\|_\Sigma\delta(\theta_0-\theta_1)=\overline{M}-\overline{M}_0<\overline{M}_1.$$

As a result, the relations of dependence for eigenvectors which define a general L_g-stage reducible LSM are given by

$$\int d\theta'\hat{\mathcal{Z}}^{\mathcal{A}_{s-2}}_{\mathcal{A}_{s-1}}(\theta_{s-2};\theta')\hat{\mathcal{Z}}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}(\theta';\theta_s)=\int d\theta'\Theta_J(\theta')\mathcal{L}^{\mathcal{A}_{s-2}J}_{\mathcal{A}_s}\left((\mathcal{A},\partial_\theta\mathcal{A})(\theta_{s-2}),\theta_{s-2},\theta';\theta_s\right),$$
$$\overline{M}_{s-1}>\sum_{k=0}^{s-1}(-1)^k\overline{M}_{s-k-2}=\operatorname{rank}\left\|\sum_{k\geq 0}\hat{\mathcal{Z}}^{\mathcal{A}_{s-2}}_{\mathcal{A}_{s-1}}(\theta_{s-2})\left(\partial_{\theta_{s-2}}\right)^k\right\|_\Sigma\delta(\theta_{s-2}-\theta_{s-1}),$$
$$\overline{M}_{L_g}=\sum_{k=0}^{L_g}(-1)^k\overline{M}_{L_g-k-1}=\operatorname{rank}\left\|\sum_{k\geq 0}\hat{\mathcal{Z}}^{\mathcal{A}_{L_g-1}}_{\mathcal{A}_{L_g}}(\theta_{L_g-1})\left(\partial_{\theta_{L_g-1}}\right)^k\right\|_\Sigma\delta(\theta_{L_g-1}-\theta_{L_g}),$$
$$\vec{\varepsilon}(\hat{\mathcal{Z}}^{\mathcal{A}_s}_{\mathcal{A}_{s+1}})=\vec{\varepsilon}_{\mathcal{A}_s}+\vec{\varepsilon}_{\mathcal{A}_{s+1}}+(1,0,1),\ \hat{\mathcal{Z}}^{\mathcal{A}_{-1}}_{\mathcal{A}_0}(\theta_{-1};\theta_0)\equiv\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta_{-1};\theta_0),$$
$$\mathcal{L}^{\mathcal{A}_{-1}J}_{\mathcal{A}_1}(\theta_{-1},\theta';\theta_1)\equiv\mathcal{K}^{IJ}_{\mathcal{A}_1}(\theta_{-1},\theta';\theta_1)=-(-1)^{(\varepsilon_I+1)(\varepsilon_J+1)}\mathcal{K}^{JI}_{\mathcal{A}_1}(\theta',\theta_{-1};\theta_1). \tag{8}$$

for $s=1,...,L_g$, $\mathcal{A}_s=1,...,M_s=M_{s+}+M_{s-}$, $\overline{M}\equiv\overline{M}_{-1}$. For $L_g=0$, the LSM is an irreducible *general gauge theory*.

In case an LSM can be presented as $S_L(\theta) = T(\partial_\theta \mathcal{A}(\theta)) - S(\mathcal{A}(\theta), \theta)$, the functions $\Theta_I(\theta)$ are given, on the extended configuration space $\mathcal{M}_{\rm CL} \times \{\theta\}$, by the relations

$$\Theta_I(\theta) = -S_{,I}(\mathcal{A}(\theta), \theta)(-1)^{\varepsilon_I} = 0, \tag{9}$$

being the usual extremals of the functional $S_0(A) = S(\mathcal{A}(0), 0)$, corresponding to $\theta = 0$. In case $\theta = 0$, condition (6) and identities (7) take the usual form

$$\operatorname{rank}\|S_{,IJ}(\mathcal{A}(\theta),\theta))\|_\Sigma = \overline{N} - \overline{M},\ S_{,I}(\mathcal{A}(\theta),\theta)\mathcal{R}_0{}^I_{\mathcal{A}_0}(\mathcal{A}(\theta),\theta) = 0, \tag{10}$$

with linearly-dependent (for $\overline{M}_0 > \overline{M}$) generators of *special gauge transformations*,

$$\delta\mathcal{A}^I(\theta) = \mathcal{R}_0{}^I_{\mathcal{A}_0}(\mathcal{A}(\theta),\theta)\,\xi_0^{\mathcal{A}_0}(\theta),$$

with leave invariant only $S(\theta)$, in contrast to $T(\theta)$. The dependence of generators $\mathcal{R}_0{}^I_{\mathcal{A}_0}(\theta)$, as well as of their zero-eigenvalue eigenvectors $\mathcal{Z}^{\mathcal{A}_0}_{\mathcal{A}_1}(\mathcal{A}(\theta),\theta)$, and so on, can also be expressed by special relations of reducibility, for $s = 1, ..., L_g$, namely,

$$\begin{gathered}\mathcal{Z}^{\mathcal{A}_{s-2}}_{\mathcal{A}_{s-1}}(\mathcal{A}(\theta),\theta)\mathcal{Z}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}(\mathcal{A}(\theta),\theta) = S_{,J}(\theta)\mathcal{L}^{\mathcal{A}_{s-2}J}_{\mathcal{A}_s}(\mathcal{A}(\theta),\theta),\ \vec{\varepsilon}(\mathcal{Z}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}) = \vec{\varepsilon}_{\mathcal{A}_{s-1}} + \vec{\varepsilon}_{\mathcal{A}_s},\\ \mathcal{Z}^{\mathcal{A}_{-1}}_{\mathcal{A}_0}(\theta) \equiv \mathcal{R}_0{}^I_{\mathcal{A}_0}(\theta),\ \mathcal{L}^{\mathcal{A}_{-1}J}_{\mathcal{A}_1}(\theta) \equiv \mathcal{K}^{IJ}_{\mathcal{A}_1}(\theta) = -(-1)^{\varepsilon_I\varepsilon_J}\mathcal{K}^{JI}_{\mathcal{A}_1}(\theta).\end{gathered} \tag{11}$$

For $\overline{M}_{L_g} = \sum_{k=0}^{L_g}(-1)^k\overline{M}_{L_g-k-1} = \operatorname{rank}\left\|\mathcal{Z}^{\mathcal{A}_{L_g-1}}_{\mathcal{A}_{L_g}}\right\|_\Sigma$, relations (9)–(11) determine a *special gauge theory* of L_g-stage reducibility. The gauge algebra of such a theory is θ-locally embedded into the gauge algebra of a general gauge theory with the functional $Z[\mathcal{A}] = \partial_\theta(T(\theta) - S(\theta))$, which implies the following relation between the eigenvectors:

$$\hat{\mathcal{Z}}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}(\mathcal{A}(\theta_{s-1}), \theta_{s-1}; \theta_s) = -\delta(\theta_{s-1} - \theta_s)\mathcal{Z}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}(\mathcal{A}(\theta_{s-1}), \theta_{s-1}), \tag{12}$$

and, besides, the fact that the structure functions of the gauge algebra of a special gauge theory may depend on $\partial_\theta\mathcal{A}^I(\theta)$ only parametrically. Note that an extended (as compared to $\{P_a(\theta)\}$, $a = 1, 2$) system of projectors onto $C^\infty(\Pi T\mathcal{M}_{\rm CL}) \times \{\theta\}$, $\{P_0(\theta), \theta\partial/\partial\theta, U_+(\theta)\}$, selects from (11) two kinds of gauge algebras: one with structure equations and functions $S(\mathcal{A}(\theta))$, $\mathcal{Z}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}(\mathcal{A}(\theta))$ not depending on θ in an explicit form, the other with the standard relations of the gauge algebra for a reducible model with the quantities $S_0(A)$, $\mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(A)$, in case $\theta = 0$, $(\varepsilon_P)_I = (\varepsilon_P)_{\mathcal{A}_s} = 0$, $s = 1, ..., L_g$, and under the assumption of completeness of the reduced generators $\mathcal{R}^i_{\alpha_0}(\mathcal{A}(\theta))$ and eigenvectors $\mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(\mathcal{A}(\theta))$.

An extension of a usual field theory to a θ-local LSM permits one to apply Noether's first theorem [27] to the invariance of the density $d\theta S_L(\theta)$ with respect to global θ-translations, as symmetry transformations of the superfields $\mathcal{A}^I(\theta)$ and coordinates (z^M, θ), $(\mathcal{A}^I, z^M, \theta) \to (\mathcal{A}^I, z^M, \theta + \mu)$. By direct verification, one establishes that the function

$$S_E((\mathcal{A}, \partial_\theta\mathcal{A})(\theta), \theta) \equiv \frac{\partial S_L(\theta)}{\partial(\partial^r_\theta\mathcal{A}^I(\theta))}\partial^r_\theta\mathcal{A}^I(\theta) - S_L(\theta) \tag{13}$$

is an LS integral of motion, i.e., a conserved quantity under the θ-evolution, in case there holds the equation

$$\frac{\partial}{\partial\theta}S_L(\theta) + 2(\partial_\theta U_+(\theta))S_L(\theta)\Big|_{\mathcal{L}^l_I S_L = 0} = 0. \tag{14}$$

In contrast to its analogue in a t-local field theory, the energy $E(t)$, the function $S_E(\theta)$ is an LS integral also in the case of an explicit dependence on θ. This fact takes place in case $S_L(\theta)$ admits the structure

$$S_L((\mathcal{A}, \partial_\theta\mathcal{A})(\theta), \theta) = S^0_L(\mathcal{A}, \partial_\theta\mathcal{A})(\theta) - 2\theta[\partial_\theta U_+(\theta)]S^0_L(\theta),\ \vec{\varepsilon}(S^0_L) = \vec{0}. \tag{15}$$

3 Hamiltonian Formulation

Independently, the LSM description can be formulated, without an $\mathcal{M}_{\rm CL}$-extension, in terms of a *Hamiltonian action*, being a $C^\infty(\Pi T^*\mathcal{M}_{\rm CL})$-function, $S_H : \Pi T^*\mathcal{M}_{\rm CL} \times \{\theta\} \to \Lambda_1(\theta; \mathbb{R})$, depending on super-antifields $\mathcal{A}^*_I(\theta) = (A^*_I - \theta J_I)$, included in the local coordinates of $\Pi T^*\mathcal{M}_{\rm CL}$: $\Gamma^P_{\rm CL}(\theta) = (\mathcal{A}^I, \mathcal{A}^*_I)(\theta)$,

$\vec{\varepsilon}(\mathcal{A}^*_I) = \vec{\varepsilon}(\mathcal{A}^I) + (1,0,1)$. The equivalence of the Lagrangian and Hamiltonian formulations is guaranteed by the nondegeneracy of the supermatrix $\|(S''_L)_{IJ}(\theta)\|$ in (5), in the framework of a Legendre transformation of $S_L(\theta)$ with respect to $\partial^r_\theta \mathcal{A}^I(\theta)$,

$$S_H(\Gamma_{\rm CL}(\theta),\theta) = \mathcal{A}^*_I(\theta)\partial^r_\theta\mathcal{A}^I(\theta) - S_L(\theta),\ \mathcal{A}^*_I(\theta) = \frac{\partial S_L(\theta)}{\partial(\partial^r_\theta \mathcal{A}^I(\theta))}, \tag{16}$$

where $S_H(\Gamma_{\rm CL}(\theta),\theta)$ coincides with $S_E(\theta)$ in terms of the $\Pi T^*\mathcal{M}_{\rm CL}$-coordinates.

The dynamics of an LSM is given by a *generalized Hamiltonian system* of $3N$ first-order equations in θ, equivalent to the LS equations in (5), and expressed through a θ-local antibracket $(\cdot\,,\cdot)_\theta$, namely,

$$\begin{aligned}&\partial^r_\theta\Gamma^P_{\rm CL}(\theta) = \left(\Gamma^P_{\rm CL}(\theta), S_H(\theta)\right)_\theta,\ \Theta^H_I(\Gamma_{\rm CL}(\theta),\theta) = \Theta_I(\mathcal{A}(\theta),\partial_\theta\mathcal{A}(\Gamma_{\rm CL}(\theta),\theta),\theta) = 0,\\ &(\mathcal{F}_1,\mathcal{F}_2)_\theta \equiv \frac{\partial\mathcal{F}_1}{\partial\mathcal{A}^I(\theta)}\frac{\partial\mathcal{F}_2}{\partial\mathcal{A}^*_I(\theta)} - \frac{\partial_r\mathcal{F}_1}{\partial\mathcal{A}^*_I(\theta)}\frac{\partial_l\mathcal{F}_2}{\partial\mathcal{A}^I(\theta)},\end{aligned} \tag{17}$$

with the *Hamiltonian constraints* $\Theta^H_I(\Gamma_{\rm CL}(\theta),\theta)$. The latter coincide with half of the equations of the HS proper, due to transformations (16) and their consequences:

$$\Theta^H_I(\Gamma_{\rm CL}(\theta),\theta) = -\partial^r_\theta\mathcal{A}^*_I(\theta) - S_{H},_I(\theta)(-1)^{\varepsilon_I}. \tag{18}$$

Formula (18) establishes the equivalence of an HS with a generalized HS, and, therefore, with an LS, within the corresponding setting ($\theta = 0$, $k = $ CL) of the Cauchy problem for integral curves $\hat{\mathcal{A}}^I(\theta)$, $\hat{\Gamma}^P_k(\theta)$

$$\left(\hat{\mathcal{A}}^I,\partial^r_\theta\hat{\mathcal{A}}^I\right)(0) = \left(\overline{\mathcal{A}}^I,\overline{\partial^r_\theta\mathcal{A}}^I\right),\ \hat{\Gamma}^P_k(0) = \left(\overline{\mathcal{A}}^I,\overline{\mathcal{A}}^*_I\right) :\ \overline{\mathcal{A}}^*_I = P_0\left[\frac{\partial S_L(\theta)}{\partial(\partial^r_\theta\mathcal{A}^I(\theta))}\right]\left(\overline{\mathcal{A}}^I,\overline{\partial^r_\theta\mathcal{A}}^I\right) \tag{19}$$

(we ignore the continuous part of the indices I). The equivalence between an HS and a generalized HS holds due to the coincidence (mutual inclusion) of the corresponding sets of solutions. Indeed, the solutions of an HS are included into those of a generalized HS by construction, while the reverse is valid due to (18).

The HS is defined through a variational problem for a functional identical with $Z[\mathcal{A}]$,

$$\begin{aligned}Z_H[\Gamma_k] &= \int d\theta\left[\frac{1}{2}\Gamma^P_k(\theta)\omega^k_{PQ}(\theta)\partial^r_\theta\Gamma^Q_k(\theta) - S_H(\Gamma_k(\theta),\theta)\right],\\ \omega^{PQ}_k(\theta) &\equiv \left(\Gamma^P_k(\theta),\Gamma^Q_k(\theta)\right)_\theta,\ \omega^{PD}_k(\theta)\omega^k_{DQ}(\theta) = \delta^P{}_Q.\end{aligned} \tag{20}$$

Definitions (9)–(11) remain the same for special gauge theories, while definitions (7), (8), in the case of general gauge theories of L_g-stage reducibility, are transformed by the rule

$$\hat{\mathcal{Z}}_H{}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s}\left(\Gamma_k(\theta_{s-1}),\theta_{s-1};\theta_s\right) = \hat{\mathcal{Z}}^{\mathcal{A}_{s-1}}_{\mathcal{A}_s} = \left(\mathcal{A}(\theta_{s-1}),\partial_{\theta_{s-1}}\mathcal{A}(\Gamma_k(\theta_{s-1}),\theta_{s-1}),\theta_{s-1};\theta_s\right)\ ,\ s = 0,...,L_g\,. \tag{21}$$

From eqs. (14), as well as from the validity of transformations (16) and of their consequence $\frac{\partial}{\partial\theta}(S_L + S_H)(\theta) = 0$, there follows the invariance of $S_H(\theta)$ under θ-shifts along arbitrary solutions $\hat{\Gamma}^P_k(\theta)$, or, equivalently, along an $(\varepsilon_P,\varepsilon)$-odd vector field $\mathbf{Q}(\theta) = \mathrm{ad}S_H(\theta) \equiv (S_H(\theta),\cdot)_\theta$. Thus,

$$\delta_\mu S_H(\theta)|_{\hat{\Gamma}_k(\theta)} = \mu\left[\frac{\partial}{\partial\theta}S_H(\theta) - \left((S_H(\theta),S_H(\theta)\right)_\theta\right] = 0,\ \delta_\mu S_H(\theta) = \mu\partial_\theta S_H(\theta) \tag{22}$$

holds true, provided that $S_H(\theta)$ can be presented, according to formula (14), in the form

$$S_H\left(\Gamma_k(\theta),\theta\right) = S^0_H\left(\Gamma_k(\theta)\right) + \theta\left(\left(S^0_H\left(\Gamma_k(\theta)\right), S^0_H\left(\Gamma_k(\theta)\right)\right)_\theta\right., \tag{23}$$

where $\partial_\theta U_+(\theta)S_L(\theta) = 1/2\left(S_H(\theta),S_H(\theta)\right)_\theta$ and $S^0_H\left(\Gamma_k(\theta)\right)$ is the Legendre transform of $S^0_L(\theta)$ defined by (15).

If $S_H(\theta)$, or $S_L(\theta)$, does not depend on θ explicitly, then eq. (22), or (14), implies the fulfilment of the equation $\left(S_H(\theta),S_H(\theta)\right)_\theta = 0$, or $(\partial_\theta U_+(\theta))S_L(\theta)|_{\hat{\mathcal{A}}(\theta)} = 0$, which has no analogies in a t-local field theory, and imposes the known condition [3] that $S_H(\theta)$, or $S_L(\theta)$, be proper, although for an LSM at the classical level. In this case, a θ-superfield integrability[7] of the HS in (17) is guaranteed, due to the standard properties of the antibracket, including the Jacobi identity:

$$(\partial^r_\theta)^2\Gamma^P_k(\theta) = \frac{1}{2}\left(\Gamma^P_k(\theta), (S_H(\Gamma_k(\theta)), S_H(\Gamma_k(\theta)))_\theta\right)_\theta = 0. \tag{24}$$

This fact provides the validity on $C^\infty(\Pi T^*\mathcal{M}_{\rm CL}\times\{\theta\})$ of the θ-translation formula

$$\delta_\mu\mathcal{F}(\theta)|_{\hat{\Gamma}_k(\theta)} = \mu\left(\frac{\partial}{\partial\theta} - \mathrm{ad}S_H(\theta)\right)\mathcal{F}(\theta) \equiv \mu\check{s}^l(\theta)\mathcal{F}(\theta), \tag{25}$$

as well as the nilpotency of a BRST-like generator of θ-shifts along $\mathbf{Q}(\theta)$, $\check{s}^l(\theta)$.

Depending on the realization of additional properties of a gauge theory (see Section 4), we shall henceforth assume the fulfillments of the equation

$$\Delta^k(\theta)S_H(\theta) = 0,\ \Delta^k(\theta) \equiv \frac{1}{2}(-1)^{\varepsilon(\Gamma^Q)}\omega^k_{QP}(\theta)\left(\Gamma^P_k(\theta), \left(\Gamma^Q_k(\theta), \cdot\right)_\theta\right)_\theta. \tag{26}$$

Eq. (26) is equivalent to a vanishing divergence of the vector field $\mathbf{Q}(\theta)$, namely,

$$\mathrm{div}\left(\partial^r_\theta\Gamma_k(\theta)|_{\hat{\Gamma}_k(\theta)}\right) = \frac{\partial_r}{\partial\Gamma^P_k(\theta)}\left(\partial^r_\theta\Gamma^P_k(\theta)|_{\hat{\Gamma}_k(\theta)}\right) = 2\Delta^k(\theta)S_H(\theta) = 0. \tag{27}$$

This condition holds trivially for the symplectic analogue of formula (27). The validity of the *Hamiltonian master equation* $(S_H(\theta), S_H(\theta))_\theta = 0$ for $\frac{\partial}{\partial\theta}S_H(\theta) = 0$ justifies the interpretation of the equivalent equation in (14), for $\frac{\partial}{\partial\theta}S_L(\theta) = 0$, $(\partial_\theta U_+(\theta))S_L(\theta)|_{\mathcal{L}^l_I S_L=0} = 0$, as a *Lagrangian master equation.*

4 Local Superfield Quantization

4.1 Superfield Quantum Action in Initial Coordinates

With the standard distribution of ghost number [3] for $\Gamma^P_{\rm CL}(\theta)$, $\mathrm{gh}(\mathcal{A}^*_I) = -1 - \mathrm{gh}(\mathcal{A}^I) = -1$, the choice $\mathrm{gh}(\theta, \partial_\theta) = (-1, 1)$ implying the absence of ghosts among $\mathcal{A}^I$, and, in particular, the relations $(\varepsilon_P)_I = 0$, consists in restricting an LSM (in both Lagrangian and Hamiltonian formulations) by the equations

$$\left(\mathrm{gh}, \frac{\partial}{\partial\theta}\right)S_{H(L)}(\theta) = (0,0). \tag{28}$$

Their solutions, given the existence of a potential term in $S_{H(L)}(\theta)$, $S(\mathcal{A}(\theta),0) = \mathcal{S}(\mathcal{A}(\theta))$, and the absence in $S_{H(L)}(\theta)$ of a dimensional constant with a nonzero ghost number, select from an LSM a standard field theory model with a classical action $S_0(A)$, in which the fields A^i are extended to $\mathcal{A}^i(\theta)$. Then an extended HS in (17) is transformed into a θ-integrable system in $\Pi T^*\mathcal{M}_{\rm cl} = \{\Gamma^p_{\rm cl}(\theta)\} = \{(\mathcal{A}^i, \mathcal{A}^*_i)(\theta)\}$, with functions $\Theta^H_i(\mathcal{A}(\theta)) = \Theta_i(\mathcal{A}(\theta))$,

$$\partial^r_\theta\Gamma^p_{\rm cl}(\theta) = \left(\Gamma^p_{\rm cl}(\theta), S_0(\mathcal{A}(\theta))\right)_\theta,\ \Theta^H_i(\mathcal{A}(\theta)) = -(-1)^{\varepsilon_i}S_{0,i}(\mathcal{A}(\theta)). \tag{29}$$

The restricted special gauge transformations $\delta\mathcal{A}^i(\theta) = \mathcal{R}^i_{0\alpha_0}(\mathcal{A}(\theta))\,\xi^{\alpha_0}_0(\theta)$, $\vec{\varepsilon}(\xi^{\alpha_0}_0(\theta)) = \vec{\varepsilon}_{\alpha_0}$, with the condition $(\varepsilon_P)_{\alpha_0} = 0$, are embedded, under the substitution $\xi^{\alpha_0}_0(\theta) = d\tilde{\xi}^{\alpha_0}_0(\theta) = \mathcal{C}^{\alpha_0}(\theta)d\theta$, $\alpha_0 = 1, ...$, $m_0 = m_{0-} + m_{0+}$, into a Hamiltonian system with $2n$ equations for unknown $\Gamma^p_{\rm cl}(\theta)$, with the Hamiltonian $S^0_1(\Gamma_{\rm cl}, C_0)(\theta) = (\mathcal{A}^*_i\mathcal{R}_0{}^i_{\alpha_0}(\mathcal{A})\mathcal{C}^{\alpha_0})(\theta)$. A union of this system with the HS in (29), extended to $2(n+m_0)$ equations, has the form

$$\partial^r_\theta\Gamma^{p[0]}_{[0]}(\theta) = \left(\Gamma^{p[0]}_{[0]}(\theta), S^0_{[1]}(\theta)\right)_\theta,\ S^0_{[1]}(\theta) = (S_0 + S^0_1)(\theta),\ \Gamma^{p[0]}_{[0]} \equiv (\Gamma^p_{\rm cl}, \Gamma^{p_0}_0),\ \Gamma^{p_0}_0 \equiv (\mathcal{C}^{\alpha_0}, \mathcal{C}^*_{\alpha_0}). \tag{30}$$

By virtue of (11), the function $S^0_1(\theta)$ is invariant, modulo $S_{0,i}(\theta)$, under special gauge transformations of ghost superfields $\mathcal{C}^{\alpha_0}(\theta)$, with arbitrary functions $\xi^{\alpha_1}_1(\theta)$, $(\varepsilon_P)_{\alpha_1} = 0$, on the manifold $\mathcal{M}$:

$$\delta\mathcal{C}^{\alpha_0}(\theta) = \mathcal{Z}^{\alpha_0}_{\alpha_1}(\mathcal{A}(\theta))\xi^{\alpha_1}_1(\theta),\ (\vec{\varepsilon}, \mathrm{gh})\xi^{\alpha_1}_1(\theta) = (\vec{\varepsilon}_{\alpha_1} + (1,0,1), 1). \tag{31}$$

[7]The notion of θ-superfield integrability is used by analogy with the treatment of Ref. [16].

After the substitution $\xi_1^{\alpha_1}(\theta) = d\tilde{\xi}_1^{\alpha_1}(\theta) = \mathcal{C}^{\alpha_1}(\theta)d\theta$, $\alpha_1 = 1,...,m_1$, and an enlargement of m_0 first-order equations in θ, with respect to the unknowns $\mathcal{C}^{\alpha_0}(\theta)$ in transformations (31), to an HS of $2m_0$ equations with the Hamiltonian $S_1^1(\mathcal{A}, \mathcal{C}_0^*, \mathcal{C}_1)(\theta) = (\mathcal{C}^*_{\alpha_0}\mathcal{Z}^{\alpha_0}_{\alpha_1}(\mathcal{A})\mathcal{C}^{\alpha_1})(\theta)$, we obtain a system of the form (30), written for $\partial^r_\theta\Gamma_0^{p_0}(\theta)$. The enlargement of the union of the latter HS with eqs. (30) formally coincides with the system (30) under the replacement

$$(\Gamma^{p[0]}_{[0]}, S^0_{[1]}) \to (\Gamma^{p[1]}_{[1]}, S^1_{[1]}):\ \left\{\Gamma^{p[1]}_{[1]} = (\Gamma^{p[0]}_{[0]}, \Gamma_1^{p_1}),\ \Gamma_1^{p_1} = (\mathcal{C}^{\alpha_1}, \mathcal{C}^*_{\alpha_1}),\ S^1_{[1]} = S^0_{[1]} + S^1_1\right\}.$$

The iteration sequence related to a reformulation of the special gauge transformations of ghosts $\mathcal{C}^{\alpha_0}, ..., \mathcal{C}^{\alpha_{s-2}}$, obtained from (possibly) enhanced[8] relations (11), leads, for an L-stage-reducible restricted LSM at the s-th step, with $0 < s \leq L$ and $\Gamma^p_{\rm cl} \equiv \Gamma^{p_{-1}}_{-1}$, to invariance transformations of $S_1^{s-1}(\theta)$, modulo $S_{0,i}(\theta)$, namely,

$$\begin{aligned}
&\delta\mathcal{C}^{\alpha_{s-1}}(\theta) = \mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(\mathcal{A}(\theta))\xi_s^{\alpha_s}(\theta),\ (\vec{\varepsilon}, \mathrm{gh})\xi_s^{\alpha_s}(\theta) = (\vec{\varepsilon}_{\alpha_s} + s(1,0,1), s),\ (\varepsilon_P)_{\alpha_s} = 0,\\
&S_1^{s-1}(\theta) = (\mathcal{C}^*_{\alpha_{s-2}}\mathcal{Z}^{\alpha_{s-2}}_{\alpha_{s-1}}(\mathcal{A})\mathcal{C}^{\alpha_{s-1}})(\theta),\ \left(\mathrm{gh}, \frac{\partial}{\partial\theta}\right) S_1^{s-1}(\theta) = (0,0).
\end{aligned} \tag{32}$$

The substitution $\xi_s^{\alpha_s}(\theta) = d\tilde{\xi}_s^{\alpha_s}(\theta) = \mathcal{C}^{\alpha_s}(\theta)d\theta$, $\alpha_s = 1,..., m_s = m_{s-} + m_{s+}$, transforms special gauge transformations (32) into m_{s-1} equations with respect to unknown $\partial^r_\theta\mathcal{C}^{\alpha_{s-1}}(\theta)$, extended by the introduction to an HS of superantifields $\mathcal{C}^*_{\alpha_{s-1}}(\theta)$:

$$\partial^r_\theta\Gamma^{p_{s-1}}_{s-1}(\theta) = \left(\Gamma^{p_{s-1}}_{s-1}(\theta), S_1^s(\theta)\right)_\theta,\ S_1^s(\theta) = (\mathcal{C}^*_{\alpha_{s-1}}\mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(\mathcal{A})\mathcal{C}^{\alpha_s})(\theta),\ \Gamma^{p_{s-1}}_{s-1} = (\mathcal{C}^{\alpha_{s-1}}, \mathcal{C}^*_{\alpha_{s-1}}). \tag{33}$$

Having combined the system (33) with an HS of the same form, although with $\partial^r_\theta\Gamma^{p[s-1]}_{[s-1]}(\theta)$ and the Hamiltonian $S^{s-1}_{[1]}(\theta) = (S_0 + \sum_{r=0}^{s-1} S_1^r)(\theta)$, and having written the result for $2\,(n + \sum_{r=0}^{s} m_r)$ equations with $S^s_{[1]}(\theta) = (S^{s-1}_{[1]} + S^s_1)(\theta)$, we obtain, by induction, the following HS:

$$\partial^r_\theta\Gamma^{p[L]}_{[L]}(\theta) = \left(\Gamma^{p[L]}_{[L]}(\theta), S^L_{[1]}(\theta)\right)_\theta,\ S^L_{[1]}(\theta) = S_0(\mathcal{A}(\theta)) + \sum_{s=0}^{L}(\mathcal{C}^*_{\alpha_{s-1}}\mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(\mathcal{A})\mathcal{C}^{\alpha_s})(\theta). \tag{34}$$

The function $S^L_{[1]}(\theta)$, obeying the condition of a proper θ-local solution of the classical master equation [3], with the antibracket extended in $\Pi T^*\mathcal{M}_k{=}\{\Gamma^{p[L]}_{[L]}(\theta) \equiv \Gamma_k^{p_k}(\theta){=}(\Phi^{A_k}, \Phi^*_{A_k})(\theta),\ A_k = n + \sum_{r=0}^{L} m_r,\ k{=}\min\}$, is a solution with accuracy up to $O(\mathcal{C}^{\alpha_s})$, modulo $S_{0,i}$. The integrability of the HS in (34) is guaranteed by a double deformation of the function $S^L_{[1]}(\theta)$: first in powers of $\Phi^*_{A_k}(\theta)$, and then in powers of $\mathcal{C}^{\alpha_s}(\theta)$, in the framework of the existence theorem [28] for the classical master equation in the minimal sector:

$$((S_{Hk}(\Gamma_k(\theta)), S_{H\,k}(\Gamma_k(\theta)))_\theta = 0,\ \left(\vec{\varepsilon}, \mathrm{gh}, \frac{\partial}{\partial\theta}\right) S_{H\,k}(\Gamma_k(\theta)) = \left(\vec{0}, 0, 0\right),\ k = \min. \tag{35}$$

The proposed superfield algorithm for constructing the function $S_{H\min}(\theta)$ may be considered as a superfield version of the Koszul–Tate complex resolution [29]. We remind that the enlargement of $S_{H\min}(\theta)$ to $S_{Hk}(\Gamma_k(\theta))$, $S_{Hk}(\theta) = S_{H\min}(\theta) + \sum_{s=0}^{L}\sum_{s'=0}^{s}(\mathcal{C}^*_{s'\alpha_s}\mathcal{B}^{\alpha_s}_{s'})(\theta)$, being a proper solution [3] in $\Pi T^*\mathcal{M}_k = \{\Gamma_k^{p_k}(\theta)\}$,

$$\begin{aligned}
&\Gamma_k^{p_k}(\theta) = (\Gamma^{p_{\min}}_{\min}, \mathcal{C}^{\alpha_s}_{s'}, \mathcal{B}^{\alpha_s}_{s'}, \mathcal{C}^*_{s'\alpha_s}, \mathcal{B}^*_{s'\alpha_s})(\theta),\ s' = 0,...,s,\ s = 0,...,L,\\
&(\vec{\varepsilon}, \mathrm{gh})\mathcal{C}^{\alpha_s}_{s'}(\theta) = (\vec{\varepsilon}_{\alpha_s} + (s+1)(1,0,1), 2s' - s - 1) = (\vec{\varepsilon}, \mathrm{gh})\mathcal{B}^{\alpha_s}_{s'}(\theta) + ((1,0,1), -1),
\end{aligned}$$

(we assume henceforth $k = \mathrm{ext}$, and take into account that $(\vec{\varepsilon}, \mathrm{gh})\Phi^*_{A_k}(\theta) = -((1,0,1),1) - (\vec{\varepsilon}, \mathrm{gh})\Phi^{A_k}(\theta))$ with the pyramids of ghosts and Nakanishi–Lautrup superfields, and with a deformation in the Planck

[8] From $\mathrm{gh}(\mathcal{A}^I) = 0$ in eqs. (28), with $(\varepsilon_P)_{\mathcal{A}_s} = (\varepsilon_P)_I = 0$, $s = 0,...,L_g$, it follows that the values of $\overline{m}$, $\overline{m}_s$ may be both larger and smaller than the corresponding values $\overline{M}$, $\overline{M}_s$, in contrast to the values of $\overline{n}$, $\overline{N}$. Indeed, for a restricted LSM, the presence of additional gauge symmetries is possible; therefore, we suppose that (possibly) enhanced sets of restricted functions $\mathcal{R}_0{}^i_{\alpha_0}(\theta)$, $\mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}(\theta)$ exhaust, correspondingly, on the surface $S_{0,i}(\theta) = 0$, the zero-modes of both the Hessian $S_{0,ij}(\theta)$ and $\mathcal{Z}^{\alpha_{i-2}}_{\alpha_{i-1}}(\theta)$. As a consequence, this implies that the final stage of reducibility for a restricted model L is different from L_g.

constant $\hbar$, determines the quantum action $S_H^\Psi(\Gamma(\theta),\hbar)$, e.g., in case of an Abelian hypergauge defined as an anticanonical phase transformation:

$$\Gamma_k^{p_k}(\theta)\to\Gamma_k^{\prime p_k}(\theta)=\left(\Phi^{A_k}(\theta),\Phi^*_{A_k}(\theta)-\frac{\partial\Psi(\Phi(\theta))}{\partial\Phi^{A_k}(\theta)}\right):\ S_H^\Psi(\Gamma(\theta),\hbar)=e^{\mathrm{ad}\Psi}S_{Hk}(\Gamma_k(\theta),\hbar). \quad (36)$$

The functions $(S_H^\Psi,S_{Hk})(\theta,\hbar)$ obey eqs. (26), (35) in case the $\hbar$-deformation of $S_{H\min}(\theta)$ is their solution. It is well-known that this choice of equations ensures the integrability of a non-equivalent HS, constructed from S_H^Ψ, S_{Hk}, as well as the anticanonical character, preserving the volume element $dV_k(\theta)=\prod_{p_k}d\Gamma_k^{p_k}(\theta)$, of this change of variables, related to a θ-shift, by a constant parameter μ, along the corresponding HS solutions. In its turn, the quantum master equation

$$\Delta^k(\theta)\exp\left[\frac{i}{\hbar}E(\theta,\hbar)\right]=0,\ E\in\{S_H^\Psi,S_{Hk}\} \quad (37)$$

determines a non-integrable HS, with the corresponding anticanonical change of variables preserving $d\hat{V}_k(\theta)=\exp\left[(i/\hbar)\,E(\theta,\hbar)\right]dV_k(\theta)$. It is the latter nonintegrable HS with the Hamiltonian $S_H^\Psi(\theta,\hbar)$ that is crucial, for $\theta=0$, in the BV formalism. This HS determines on $\Pi T^*\mathcal{M}_k$, a θ-local, but not nilpotent, generator of BRST transformations, $\breve{s}^{l(\Psi)}(\theta)$, which is associated with its θ-nonintegrable consequence

$$\partial_\theta^r\left(\Phi^{A_k},\Phi^*_{A_k}\right)(\theta)=\left(\left(\Phi^{A_k}(\theta),S_H^\Psi(\theta,\hbar)\right)_\theta,0\right),\ \breve{s}^{l(\Psi)}(\theta)=\frac{\partial}{\partial\theta}+\frac{\partial_r S_H^\Psi(\theta,\hbar)}{\partial\Phi^*_{A_k}(\theta)}\frac{\partial_l}{\partial\Phi^{A_k}(\theta)}. \quad (38)$$

4.2 Duality between the BV and BFV Superfield Quantities

An embedding of a restricted LSM gauge algebra, by the action $S_{H\min}(\theta)$ and eq. (35), into the gauge algebra of a general gauge theory in Lagrangian formalism, see eqs. (7)–(12), can be effectively realized by means of dual functional analogues, with the opposite $(\varepsilon_P,\varepsilon)$-parity, of the action and antibracket, following, in part, the approach of Refs. [12,18]. To this end, we consider the functional

$$Z_k[\Gamma_k]=-\partial_\theta S_{Hk}(\theta),\ (\vec{\varepsilon},\mathrm{gh})Z_k=((1,0,1),1)$$

on the supermanifold $\Pi T(\Pi T^*\mathcal{M}_k)=\{(\Gamma_k^{p_k},\partial_\theta\Gamma_k^{p_k})(\theta),k=\min\}$ with natural, $(\varepsilon_P,\varepsilon)$-even, symplectic and odd Poisson structures. These structures define an $(\varepsilon_P,\varepsilon)$-even functional $\{\cdot,\cdot\}$ with canonical pairs $\{(\Phi_k^{A_k},\partial_\theta\Phi^*_{A_k}),\ (\partial_\theta\Phi_k^{A_k},\Phi^*_{A_k})\}(\theta)$, and $(\varepsilon_P,\varepsilon)$-odd θ-local, $(\cdot,\cdot)_\theta^{(\Gamma_k,\partial_\theta\Gamma_k)}$, Poisson brackets. The latter act on the superalgebra $C^\infty(\Pi T(\Pi T^*\mathcal{M}_k)\times\theta)$ and provide the lifting of the antibrackets $(\cdot,\cdot)_\theta$, defined on $\Pi T^*\mathcal{M}_k$. For arbitrary functionals $F_i[\Gamma_k]=\partial_\theta\mathcal{F}_i\left((\Gamma_k,\partial_\theta\Gamma_k)(\theta),\theta\right)$, $i=1,2$, we have the following representation and correspondence between the Poisson brackets of opposite Grassmann grading:

$$\{F_1,F_2\}=\int d\theta\left[\frac{\delta F_1}{\delta\Phi^{A_k}(\theta)}\frac{\delta F_2}{\delta\Phi^*_{A_k}(\theta)}-\frac{\delta_r F_1}{\delta\Phi^*_{A_k}(\theta)}\frac{\delta_l F_2}{\delta\Phi^{A_k}(\theta)}\right]=\int d\theta(\mathcal{F}_1(\theta),\mathcal{F}_2(\theta))_\theta^{(\Gamma_k,\partial_\theta\Gamma_k)},$$
$$(\mathcal{F}_1(\theta),\mathcal{F}_2(\theta))_\theta^{(\Gamma_k,\partial_\theta\Gamma_k)}\equiv\left[(\mathcal{L}_{A_k}\mathcal{F}_1)\,\mathcal{L}^{*A_k}\mathcal{F}_2-(\mathcal{L}_r^{*A_k}\mathcal{F}_1)\mathcal{L}^l_{A_k}\mathcal{F}_2\right](\theta), \quad (39)$$

where the Euler–Lagrange superfield derivative, e.g., with respect to $\Phi^*_{A_k}(\theta)$, for fixed θ, $\mathcal{L}^{*A_k}(\theta)$, has the form $\mathcal{L}^{*A_k}(\theta)=\partial/\partial\Phi^*_{A_k}(\theta)-(-1)^{\varepsilon_{A_k}+1}\partial_\theta\cdot\partial/\partial\left(\partial_\theta\Phi^*_{A_k}(\theta)\right)$.

By construction, the functional Z_k is nilpotent:

$$\{Z_k,Z_k\}=\int d\theta(S_{H\,k}(\theta),S_{Hk}(\theta))_\theta=0,\ k=\min, \quad (40)$$

and, due to the absence of the time coordinate, is formally related to the BRST charge of a dynamical system with first-class constraints [1]. Indeed, after identifying the fields $(\Gamma_k,\partial_\theta\Gamma_k)(0)$ with the phase-space coordinates of the minimal sector, canonical with respect to the $(\varepsilon_P,\varepsilon)$-even brackets in the framework of the BFV method [1] for first-class constrained systems of $(L+1)$-stage reducibility,

$$(q^i,p_i)=(\mathcal{A}^i,\partial_\theta\mathcal{A}_i^*)(0),\ \left(\mathcal{C}^{A_s},\mathcal{P}_{A_s}\right)=\left((\partial_\theta^r\mathcal{C}^{\alpha_{s-1}},\mathcal{C}^{\alpha_s}),(\mathcal{C}^*_{\alpha_{s-1}},\partial_\theta\mathcal{C}^*_{\alpha_s})\right)(0),$$
$$A_s=(\alpha_{s-1},\alpha_s),\ s=0,...,L,\ \left(\mathcal{C}^{A_{L+1}},\mathcal{P}_{A_{L+1}}\right)=\left(\partial_\theta^r\mathcal{C}^{\alpha_L},\mathcal{C}^*_{\alpha_L}\right)(0), \quad (41)$$

the functional Z_k takes the form

$$Z_k[\Gamma_k] = T_{A_0}(q,p)C^{A_0} + \sum_{s=1}^{L+1} \mathcal{P}_{A_{s-1}} Z^{A_{s-1}}_{A_s}(q) C^{A_s} + O(C^2). \tag{42}$$

The constraints $T_{A_0}(q,p)$ and the set of $(L+1)$-stage-reducible eigenvectors $Z^{A_{s-1}}_{A_s}(q)$ are defined – through the gauge algebra structure functions of the original L-stage-reducible restricted LSM in the enhanced eqs. (11) – by the relations ("T" stands for transposition)

$$T_{A_0}(q,p) = \left(S_{0,i}(q), -p_i \mathcal{R}_0{}^i_{\alpha_0}(q)\right),\; Z^{A_{s-1}}_{A_s}(q) = \mathrm{diag}\left(\mathcal{Z}^{\alpha_{s-2}}_{\alpha_{s-1}}, \mathcal{Z}^{\alpha_{s-1}}_{\alpha_s}\right)(q),$$

$$s = 1,...,L,\; \left(Z^{A_L}_{A_{L+1}}\right)^T(q) = \left(\mathcal{Z}^{\alpha_{L-1}}_{\alpha_L}, 0\right)^T(q), \tag{43}$$

$$Z^{A_{s-2}}_{A_{s-1}} Z^{A_{s-1}}_{A_s} = T_{B_0} L^{A_{s-2}B_0}_{A_s}(q,p),\; s = 1,...,\ L+1,\; Z^{A_{-1}}_{A_0} \equiv T_{A_0},\; L^{A_{s-2}\beta_0}_{A_s} = 0,$$

$$L^{A_{s-2}j}_{A_s} = \mathrm{diag}\left(\mathcal{L}^{\alpha_{s-3}j}_{\alpha_{s-1}},\ \mathcal{L}^{\alpha_{s-2}j}_{\alpha_s}\right),\; \mathcal{L}^{\alpha_{-2}j}_{\alpha_0} = \mathcal{L}^{\alpha_{L-1}j}_{\alpha_{L+1}} = 0,\; \mathcal{L}^{\alpha_{-1}j}_{\alpha_1}(q,p) = (-1)^{\varepsilon_j+1} p_i \mathcal{K}^{ji}_{\alpha_1}(q). \tag{44}$$

Formulae (39)–(44) generalize, to the case of arbitrary reducible theories, the results of Ref. [18], concerning a dual description (for $\varepsilon_i = \varepsilon_{\alpha_0} = L = 0$) of the quantum action and classical master equation by means of a nilpotent BRST charge.

A comparison of the superfields $\mathcal{C}^{\alpha_s}_{s'}(\theta)$, $s' = 0,...,s$, selected from the non-minimal configuration space of an L-stage-reducible LSM, with and the coordinates $C^{A_s}_{s'}$ selected from the non-minimal phase space of the corresponding $(L+1)$-stage-reducible dynamical system [1] – and with the rest of the variables $(\mathcal{C}^*_{s'\alpha_s}, \mathcal{B}^*_{s'\alpha_s}, \mathcal{B}^{\alpha_s}_{s'})(\theta)$, identical, by the rule (41), to the respective ghost momenta $\mathcal{P}_{s'A_s}$, Lagrangian multipliers $\lambda_{s'A_s}$ and their conjugate momenta $\pi^{A_s}_{s'}$ in [1] – shows the only embedding of $\Pi T(\Pi T^*\mathcal{M}_{\rm ext})$ into the phase space of the BFV method. Indeed, for the coordinates $C^{A_{L+1}}_0$, $\mathrm{gh}(C^{A_{L+1}}_0) = -L-2$, there exists no pre-image among $(\mathcal{C}^{\alpha_s}_{s'}, \partial_\theta \mathcal{C}^{\alpha_s}_{s'})(0)$, because the ghost number spectrum for the latter variables is bounded from below, by the value

$$\min \mathrm{gh}(\mathcal{C}^{\alpha_s}_{s'}, \partial_\theta \mathcal{C}^{\alpha_s}_{s'}) = \mathrm{gh}(\mathcal{C}^{\alpha_L}_0) = -L-1.$$

As a consequence, the nilpotent functional $Z_k[\Gamma_k] = -\partial_\theta S_{Hk}(\theta)$, $k = \mathrm{ext}$, is embedded into the complete BRST charge constructed by the prescriptions of Ref. [1].

It should be noted that the systems constructed with respect to the Hamiltonians $S^\Psi_H(\Gamma(\theta),\hbar)$ and $S_{Hk}(\theta)$, $k = \min, \mathrm{ext}$, are equivalently described by dual fermion functionals $Z_k[\Gamma_k]$ and $Z^\Psi[\Gamma] = -\partial_\theta S^\Psi_H(\Gamma(\theta),\hbar)$, in terms of even Poisson brackets, for instance,

$$\partial^r_\theta \Gamma^p(\theta) = \left(\Gamma^p(\theta), S^\Psi_H(\Gamma(\theta),\hbar)\right)_\theta = -\left\{\Gamma^p(\theta), Z^\Psi[\Gamma]\right\}. \tag{45}$$

Thereby, BRST transformations in Lagrangian formalism with Abelian hypergauges can be encoded by a formal BRST charge, $Z^\Psi[\Gamma]$, related to $Z_k[\Gamma_k]$, $k = \mathrm{ext}$, by means of a phase canonical transformation with the $(\varepsilon_P, \varepsilon)$-even phase $F^\Psi[\Phi] = \partial_\theta \Psi(\Phi(\theta))$,

$$Z^\Psi[\Gamma] = e^{\overline{\mathrm{ad}}\, F^\Psi} Z_k[\Gamma_k],\; \overline{\mathrm{ad}}\, F^\Psi \equiv \left\{F^\Psi, \cdot\right\}. \tag{46}$$

The problem of including the restricted LSM gauge algebra into that of the initial general gauge theory, defined by relations (2), (7), (8) – given the assumption that an additional gauge invariance does not appear in deriving the former model from the latter, i.e., $\overline{m}_s \le \overline{M}_s$, and, therefore, $L \le L_g$, cf. footnote 4 – is solved with the help of a nilpotent functional defined on $\Pi T(\Pi T^*\mathcal{M}_k) = \{(\Gamma^{P_k}_k, \partial_\theta \Gamma^{P_k}_k)(\theta), \Gamma^{P_k}_k(\theta) = \left(\Gamma^{P_{\rm CL}}_{\rm CL}, \mathcal{C}^{A_s}, \mathcal{C}^*_{A_s}\right)(\theta),\ s = 0,1,...,L_g,\ k = \mathrm{MIN}\}$, namely,

$$\begin{aligned}\hat{Z}_k[\Gamma_k] &= Z[\mathcal{A}] + \sum_{s=0}^{L_g}\left(\int d\theta_{s-1} d\theta_s \mathcal{C}^*_{A_{s-1}}(\theta_{s-1}) \hat{\mathcal{Z}}^{A_{s-1}}_{A_s}(\theta_{s-1};\theta_s) \mathcal{C}^{A_s}(\theta_s)(-1)^{\varepsilon_{A_{s-1}}+s}\right.\\ &\left. + O(\mathcal{C}^{A_s})\right) = \int d\theta S_{Lk}\left((\Gamma_k, \partial_\theta \Gamma_k)(\theta), \theta\right).\end{aligned} \tag{47}$$

Note that the representation of solutions to the generating equation $\left\{\hat{Z}_k, \hat{Z}_k\right\} = 0$ as expansions in powers of superfields $\mathcal{C}^{A_s}$ – introduced as simple ghosts $\mathcal{C}^{\alpha_s}$ although used for a description of a general gauge algebra – can be controlled by an additional *generalized ghost number*, gh_g, $\mathrm{gh}_g(\hat{Z}_k) = 0$, coinciding with the standard ghost number only in the sector of $(\Phi^{A_{\mathrm{MIN}}}, \Phi^*_{A_{\mathrm{MIN}}})(0)$, for $(\varepsilon_P)_{\mathcal{A}_s} = (\varepsilon_P)_I = 0$, and having the spectrum

$$\mathrm{gh}_g(\mathcal{A}^I, C^{\mathcal{A}_s}) = (0, 1+s),\ \mathrm{gh}_g(\Phi^*_{A_{\mathrm{MIN}}}) = -1 - \mathrm{gh}_g(\Phi^{A_{\mathrm{MIN}}}),\ \mathrm{gh}_g(\theta, \partial_\theta) = (0,0).$$

Conditions (28), applied to $S_{Lk}(\theta)$ and to $(\varepsilon_P)_{\mathcal{A}_s} = (\varepsilon_P)_I = 0$, $s = 0, ..., L_g$, select from $\hat{Z}_k$ the functional Z_k in (42), so that an $(\varepsilon_P, \varepsilon)$-even θ-density $S_{Lk}(\theta)$ lifts the function $S_{Hk}(\theta) \in C^\infty(\Pi T^*\mathcal{M}_{\min})$ to $C^\infty(\Pi T(\Pi T^*\mathcal{M}_{\mathrm{MIN}}) \times \theta)$. In general, $S_{Lk}(\theta)$ does not satisfy the generalized master equation (35) with antibracket (39) acting on $C^\infty(\Pi T(\Pi T^*\mathcal{M}_{\mathrm{MIN}}) \times \theta)$,

$$(S_{Lk}(\theta), S_{Lk}(\theta))_\theta^{(\Gamma_k, \partial_\theta\Gamma_k)} = \tilde{f}\left((\Gamma_k, \partial_\theta\Gamma_k)(\theta), \theta\right),\ \tilde{f}(\theta) \in \ker\{\partial_\theta\},\ k = \mathrm{MIN}. \tag{48}$$

4.3 Local Quantization

Leaving aside the realization of a reducible LSM on $\Pi T^*\mathcal{M}_{\mathrm{ext}}$, we now suppose that the model is described by a quantum action, $W(\theta, \hbar) = W(\theta)$, defined on an arbitrary antisymplectic manifold $\mathcal{N}$ without connection, $\dim \mathcal{N} = \dim \Pi T^*\mathcal{M}_{\mathrm{ext}} = (\overline{n} + (n_-, n_+) + \sum_{r=0}^{L}(2r+3)(\overline{m}_r + (m_{r-}, m_{r+}))$, with local coordinates $\Gamma^p(\theta)$ and a density function $\rho(\Gamma(\theta))$. A local antibracket, an invariant volume element, $d\mu(\Gamma(\theta))$, and a nilpotent second-order operator, $\Delta^{\mathcal{N}}(\theta)$, are defined with the help of an $(\varepsilon_P, \varepsilon)$-odd Poisson bivector, $\omega^{pq}(\Gamma(\theta)) = (\Gamma^p(\theta),\ \Gamma^q(\theta))_\theta^{\mathcal{N}}$, namely,

$$d\mu(\Gamma(\theta)) = \rho(\Gamma(\theta))d\Gamma(\theta),\ \Delta^{\mathcal{N}}(\theta) = \frac{1}{2}(-1)^{\varepsilon(\Gamma^q)}\rho^{-1}\omega_{qp}(\theta)\left(\Gamma^p(\theta), \rho\left(\Gamma^q(\theta),\ \ \right)_\theta^{\mathcal{N}}\right)_\theta^{\mathcal{N}}. \tag{49}$$

The definition of a generating functional of Green's functions, $\mathcal{Z}\left((\partial_\theta\varphi^*, \varphi^*, \partial_\theta\varphi, I)(\theta)\right) = \mathcal{Z}(\theta)$, as a path integral, for a fixed θ, is possible, within perturbation theory, by introducing on $\mathcal{N}$ the Darboux coordinates, $\Gamma^p(\theta) = (\varphi^a, \varphi^*_a)(\theta)$, in a vicinity of solutions of the equations $\partial W(\theta)/\partial\Gamma^p(\theta) = 0$, so that, $\rho = 1$ and $\omega^{pq}(\theta) = \mathrm{antidiag}(-\delta^a_b, \delta^a_b)$. The function

$$\begin{aligned}\mathcal{Z}(\theta) \quad = \quad & \int d\lambda(\theta)\, d\mu\left(\tilde{\Gamma}(\theta)\right) \exp\left\{(i/\hbar)\left[W\left(\tilde{\Gamma}(\theta), \hbar\right) + X\left((\tilde{\varphi}, \tilde{\varphi}^* - \varphi^*, \lambda, \lambda^*)(\theta), \hbar\right)|_{\lambda^*=0}\right.\right.\\ & \left.\left. - \left((\partial_\theta\varphi^*_a)\tilde{\varphi}^a + \tilde{\varphi}^*_a\partial^r_\theta\varphi^a - I_a\lambda^a\right)(\theta)\right]\right\}\end{aligned} \tag{50}$$

depends on an extended set of sources,

$$\begin{aligned}&(\partial_\theta\varphi^*_a, \partial^r_\theta\varphi^a, I_a)(\theta) = (-J_a, \lambda^a, I_{0a} + I_{1a}\theta),\\ &(\vec{\varepsilon}, \mathrm{gh})\partial_\theta\varphi^*_a = (\vec{\varepsilon}, \mathrm{gh})I_a + ((1,0,1),1) = (\vec{\varepsilon}, -\mathrm{gh})\varphi^a,\end{aligned}$$

to the superfields $(\varphi^a, \varphi^*_a, \lambda^a)(\theta)$, where $\lambda^a(\theta) = (\lambda^a_0 + \lambda^a_1\theta)$ are Lagrangian multipliers to independent non-Abelian hypergauges, see [21],

$$\begin{aligned}&G_a(\Gamma(\theta)), a = 1, ..., \ k = n + \sum_{r=0}^{L}(2r+3)m_r,\ k = k_+ + k_-,\\ &\mathrm{rank}\,\|\partial G_a(\theta)/\partial\Gamma^p(\theta)\|_{\partial W/\partial\Gamma = G = 0} = \bar{l},\ l = l_+ + l = k.\end{aligned}$$

The functions $G_a(\Gamma(\theta))$, $(\vec{\varepsilon}, \mathrm{gh})G_a = (\vec{\varepsilon}, \mathrm{gh})I_a$, determine a boundary condition for the gauge-fixing action, $X(\theta) = X\left((\Gamma, \lambda, \lambda^*)(\theta), \hbar\right)$,

$$\partial_r X(\theta)/\partial\lambda^a(\theta)|_{\lambda^* = \hbar = 0} = G_a(\theta),$$

defined on the direct sum $\mathcal{N}_{\mathrm{tot}} = \mathcal{N} \oplus \Pi T^*\mathcal{K}$ of the manifolds $\mathcal{N}$ and $\Pi T^*\mathcal{K} = \{(\lambda^a, \lambda^*_a)(\theta)\}$. Hypergauges in involution, $(G_a(\theta), G_b(\theta))_\theta^{\mathcal{N}} = G_c(\theta)U^c_{ab}(\Gamma(\theta))$, obey different types of unimodularity relations [21], depending on a set of equations for which $X(\theta)$ may be a solution, independently from $W(\theta)$, in terms of the antibracket $(\cdot\,,\cdot)_\theta = (\cdot\,,\cdot)_\theta^{\mathcal{N}} + (\cdot\,,\cdot)_\theta^{\mathcal{K}}$ and the operator $\Delta(\theta) = (\Delta^{\mathcal{N}} + \Delta^{\mathcal{K}})(\theta)$, trivially lifted from $\mathcal{N}$ to $\mathcal{N}_{\mathrm{tot}}$,

$$1)\ (E(\theta), E(\theta))_\theta = 0,\ \Delta(\theta)E(\theta) = 0;\ 2)\ \Delta(\theta)\exp\left[\frac{i}{\hbar}E(\theta)\right] = 0,\ E \in \{W, X\}. \tag{51}$$

The functions $G_a(\theta)$, assumed to be solvable with respect to $\varphi_a^*(\theta)$, determine a Lagrangian surface, $\Lambda_g = \{(\varphi^*, \lambda)(\theta)\} \subset \mathcal{N}_{\rm tot}$, on which the restriction $X(\theta)|_{\Lambda_g}$ is non-degenerate. Given this, integration over $(\tilde{\varphi}^*, \lambda)(\theta)$ in eq. (50) determines a function, for $\partial_\theta^r \varphi^a = I_a = 0$, whose restriction to the Lagrangian surface $\Lambda = \{\varphi(\theta)\} \subset \mathcal{N}$ is also non-degenerate.

In view of the properties of $(W, X)(\theta)$, one can define an effective action, $\Gamma(\theta) = \Gamma(\varphi, \varphi^*, \partial_\theta^r \varphi, I)(\theta)$, introduced in the usual manner, i.e., by means of a Legendre transformation of $\mathcal{Z}(\theta)$ with respect to $\partial_\theta \varphi_a^*(\theta)$,

$$\Gamma(\theta) = \frac{\hbar}{i} \ln \mathcal{Z}(\theta) + ((\partial_\theta \varphi_a^*)\varphi^a))(\theta),\ \varphi^a(\theta) = -\frac{\hbar}{i} \frac{\partial_l \ln \mathcal{Z}(\theta)}{\partial(\partial_\theta \varphi_a^*(\theta))}. \tag{52}$$

The analysis of the properties of $(\mathcal{Z}, \Gamma)(\theta)$ is based on the following θ-nonintegrable Hamiltonian-like system, which contains an arbitrary $(\varepsilon_P, \varepsilon)$-even $C^\infty(\mathcal{N}_{\rm tot})$-function, $R(\theta) = R\left((\tilde{\Gamma}, \lambda, \lambda^*)(\theta), \hbar\right)$, with a vanishing ghost number:

$$\begin{aligned}
&\partial_\theta^r \tilde{\Gamma}^p(\theta) = -i\hbar\, T^{-1}(\theta)\left(\tilde{\Gamma}^p(\theta), T(\theta)R(\theta)\right)_\theta\Big|_{\lambda^*=0}\,,\\
&\partial_\theta^r \lambda^a(\theta) = -2i\hbar\, T^{-1}(\theta)(\lambda^a(\theta), T(\theta)R(\theta))_\theta|_{\lambda^*=0}\,,\\
&\partial_\theta^r\left(\varphi_a^*, \lambda_a^*\right)(\theta) = 0\,,
\end{aligned} \tag{53}$$

where the function $T\left((\tilde{\Gamma}, \lambda, \lambda^*)(\theta), \hbar\right) = T(\theta)$ has the form $T(\theta) = \exp\left[(i/\hbar)\,(W - X)(\theta)\right]$. Let us list the properties of $(\mathcal{Z}, \Gamma)(\theta)$.

1. The integrand in (50) is invariant, for $\partial_\theta \varphi^* = \partial_\theta^r \varphi = I = 0$, with respect to the *superfield BRST transformations*

$$\tilde{\Gamma}_{\rm tot}(\theta) = (\tilde{\Gamma}, \lambda, \lambda^*)(\theta) \to \left(\tilde{\Gamma}_{\rm tot} + \delta_\mu \tilde{\Gamma}_{\rm tot}\right)(\theta),\ \delta_\mu \tilde{\Gamma}_{\rm tot}(\theta) = \left(\partial_\theta^r \tilde{\Gamma}_{\rm tot}\right)\Big|_{\check{\Gamma}_{\rm tot}} \mu, \tag{54}$$

having the form of θ-shifts by a constant parameter μ, along an arbitrary solution $\check{\Gamma}_{\rm tot}(\theta)$ of the system (53), or, equivalently, along a vector field determined by the r.h.s. of (53), for $R(\theta) = 1$. Here, the arguments of $(W, X)(\theta)$ are the same as in definition (50). The above statement can be verified with the help of the identities

$$\partial_r X(\theta)/\partial F(\theta)|_{\lambda^*=0} = \partial_r (X(\theta)|_{\lambda^*=0})/\partial F(\theta),\ F = (\Gamma, \lambda).$$

Notice that the system (53), for $R(\theta) = {\rm const}$, has the integral $(W + X)(\theta)$, in case W and X obey the first system in (51).

2. The vacuum function $\mathcal{Z}_X(\theta) = \mathcal{Z}(0, \varphi^*, 0, 0)(\theta)$ is gauge-independent in changing $X(\theta)$ by an action $(X + \Delta X)(\theta)$ which obeys the same system in (51) that is valid for $X(\theta)$ and conforms to the condition of nondegeneracy on the surface Λ_g. Indeed, from this hypothesis it follows that the variation $\Delta X(\theta)$ obeys a linearized equation with a nilpotent operator $Q_j(X)$, $j = 1, 2$,

$$Q_j(X)\Delta X(\theta) = 0,\ Q_j(X) = {\rm ad}\, X(\theta) - \delta_{j2}(i\hbar\Delta(\theta)), \tag{55}$$

where j is identical to the number that labels the system in eqs. (51) for which $X(\theta)$ is a solution. Using the fact that solutions $X(\theta)$ of every system in (51) are proper, one can prove, by analogy with the theorems of Ref. [30], that the cohomologies of the operator $Q_j(X)$ on functions $f(\Gamma_{\rm tot}(\theta)) \in C^\infty(\mathcal{N}_{\rm tot})$, vanishing for $\Gamma_{\rm tot}(\theta) = 0$, are trivial. Hence, the general solution of eq. (55) has the form

$$\Delta X(\theta) = Q_j(X)\Delta Y(\theta),\ \left(\vec{\varepsilon}, {\rm gh}, \frac{\partial}{\partial\theta}\right)\Delta Y(\theta) = ((1, 0, 1), -1, 0)\,,\ \Delta Y(\theta)|_{\Gamma_{\rm tot}=0} = 0, \tag{56}$$

with a certain $\Delta Y(\theta)$. Now, having performed in $\mathcal{Z}_{X+\Delta X}(\theta)$ a change of variables related to a θ-shift by a constant μ, corresponding to the system (53), and choosing

$$2R(\theta)\mu = \Delta Y(\theta),$$

we find $\mathcal{Z}_{X+\Delta X}(\theta) = \mathcal{Z}_X(\theta)$ and conclude that the S-matrix is gauge-independent, in view of the equivalence theorem [31][9].

[9]Properties 1, 2 of $\mathcal{Z}_X(\theta)|_{\varphi^*=0}$ are valid for arbitrary $\rho(\theta)$, $\Gamma^p(\theta)$ on the manifold $\mathcal{N}$.

The above proof shows that the system (53) encodes, due to the convention (54), the BRST transformations for $R(\theta) = \text{const}$ (without contributions from Z^R), and, at the same time, the continuous anticanonical transformations in an infinitesimal form, with a scalar fermionic generating function, $R(\theta)\mu$, for an arbitrary $R(\theta)$ and a constant μ.

Equivalently, following the ideas of Subsection 4.2, the above characteristics of the generating functional of Green's functions can be derived from a Hamiltonian-like system, presented in terms of a superfield even Poisson bracket in general coordinates (see footnote 9),

$$\partial_\theta^r \tilde{\Gamma}^p(\theta) = -\left\{\tilde{\Gamma}^p(\theta), Z^W[\tilde{\Gamma}] - (Z^X + i\hbar Z^R)[\tilde{\Gamma}_{\rm tot}]\right\}_{\lambda^*=0},$$
$$\partial_\theta^r \lambda^a(\theta) = -2\left\{\lambda^a(\theta), Z^W[\tilde{\Gamma}] - (Z^X + i\hbar Z^R)[\tilde{\Gamma}_{\rm tot}]\right\}_{\lambda^*=0},$$
$$\partial_\theta^r \left(\varphi_a^*, \lambda_a^*\right)(\theta) = 0, \tag{57}$$

with a linear combination of fermionic functionals, corresponding to the above actions, and a bosonic function, by the rule

$$Z^E[\Gamma_{\rm tot}] = -\partial_\theta E(\Gamma_{\rm tot}(\theta), \hbar),\ E \in \{W, X, R\}. \tag{58}$$

If the actions $(W, X)(\theta)$ obey the first system in (51), then the functionals Z^W, Z^X, formally playing the role of the usual and *gauge-fixing* BRST charges, are nilpotent with respect to the even Poisson bracket $\{\cdot,\cdot\} = \{\cdot,\cdot\}^{\Pi T\mathcal{N}} + \{\cdot,\cdot\}^{\Pi T\mathcal{K}}$. Here, for instance, the first bracket in the sum is defined on arbitrary functionals over $\Pi T\mathcal{N} \times \{\theta\}$, via a θ-local extension of the odd bracket $(\cdot,\cdot)_\theta^{\Pi T\mathcal{N}}$ in (39), as follows:

$$\{F_1, F_2\}^{\Pi T\mathcal{N}} \equiv \int d\theta \frac{\delta_r F_1}{\delta\Gamma^p(\theta)} \omega^{pq}(\Gamma(\theta)) \frac{\delta_l F_2}{\delta\Gamma^q(\theta)} = \partial_\theta (\mathcal{F}_1(\theta), \mathcal{F}_2(\theta))_\theta^{\Pi T\mathcal{N}},$$
$$(\mathcal{F}_1(\theta), \mathcal{F}_2(\theta))_\theta^{\Pi T\mathcal{N}} \equiv ((\mathcal{L}_p^r \mathcal{F}_1)\omega^{pq}(\Gamma(\theta))\mathcal{L}_q^l \mathcal{F}_2)(\theta),\ F_i[\Gamma] = \partial_\theta \mathcal{F}_i((\Gamma, \partial_\theta\Gamma)(\theta), \theta), \tag{59}$$

where $\mathcal{L}_q^l(\theta)$ is the left-hand Euler–Lagrange superfield derivative with respect to $\Gamma^q(\theta)$[10]

Therefore, as in the case of the HS in (45), we arrive at the interpretation of the BRST transformations, for a gauge theory with non-Abelian hypergauges in Lagrangian formalism, in terms of the formal "BRST charges" Z^W, Z^X, as well as in terms of the functional Z^R and the even Poisson bracket[11]. The system (57) encodes the BRST transformations, for $Z^R = 0$, and, at the same time, the BRST and continuous canonical transformations with the bosonic generating functional $Z^R\mu$, for an arbitrary Z^R and a constant μ.

3. The functions $(\mathcal{Z}, \Gamma)(\theta)$ obey the Ward identities

$$\partial_\theta \varphi_a^*(\theta) \frac{\partial_l \mathcal{Z}(\theta)}{\partial \varphi_a^*(\theta)} + \frac{i}{\hbar} I_a(\theta) \frac{\partial_l}{\partial \lambda_a^*(\theta)} X \left(i\hbar \frac{\partial_l}{\partial(\partial_\theta \varphi^*)}, i\hbar \frac{\partial_r}{\partial(\partial_\theta^r \varphi)} - \varphi^*, \frac{\hbar}{i}\frac{\partial_l}{\partial I}, \lambda^*\right)\bigg|_{\lambda_a^*=0} \mathcal{Z}(\theta) = 0, \tag{60}$$

$$I_a(\theta) \frac{\partial_l}{\partial \lambda_a^*(\theta)} X \left(\varphi^b + i\hbar (\Gamma''^{-1})^{bc} \frac{\partial_l}{\partial \varphi^c}, i\hbar \frac{\partial_r}{\partial(\partial_\theta^r \varphi)} - \frac{\partial_r}{\partial(\partial_\theta^r \varphi)} - \varphi^*, \frac{\partial_l \Gamma}{\partial I} + \frac{\hbar}{i}\frac{\partial_l}{\partial I}, \lambda^*\right)\bigg|_{\lambda_a^*=0}$$
$$+ \frac{1}{2} (\Gamma(\theta), \Gamma(\theta))_\theta^{(\Gamma)} = 0, \tag{61}$$

with the notation $\Gamma''_{ab}(\theta) = \frac{\partial_l}{\partial \varphi^a(\theta)} \frac{\partial_r}{\partial \varphi^b(\theta)} \Gamma(\theta)$, $\Gamma''_{ac}(\theta)(\Gamma''^{-1})^{cb}(\theta) = \delta_a{}^b$. Namely, in the symmetric form of the above identities, we have extended the standard set of sources $\partial_\theta \varphi_a^*(\theta)$ used in the definition of the generating functional of Green's functions in Abelian hypergauges.

Identities (60) and (61) follow from the corresponding system in (51) for $(W, X)(\theta)$. For instance, making the functional averaging of the second system in (51), for $X(\theta)$,

$$\int d\mu \left(\tilde{\Gamma}(\theta)\right) d\lambda(\theta) \exp\left[\frac{i}{\hbar}\left(W - (\partial_\theta \varphi_a^*)\tilde{\varphi}^a - \tilde{\varphi}_a^* \partial_\theta^r \varphi^a + I_a \lambda^a\right)(\theta)\right]$$
$$\times \left\{\Delta(\theta) \exp\left[\frac{i}{\hbar} X\left((\tilde{\varphi}, \tilde{\varphi}^* - \varphi^*, \lambda, \lambda^*)(\theta), \hbar\right)\right]\right\}_{\lambda^*=0} = 0, \tag{62}$$

[10] The antibracket $(\cdot,\cdot)_\theta^{\Pi T\mathcal{N}}$ coinciding, for $\mathcal{N} = \Pi T\mathcal{M}_k$, with $(\cdot,\cdot)_\theta^{(\Gamma_k, \partial_\theta \Gamma_k)}$, $k = \text{ext}$, in (39) lifts the operator $\Delta^{\mathcal{N}}$ in (49) to the nilpotent operator $\Delta^{\Pi T\mathcal{N}}$ on $C^\infty(\Pi T\mathcal{N} \times \{\theta\})$, defined exactly as $\Delta^{\mathcal{N}}(\theta)$, although in terms of the antibracket (59).

[11] The construction of the latter bracket is different from that of [5], where an odd superfield Poisson bracket was derived from a (t, θ)-local even bracket; however, it is similar to the construction of Ref. [18], see eqs. (27).

and integrating by parts in (62), with allowance for $(\partial/\partial\tilde{\varphi}^* + \partial/\partial\varphi^*)X(\theta) = 0$, we obtain eq. (60). Identities (60) and (61) take the standard form for $\partial^r_\theta\varphi^a = I_a(\theta) = \theta = 0$, although in the case of non-Abelian hypergauges.

In the special case of Abelian hypergauges, $G_A\left((\Phi,\Phi^*)(\theta)\right) = \Phi^*_A(\theta) - \partial\Psi(\Phi(\theta))/\partial\Phi^A(\theta) = 0$, related to the change of variables (36), for $(\varphi,\varphi^*,W) = (\Phi,\Phi^*,S_{H\,\text{ext}})$, $\partial^r_\theta\Phi^A = I_A = 0$ (locally, $\mathcal{N} = \Pi T^*\mathcal{M}_{\text{ext}}$), the object $\mathcal{Z}(\partial_\theta\Phi^*,\Phi^*)(\theta)$ takes the form

$$\mathcal{Z}\left(\partial_\theta\Phi^*,\Phi^*\right)(\theta) = \int d\Phi(\theta)\exp\left\{\frac{i}{\hbar}\left[S^\Psi_H\left(\Gamma(\theta),\hbar\right) - \left((\partial_\theta\Phi^*_A)\Phi^A\right)(\theta)\right]\right\}. \tag{63}$$

A θ-local BRST transformation for $\mathcal{Z}\left(\partial_\theta\Phi^*,\Phi^*\right)(\theta)$ is given, for an HS defined on $\Pi T^*\mathcal{M}_{\text{ext}}$ with the Hamiltonian $S^\Psi_H(\theta,\hbar)$ and a solution $\check{\Gamma}(\theta)$, by the change of variables

$$\Gamma^p(\theta) \to \Gamma^{(1)p}(\theta) = \exp\left[\mu s^{l(\Psi)}(\theta)\right]\Gamma^p(\theta),\ s^{l(\Psi)}(\theta) \equiv \frac{\partial}{\partial\theta} - \text{ad}S^\Psi_H(\theta,\hbar). \tag{64}$$

Transformation (64) with a constant μ is anticanonical, with $\text{Ber}\|\frac{\partial\Gamma^{(1)}(\theta)}{\partial\Gamma(\theta)}\| = \text{Ber}\|\frac{\partial\Phi^{(1)}(\theta)}{\partial\Phi(\theta)}\| = 1$, if $S^\Psi_H(\theta,\hbar)$ is subject to the first system in (51).

The obvious permutation rule of the functional integral, $\varepsilon(d\Phi(\theta)) = 0$,

$$\partial_\theta\int d\Phi(\theta)\mathcal{F}\left((\Phi,\Phi^*)(\theta),\theta\right) = \int d\Phi(\theta)\left[\frac{\partial}{\partial\theta} + (\partial_\theta V_+)(\theta)\right]\mathcal{F}(\theta),\ \partial_\theta V_+(\theta) = \partial_\theta\Phi^*_A(\theta)\frac{\partial}{\partial\Phi^*_A(\theta)},$$

yields, for $i\hbar\partial^r_\theta\ln\mathcal{Z}(\theta) = (\partial_\theta\Phi^*_A\partial^r_\theta\Phi^A)(\theta) - \partial^r_\theta\Gamma(\theta)$, the following relations:

$$\partial_\theta\mathcal{Z}(\theta)|_{\check{\Gamma}(\theta)} = (\partial_\theta V_+)(\theta)\mathcal{Z}(\theta) = 0,\ \partial^r_\theta\Gamma(\theta)|_{\check{\Gamma}(\theta)} = \left(\Gamma(\Gamma(\theta)),\Gamma(\Gamma(\theta))\right)_\theta = 0. \tag{65}$$

When deriving eqs. (65), we have taken into account that the functional averaging of the HS with respect to $\mathcal{Z}(\theta)$, $\Gamma(\theta)$ has the form

$$\langle\partial^r_\theta\Gamma^p\rangle|_{\mathcal{Z}} = \left(\frac{\hbar}{i}\mathcal{Z}^{-1}\frac{\partial\mathcal{Z}(\theta)}{\partial\Phi^*_A(\theta)}, -\partial_\theta\Phi^*_A(\theta)\right),\ \langle\partial^r_\theta\Gamma^p\rangle = \left(\langle\Gamma^p(\theta)\rangle,\Gamma(\langle\Gamma(\theta)\rangle)\right)_\theta = \partial^r_\theta\langle\Gamma^p\rangle, \tag{66}$$

without the sign of average in (65) for $\check{\Gamma}(\theta)$ and $\Gamma^p(\theta)$. Expressions (65) relate the explicit form of the Ward identities, in a theory with Abelian hypergauges, to the invariance of the generating functional of Green's functions with respect to the superfield BRST transformations.

5 Connections between Lagrangian Quantizations

5.1 Component Formulation and its Relation to Batalin–Vilkovisky, Batalin–Tyutin and Superfield Methods

The relation of the objects and quantities of θ-local quantization in the Lagrangian and Hamiltonian formulations of an LSM with the conventional description of gauge field theory is established through a component representation of the variables $\Gamma^{P_{\text{MIN}}}_{\text{MIN}}(\theta)$, $\Gamma^{p_k}_k(\theta)$, $I_a(\theta)$, $\Gamma^{p_k}_k(\theta) = \Gamma^{p_k}_{0k} + \Gamma^{p_k}_{1k}\theta$, $k = \text{tot}$, under the restriction $\theta = 0$, for instance, $(\mathcal{M},\mathcal{N}_k,I_a) \to (\widetilde{\mathcal{M}},\mathcal{N}_k|_{\theta=0} = \{\Gamma^{p_k}_{0k}\},I_{0a})$. The extraction of a standard field model from a classical formulation of a genera gauge theory is realized, in addition to $\theta = 0$, by different kinds of eliminating the functions $\partial_\theta\mathcal{A}^I(\theta)$, $\mathcal{A}^*_I(\theta)$, and those superfields among $\mathcal{A}^I(\theta)$ which contain functions with an incorrect spin-statistics relation, $\varepsilon_P(\mathcal{A}^I) \neq 0$. A first possibility of such elimination is given by the conditions $\text{gh}(\mathcal{A}^I) = -1 - \text{gh}(\mathcal{A}^*_I) = 0$, $(\varepsilon_P)_I = 0$, and $(\text{gh},\partial/\partial\theta)\,S_{L(H)}(\theta) = (0,0)$, mentioned in Subsection 4.1.

A second possibility is related to the superfield BRST transformations of Yang–Mills type theories [10, 32, 33], for which, a Lagrangian classical action $S_{\text{LYM}}(\theta) = S_L\left(\mathcal{A},\mathcal{D}_\theta\mathcal{A},\tilde{\mathcal{A}},\mathcal{D}_\theta\tilde{\mathcal{A}}\right)(\theta)$ is defined in terms of generalized Yang–Mills superfields, $\mathcal{A}^{Bs}(z)$, $\mathcal{A}^{Bs} = (\mathcal{A}^{\mu s},\mathcal{C}^s)$, $s = 1,...,r$, and matter superfields, $\tilde{\mathcal{A}}(z) = (\Psi^\delta,\overline{\Psi}^\epsilon,\varphi^f,\varphi^{+g})(z)$ – with spinor, Ψ^δ, $\overline{\Psi}^\epsilon$, $\delta,\epsilon = 1,...,k_1$, and spinless, φ^f, φ^{+g}, $f,g = 1,...,k_2$, superfields – defined on the superspace $M = R^{1,3}\times\tilde{P} = \{z^B = (x^\mu,\theta)\}$, and taking values, respectively, in the adjoint and vector representation spaces of an r-parametric Lie group. The action $S_{\rm LYM}(\theta)$ can be written as

$$S_{\rm LYM}(\theta) = \int d^4x \left[\frac{1}{4}\mathcal{G}_{BC}{}^s\mathcal{G}^{CBs}(-1)^{\varepsilon_B} - i\overline{\Psi}^\epsilon\gamma^B\nabla_B{}^\epsilon_\delta\Psi^\delta - \overline{\nabla}_B{}^h_g\varphi^{+g}\nabla^B{}^h_f\varphi^f + M(\tilde{\mathcal{A}})\right](z), \tag{67}$$

with an $\tilde{\mathcal{A}}(z)$-local gauge-invariant polynomial $M(\tilde{\mathcal{A}})$, containing no derivatives with respect to z^B. In expression (67), we have introduced the superfield strength $\mathcal{G}_{BC}{}^s{=}i[\mathcal{D}_B,\mathcal{D}_C]^s{=}\partial_B\mathcal{A}^s_C-(-1)^{\varepsilon_B\varepsilon_C}\partial_C\mathcal{A}^s_B+f^{sut}\mathcal{A}^u_B\mathcal{A}^t_C$, $\partial_B = (\partial_\mu,\partial_\theta)$ and the following covariant derivatives, expressed through the matrix elements of the Hermitian generators $\Gamma^u = \mathrm{diag}\left(T^u,\overline{T}^u,\tau^u,\overline{\tau}^u\right)$ of the corresponding Lie algebra:

$$\left(\mathcal{D}^{st}_B, \nabla_B{}^\epsilon_\delta, \nabla_B{}^e_f, \overline{\nabla}_B{}^g_h\right) = \partial_B\left(\delta^{st},\delta^\epsilon_\delta,\delta^e_f,\delta^g_h\right) + \left(f^{sut}, -i(T^u)^\epsilon_\delta, -i(\tau^u)^e_f, -i(\overline{\tau}^u)^g_h\right)\mathcal{A}^u_B, \tag{68}$$

where the coupling constant is absorbed into the completely antisymmetric structure coefficients f^{uts}. We have also used a generalization of Dirac's matrices, $\gamma^B = (\gamma^\mu,\gamma^\theta)$, $\gamma^\theta = (\gamma^\theta)^+ = \xi 1_4$, with a Grassmann scalar ξ. The $\vec{\varepsilon}$-grading and ghost number are nonvanishing for the superfields $(\Psi,\overline{\Psi},\mathcal{C}^s)$, namely, $\vec{\varepsilon}(\Psi,\overline{\Psi}) = (0,1,1)$, $\vec{\varepsilon}(\mathcal{C}^s) = (1,0,1)$, $\mathrm{gh}(\mathcal{C}^s) = 1$. The functional $Z[\mathcal{A},\tilde{\mathcal{A}}] = \partial_\theta S_{\rm LYM}(\theta)$ is invariant under the infinitesimal general gauge transformations

$$\delta_g\mathcal{A}^I(\theta) = \delta_g(\mathcal{A}^{Bs};\tilde{\mathcal{A}})(z) = -\int d^5z_0\left(\mathcal{D}^{Bst}(z)(-1)^{\varepsilon_B}; i\Gamma^t\tilde{\mathcal{A}}(-1)^{\varepsilon(\tilde{\mathcal{A}})}\right)\delta(z-z_0)\xi^t(z_0), \tag{69}$$

with arbitrary bosonic ($\vec{\varepsilon}_{\mathcal{A}_0} = 0$) functions $\xi^t(z_0)$ on $\mathcal{M}$, and with functionally-independent generators $\hat{\mathcal{R}}^I_{\mathcal{A}_0}(\theta,\theta_0) \equiv \hat{\mathcal{R}}^I_{\mathcal{A}_0}(\mathcal{A}(z),z,z_0)$. The condensed indices I, $\mathcal{A}_0$ of the theory in question, $(I;\mathcal{A}_0) = ((B,s,\delta,\epsilon,f,h,x);(t,x_0))$, conform to the relations, $\overline{N} > \overline{n}$, $\overline{M} = \overline{m}$, $(\overline{m},\overline{M}) = (\overline{m}_0,\overline{M}_0)$, in case

$$\overline{N} = (4r+2k_2, r+8k_1),\ \overline{M} = (r,0),\ \overline{n} = \overline{N} - (0,r),$$

which hold for a reduced theory with the action $S_{\rm YM}(\theta) = -S_L\left(\mathcal{A},0,\tilde{\mathcal{A}},0\right)(\theta)$ on $\mathcal{M}_{\rm cl} = \{\mathcal{A}^{\mu s},\tilde{\mathcal{A}}\}(z)$[12], in view of special *horizontality conditions* for the strength $\mathcal{G}_{BC}{}^s$ and certain subsidiary conditions for the matter superfields $\tilde{\mathcal{A}}(z)$ in [10,32],

$$\mathcal{G}_{BC}{}^s(z) = \mathcal{G}_{\mu\nu}{}^s(z),\ \left(\nabla_\theta{}^\eta_\delta\Psi^\delta, \overline{\nabla}_\theta{}^\vartheta_\epsilon\overline{\Psi}^\epsilon, \nabla_\theta{}^e_f\varphi^f, \overline{\nabla}_\theta{}^g_h\overline{\varphi}^g\right)(z) = (0,0,0,0). \tag{70}$$

To extract a standard component model defined on $\mathcal{M}_{\rm cl}|_{\theta=0}$ from a Hamiltonian LSM description, it is sufficient to eliminate, for $\theta=0$, the antifields $\mathcal{A}^*_I(\theta)$ of a Yang–Mills type theory, by analogy with the prescription (70), i.e., by taking into account the relation between $\mathcal{A}^*_I(\theta)$ and $\partial_\theta\mathcal{A}^I(\theta)$: see Section 3 and the final remarks (see item A) in the Conclusion.

For the restricted LSM used in the Feynman rules of Section 4, the reduction to the model in the framework of the multilevel formalism of Ref. [21] is provided by the conditions

$$\theta = 0,\ \partial_\theta\varphi^*_a = \partial^r_\theta\varphi^a = \varphi^*_a = I_a = 0. \tag{71}$$

In this case, the first-level functional integral $Z^{(1)}$ and its symmetry transformations [21], with the notation λ^a_0, instead of π^a, for Lagrangian multipliers in [21],

$$Z^{(1)} = \int d\lambda_0 d\Gamma_0 M(\Gamma_0)\exp\left\{\frac{i}{\hbar}\left(W(\Gamma_0)+G_a(\Gamma_0)\lambda^a_0\right)\right\},$$

$$\left\{\begin{array}{l}\delta\Gamma^p_0 = (\Gamma^p_0, -W+G_a\lambda^a_0)\mu,\\ \delta\lambda^a_0 = \left(-U^a_{cb}\lambda^b\lambda^c(-1)^{\varepsilon_c} + 2i\hbar V^a_b\lambda^b + 2(i\hbar)^2\tilde{G}^a\right)\mu,\end{array}\right.$$

under the identification $(\rho,\omega^{pq})(\Gamma_0) = (M,E^{pq})(\Gamma_0)$, implying the coincidence of $(\cdot,\cdot)_\theta|_{\theta=0}$ and $\Delta(0)$ with their counterparts of [21], coincide with $\mathcal{Z}_X(0)|_{\varphi^*_0=0}$ and with the BRST transformations $\delta_\mu\Gamma_{0\rm tot}$ (having the opposite signs), generated by the system (53), for $R(\theta)=1$. This coincidence is guaranteed by the choice of $X(\theta)$ in the form

$$X(\theta) = \left\{G_a(\Gamma)\lambda^a - \lambda^*_a\left[\frac{1}{2}U^a_{cb}(\Gamma)\lambda^b\lambda^c(-1)^{\varepsilon_c} - i\hbar V^a_b(\Gamma)\lambda^b - (i\hbar)^2\tilde{G}^a(\Gamma)\right]\right\}(\theta) + o(\lambda^*), \tag{72}$$

[12]For $\theta=0$, the functional $S_{\rm YM}(0) = S_{0\rm YM}$ coincides with the corresponding classical action of [35].

where $(V_b^a, \tilde{G}^a)(\theta)$, together with $(U_{cb}^a, G_a)(\theta)$, define the unimodularity relations [21]. The relation to the generating functional of Green's function $Z[J,\phi^*]$ of the BV method [3] is evident after identifying $\mathcal{Z}(\partial_\theta\Phi^*, \Phi^*)(0) = Z[J,\phi^*]$ in (63), where the action $S_H^\Psi(\Gamma_0, \hbar)$ in (36) satisfies eq. (37).

The following aspect of the restriction $\theta = 0$ consists in the representation of an arbitrary function $\mathcal{F}(\theta) = \mathcal{F}((\Gamma, \partial_\theta\Gamma)(\theta), \theta) \in C^\infty(\Pi T\mathcal{N} \times \{\theta\})$ by a functional $F[\Gamma]$ of the superfield methods [6,7] (in case $\Gamma^p = (\Phi^A, \Phi_A^*)$, see the Introduction)

$$F[\Gamma] = \int d\theta\theta\mathcal{F}(\theta) = \mathcal{F}(\Gamma(0), \partial_\theta\Gamma, 0) \equiv \mathcal{F}(\Gamma_0, \Gamma_1). \tag{73}$$

Formula (73) implies, in the first place, the independence of $F[\Gamma]$ from $\partial_\theta^r\Gamma^p(\theta) = \Gamma_1^p$, in case $F(\theta) = F(\Gamma(\theta), \theta)$. Secondly, it is fundamental in establishing a relation between the θ-local antibracket $(\,\cdot\,,\,\cdot\,)_\theta^{\mathcal{N}}$ and operator $\Delta^{\mathcal{N}}(\theta)$, acting on $C^\infty(N \times \{\theta\})$, with a generalization, to the case of arbitrary $(\Gamma, \omega^{pq}, \rho)(\theta)$, of the flat functional operations $(\,\cdot\,,\,\cdot\,)$, Δ of Refs. [6,7], coinciding with their representations in the BV method, for $\Gamma^p = (\Phi^A, \Phi_A^*)$, $\omega^{pq}(\Gamma(\theta)) = \text{antidiag}\left(-\delta_B^A, \delta_B^A\right)$, $\rho(\theta) = 1$, and for a different odd Poisson bivector, $\tilde{\omega}^{pq}(\Gamma(\theta), \theta') = (1 + \theta'\partial_\theta)\omega^{pq}(\theta)$. The correspondence follows from

$$\begin{aligned}
&(\mathcal{F}(\theta), \mathcal{G}(\theta))_\theta^{\mathcal{N}}\big|_{\theta=0} = \frac{\delta_r\mathcal{F}(\Gamma_0)}{\delta\Gamma_0^p}\omega^{pq}(\Gamma_0)\frac{\delta_l\mathcal{G}(\Gamma_0)}{\delta\Gamma_0^q} = (F[\Gamma], G[\Gamma])^{\mathcal{N}}, \\
&(F[\Gamma], G[\Gamma])^{\mathcal{N}} = \partial_\theta\left[\frac{\delta_r F[\Gamma]}{\delta\Gamma^p(\theta)}\partial_{\theta'}\left(\tilde{\omega}^{pq}(\Gamma(\theta), \theta')\frac{\delta_l G[\Gamma]}{\delta\Gamma^q(\theta')}\right)\right](-1)^{\varepsilon(\Gamma^p)+1},
\end{aligned}\tag{74}$$

$$\begin{aligned}
&\Delta^{\mathcal{N}}(\theta)\mathcal{F}(\theta)\big|_{\theta=0} = \Delta^{\mathcal{N}}(0)\mathcal{F}(\Gamma_0) = \Delta^{\mathcal{N}}F[\Gamma], \\
&\Delta^{\mathcal{N}} = \frac{1}{2}(-1)^{\varepsilon(\Gamma^q)}\partial_\theta\partial_{\theta'}\left[\rho^{-1}[\Gamma]\tilde{\omega}_{qp}(\theta', \theta)\left(\Gamma^p(\theta), \rho[\Gamma]\left(\Gamma^q(\theta'), \cdot\right)^{\mathcal{N}}\right)^{\mathcal{N}}\right],
\end{aligned}\tag{75}$$

where $\left(\rho[\Gamma], \tilde{\omega}_{pq}(\theta', \theta)\right) = \left(\rho(\Gamma_0), \theta'\theta\omega_{pq}(\theta)\right)$ and

$$\int d\theta''\tilde{\omega}^{pd}(\theta', \theta'')\tilde{\omega}_{dq}(\theta'', \theta) = \theta\delta^p{}_q.$$

When establishing the correspondence with the operations $(\,\cdot\,,\,\cdot\,)$, Δ of [6,7] in (74), (75), we have used a relation between the superfield and component derivatives:

$$\delta_l/\delta\Gamma^p(\theta) = (-1)^{\varepsilon(\Gamma^p)}\left(\theta\delta_l/\delta\Gamma_0^p - \delta_l/\delta\Gamma_1^p\right), \; \Gamma_1^p = (\lambda^A, -(-1)^{\varepsilon_A}J_A).$$

In general coordinates, the action of the sum and difference of the operators $\partial_\theta(V_+ \pm U_+)^{\mathcal{N}}(0)$, reduced, in the case $\mathcal{N} = \Pi T^*\mathcal{M}'_{\text{ext}}|_{\theta=0}$, to

$$\partial_\theta(V_+ \mp U_+)(0) = \partial_\theta\Phi_A^*(\theta)\partial/\partial\Phi_A^*(\theta) \pm \partial_\theta\Phi^A(\theta)\partial_l/\partial\Phi^A(\theta),$$

is identical to the action of the generalized sum and difference of the functional analogies V, U in [6]:

$$\begin{aligned}
&\partial_\theta(V_+ - (-1)^t U_+)^{\mathcal{N}}(\theta)\mathcal{F}(\theta)\big|_{\theta=0} = \left(\mathcal{S}^t(\theta), \mathcal{F}(\theta)\right)_\theta^{\mathcal{N}}{}_{|\theta=0} = (V - (-1)^t U)^{\mathcal{N}}F[\Gamma], \; t = 1, 2, \\
&\mathcal{S}^t(\theta) = (\partial_\theta\Gamma^p)\omega_{pq}^t(\Gamma(\theta))\Gamma^q(\theta), \; \omega_{pq}^t(\theta) = (-1)^{\varepsilon(\Gamma^p)\varepsilon(\Gamma^q)+t}\omega_{qp}^t(\theta), \omega_{pq}^1(\theta) \equiv \omega_{pq}(\theta), \\
&(V - (-1)^t U)^{\mathcal{N}} = (S^t[\Gamma], \cdot)^{\mathcal{N}}, \; S^t[\Gamma] = \mathcal{S}^t(0) = \partial_\theta\left\{\Gamma^p(\theta)\partial_{\theta'}\partial_\theta\left[\tilde{\omega}_{pq}^t(\theta, \theta')\Gamma^q(\theta')\right]\right\}, \\
&\tilde{\omega}_{pq}^t(\theta, \theta') = -(-1)^{t+\varepsilon(\Gamma^p)\varepsilon(\Gamma^q)}\tilde{\omega}_{qp}^t(\theta', \theta) = \theta\theta'\omega_{pq}^t(\theta'), \; (\vec{\varepsilon}, \text{gh})\left(\mathcal{S}^t(\theta)\right) = (\vec{0}, 0).
\end{aligned}\tag{76}$$

The quantities $\mathcal{S}^t(\theta)$ and $S^t[\Gamma]$ play the role of the symmetric Sp(2)-tensor S_{ab} $(a, b = 1, 2)$ and anti-Hamiltonian S_0 of Ref. [34], which determine (through extended antibrackets) the first-order operators of the modified triplectic algebra. In this case, the additional functions $\omega_{pq}^2(\theta)$, $\tilde{\omega}_{pq}^2(\theta, \theta')$ may be considered as quantities that define another non-antisymplectic (non-Riemannian) nondegenerate structure on $\mathcal{N}$. The θ-local functional operators $\{\Delta^{\mathcal{N}}, V^{\mathcal{N}}, U^{\mathcal{N}}\}(\theta)$ anticommute for a fixed θ,

$$[E_i^{\mathcal{N}}(\theta), E_j^{\mathcal{N}}(\theta)]_+ = 0, \; i, j = 1, 2, 3, \; (E_1, E_2, E_3) = (\Delta, V, U), \tag{77}$$

provided that $\mathcal{S}^t(\theta)$, or $S^t[\Gamma]$, is subject to

$$\left(\mathcal{S}^t(\theta), \mathcal{S}^u(\theta)\right)_\theta^{\mathcal{N}} = 0, \; \Delta^{\mathcal{N}}(\theta)\mathcal{S}^t(\theta) = 0, \; t, u = 1, 2. \tag{78}$$

Relations (78), which hold, due to eqs. (74)–(77), also for functional objects (i.e., those without θ-dependence), follow from the well-known properties of the antibracket (*bilinearity*, *graded antisymmetry*, *Leibniz rule*, *Jacobi identity*), and from the rule of the generation of the antibracket by the operator $\Delta^{\mathcal{N}}(\theta)$. The system (78) specifies the geometry of $\mathcal{N}$ by restricting the choice of both quantities $\omega^t_{pq}(\theta)$, $\tilde{\omega}^t_{pq}(\theta,\theta')$. Notice that a solution of eqs. (78) always exists, e.g., $\omega^t_{pq}(\theta) = \text{antidiag}\left(\delta^A_B, (-1)^t\delta^A_B\right)$.

5.2 Superfield Functional Quantization in General Coordinates

Let us consider a generalization of the vacuum functional of the superfield method [6, 7], namely,

$$Z^{\mathcal{N}}_{X'} = \int d\mu[\Gamma] q^{\mathcal{N}}[\Gamma] \exp\left\{\frac{i}{\hbar}\left(W' + X' + \varkappa_2 S^2\right)[\Gamma]\right\}, \tag{79}$$

where $\varkappa_2$ is an arbitrary real number; W', X' are the quantum and gauge-fixing actions, defined on $\mathcal{N}$ and subject to the equations

$$\frac{1}{2}(W', W')^{\mathcal{N}} + \mathcal{V}W' = i\hbar\Delta^{\mathcal{N}}W',\ \frac{1}{2}(X', X')^{\mathcal{N}} + \mathcal{U}X' = i\hbar\Delta^{\mathcal{N}}X', \tag{80}$$

while the integration measure and the weight functional $q^{\mathcal{N}}[\Gamma]$ have the form

$$d\mu[\Gamma] = \rho[\Gamma]\tilde{d}\Gamma,\ \tilde{d}\Gamma = d\Gamma_0 d\Gamma_1,\ q^{\mathcal{N}}[\Gamma] = \delta(\mathcal{V}\Gamma(\theta)). \tag{81}$$

In (81), we have introduced a two-parametric set, $\mathcal{U}(t,\varkappa)$, $\mathcal{V}(t,\varkappa)$, of anti-commuting generalized operators,

$$\mathcal{U} = \frac{1}{2}(-1)^t\varkappa_t(S^t[\Gamma], \cdot)^{\mathcal{N}},\ \mathcal{V} = \frac{1}{2}\varkappa_t(S^t[\Gamma], \cdot,)^{\mathcal{N}}, \tag{82}$$

satisfying, together with $\Delta^{\mathcal{N}}$, the algebra (77) – for arbitrary real numbers $\varkappa_t$, whose choice fixes the form of $Z^{\mathcal{N}}_{X'}$ – as well as equations (80) and the boundary conditions for W' and X'.

The basic properties of the functional $Z^{\mathcal{N}}_{X'}$ are analogous to those of 1, 2 for $\mathcal{Z}(\theta)$ in (50), encoded by a Hamiltonian-like system with an arbitrary functional $R[\Gamma]$, $(\vec{\varepsilon}, \text{gh})\, R = (\vec{0}, 0)$,

$$\partial^r_\theta\Gamma^p(\theta) = \frac{\hbar}{i}T^{-1}[\Gamma]\left(\Gamma^p(\theta), T[\Gamma]R\right)^{\mathcal{N}},\ \ T[\Gamma] = \exp\left[\frac{i}{\hbar}\left(W' - X' + \varkappa_1 S^1\right)\right]. \tag{83}$$

For instance, the *superfield BRST transformations*, $\delta_\mu\Gamma^p(\theta) = \partial^r_\theta\Gamma^p(\theta)\mu$, for $Z^{\mathcal{N}}_{X'}$ are derived from (80), for $R = 1$, and from the additional equations

$$\varkappa_t\left(\partial_{\theta'}\left(\tilde{\omega}^{pq}(\theta,\theta')\frac{\delta_l S^t}{\delta\Gamma^q(\theta')}\right), W' - X' + \varkappa_1 S^1\right)^{\mathcal{N}} = 0 \Longleftrightarrow \delta_\mu(\mathcal{V}\Gamma^p(\theta)) = 0, \tag{84}$$

providing the BRST invariance of $q^{\mathcal{N}}$. In order to be valid for any gauge theory with an admissible action, eqs. (84) impose strong restrictions on the quantities $\tilde{\omega}^t_{pq}(\theta,\theta')$, and, therefore, on the geometry of $\mathcal{N}$. For example, the constant functions $\tilde{\omega}^t_{pq}(\theta,\theta')$ belong to the set of solutions to eqs. (84). However, more generally, we do not restrict the consideration to this special case, assuming that eqs. (84) are fulfilled for any W', X'.

Setting

$$\left(\varkappa_t, \Gamma^p, \rho, \tilde{\omega}^t_{pq}(\theta,\theta')\right) = \left(1, (\Phi^A, \Phi^*_A), 1, \theta\theta'\text{antidiag}\left(\delta^A_B, (-1)^t\delta^A_B\right)\right), \tag{85}$$

we obtain

$$\left((\mathcal{V},\mathcal{U})F, S^2\right) = \left((V, -U)F(-1), \partial_\theta(\Phi^*_A\Phi^A)(\theta)\right), \tag{86}$$

where $(V, U) = (-1)^{\varepsilon_A}\partial_\theta\left(-\Phi^*_A(\theta)\partial_\theta\delta/\delta\Phi^*_A(\theta), \Phi^A(\theta)\partial_\theta\delta_l/\delta\Phi^A(\theta)\right)$, according to [6], and, therefore, we arrive at the coincidence of $Z^{\mathcal{N}}_{X'}$, and, besides, of equations (80) and BRST transformations, that follow from (83) in case $R = 1$, respectively, with the vacuum functional Z,

$$Z = \int d\Phi d\Phi^*\delta(\partial_\theta\Phi^*(\theta))\exp\left\{\frac{i}{\hbar}\left(W[\Phi,\Phi^*] + X[\Phi,\Phi^*] + \partial_\theta(\Phi^*_A\Phi^A)\right)\right\},$$

and with the equations $(W,X) = (W',X')$, $1/2(W,W) + VW = i\hbar\Delta W$, $1/2(X,X) - UX = i\hbar\Delta X$ and BRST symmetry transformations [7] for Z (having the opposite signs in the r.h.s.)

$$\delta\Phi^A(\theta) = \mu U\Phi^A(\theta) + (\Phi^A(\theta), X - W)\mu\,,\ \delta\Phi^*_A(\theta) = \mu V\Phi^*_A(\theta) + (\Phi^*_A(\theta), X - W)\mu\,.$$

In particular, choosing the action X in terms of the gauge fermion $\Psi[\Phi] = \Psi[\phi,\lambda]$, $X[\Phi,\Phi^*] = U\Psi[\Phi]$, first realized in [6], we arrive at the generating functional of Green's functions $Z[\Phi^*]$ used in Section 1 in order to determine the superfield effective action in Abelian hypergauges.

The maximal correspondence of the functional $\mathcal{Z}_X(0)|_{\varphi^*_0=0}$ in (50) with $Z^{\mathcal{N}}_{X'}$ follows, in the first place, from the representation $(1/2)(1+(-1)^t)\varkappa_t S^t$ for the functional $\varkappa_2 S^2$, so that the redefined actions

$$W'' = W' + \frac{1}{2}\varkappa_t S^t,\ X'' = X' + \frac{1}{2}(-1)^t\varkappa_t S^t \tag{87}$$

obey eqs. (80), without the operators $\mathcal{V}$ and $\mathcal{U}$. Second, let the actions $W(\theta)$ in (50) and $W''[\Gamma]$, as well as the quantities $X(\theta)|_{\lambda^*=0}$ in (50) and $X''[\Gamma]$, be related by formula (73). Third, the solvability of the hypergauges $G_a[\Gamma]$ with respect to the fields $\varphi^*_a(\theta)$, on condition that $\lambda^a(\theta) = \partial^r_\theta\varphi^a(\theta)$, implies, together with the previous restriction, a linear dependence of $X''[\Gamma]$ on $\lambda^a(\theta)$, and its independence from $\partial_\theta\varphi^*_a(\theta)$. Next, the structure of the generating equation for $X''[\Gamma]$, as well as the second system for $X(\theta)$ in (51), having consequently the form (72), and, finally, the fact that the corresponding systems (83), (53), encoding the BRST transformations, coincide with each other, require the commutativity of $G_a[\Gamma]$ and the triviality of the unimodularity relations, i.e., $\Delta^{\mathcal{N}}G_a = V^a_b = \tilde{G}^a = 0$. Finally, the measure $d\mu[\Gamma]q^{\mathcal{N}}$ in (79) is identical to $d\mu(\Gamma(\theta))d\lambda(\theta)|_{\theta=0}$ in (50), with the choice of $q^{\mathcal{N}}$ as $q^{\mathcal{N}} = \delta(\partial_\theta\varphi^*(\theta))$. The latter choice can be realized by $(\varkappa_t, \tilde{\omega}^t_{pq}(\theta,\theta')) = \left(1, \theta\theta'\mathrm{antidiag}(\delta^a_b, (-1)^t\delta^a_b)\right)$.

6 Conclusion

Let summarize the main results of the present work:

1. We have proposed a θ-local description of an arbitrary reducible superfield model, as a natural extension of a standard gauge field theory, defined on a configuration space $\mathcal{M}_{\mathrm{cl}}|_{\theta=0}$ of classical superfields A^i, to a superfield theory defined on extended cotangent, $\Pi T^*\mathcal{M}_{\mathrm{cl}}\times\{\theta\}$, and tangent, $\Pi T\mathcal{M}_{\mathrm{cl}}\times\{\theta\}$, odd bundles, in the respective Hamiltonian and Lagrangian formulations. It is shown that the conservation, under the θ-evolution, of the Hamiltonian action $S_H\left((\mathcal{A},\mathcal{A}^*)(\theta),\theta\right)$, being an odd analogue of the energy $S_E\left((\mathcal{A},\partial_\theta\mathcal{A})(\theta),\theta\right)$, is equivalent, due to Noether's first theorem, to the Lagrangian (Hamiltonian) master equation, i.e., the Lagrangian (Hamiltonian) system for superfield extensions of the usual extremals.

2. Using non-Abelian hypergauges, we have constructed a θ-local superfield formulation of Lagrangian quantization of a reducible gauge model, selected from a general superfield model by conditions of the explicit θ-independence of the classical action and the vanishing of ghost number and auxiliary Grassmann parity (associated with θ) for the action and $\mathcal{A}^i(\theta)$. In particular, we have proposed a new superfield algorithm for constructing a first approximation to the quantum action in powers of ghosts of the minimal sector, on the basis of interpreting the reducibility relations as special gauge transformations of ghosts, transformed in an HS with the Hamiltonian chosen as the quantum action. To investigate the properties of BRST invariance and gauge-independence in a superfield form, for the introduced generating functionals of Green's functions (including the effective action), we have used *two equivalent* Hamiltonian-like systems. The first system is defined by a θ-local antibracket, in terms of a quantum action, a gauge-fixing action, and an arbitrary θ-local boson function, while the second (dual) system is defined by an even Poisson bracket, in terms of fermion functionals corresponding to the above functions. The two systems permit one to describe the BRST transformations and the continuous (anti)canonical transformations in a manner analogous to the the relation between these transformations in the superfield Hamiltonian formalism [5]. We emphasize that, as a basis for the local quantization, we have intensely used the first-level formalism of [21], whose central ingredient is the vacuum functional (however, without recourse to the gauge-fixing action in an explicit form).

3. We have considered the problem of a *dual description* of L-stage-reducible gauge theory, in terms of a BRST charge for a formal dynamical system with first-class constraints of $(L+1)$-stage-reducibility. It is shown that this problem is a particular case of describing the embedding of a reducible special gauge theory into a general gauge theory of the same stage of reducibility.

4. We have proposed an extension of the functional superfield quantization [6,7] to the case of general antisymplectic manifold without connection. We have found that the requirements of anti-commutativity

for all operators and of the correct transformation of the path integral measure impose strong restrictions on the geometry on the manifold.

5. We have established the coincidence of the first-level functional integral $Z^{(1)}$ in [21] with the local vacuum function of the proposed quantization scheme, in case $\theta = 0$ and $\varphi^*(\theta) = 0$, $\mathcal{Z}_X(0)|_{\varphi_0^*=0}$. We have found a correspondence between $\mathcal{Z}_X(0)|_{\varphi_0^*=0}$ and the vacuum functional $Z_{X'}^{\mathcal{N}}$ of the proposed extension of the superfield quantization [6, 7]. We have shown that the above functionals coincide only in the case of Abelian hypergauges.

From the obtained results follow the generating functional of Green's functions and the effective action defined in the first-level formalism [21]. We also observe that, in the case of the dependence of the quantum action $W'[\Gamma]$, or more-than-linear dependence of the gauge-fixing action $X'[\Gamma]$, on the superfields $\partial_\theta \Gamma^p(\theta)$, the functional $Z_{X'}^{\mathcal{N}}$ differs from $\mathcal{Z}_X(0)|_{\varphi_0^*=0}$ exactly as the functional Z in [7] differs from $Z^{(1)}$ in [21].

In relation with the above points, the following open problems seem to be of interest:

A. A construction of a Hamiltonian formulation of an LSM from a Lagrangian formulation in the case of a degenerate Hessian supermatrix $(S''_L)_{ij}(\theta)$ in (5) and its relation to the standard component description of a model. In this case, the use of Dirac's algorithm in terms of a θ-local antibracket, under the conservation of primary constraints in the course of the θ-evolution along a vector field defined by an HS with the primary Hamiltonian expressed in terms of antifields, would determine all the antisymplectic constraints on the classical superfields $\Gamma^p_{\rm cl}(\theta)$. Among these constraints, there may be a subsystem of second-class ones, in the case of the degeneracy of the supermatrix $\left\|\mathcal{L}^l_J(\theta_1)\left[\mathcal{L}^l_I(\theta_1)S_L(\theta_1)(-1)^{\varepsilon_i}\right]\right\|_\Sigma$ in (6). It is interesting to apply the BFV method to construct, in terms of a θ-local Dirac's antibracket, $(\,\cdot\,,\,\cdot\,)_{\theta D}$[13], a triplet of θ-local quantities $\tilde{S}_H(\theta)$, $\tilde{\Omega}(\theta)$, $\tilde{\Psi}(\theta)$: $(\varepsilon_P, \varepsilon)$-even functions $\tilde{S}_H(\theta)$, $\tilde{\Omega}(\theta)$, commuting with respect to $(\,\cdot\,,\,\cdot\,)_{\theta D}$, – by analogy with the Hamilton function and the BFV–BRST charge in a t-local field theory – and an $(\varepsilon_P, \varepsilon)$-odd function $\tilde{\Psi}(\theta)$, which encodes the dynamics of an LSM, its first-class constraint algebra, and the fixing of "gauge" arbitrariness in a wider space than $\Pi T^*\mathcal{M}_{\rm cl} \times \{\theta\}$. In this connection, it appears of interest to consider the question: "How will the construction of the 'unitarizing Hamiltonian', $\tilde{\mathcal{S}}_H(\theta) = \tilde{S}_H(\theta) + (\tilde{\Omega}(\theta), \tilde{\Psi}(\theta))_{\theta D}$, and of $\tilde{S}_H(\theta)$, $\tilde{\Omega}(\theta)$, be related to the quantum action of the BV method?"

B. From the solution of the dual problem of Subsection 4.2, obtained within the classical description, there arise two natural questions: "How will the operator formulation of a formal dynamical system with a nilpotent BRST charge and a quantum analogue of the even Poisson bracket correspond to the Lagrangian quantization of a gauge model?" and "What will be related, in a Lagrangian formulation, to the formal supercommutator and the Hilbert space of states?" The mentioned problems seem to be related with the correspondence found in Ref. [36] between Poisson brackets and their operator analogues of the opposite parity.

C. Notice that one of the possibilities of describing theories with non-Abelian hypergauges within the superfield method [6, 7] consists in an enlargement of the component spectrum of superfields $(\Phi^A, \Phi^*_A)(\theta)$ by a Grassmann parameter $\tilde{\theta}$, not related to an additional antiBRST symmetry. In this case, the inclusion of $(\Phi^A, \Phi^*_A)(\theta)$ and the fields λ^*_A, anticanonically conjugate to λ^A, into the general superfields $(\Phi^A, \overline{\Phi}_A)(\theta, \tilde{\theta})$ is provided by the relations

$$(\Phi^A, \partial_{\tilde{\theta}}\overline{\Phi}_A)(\theta, 0) = (\Phi^A, \Phi^*_A)(\theta),\ \ \overline{\Phi}_A(0,0) = \lambda^*_A.$$

Finally, note that the procedure of $N = 1$ local quantization has been recently developed in [37], as applied to the case of reducible general hypergauges when it is impossible to determine hypergauge conditions in a covariant manner on an antisymplectic manifold.

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