Ñ IFUSP

Instituto de Física
Universidade de São Paulo

NON-VOLKOV SOLUTIONS FOR A CHARGE IN A PLANE WAVE

Bagrov, V

Instituto de Física, Universidade de São Paulo,
Caixa Postal 66318-CEP, 05315-970 São Paulo,
SP, Brazil

Publicação IF – 1606/2005

UNIVERSIDADE DE SÃO PAULO
Instituto de Física
Cidade Universitária
Caixa Postal 66.318
05315-970 - São Paulo - Brasil

1606

Non-Volkov solutions for a charge in a plane wave

V. Bagrov* and D. Gitman†

Instituto de Física, Universidade de São Paulo,
Caixa Postal 66318-CEP, 05315-970 São Paulo, S.P., Brazil

January 26, 2005

Abstract

We focus our attention, once again, on the Klein–Gordon and Dirac equations with a plane-wave field. We recall that for the first time a set of solutions of these equations was found by Volkov. The Volkov solutions are widely used in calculations of quantum effects with electrons and other elementary particles in laser beams. We demonstrate that one can construct sets of solutions which differ from the Volkov solutions and which may be useful in physical applications. For this purpose, we show that the transversal charge motion in a plane wave can be mapped by a special transformation to transversal free particle motion. This allows us to find new sets of solutions where the transversal motion is characterized by quantum numbers different from Volkov's (in the Volkov solutions this motion is characterized by the transversal momentum). In particular, we construct solutions with semiclassical transversal charge motion (transversal squeezed coherent states). In addition, we demonstrate how the plane-wave field can be eliminated from the transversal charge motion in a more complicated case of the so-called combined electromagnetic field (a combination of a plane-wave field and constant colinear electric and magnetic fields). Thus, we find new sets of solutions of the Klein–Gordon and Dirac equations with the combined electromagnetic field.

1 Introduction

Relativistic wave equations (Dirac and Klein–Gordon) provide a basis for relativistic quantum mechanics and QED of spinor and scalar particles. In relativistic quantum mechanics, solutions of relativistic wave equations are referred to as one-particle wave functions of fermions and bosons in external electromagnetic fields. In QED, such solutions permit the development of the perturbation expansion known as the Furry picture, which incorporates the interaction with the

*On leave from Tomsk State University and Tomsk Institute of High Current Electronics, Russia; e-mail: bagrov@phys.tsu.ru

†E-mail: gitman@dfn.if.usp.br

external field exactly, while treating the interaction with the quantized electromagnetic field perturbatively [1, 2, 3, 4, 5]. The most important exact solutions of the Klein–Gordon and Dirac equations are: solutions with the Coulomb field, which allow one to construct the relativistic theory of atomic spectra [6], solutions with a uniform magnetic field, which provide the basis of synchrotron radiation theory [7], and solutions in the field of a plane wave, which are widely used for calculations of quantum effects involving electrons and other elementary particles in laser beams [8]. This is why any progress in studying these basic solutions can result in new physical applications and seems to be important. In the present article, we focus our attention, once again, on the Klein–Gordon and Dirac equations with a plane-wave field. We recall that for the first time a set of solutions of these equations was obtained by Volkov in [10], see Sec. 2. It is Volkov's solutions that have been used in all of the above-mentioned calculations. However, we demonstrate below that one can construct sets of solutions which differ from the Volkov solutions, and which may be useful in physical applications. It is known that the transversal charge motion in the Volkov solutions is characterized by a definite transversal momentum. We show that the transversal charge motion in a plane wave can be mapped by a special transformation to transversal free particle motion. This allows us to find sets of solutions where the transversal charge motion is characterized by different (from the Volkov case) quantum numbers, see Sec. 3. The importance of constructing solutions with different quantum numbers is related to possible different experimental conditions, e.g., special initial charge states, or specially prepared charge states in a plane-wave. In particular, we construct solutions with semiclassical transversal charge motion (transversal coherent states). In Sec. 4, we demonstrate how the plane-wave field can be eliminated from the transversal charge motion in a more complicated case of the so-called combined electromagnetic field (a combination of a plane-wave field and colinear constant electric and magnetic fields). One ought to say that Volkov-like solutions with the combined field were first obtained in [11, 12]. Using the above-mentioned transformation, we find new sets of solutions of the Klein–Gordon and Dirac equations with the combined electromagnetic field.

2 Volkov solutions for a charge in a plane-wave

An electromagnetic field of a plane-wave propagating along a unit vector $\mathbf{n}$ (here and elsewhere, we choose $\mathbf{n} = (0,0,1)$) can be described by potentials $A^\mu = A^\mu(\xi)$, where $\xi = x^0 - x^3 = nx$ is a light-cone variable[1] $(n^\mu = (1,0,0,1)\,,\ n^2 = 0)$. Choosing the Lorentz gauge $\partial_\mu A^\mu = 0$, which implies that $nA = 0$, and the gauge condition $A_0 = 0$, we have $A^\mu = (0,\mathbf{A})\ ,\ \mathbf{nA} = 0$. The electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields are expressed through the potentials as $\mathbf{E} = -\mathbf{A}'$, $\mathbf{H} = [\mathbf{A}'\times\mathbf{n}]$. Here and elsewhere, primes stand for derivatives with respect to ξ, i.e., $\mathbf{A}' = d\mathbf{A}/d\xi$.

[1] As usual, we denote the Minkowski coordinates by $x = (x^\mu) = (x^0, \mathbf{r}), \mathbf{r} = (x^i,\ i = 1,2,3)$.

In the case under consideration, the Lorentz equations have the form

$$m\ddot{x}^{\mu} = e\,(\dot{x}A')\,n^{\mu} - e\dot{A}^{\mu} \Longleftrightarrow \left\{ \begin{array}{l} m\ddot{x}^{0} = -e\,(\mathbf{A}'\dot{\mathbf{r}}) \\ m\ddot{\mathbf{r}} = -e\left[\mathbf{A}'\dot{\xi} + \mathbf{n}\,(\mathbf{A}'\dot{\mathbf{r}})\right] \end{array} \right. , \tag{1}$$

where dots stand for derivatives with respect to the proper time τ. After multiplying these equations by the vector n^{μ}, we obtain $m\ddot{\xi} = 0 \Longrightarrow m\dot{\xi} = \lambda$, where λ is an integral of motion. The classical action (a solution of the Hamilton–Jacobi equation) that depends on the integrals of motion λ and $\mathbf{p}_{\perp}$ has the form [9]

$$S\,(x) = -\frac{\lambda}{2}\left(x^{0} + x^{3}\right) + \mathbf{p}_{\perp}\mathbf{r} - \frac{1}{2\lambda}\int\left(m^{2} + \mathbf{P}_{\perp}^{2}\right)d\xi\,, \tag{2}$$

where

$$\lambda = np = nP\,,\ p_{\mu} = -\partial_{\mu}S\,,\ P_{\mu} = p_{\mu} - eA_{\mu}\,,\ \mathbf{p} = (-p_{1}, -p_{2}, -p_{3})\,,$$
$$\mathbf{P}_{\perp} = \mathbf{p}_{\perp} - e\mathbf{A}\,(\xi)\,,\ \mathbf{p}_{\perp} = (-p_{1}, -p_{2}, 0)\,,\ \mathbf{p} = \nabla S = \mathbf{p}_{\perp} + \frac{m^{2} + \mathbf{P}_{\perp}^{2} - \lambda^{2}}{2\lambda}\mathbf{n}\,. \tag{3}$$

Considering the Klein–Gordon and Dirac equations with a plane-wave field,

$$\hat{\mathcal{K}}\varphi\,(x) = 0\,,\ \hat{\mathcal{K}} = \hat{P}^{2} - m^{2}\,,\ \hat{P}_{\mu} = \hat{p}_{\mu} - eA_{\mu}\,, \tag{4}$$
$$\hat{\mathcal{D}}\psi\,(x) = 0\,,\ \hat{\mathcal{D}} = \hat{P}_{\mu}\gamma^{\mu} - m\,, \tag{5}$$

one usually seeks for such solutions that are eigenvectors of the operators (quantum integrals of motion) $\hat{\lambda}$ and $\hat{\mathbf{p}}_{\perp}$, which commute both with $\hat{\mathcal{K}}$ and $\hat{\mathcal{D}}$ and between themselves:

$$\hat{\lambda} = (n\hat{p})\,,\ \hat{p}_{\mu} = i\partial_{\mu}\,,\ \hat{\mathbf{p}} = -i\nabla,\ \hat{\mathbf{p}}_{\perp} = -i\,(\partial_{1}, \partial_{2}, 0)\,,$$
$$\left[\hat{\lambda}, \hat{\mathbf{p}}_{\perp}\right] = \left[\hat{\lambda}, \hat{\mathcal{K}}\right] = \left[\hat{\lambda}, \hat{\mathcal{D}}\right] = \left[\hat{\mathbf{p}}_{\perp}, \hat{\mathcal{K}}\right] = \left[\hat{\mathbf{p}}_{\perp}, \hat{\mathcal{D}}\right] = 0\,. \tag{6}$$

Such solutions where first obtained by Volkov, see [10], and are referred to as the Volkov solutions in what follows. The Volkov solutions are subject to the conditions

$$\hat{\lambda}\varphi_{\lambda,\mathbf{p}_{\perp}}(x) = \lambda\varphi_{\lambda,\mathbf{p}_{\perp}}(x)\,,\ \hat{\mathbf{p}}_{\perp}\varphi_{\lambda,\mathbf{p}_{\perp}}(x) = \mathbf{p}_{\perp}\varphi_{\lambda,\mathbf{p}_{\perp}}(x)\,,$$
$$\hat{\lambda}\psi_{\lambda,\mathbf{p}_{\perp}}(x) = \lambda\psi_{\lambda,\mathbf{p}_{\perp}}(x)\,,\ \hat{\mathbf{p}}_{\perp}\psi_{\lambda,\mathbf{p}_{\perp}}(x) = \mathbf{p}_{\perp}\psi_{\lambda,\mathbf{p}_{\perp}}(x)\,, \tag{7}$$

without restrictions on λ, and have the form[2]

$$\varphi_{\lambda,\mathbf{p}_{\perp}}(x) = N\exp iS\,(x)\,, \tag{8}$$
$$\psi_{\lambda,\mathbf{p}_{\perp}}(x) = N\exp iS\,(x)\begin{pmatrix} m + \lambda + \sigma_{3}\,(\sigma\mathbf{P}_{\perp}) \\ (m - \lambda)\,\sigma_{3} + \sigma\mathbf{P}_{\perp} \end{pmatrix}\vartheta\,, \tag{9}$$

where $S\,(x)$ is the classical action (2), N is a normalization constant, and ϑ is an arbitrary two-component constant spinor.

The set of solutions (8) and (9) is orthonormal, with respect to the scalar products on the null-plane $\xi = \text{const}$, and with respect to the scalar products on the plane $x^{0} = \text{const}$.

[2]We denote the Pauli matrices as $\sigma = (\sigma_{i},\ i = 1, 2, 3)$.

3 Non-Volkov solutions

The Volkov solutions are subject to the conditions (7), i.e., they represent quantum states with the conserved integrals of motion λ and $\mathbf{p}_\perp$. We present below a different way of solving the Klein–Gordon and Dirac equations with a plane-wave field. In such a way, we obtain a wider class of solutions. In particular, the latter do not have to be eigenfunctions of the operators $\hat{\mathbf{p}}_\perp$.

3.1 Exclusion of plane-wave field from transversal motion

Consider, first of all, the Klein–Gordon equation with a plane-wave field (4). We shall be interested in such solutions of this equation that are eigenfunctions of the operator $\hat{\lambda} = (n\hat{p})$,

$$\hat{\lambda}\varphi_\lambda(x) = \lambda\varphi_\lambda(x) \ . \tag{10}$$

However, as was already mentioned, we do not demand that the wave functions $\varphi_\lambda(x)$ be eigenfunctions of the operator $\hat{\mathbf{p}}_\perp$. The general solution of equation (10) is

$$\varphi_\lambda(x) = \exp\left(-i\lambda x^0 + i\frac{\lambda}{2}\xi\right)\Phi_\lambda(\xi, \mathbf{r}_\perp) \ , \ \mathbf{r}_\perp = \left(x^1, x^2, 0\right) \ , \tag{11}$$

where $\Phi_\lambda(\xi, \mathbf{r}_\perp)$ is an arbitrary function of the indicated arguments. This function has to obey the equation

$$i\frac{\partial\Phi_\lambda}{\partial\xi} = \hat{H}\Phi_\lambda \ , \ \hat{H} = \frac{1}{2\lambda}\left(\hat{\mathbf{P}}_\perp^2 + m^2\right) \ , \ \hat{\mathbf{P}}_\perp^2 = \hat{\mathbf{p}}_\perp^2 - e\mathbf{A}(\xi) \ . \tag{12}$$

Equation (12) is a nonstationary two-dimensional Schrödinger equation with respect to the "time" ξ.

We can see that there exists a transformation that eliminates the plane-wave field $\mathbf{A}(\xi)$ from equation (12) and reduces the latter to a free two-dimensional Schrödinger equation. The transformation consists of a change of variables $\mathbf{r}_\perp \to \mathbf{x}_\perp$,

$$\mathbf{r}_\perp = \mathbf{x}_\perp - \frac{e}{\lambda}\int \mathbf{A}(\xi)\, d\xi \, , \tag{13}$$

and a function replacement, $\Phi_\lambda(\xi, \mathbf{r}_\perp) \to \Psi_\lambda(\xi, \mathbf{r}_\perp)$,

$$\begin{aligned} &\Phi_\lambda(\xi, \mathbf{r}_\perp) = \exp\left[-i\alpha(\xi)\right]\Psi_\lambda\left(\xi, \mathbf{r}_\perp + \frac{e}{\lambda}\int \mathbf{A}(\xi)\, d\xi\right) \ , \\ &\alpha(\xi) = \frac{1}{2\lambda}\int \left[e^2\mathbf{A}^2(\xi) + m^2\right] d\xi \, . \end{aligned} \tag{14}$$

It is a simple task to verify that the function $\Psi_\lambda(\xi, \mathbf{r}_\perp)$ is a solution of a free two-dimensional Schrödinger equation of the form

$$i\frac{\partial\Psi_\lambda(\xi, \mathbf{r}_\perp)}{\partial\xi} = \hat{H}_0\Psi_\lambda(\xi, \mathbf{r}_\perp) \ , \ \hat{H}_0 = \frac{\hat{\mathbf{p}}_\perp^2}{2\lambda} \, . \tag{15}$$

Finally, we have a set of solutions to the Klein–Gordon equation with a plane-wave field in the form

$$\varphi_\lambda(x) = \exp(-i\Gamma)\,\Psi_\lambda\left(\xi, \mathbf{r}_\perp + \frac{e}{\lambda}\int \mathbf{A}(\xi)\,d\xi\right),$$
$$\Gamma = \lambda x^0 - \frac{\lambda}{2}\xi + \frac{1}{2\lambda}\int\left[e^2\mathbf{A}^2(\xi) + m^2\right]d\xi\,, \tag{16}$$

where the function $\Psi_\lambda(\xi, \mathbf{r}_\perp)$ is a solution of the free two-dimensional Schrödinger equation (15) that does not contain the plane-wave field.

It is interesting to note that the change of variables (13) also eliminates the plane-wave field from the classical equations of motion. Indeed, this follows from the Lorentz equations

$$m\ddot{\mathbf{r}}_\perp = -e\mathbf{A}'(\xi)\,\dot{\xi}\,. \tag{17}$$

In terms of the variables $\mathbf{x}_\perp$, related to $\mathbf{r}_\perp$ by (13), we have the equations of free motion $m\ddot{\mathbf{x}}_\perp = 0$.

Similarly, we find that there exist solutions of the Dirac equation with a plane-wave field in the form

$$\psi_\lambda(x) = \exp(-i\Gamma)\,\hat{R}\Psi_\lambda\left(\xi, \mathbf{r}_\perp + \frac{e}{\lambda}\int \mathbf{A}(\xi)\,d\xi\right)\vartheta\,, \tag{18}$$

where ϑ is an arbitrary two-component spinor, while the operator $\hat{R}$ is

$$\hat{R} = \begin{pmatrix} m + \lambda + \sigma_3\sigma\hat{\mathbf{P}}_\perp \\ (m-\lambda)\,\sigma_3 + \sigma\hat{\mathbf{P}}_\perp \end{pmatrix}, \tag{19}$$

and the function $\Psi_\lambda(\xi, \mathbf{r}_\perp)$ is a solution of the free two-dimensional Schrödinger equation (15).

3.2 Examples of non-Volkov solutions

We now present some specific examples of non-Volkov solutions, considering different solutions of Eq. (15).

3.2.1 First example

Let us consider Eq. (15). We now introduce the dimensionless variables r_s, $s = 1, 2$, and τ that are related to the variables x^s and ξ as follows:

$$\eta_s = 2\lambda x^s + 2e\int A^s(\xi)\,d\xi\,,\ \tau = 2\lambda\xi\,,\ A^1(\xi) = A_x(\xi)\,,\ A^2(\xi) = A_y(\xi)\,. \tag{20}$$

In terms of the new variables, we have

$$\hat{K}\Psi(\eta_1, \eta_2, \tau) = 0\,,\ \hat{K} = i\frac{\partial}{\partial\tau} + \frac{\partial^2}{\partial\eta_1^2} + \frac{\partial^2}{\partial\eta_2^2}\,,$$
$$\Psi_\lambda(\xi, \mathbf{r}_\perp) = \Psi(\eta_1, \eta_2, \tau)\,. \tag{21}$$

We introduce the creation and annihilation operators as

$$a_s = \frac{1}{\sqrt{2}}\left(\eta_s + \frac{\partial}{\partial\eta_s}\right)\,,\ a_s^+ = \frac{1}{\sqrt{2}}\left(\eta_s - \frac{\partial}{\partial\eta_s}\right)\,, \tag{22}$$

$$[a_s, a_{s'}] = \left[a_s^+, a_{s'}^+\right] = 0\,,\ \left[a_s, a_{s'}^+\right] = \delta_{ss'}\,,\ s = 1,2\,. \tag{23}$$

In terms of these operators,

$$\hat{K} = i\frac{\partial}{\partial\tau} - \sum_{s=1,2} H_s\,,\ 2H_s = a_s a_s^+ + a_s^+ a_s - a_s^2 - a_s^{+2}\,. \tag{24}$$

Let us construct such integrals of motion A_s for equation (21) that are linear in the creation and annihilation operators of the same kind (the same s),

$$A_s = f_s(\tau)\,a_s + g_s(\tau)\,a_s^+\,, \tag{25}$$

$$\left[A_s, \hat{K}\right] = 0\,,\ s = 1,2\,. \tag{26}$$

It follows from (26) that the functions f_s and g_s obey the equations

$$i\dot{f}_s + f_s + g_s = 0\,,\ i\dot{g}_s - f_s - g_s = 0\,. \tag{27}$$

The general solution of the set (27) has the form

$$f_s(\tau) = c_1^{(s)} + i\left(c_1^{(s)} + c_2^{(s)}\right)\tau\,,\ g_s(\tau) = c_2^{(s)} - i\left(c_1^{(s)} + c_2^{(s)}\right)\tau\,,\ s = 1,2\,, \tag{28}$$

where $c_1^{(s)}$ and $c_2^{(s)}$ are arbitrary complex numbers. One can easily check (taking (23) into account) that the above-introduced integrals of motion obey the following commutation relations:

$$[A_s, A_{s'}] = \left[A_s^+, A_{s'}^+\right] = 0\,,\ \left[A_s, A_{s'}^+\right] = \Delta_s\delta_{ss'}\,,$$

$$\Delta_s = |f_s|^2 - |g_s|^2 = \left|c_1^{(s)}\right|^2 - \left|c_2^{(s)}\right|^2\,,\ s = 1,2\,. \tag{29}$$

Since A_s are integrals of motion, we can look for such solutions of equation (21) that are eigenvectors of A_s,

$$A_s\Psi_z(\eta_1,\eta_2,\tau) = z_s\Psi_z(\eta_1,\eta_2,\tau)\,,\ s = 1,2\,. \tag{30}$$

Here, z_s are arbitrary complex numbers. Such solutions can be chosen as

$$\Psi_{z_1 z_2}(\eta_1,\eta_2,\tau) = \Psi_{z_1}(\eta_1,\tau)\,\Psi_{z_2}(\eta_2,\tau)\,, \tag{31}$$

where the function $\Psi_{z_s}(\eta_s,\tau)$ obeys similar (for each s) equations:

$$\left(i\frac{\partial}{\partial\tau} - H_s\right)\Psi_{z_s}(\eta_s,\tau) = 0\,, \tag{32}$$

$$A_s\Psi_{z_s}(\eta_s,\tau) = z_s\Psi_{z_s}(\eta_s,\tau)\,,\ s = 1,2\,. \tag{33}$$

In what follows, we analyze these equations for a fixed s (s is then omitted in all the quantities).

We first consider the case $\Delta = 0$. In this case, the operator A can be chosen, without loss of generality, as self-adjoint, $A^+ = A$. Then $z = z^*$, $g = f^*$, $f(\tau) = c + i(c + c^*)\tau$, and solutions of the set (32), (33) have the form

$$\Psi_z(\eta,\tau) = \left[\sqrt{2\pi}\,(f - f^*)\right]^{-1/2} \exp\left[\frac{f+f^*}{2(f^* - f)}\left(\eta - \frac{\sqrt{2}z}{f+f^*}\right)^2\right]. \tag{34}$$

The functions (34) obey the following relations of orthogonality and completeness:

$$\int_{-\infty}^{\infty} \Psi_z^*(\eta,\tau)\,\Psi_{z'}(\eta,\tau)\,d\eta = \delta(z - z')\,,\ \int_{-\infty}^{\infty} \Psi_z^*(\eta',\tau)\,\Psi_z(\eta,\tau)\,dz = \delta(\eta - \eta')\,. \tag{35}$$

If $\Delta > 0$, then, without loss of generality, we can choose $\Delta = 1$, multiplying A by a complex number. In this case, A and A^+ are annihilation and creation operators, and solutions $\Psi_{\lambda z_s}$ of equation (33) are coherent states. In order to obey equation (32), these states must take the form

$$\begin{aligned}
&\Psi_z(\eta,\tau) = (f-g)^{-1/2}\,U_0(q)\exp\Theta\,,\\
&q = \left[2\eta - \sqrt{2}z(f^* - g^*) - \sqrt{2}z^*(f-g)\right][2|f-g|]^{-1}\,,\\
&\Theta = \Big\{2\eta^2(fg^* - gf^*) + 2\sqrt{2}\eta\,[z(f^*-g^*) - z^*(f-g)]\\
&+z^{*2}(f-g)^2 - z^2(f^*-g^*)^2\Big\}[4|f-g|]^{-1}\,.
\end{aligned} \tag{36}$$

Here, $U_0(x)$ is the zero function from the set of Hermite functions $U_n(x) = (2^n n!\sqrt{\pi})^{-1/2}\exp(-x^2/2)\,H_n(x)$, where $H_n(x)$ are Hermite polynomials [13].

Finally, if $\Delta < 0$, then the operator A has no eigenfunctions that can be normalized, even as distributions. However, in this case the operator $B = A^+$ is indeed an annihilation operator, and the above consideration is applicable here.

Having the expressions for the functions $\Psi_z(\eta_1,\eta_2,\tau)$, we can construct the corresponding solutions of the Klein–Gordon equation (4) and of the Dirac equation (5), with the help of formulae (16) and (18). Such solutions are eigenfunctions of the integrals of motion $\mathcal{A}_s$, $s = 1,2$, which can be constructed from the operators A_s, with allowance for the transformation (20),

$$\begin{aligned}
&\mathcal{A}_s = \frac{f_s\left(2\hat{\lambda}\xi\right)}{\sqrt{2}}\left[2\hat{\lambda}x^s + 2e\int A^s(\xi)\,d\xi + \frac{1}{2\hat{\lambda}}\frac{\partial}{\partial x^s}\right]\\
&+\frac{g_s\left(2\hat{\lambda}\xi\right)}{\sqrt{2}}\left[2\hat{\lambda}x^s + 2e\int A^s(\xi)\,d\xi - \frac{1}{2\hat{\lambda}}\frac{\partial}{\partial x^s}\right],\\
&[\mathcal{A}_s, \mathcal{A}_{s'}] = [\mathcal{A}_s^+, \mathcal{A}_{s'}^+] = 0\,,\ [\mathcal{A}_s, \mathcal{A}_{s'}^+] = \Delta_s\delta_{ss'}\,,\\
&\left[\mathcal{A}_s, \hat{\mathcal{K}}\right] = \left[\mathcal{A}_s^+, \hat{\mathcal{K}}\right] = 0\,,\ \left[\mathcal{A}_s, \hat{\mathcal{D}}\right] = \left[\mathcal{A}_s^+, \hat{\mathcal{D}}\right] = 0\,.
\end{aligned} \tag{37}$$

For example, in the case $\Delta_s = 1$, $s = 1, 2$, we obtain squeezed coherent states describing the transversal motion of a charge in a plane-wave field (transversal squeezed coherent states). The squeezing of the states is determined by the variation of the constants c in formulae (28). One can easily verify that in such a case the mean values of the transversal coordinates obey the classical equations of motion (see the following example).

3.2.2 Second example

Consider the case $\Delta = 1$. Since the operator A^+ is an integral of motion, we can construct solutions of equation (32) (which no longer obey Eq. (34)) as follows:

$$\Psi_{z,n}(\eta,\tau) = \left(A^+ - z^*\right)^n \Psi_z(\eta,\tau)\ ,\ n = 0, 1, 2, \ldots . \tag{38}$$

For $n = 0$, they coincide with the solutions considered in the first example. The functions (38) may be called generalized squeezed coherent states. They have the explicit form

$$\Psi_{z,n}(\eta,\tau) = (f - g)^{-1/2} \left(\frac{f^* - g^*}{f - g}\right)^{\frac{n}{2}} U_n(q) \exp\Theta\,, \tag{39}$$

and the following properties:

$$\begin{aligned}
&(A - z)\,\Psi_{z,n} = \sqrt{n}\Psi_{z,n-1}\,,\\
&\left(A^+ - z^*\right)\Psi_{z,n} = \sqrt{n+1}\Psi_{z,n+1}\,,\\
&\int_{-\infty}^{\infty} \Psi^*_{z,n}(\eta,\tau)\,\Psi_{z,n'}(\eta,\tau)\,d\eta = \delta_{nn'}\,,\\
&\sum_{n=0}^{\infty} \Psi^*_{z,n}(\eta',\tau)\,\Psi_{z,n}(\eta,\tau) = \delta(\eta - \eta')\\
&\int_{-\infty}^{\infty} \Psi^*_{z,n}(\eta',\tau)\,\Psi_{z,n'}(\eta,\tau)\,d^2z = \pi\delta_{nn'}\delta(\eta - \eta')\,.
\end{aligned} \tag{40}$$

Using (39), we can now construct solutions of the Klein–Gordon and Dirac equations by means of formulae (16) and (18) (transversal generalized squeezed coherent states). Calculating the mean values of the operators $\hat{\mathbf{r}}_\perp$and $\hat{\mathbf{p}}_\perp$ on such states, we obtain (the results are the same for both the Klein–Gordon and the Dirac solutions, and do not depend on the quantum number n)

$$\langle\hat{\mathbf{r}}_\perp\rangle = \mathbf{r}_\perp(0) + \lambda^{-1}\int_0^{\xi} \mathbf{P}_\perp(\xi)\,d\xi\,,\ \langle\hat{\mathbf{p}}_\perp\rangle = \mathbf{p}_\perp\,, \tag{41}$$

where $\mathbf{r}_\perp(0) = \left(x_0^1, x_0^2, 0\right)$, $\mathbf{p}_\perp = \left(p^1, p^2, 0\right)$, and

$$\begin{aligned}
x_0^s &= \left(2\sqrt{2}\lambda\right)^{-1}\left[\left(c_1^{s*} - c_2^{s*}\right) z_s + \left(c_1^s - c_2^s\right) z_s^*\right]\,,\\
p^s &= i\sqrt{2}\lambda\left[\left(c_1^s + c_2^s\right) z_s^* - \left(c_1^{s*} + c_2^{s*}\right) z_s\right]\,,\ s = 1, 2\,.
\end{aligned} \tag{42}$$

It is easy to see from (3) and (17) that the mean values (41) follow the classical trajectory of transversal motion with the initial data (at $\xi = 0$) given by $\mathbf{r}_\perp(0)$. To provide such initial data, one has to select states with

$$z_s = \sqrt{2}\lambda x_0^s \left(c_1^s + c_2^s\right) + \frac{ip^s}{2\sqrt{2}\lambda}\left(c_1^s - c_2^s\right) .$$

3.2.3 Third example

Consider the operator

$$\hat{L}_z = i\left(\eta_2 \frac{\partial}{\partial \eta_1} - \eta_1 \frac{\partial}{\partial \eta_2}\right) = -i\frac{\partial}{\partial \varphi},$$
$$\eta_1 = \rho\cos\varphi\,,\ \eta_2 = \rho\sin\varphi\,,\ \rho = \sqrt{\eta_1^2 + \eta_2^2}\,, \tag{43}$$

which is an integral of motion for equation (21)

$$\left[\hat{L}_z, \hat{K}\right] = 0\,.$$

It can be interpreted as the z-projection of the angular momentum operator. The corresponding integral of motion for the Klein–Gordon equation can be easily recovered. It has the form

$$\hat{\mathcal{L}}_z = \left(\left[\mathbf{r}_\perp + \frac{e}{\lambda}\int \mathbf{A}(\xi)\,d\xi\,, \hat{\mathbf{p}}_\perp\right]\mathbf{k}\right)\,,\ \left[\hat{\mathcal{L}}_z, \hat{\mathcal{K}}\right] = 0\,, \tag{44}$$

where $\mathbf{k}$ is a unit vector in the z-direction.

Let us seek for such solutions of equation (21) that are eigenvectors of $\hat{L}_z$,

$$\hat{L}_z\Psi_l(\eta_1,\eta_2,\tau) = l\Psi_l(\eta_1,\eta_2,\tau)\ ,\ l = 0, \pm 1, \pm 2, \ldots\,. \tag{45}$$

The functions $\Psi_l(\eta_1,\eta_2,\tau)$ have the form

$$\Psi_l(\eta_1,\eta_2,\tau) = \chi_l(\rho,\tau)\exp il\varphi\ ,\ \left(\exp i\varphi = (\eta_1 + i\eta_2)/\rho\right)\,, \tag{46}$$

where the functions $\chi_l(\rho,\xi)$ obey the equation

$$\left(i\frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{l^2}{\rho^2}\right)\chi_l(\rho,\tau) = 0 \tag{47}$$

A general solution of the latter equation, with the initial condition $\chi_l(\rho,0) = \chi_l^{(0)}(\rho)$, has the form

$$\chi_l(\rho,\tau) = \int_0^\infty G_l(\rho,\rho';\tau)\,\chi_l^{(0)}(\rho')\,d\rho'\,,$$
$$G_l(\rho,\rho';\tau) = \frac{(-i)^{l+1}\rho'}{2\tau}J_l\left(\frac{\rho\rho'}{2\tau}\right)\exp\left(i\frac{\rho^2+\rho'^2}{4\tau}\right)\,, \tag{48}$$

where $J_l(x)$ are Bessel functions [13]. One can consider stationary states, choosing $\chi_l(\rho,\tau) = f_{lp}(\rho)\exp(-ip\tau)$. Then $f_{lp}(\rho) = J_l(\sqrt{p}\rho)$.

The corresponding solutions of the Klein–Gordon equations can be easily found with the help of formulae (16). Switching on the plane-wave field in these solutions, we obtain the states of a free relativistic particle with a definite z-projection of the angular momentum.

4 Combined field

Consider now a combination of a plane-wave field and colinear constant electric and magnetic fields. The covariant components of the corresponding electromagnetic potentials are chosen as

$$A_0 = A_3 = A_0(\xi)\ ,\ A_1 = \frac{H}{2}x^2 + A_1(\xi)\ ,\ A_2 = -\frac{H}{2}x^1 + A_2(\xi)\ , \tag{49}$$

where $A_k(\xi)$, $k = 1,2$, are arbitrary functions of ξ, and $H = \text{const}$. The corresponding components of the electric and magnetic fields are

$$E_x = H_y = A_1'(\xi),\ E_y = -H_x = A_2'(\xi)\ ,\ E_z = 2A_0'(\xi)\ ,\ H_z = H. \tag{50}$$

In what follows, we call the electromagnetic field (50) the combined electromagnetic field.

The Lorentz equations with the combined electromagnetic field can be written as

$$\begin{aligned} m\ddot{x}^0 &= e\left(A_1'\dot{x}^1 + A_2'\dot{x}^2 + A_0'\dot{x}^3\right),\ m\ddot{x}^1 = e\left(A_1'\dot{\xi} + H\dot{x}^2\right)\ , \\ m\ddot{x}^2 &= e\left(A_2'\dot{\xi} - H\dot{x}^1\right)\ ,\ m\ddot{x}^3 = e\left(A_1'\dot{x}^1 + A_2'\dot{x}^2 + 2A_0'\dot{x}^0\right)\ . \end{aligned} \tag{51}$$

Dots stand for derivatives with respect to the proper time τ. Solutions of equations (51) are known, see [12]. We only remark here that these equations imply the conservation of the quantity λ,

$$\begin{aligned} \lambda &= m\left(\dot{x}^0 - \dot{x}^3\right) + 2eA_0(\xi) = m\dot{\xi} + 2eA_0(\xi) \\ &= P_0 + P_3 + 2eA_0(\xi) = p_0 + p_3\,. \end{aligned} \tag{52}$$

Here, p_μ are components of the generalized momentum, and $P_\mu = m\dot{x}_\mu = p_\mu - eA_\mu$ are components of the kinetic momentum. We introduce the notation

$$\begin{aligned} &2eA_0(\xi) = -a_0(\xi)\ ,\ eA_k(\xi) = a_k(\xi)\ , \\ &eH = \gamma,\ \mathcal{P} = \mathcal{P}(\xi) = \lambda + a_0(\xi)\ . \end{aligned} \tag{53}$$

Then (52) takes the form

$$m\dot{\xi} = \mathcal{P}(\xi) = \lambda + a_0(\xi)\ . \tag{54}$$

Integrating (54), we obtain a relation between the proper time τ and the variable ξ,

$$\tau(\xi) = m \int \frac{d\xi}{\mathcal{P}(\xi)} \,. \tag{55}$$

The Lorentz equations for x^k, $k = 1, 2$, take the form

$$m\ddot{x}^1 = a_1'(\xi)\,\dot{\xi} + \gamma\dot{x}^2 \,, \; m\ddot{x}^2 = a_2'(\xi)\,\dot{\xi} - \gamma\dot{x}^1 \,. \tag{56}$$

in terms of notation (53). We make the change of variables

$$x^1 = X - q_1(\xi) \,,\; x^2 = Y - q_2(\xi) \,, \tag{57}$$

where the real functions $q_k(\xi)$, $k = 1, 2$, are defined as

$$q(\xi) = q_1(\xi) + iq_2(\xi) = -\exp\left[-\frac{i\gamma\tau(\xi)}{m}\right] \int \exp\left[\frac{i\gamma\tau(\xi)}{m}\right] \frac{a(\xi)}{\mathcal{P}(\xi)} d\xi \,, \tag{58}$$

with $a(\xi) = a_1(\xi) + ia_2(\xi)$. The complex function $q(\xi)$ obeys the equation

$$\mathcal{P}(\xi)\, q'(\xi) + i\gamma q(\xi) + a(\xi) = 0 \,. \tag{59}$$

In terms of the new variables u and v, equations (56) take the form

$$m\ddot{X} - \gamma\dot{Y} = 0 \,,\; m\ddot{Y} + \gamma\dot{X} = 0 \,. \tag{60}$$

The set (60) contains neither the plane-wave field nor the colinear electric field. It describes the motion of a charged particle in a constant magnetic field in the plane $x^3 = \text{const}$. The above consideration indicates that the plane-wave field and the colinear electric field can be eliminated from the quantum equations of motion as well.

We now consider the Klein–Gordon and Dirac equations with the combined electromagnetic field. Exact solutions of these equations were first found in [11, 12]. As in the case of a plane-wave, we present below new classes of solutions of these equations. To this end, we are going to represent a transformation that eliminates the plane-wave field and the colinear electric field from the transversal motion. Thus, the transversal motion in the combined electromagnetic field is mapped to the nonrelativistic transversal motion in the constant uniform magnetic field.

In quantum theory, the operator $\widehat{\lambda}$ corresponding to the classical quantity (52) is an integral of motion. We seek solutions of the Klein–Gordon equation as eigenfunctions of the operator $\widehat{\lambda}$. Such solutions have the form

$$\varphi_\lambda(x) = \frac{1}{\sqrt{\mathcal{P}(\xi)}} \exp\left(-i\lambda x^0 + i\frac{\lambda}{2}\xi\right) \Phi_\lambda\left(\xi, x^1, x^2\right) ,$$

where the function $\Phi_\lambda\left(\xi, x^1, x^2\right)$ has to obey the equation

$$\begin{aligned} &2i\mathcal{P}(\xi)\,\frac{\partial\Phi_\lambda}{\partial\xi} = \left(\hat{P}_1^2 + \hat{P}_2^2 + m^2\right)\Phi_\lambda \,, \\ &\hat{P}_k = i\partial_k - eA_k(\xi) \,,\; k = 1, 2 \,. \end{aligned} \tag{61}$$

We now use the variables X, Y and $\tau(\xi)$, see (55) and (57), and a function replacement, $\Phi_\lambda \to \Psi_\lambda$, for eliminating the plane-wave field and the colinear electric field from the equations,

$$\begin{aligned}
&\Phi_\lambda\left(\xi, x^1, x^2\right) = \exp\left(-i\Gamma\right)\Psi_\lambda\left(\tau, X, Y\right), \\
&\Gamma = \frac{i\gamma}{4}\left[q^*\left(\xi\right)\left(x^1 + ix^2\right) - q\left(\xi\right)\left(x^1 - ix^2\right)\right] \\
&+ \int \frac{d\xi}{2\mathcal{P}\left(\xi\right)}\left\{\left|a\left(\xi\right)\right|^2 + m^2 + \frac{i\gamma}{2}\left[q\left(\xi\right)a^*\left(\xi\right) - q^*\left(\xi\right)a\left(\xi\right)\right]\right\}. \qquad (62)
\end{aligned}$$

The function $\Psi_\lambda(\tau, X, Y)$ obeys the equation

$$\begin{aligned}
&i\frac{\partial\Psi_\lambda\left(\tau, X, Y\right)}{\partial\tau} = \frac{\left(\hat{\pi}_1^2 + \hat{\pi}_2^2\right)}{2m}\Psi_\lambda\left(\tau, X, Y\right), \\
&\hat{\pi}_1 = i\partial_X - \frac{\gamma}{2}Y, \ \hat{\pi}_2 = i\partial_Y + \frac{\gamma}{2}X. \qquad (63)
\end{aligned}$$

This is the two-dimensional Schrödinger equation for a charged particle in a constant uniform magnetic field. In this equation, the plane-wave field and the colinear electric field are already eliminated.

We now pass to the Dirac equation with the combined electromagnetic field. We shall seek solutions of the Dirac equation as eigenfunctions of the operator $\widehat{\lambda}$. Such solutions have the form

$$\psi_\lambda\left(x\right) = \exp\left(-i\lambda x^0 + i\frac{\lambda}{2}\xi\right)\left(\begin{array}{c} V + U \\ \sigma_3\left(V - U\right)\end{array}\right). \qquad (64)$$

Here, $V = V\left(\xi, x^1, x^2\right)$ and $U = U\left(\xi, x^1, x^2\right)$ are two-component spinors that have to obey the equations

$$\begin{aligned}
&2i\frac{\partial U}{\partial\xi} = \left[m + \left(\sigma\hat{\mathbf{P}}_\perp\right)\sigma_3\right]V, \\
&V = \mathcal{P}^{-1}\left(\xi\right)\left[m - \left(\sigma\hat{\mathbf{P}}_\perp\right)\sigma_3\right]U, \qquad (65)
\end{aligned}$$

where $\hat{\mathbf{P}}_\perp = -\left(\hat{P}_1, \hat{P}_2, 0\right)$. Obviously, it is sufficient to know the spinor U. The latter spinor obeys the equation

$$2i\mathcal{P}\left(\xi\right)\frac{\partial U}{\partial\xi} = \left(\hat{\mathbf{P}}_\perp^2 + m^2 - \gamma\sigma_3\right)U. \qquad (66)$$

We transform this equation by the spinor replacement $U \to \Theta$,

$$U\left(\xi, x^1, x^2\right) = \exp\left[i\frac{\gamma\tau\left(\xi\right)\sigma_3}{2m}\right]\Theta\left(\xi, x^1, x^2\right).$$

The new spinor $\Theta\left(\xi, x^1, x^2\right)$ has to obey the following equation:

$$2i\mathcal{P}\left(\xi\right)\frac{\partial\Theta}{\partial\xi} = \left(\hat{\mathbf{P}}_\perp^2 + m^2\right)\Theta. \qquad (67)$$

We can see that the spinor $\Theta\left(\xi, x^1, x^2\right)$ may be chosen as

$$\Theta\left(\xi, x^1, x^2\right) = \Phi_\lambda\left(\xi, x^1, x^2\right)\vartheta\,, \tag{68}$$

where the function $\Phi_\lambda\left(\xi, x^1, x^2\right)$ obeys equation (61) and ϑ is an arbitrary constant two-component spinor. Then the transformation (62) allows one to reduce the problem to the two-dimensional Schrödinger equation (63) for a charged particle in a constant uniform magnetic field. Solutions of the latter equation are studied in detail: see, for example, [14].

Acknowledgement 1 *V.B. thanks FAPESP for support and Nuclear Physics Department of São Paulo University for hospitality; he also thanks Russia President grant SS-1743.2003.2 and RFBR grant 03-02-17615 for partial support. D.G. is grateful to FAPESP and CNPq for permanent support.*

References

[1] S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, 1961)

[2] W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer-Verlag, Berlin 1985)

[3] V.G. Bagrov, D.M. Gitman, *Exact Solutions of Relativistic Wave Equations* (Kluwer, Dordrecht 1990)

[4] E.S. Fradkin, D.M. Gitman and S.M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, Berlin 1991)

[5] W. Greiner, *Relativistic Quantum Mechanics. Wave Equations* (Springer-Verlag, Berlin 1997)

[6] H.A. Bethe , E.E. Salpeter, *Quantum mechanics of atoms with one and two electrons, Handbuch der Physik*, (Bd. XXXV/1, Berlin, 1957)

[7] A.A. Sokolov and I.M. Ternov, *Synchrotron Radiation*, (Akad.Verlag, Berlin 1968); *Radiation from Relativistic Electrons,* (American Institute of Physics, New York 1986); I.M. Ternov, V.V. Mihailin and V.R. Khalilov, *Synhrotron Radiation and Its Application,* (Harwood Academic Publishers, New York 1985); *Synchrotron Radiation Theory and its Development,* Ed.: V.A. Bordovitsyn (World Scientific, Singapore 1999)

[8] *Quantum Electrodynamics of Phenomena in Intense Fields*, Proc. P.N. Lebedev Phys. Inst. **111** (Nauka, Moscow 1979); *Problems of Electrodynamics with Intense Fields*, Proc. P.N. Lebedev Phys. Inst. **168** (Nauka, Moscow 1968)

[9] L.D. Landau, E.M. Lifshits, *Field Theory* (Nauka, Moscow 1978)

[10] D. M. Volkov, *Solution exacte de l'equation de Dirac pour une onde plane de frequence determinee.* Comptes Rendus de L'Académie des Sciences de L'URSS, **1**, No 9 (1935) 608; *Über eine Klasse von Lösungen der Diracschen Gleichung,* Zeitschrift für Physik, Bd. 94, H. 3-4 (1935) 250; Zh. Eksp. Teor. Fiz., **7** (1937) 1286

[11] P. J. Redmond, *Solution of the Klein–Gordon and Dirac Equations for a Particle with a Plane Electromagnetic Wave and a Parallel Magnetic Field,* J. Math. Phys., **6** (1965) 1163

[12] V.G. Bagrov, D.M. Gitman, A.V. Jushin, *Solutions for the motion of an electron in electromagnetic field,* Phys. Rev. D **12** (1975) 3200

[13] I.S. Gradshtein and I.W. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1994).

[14] V.G. Bagrov, M.C. Baldiotti, D.M. Gitman, and I.V. Shirokov, *New solutions of relativistic wave equations in magnetic field and longitudinal fields,* J.Math. Phys. **43** (2002) 2284