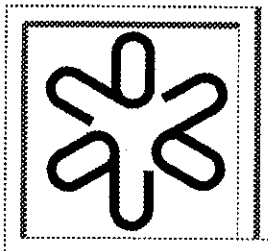

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SYMMETRIES OF DYNAMICALLY EQUIVALENT THEORIES

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Symmetries of dynamically equivalent theories

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Abstract

A natural and very important development of constrained system theory is a detail study o relation between the constraint structure in the Hamiltonian formulation with specific featur the theory in the Lagrangian formulation, especially the relation between the constraint stru with the symmetries of the Lagrangian action. An important preliminary step in this directi a strict demonstration, and this is the aim of the present article, that the symmetry structur the Hamiltonian action and of the Lagrangian action are the same. This proved, it is suffi to consider the symmetry structure of the Hamiltonian action. The latter problem is, in sense, simpler because the Hamiltonian action is a first-order action. At the same time, the s of the symmetry of the Hamiltonian action naturally involves Hamiltonian constraints as objects. One can see that the Lagrangian and Hamiltonian actions are dynamically equiva This is why, in the present article, we consider from the very beginning a more general prob how the symmetry structures of dynamically equivalent actions are related. First, we pr some necessary notions and relations concerning infinitesimal symmetries in general, as well strict definition of dynamically equivalent actions. Finally, we demonstrate that there exist isomorphism between classes of equivalent symmetries of dynamically equivalent actions.

CONTENTS

I. INTRODUCTION

The most of contemporary particle-physics theories are formulated as gauge thec It is well known that within the Hamiltonian formulation gauge theories are theories constraints. This is the main reason for a long and intensive study of the formal th of constrained systems, see [1]. It still attracts considerable attention of researchers. I the very beginning, it became clear that the presence of first-class constraints among complete set of constraints in the Hamiltonian formulation is a direct indication that theory is a gauge one, i.e., its Lagrangian action is invariant under gauge transformat A next natural, and very important, step would be a detail study of the relation betwee constraint structure and constraint dynamics in the Hamiltonian formulation with spe features of the theory in the Lagrangian formulation, especially the relation between constraint structure with the gauge transformation structure of the Lagrangian action. important problem to be solved in this direction would be a strict demonstration, and tl the aim of the present article, that the symmetry structures of the Hamiltonian action

of the Lagrangian action are the same. This proved, it is sufficient to consider the symm structure of the Hamiltonian action. The latter problem is, in some sense, simpler bec the Hamiltonian action is a first-order action. At the same time, the study of the symm of the Hamiltonian action naturally involves Hamiltonian constraints as basic objects [2, 3]. It follows from the results of the article [4] that the Lagrangian and Hamilto actions are dynamically equivalent. This is why in the present article we consider the very beginning a more general problem: how the symmetry structures of dynami equivalent actions are related. The article is organized as follows: In sec. 2, we pre some necessary notions and relations concerning infinitesimal symmetries in general strict definition of dynamically equivalent actions is given in sec. 3. Finally, in sec. 4 demonstrate that there exists an isomorphism between classes of equivalent symmetri dynamically equivalent actions.

II. SYMMETRIES

A Basic notation and relations

We consider finite-dimensional systems which are described by the generalized coordin $q \equiv \{q^a;\ a = 1, 2, ..., n\}$. The space of the variables $q^{a[l]}$,

$$q^{a[l]} = (d_t)^l\, q^a\,,\ \ l = 0, 1, ..., N_a,\ \ \left(q^{a[0]} = q^a\right)\,,\ d_t = \frac{d}{dt}\,,$$

considered as independent variables, with finite N_a , or with some infinite N_a , is c the jet space. The majority of physical quantities are described by so-called local funct (LF) which are defined on the jet space. The LF depend on $q^{a[l]}$ up to some finite or $l \leq N_a \geq 0$. The following notation is often used[6]:

$$F\left(q^{a[0]}, q^{a[1]}, q^{a[2]}, ...\right) = F\left(q^{[]}\right)$$

for the LF. In what follows, we also deal with so-called local operators (LO). LO $\hat{U}_A$ matrix operators which act on columns of LF f^a producing columns F^A of LF, $F^A = \hat{U}_A$

LO have the form

$$\hat{U}_{Aa} = \sum_{k=0}^{K<\infty} u^k_{Aa} \left(d_t\right)^k \ ,$$

where u^k_{Aa} are LF. We call the operator

$$\left(\hat{U}^T\right)_{aA} = \sum_{k=0}^{K<\infty} \left(-d_t\right)^k u^k_{Aa}$$

the transposed operator with respect to $\hat{U}_{Aa}$. The following relation holds true for any F^A and f_a:

$$F^A \hat{U}_{Aa} f^a = \left[\left(\hat{U}^T\right)_{aA} F^A\right] f^a + d_t Q \ ,$$

where Q is an LF. The LO $\hat{U}_{ab}$ is symmetric $(+)$ or antisymmetric $(-)$ respective $\left(\hat{U}^T\right)_{ab} = \pm \hat{U}_{ab}$. Thus, for any antisymmetric LO $\hat{U}_{ab}$ relation (5) is reduced to the lowing: $f^a \hat{U}_{ab} f^b = dQ/dt$, where Q is a LF.

Suppose the total time derivative of an LF vanishes. Then this LF is a constant. Nar

$$\frac{dF\left(q^{[l]}\left(t\right)\right)}{dt} \equiv 0 \Longrightarrow F\left(q^{[l]}\right) \equiv \text{const} \, .$$

Indeed, let us suppose that N_a are the orders of the coordinates q^a in the LF, i.e. $F\left(q^{[}\right.$ $F\left(\cdots q^{a[N_a]}\right)$. Then according to (6) the following relation holds true

$$\frac{\partial F}{\partial q^{a[N_a]}} q^{a[N_a+1]} \equiv - \left[\partial_t F + \sum_a \sum_{k=0}^{N_a-1} \frac{\partial F}{\partial q^{a[k]}} q^{a[k+1]}\right] \, .$$

The right hand side of the above relation does not depend on $q^{a[N_a+1]}$. Thus, $\partial F/\partial q^{a[N_a]}$ and therefore $F\left(q^{[l]}\right)$ must not depend on $q^{a[N_a]}$. In the same manner we can see $F\left(q^{[l]}\right)$ must not depend on $q^{[N-1]}$ and so on. If $F\left(q^{[l]}\right)$ does not depend on any $q^{[l]}$, $\partial_t F\left(q^{[l]}\right) \equiv 0$ as well, and we get $F\left(q^{[l]}\right) = \text{const}$.

We recall that $F_A\left(q^{[]}\right) = 0$ and $\chi_\alpha\left(q^{[]}\right) = 0$ are equivalent sets of equations whenever have the same sets of solutions. In what follows, we denote this fact as $F = 0 \Longleftrightarrow \chi$ Via $O\left(F\right)$ we denote any LF that vanishes on the equations $F_a\left(q^{[]}\right) = 0$. More exactly define $O\left(F\right) = \hat{V}^b F_b$, where $\hat{V}^b$ is an LO. Besides, we denote via $\hat{U} = \hat{O}\left(F\right)$ any LO

vanish on the equations $F_a\left(q^{[]}\right)=0$. That means that the LF u that enter into (3) va on these equations, $u=O\left(F\right)$, or equivalently $\hat{U}f=O\left(F\right)$ for any LF f.

We consider Lagrangian theories given by an action $S\left[q\right]$,

$$S\left[q\right]=\int_{t_1}^{t_2}Ldt\,,\;L=L\left(q^{[]}\right)\,,$$

where a Lagrange function L is defined as an LF on the jet space[7]. The Euler–Lagr equations are

$$\frac{\delta S}{\delta q^a}=\sum_{l=0}\left(-d_t\right)^l\frac{\partial L}{\partial q^{a[l]}}=0\,.$$

Any LF of the form $O\left(\delta S/\delta q\right)$ is called an extremal.

For any LF $F\left(q^{[]}\right)$ the operation

$$\frac{d_{\mathrm{EL}}F}{dq^a}=\sum_{l=0}^{N_a}\left(-\frac{d}{dt}\right)^l\frac{\partial F}{\partial q^{a[l]}}$$

is called the Euler–Lagrange derivative with respect to the coordinate q^a. One can see the functional derivative of the action S coincides with the Euler–Lagrange derivative o Lagrange function,

$$\frac{\delta S}{\delta q^a}=\frac{d_{\mathrm{EL}}L}{dq^a}\,.$$

The Euler–Lagrange derivative has the following property:

$$\frac{d_{\mathrm{EL}}}{dq^a}\frac{d}{dt}=0\,.$$

To prove this, one may use the relation

$$\frac{\partial}{\partial q^{a[k]}}\frac{d}{dt}=\frac{\partial}{\partial q^{a[k]}}\left(\partial_t+\sum_{l=0}q^{b[l+1]}\frac{\partial}{\partial q^{b[l]}}\right)=(1-\delta_{k0})\frac{\partial}{\partial q^{a[k-1]}}$$
$$+\left(\partial_t+\sum_{l=0}q^{b[l+1]}\frac{\partial}{\partial q^{b[l]}}\right)\frac{\partial}{\partial q^{a[k]}}=\frac{d}{dt}\frac{\partial}{\partial q^{a[k]}}+(1-\delta_{k0})\frac{\partial}{\partial q^{a[k-1]}}\,.$$

Thus, one gets

$$\frac{d_{\mathrm{EL}}}{dq^a}\frac{d}{dt}=\sum_{k=0}\left(-\frac{d}{dt}\right)^k\frac{\partial}{\partial q^{a[k]}}\frac{d}{dt}=-\sum_{k=0}\left(-\frac{d}{dt}\right)^{k+1}\frac{\partial}{\partial q^{a[k]}}+\sum_{k=1}\left(-\frac{d}{dt}\right)^k\frac{\partial}{\partial q^{a[k-1]}}$$
$$=\frac{d}{dt}\sum_{k=0}\left(-\frac{d}{dt}\right)^k\frac{\partial}{\partial q^{a[k]}}-\frac{d}{dt}\sum_{k=1}\left(-\frac{d}{dt}\right)^{k-1}\frac{\partial}{\partial q^{a[k-1]}}=\frac{d}{dt}\frac{d_{\mathrm{EL}}}{dq^a}-\frac{d}{dt}\frac{d_{\mathrm{EL}}}{dq^a}=0\,.$$

B Noether symmetries

Consider an infinitesimal inner[8] trajectory variation δq^a (inner variations vanish toge with all their time derivatives at t_1 and t_2). Namely,

$$q^a(t) \to q'^a(t) = q^a(t) + \delta q^a .$$

We suppose that $\delta q^a = \delta q^a\left(q^{[]}\right)$ is an LF. The corresponding first variation of the ac can be written as follows:

$$\delta S = \int_{t_1}^{t_2} \hat{\delta} L \, dt ,$$

where the operator $\hat{\delta}$, which will be called the transformation operator, acts on the c sponding LF as[9]

$$\hat{\delta} = \sum_{k=0} \delta q^{a[k]} \frac{\partial}{\partial q^{a[k]}} = \hat{\delta}_{\delta q} .$$

Two simple but useful relations follow from (14):

$$\hat{\delta} q^a = \delta q^a , \quad \hat{\delta}_{c^i \delta_i q} = c^i \hat{\delta}_{\delta_i q} .$$

The variation (12) is a symmetry transformation of the action S, or sir a symmetry of the action S, whenever the corresponding first variation of the Lagrange f tion is reduced to the total time derivative of a LF. Namely, δq is a symmetry if

$$\hat{\delta} L = \frac{dF}{dt} ,$$

where F is an LF. In this case the first variation (13) of the action depends on the com set of the variables $q^{[]}$ at $t = t_1$ and $t = t_2$ only,

$$\delta S = \int_{t_1}^{t_2} \hat{\delta} L dt = F|_{t_1}^{t_2} .$$

Any linear combination of symmetry transformations is a symmetry.

Indeed, let $\delta_i q$ be some symmetry transformations, and $\delta q = c^i \delta_i q$, where c^i are s constants. Then, taking into account (15), we obtain:

$$\hat{\delta}_{\delta_i q} L = \frac{dF_i}{dt} \Longrightarrow \hat{\delta}_{\delta q} L = \frac{dF}{dt} , \quad F = c^i F_i .$$

Transformation operators that correspond to symmetry transformations are called s
metry operators.

The above-described symmetry transformations are called Noether symmetries.

Below, we list some properties of the transformation operators and of the symm
transformations:

a) Any first variation of the Lagrange function can be presented as

$$\hat{\delta}L = \delta q^a \frac{d_{\mathrm{EL}} L}{dq^a} + \frac{dP}{dt} = \delta q^a \frac{\delta S}{\delta q^a} + \frac{dP}{dt}\,,$$

where P is an LF of the form

$$P = \sum\nolimits_a{}' \sum_{m=1}^{N_a} p_a^m \delta q^{a[m-1]}\,,\ p_a^m = \sum_{s=l}^{N_a} \left(-\frac{d}{dt}\right)^{s-m} \frac{\partial L}{\partial q^{a[s]}}\,.$$

One ought to remark that the sum (19) that presents P is running only over those a for w
$N_a > 0$. However, it can be extended over all a's since the momenta p_a^m that correspon
the degenerate coordinates are zero. Thus, the prime over the sum above can be omitt

b) Any transformation operator commutes with the total time derivative:

$$\left[\hat{\delta}\,, \frac{d}{dt}\right] = 0\,.$$

The latter property is justified by the following relations:

$$\frac{d}{dt}\hat{\delta} = \sum_{k=0} \left[\delta q^{a[k+1]} \frac{\partial}{\partial q^{a[k]}} + \delta q^{a[k]} \frac{\partial}{\partial q^{a[k]}}\partial_t\right] + \sum_{k,l=0} q^{b[l+1]}\, \delta q^{a[k]} \frac{\partial^2}{\partial q^{a[k]} \partial q^{b[l]}}\,,$$

$$\hat{\delta}\frac{d}{dt} = \sum_{l=0} \left[\hat{\delta} q^{b[l+1]}\right] \frac{\partial}{\partial q^{b[l]}} + \hat{\delta}\partial_t + \sum_{k,l=0} \delta q^{a[k]} q^{b[l+1]} \frac{\partial^2}{\partial q^{b[l]} \partial q^{a[k]}} = \frac{d}{dt}\hat{\delta}\,.$$

c) The commutator of any two transformation operators is a transformation operat
well.

Namely, let $\hat{\delta}_1 q = \delta q_1$, and $\hat{\delta}_2 q = \delta q_2$. Then

$$\left[\hat{\delta}_1, \hat{\delta}_2\right] = \hat{\delta}_3\,,\ \ \hat{\delta}_3 q = \hat{\delta}_1 \delta q_2 - \hat{\delta}_2 \delta q_1\,.$$

Indeed, one can write:

$$\hat{\delta}_1\hat{\delta}_2 = \sum_{l=0}\left(\hat{\delta}_1\delta q_2^{b[l]}\right)\frac{\partial}{\partial q^{b[l]}} + \sum_{k,l=0}\delta q_1^{a[k]}\delta q_2^{b[l]}\frac{\partial}{\partial q^{b[l]}}\frac{\partial}{\partial q^{a[k]}}$$

$$= \sum_{l=0}\frac{d^l(\hat{\delta}_{\epsilon_1}\delta q_2^b)}{dt^l}\frac{\partial}{\partial q^{b[l]}} + \sum_{k,l=0}\delta q_1^{a[k]}\delta q_2^{b[l]}\frac{\partial}{\partial q^{b[l]}}\frac{\partial}{\partial q^{a[k]}},$$

$$\hat{\delta}_2\hat{\delta}_1 = \sum_{k=0}\left(\hat{\delta}_{\epsilon_2}\delta q_1^{a[k]}\right)\frac{\partial}{\partial q^{a[k]}} + \sum_{l,k=0}\delta q_2^{b[l]}\delta q_1^{a[k]}\frac{\partial}{\partial q^{a[k]}}\frac{\partial}{\partial q^{b[l]}}$$

$$= \sum_{k=0}\frac{d^k\left(\hat{\delta}_{\epsilon_2}\delta q_1^b\right)}{dt^k}\frac{\partial}{\partial q^{a[k]}} + \sum_{k,l=0}\delta q_1^{a[k]}\delta q_2^{b[l]}\frac{\partial}{\partial q^{b[l]}}\frac{\partial}{\partial q^{a[k]}}.$$

Then subtracting Eq. (23) from Eq. (22), we obtain the relation (21).

In other words, the set of all transformation operators form a Lie algebra.

d) The commutator of the Euler–Lagrange derivative and a transformation operat proportional to the Euler–Lagrange derivative. Namely, if $\hat{\delta}q = \delta q^b$, then

$$\left[\frac{d_{\mathrm{EL}}}{dq^a}, \hat{\delta}\right] = \hat{Q}_a^b\frac{d_{\mathrm{EL}}}{dq^b}, \quad \hat{Q}_a^b = \sum_{k=0}\left(-\frac{d}{dt}\right)^k\frac{\partial}{\partial q^{a[k]}}\delta q^b.$$

To prove this property, one may consider a sequence of equalities,

$$\int_{t_1}^{t_2}\frac{d_{\mathrm{EL}}\left(\hat{\delta}F\right)}{dq^a}\zeta^a dt = \int_{t_1}^{t_2}\hat{\delta}_\zeta\hat{\delta}F dt = \int_{t_1}^{t_2}\hat{\delta}\hat{\delta}_\zeta F dt + \int_{t_1}^{t_2}\hat{\delta}_{\hat{\delta}_\zeta\delta q}F dt$$

$$= \int_{t_1}^{t_2}\zeta^a\sum_{k=0}\left(-\frac{d}{dt}\right)^k\hat{\delta}\frac{\partial F}{\partial q^{a[k]}}dt + \int_{t_1}^{t_2}\hat{\delta}_\zeta\delta q^b\frac{d_{\mathrm{EL}}F}{dq^b}dt$$

$$= \int_{t_1}^{t_2}\zeta^a\left(\hat{\delta}\delta_a^b + \hat{Q}_a^b\right)\frac{d_{\mathrm{EL}}F}{dq^b}dt, \quad \left(\hat{\delta}_\zeta q^a = \zeta^a\right),$$

where $\zeta(t)$ is an arbitrary inner variation, and F is an LF.

It is useful to keep in mind the following generalization of relation (24):

$$\left[\left(\frac{d}{dt}\right)^k\frac{d_{\mathrm{EL}}}{dq^a}, \hat{\delta}\right] = \left(\frac{d}{dt}\right)^k\hat{Q}_a^b\frac{d_{\mathrm{EL}}}{dq^b},$$

which follows immediately from (20) and (24).

e) The commutator of two symmetry operators is a symmetry operator as well.

Indeed, let $\hat{\delta}_1 q = \delta q_1$, and $\hat{\delta}_2 q = \delta q_2$ be symmetry transformations, i.e., $\hat{\delta}_1 L = dF_1$ and $\hat{\delta}_2 L = dF_2/dt$. Then, taking into account (20) and (21), we obtain

$$\left[\hat{\delta}_1, \hat{\delta}_2\right] L = \hat{\delta}_1 L = \frac{d}{dt} F_3\,, \quad F_3 = \hat{\delta}_1 F_2 - \hat{\delta}_2 F_1\,.$$

Thus, the set of symmetry operators of the action S forms a Lie subalgebra of the algebra of all transformation operators.

f) Symmetry transformations transform extremals into extremals.

The validity of this assertion follows from the relations proven below.

Suppose $\hat{\delta}$ is a symmetry operator; then the following relation takes place:

$$\hat{\delta} \frac{\delta S}{\delta q^a} = -\hat{Q}^b_a \frac{\delta S}{\delta q^b}\,.$$

Indeed, by virtue of (10), (11), and (24), we can write

$$\begin{aligned} \hat{\delta} \frac{\delta S}{\delta q^a} = \hat{\delta} \frac{d_{\mathrm{EL}} L}{dq^a} &= \frac{d_{\mathrm{EL}} \left(\hat{\delta} L\right)}{dq^a} - \hat{Q}^b_a \frac{d_{\mathrm{EL}} L}{dq^b} \\ &= \frac{d_{\mathrm{EL}}}{dq^a} \frac{dF}{dt} - \hat{Q}^b_a \frac{\delta S}{\delta q^b} = -\hat{Q}^b_a \frac{\delta S}{\delta q^b}\,. \end{aligned}$$

A generalization of (27) based on the relation (24) reads:

$$\hat{\delta} \frac{d^k}{dt^k} \frac{\delta S}{\delta q^a} = -\frac{d^k}{dt^k} \hat{Q}^b_a \frac{\delta S}{\delta q^b}\,.$$

g) Symmetry transformations transform genuine trajectories into genuine trajectori

Indeed, suppose that $\tilde{q}^a$ be a genuine trajectory, that is

$$\left.\frac{\delta S}{\delta q^a}\right|_{\tilde{q}} = 0\,,$$

and δq^a be a symmetry transformation. Then the transformed trajectory $\tilde{q}'^a = \tilde{q}^a + \delta$ also a genuine one. Indeed, by virtue of (27) and (29), we get:

$$\left.\frac{\delta S}{\delta q^a}\right|_{\tilde{q}'=\tilde{q}+\delta q} = \left.\frac{\delta S}{\delta q^a}\right|_{\tilde{q}} + \hat{\delta} \left.\frac{\delta S}{\delta q^a}\right|_{\tilde{q}} = \left(\delta^b_a - \hat{Q}^b_a\right) \left.\frac{\delta S}{\delta q^b}\right|_{\tilde{q}} = 0\,.$$

C Trivial symmetries

Below, we are going to describe so-called trivial symmetries transformations, which
for any action.

A symmetry transformation is called a trivial symmetry transformation whenever
corresponding trajectory variation has the form

$$\delta q^a = \hat{U}^{ab}\frac{\delta S}{\delta q^b},$$

where $\hat{U}$ is an antisymmetric LO, that is $\left(\hat{U}^T\right)^{ab} = -\hat{U}^{ab}$. Thus, trivial symmetry tran
mations do not affect genuine trajectories. (One can prove, see below, that any symm
transformation that vanishes on the equations of motion, $\delta q^a = O\left(\delta S/\delta q\right)$, is trivial, na
it has the form (30)). With the help of relations (5) and (18), we can easily verify that
is actually a symmetry transformation. Indeed,

$$\hat{\delta}L = \frac{d_{\mathrm{EL}}L}{dq^b}\hat{U}^{ab}\frac{d_{\mathrm{EL}}L}{dq^b} + \frac{dP}{dt} = \frac{dF}{dt} + \frac{dP}{dt} = \frac{d\left(F+P\right)}{dt},$$

where F and P are some LF.

Since trivial symmetry transformations are proportional to the EM, they do not ch
genuine trajectories, as was already mentioned above.

The commutator of a symmetry operator and a trivial-symmetry operator is a tri
symmetry operator. Namely, if

$$\hat{\delta}_1 L = dF_1/dt\,,\ \hat{\delta}_2 L = dF_2/dt,\ \hat{\delta}_2 q^a = \delta_2 q^a = \hat{V}^{ab}\delta S/\delta q^b\,,$$

then

$$\left[\hat{\delta}_1, \hat{\delta}_2\right] L = \hat{\delta}_3 L\,,\ \hat{\delta}_3 q^a = \delta_3 q^a = \hat{U}^{ab}\frac{\delta S}{\delta q^b},$$

where $\hat{V}^{ab}$ and $\hat{U}^{ab}$ are some antisymmetric LO.

To verify (31), we remark that, according to (21), $\hat{\delta}_3$ is a symmetry operator, with δ
$\hat{\delta}_1\delta_2 q - \hat{\delta}_2\delta_1 q$, where $\delta_1 q = \hat{\delta}_1 q^a$. The term $\hat{\delta}_1\delta_2 q$ can be calculated with the help of (14

$$\hat{\delta}_1\delta_2 q^a = \sum_{k=0}\frac{\partial\left(\delta_2 q^a\right)}{\partial q^{c[k]}}\left[\frac{d^k}{dt^k}\left(\hat{V}^{cb}\frac{\delta S}{\delta q^b}\right)\right],$$

and the term $\hat{\delta}_2\delta_1 q$ can be calculated with the help of (27),

$$\hat{\delta}_2\delta_1 q = \left(\hat{\delta}_2\hat{V}^{ab}\right)\frac{\delta S}{\delta q^b} + \hat{V}^{ab}\hat{\delta}_2\frac{\delta S}{\delta q^b} = \left(\hat{\delta}_2\hat{V}^{ab}\right)\frac{\delta S}{\delta q^b} - \hat{V}^{ab}\hat{Q}^c_b\frac{\delta S}{\delta q^c}\,.$$

Thus, we obtain: $\hat{\delta}_3 q^a = \delta_3 q^a = \hat{U}^{ab}\delta S/\delta q^b$, where $\hat{U}^{ab}$ is an antisymmetric LO of the fc

$$\hat{U}^{ab} = \sum_{k=0}\left[\frac{\partial\left(\delta_2 q^a\right)}{\partial q^{c[k]}}\left(\frac{d}{dt}\right)^k\hat{V}^{cb} + \hat{V}^{ac}\left(-\frac{d}{dt}\right)^k\frac{\partial\left(\delta_2 q^b\right)}{\partial q^{c[k]}}\right] - \hat{\delta}_2\hat{V}^{ab}.$$

We call two symmetry transformations $\delta_1 q$ and $\delta_2 q$ equivalent ($\delta_1 q \sim \delta_2 q$) whenever differ by a trivial symmetry transformation:

$$\delta_1 q \sim \delta_2 q \Longleftrightarrow \delta_1 q^a - \delta_2 q^a = \hat{U}^{ab}\frac{\delta S}{\delta q^b}\,.$$

Here $\left(\hat{U}^T\right)^{ab} = -\hat{U}^{ab}$.

Let $\mathbf{G}(S)$ be the Lie algebra of all symmetries of the action S. The trivial symme form the ideal $\mathbf{G}_{\rm tr}(S)$ in the Lie algebra $\mathbf{G}(S)$. Then the classes of equivalent symme form a Lie algebra $\mathbf{G}_{\rm Ph}(S)$ isomorphic to the quotient algebra:

$$\mathbf{G}_{\rm Ph}(S) = \mathbf{G}(S)/\mathbf{G}_{\rm tr}(S)\,.$$

III. DYNAMICALLY EQUIVALENT ACTIONS

Very often we encounter an action

$$S_{\rm E}[q,y] = \int L_{\rm E}\left(q^{[]}, y^{[]}\right)dt\,,$$

which contains two groups of coordinates $q^{[]}$ and $y^{[]}$ such that the Euler–Lagrange allow to express all y via $q^{[]}$. It is convenient to call $S_{\rm E}[q,y]$ the extended action. One car to eliminate the variables y from the extended action to get some reduced action, w depends now only on q, and ask the question: What is the relation between the exte and the reduced actions? There exist a case when this question has a definite answer [

Namely, let us suppose that the Euler–Lagrange $\delta S_{\mathrm{E}}\left[q,y\right]/\delta y = 0$ allow one to exp uniquely the variables y as LF of the variables q,

$$\frac{\delta S_{\mathrm{E}}\left[q,y\right]}{\delta y} = 0 \Longleftrightarrow y = \bar{y}\left(q^{[]}\right) .$$

Then we define the reduced action $S\left[q\right]$

$$S\left[q\right] = S_{\mathrm{E}}[q,\bar{y}] = \int L_{\mathrm{E}}\left(q^{[]},\bar{y}^{[]}\right)dt = \int L\left(q^{[]}\right)dt .$$

Let us compare the Euler–Lagrange that correspond to both actions. First consider variation of the reduced action δS under arbitrary inner variations δq,

$$\delta S[q] = \int \left(\left.\frac{\delta S_{\mathrm{E}}\left[q,y\right]}{\delta q^i}\right|_{y=\bar{y}} \delta q^i + \left.\frac{\delta S_{\mathrm{E}}\left[q,y\right]}{\delta y^\alpha}\right|_{y=\bar{y}} \delta\bar{y}^\alpha \right) dt = \int \frac{\delta S\left[q\right]}{\delta q^i}\delta q^i dt .$$

In virtue of (34), the Euler–Lagrange of the reduced action read

$$\frac{\delta S\left[q\right]}{\delta q} = \left.\frac{\delta S_{\mathrm{E}}\left[q,y\right]}{\delta q}\right|_{y=\bar{y}} = 0 .$$

On the other hand, the Euler–Lagrange of the extended action $S_{\mathrm{E}}\left[q,y\right]$ are

$$\frac{\delta S_{\mathrm{E}}\left[q,y\right]}{\delta q} = 0 , \quad \frac{\delta S_{\mathrm{E}}\left[q,y\right]}{\delta y} = 0 \Longleftrightarrow y = \bar{y}\left(q^{[]}\right) .$$

They are reduced to (37) in the q-sector. We can see that the extended action and reduced action lead to the same Euler–Lagrange for q. This is why the variables y are c the auxiliary variables. The auxiliary variables y can be eliminated from the action wit help of the Euler–Lagrange. Further, we call the actions $S_{\mathrm{E}}\left[q,y\right]$ and $S[q]$ the dynami equivalent actions.

One ought to stress that the above equivalence is a consequence of the assumption the variables y are expressed via q by means of the equations $\delta S/\delta y = 0$ only. If, for purpose, some of the equations $\delta S/\delta q = 0$ are used as well, then the above equivalence ca absent. Of course, the solutions of the Euler–Lagrange for the reduced action, together the definition $y = \bar{y}$, contain all solutions of the Euler–Lagrange for the extended actio it is easily seen from Eq. (36)). However, the reduced action can imply additional solut

Actions containing auxiliary variables and the corresponding reduced actions have sir properties, in particular, there exists a direct relation between their symmetry transfo tions.

As was mentioned above, we are going to relate the symmetry properties of the exte and reduced actions. To this end, it is convenient to make an invertible coordinate rep ment, $(q^a, y^\alpha) \to \tilde{q}^A = (q^a, z^\alpha)$, $y = z + \bar{y}\left(q^{[l]}\right)$, in the extended action. In fact, we are g to consider a modified extended action $\tilde{S}[\tilde{q}]$, which is obtained from the extended a $S_{\rm E}[q, y]$ as follows:

$$\tilde{S}[\tilde{q}] = \int \tilde{L}\left(\tilde{q}^{[l]}\right) dt = S_{\rm E}[q, z + \bar{y}] = \int L_{\rm E}\left(q^{[l]}, z^{[l]} + \bar{y}^{[l]}\right) dt\,.$$

The extended action $S_{\rm E}[q, y]$ and the modified extended action $\tilde{S}[\tilde{q}]$ are completely ec alent. They lead to completely equivalent Euler–Lagrange. Thus, it is sufficient to s the relation between the symmetry properties of the modified extended action $\tilde{S}[\tilde{q}]$ an reduced action $S\,[q]$.

Note that

$$S\,[q] = \tilde{S}[\tilde{q}]\Big|_{z=0}\,,\; L\left(q^{[l]}\right) = \tilde{L}\left(\tilde{q}^{[l]}\right)\Big|_{z=0}\,.$$

Besides, the action (38) can be presented in the form

$$\tilde{S}[\tilde{q}] = S\,[q] + \Delta S\,[\tilde{q}]\;,\;\; \Delta S\,[\tilde{q}] = \int \Delta L dt\,,$$
$$\Delta L = \tilde{L}\left(\tilde{q}^{[l]}\right) - L\left(q^{[l]}\right) = L_{\rm E}\left(q^{[l]}, z^{[l]} + \bar{y}^{[l]}\right) - L_{\rm E}\left(q^{[l]}, \bar{y}^{[l]}\right)\,.$$

The variables z are auxiliary ones for the action $\tilde{S}[\tilde{q}]$, and, in particular, $z = 0$ on Euler–Lagrange. Indeed,

$$\frac{\delta \tilde{S}[\tilde{q}]}{\delta z} = 0 \Longleftrightarrow \frac{\delta S_{\rm E}[q, y]}{\delta y} = 0 \Longrightarrow y = \bar{y}\left(q^{[l]}\right) \Longrightarrow z = 0\,.$$

The latter implies:

$$\frac{\delta \tilde{S}}{\delta z^\alpha} = \frac{\delta \Delta S}{\delta z^\alpha} = \hat{U}_{\alpha\beta} z^\beta = 0\,.$$

Since equation (41) has the unique solution $z = 0$, one can easily verify that $\hat{U}$ is an inver
LO. The equation (42) implies

$$\Delta L = z^\alpha \hat{K}_{\alpha\beta} z^\beta + \frac{d}{dt} F ,$$

where $\hat{K}$ is a symmetric LO, and F is an LF. Besides, one can write

$$z^\alpha = \left(\hat{U}^{-1}\right)^{\alpha\beta} \frac{\delta \Delta S}{\delta z^\beta} = \left(\hat{U}^{-1}\right)^{\alpha\beta} \frac{\delta \tilde{S}}{\delta z^\beta} .$$

On the other hand, due to the property (11), one can write

$$\frac{\delta \Delta S}{\delta q^a} = \frac{d_{\rm EL} \Delta L}{dq^a} = \frac{d_{\rm EL}}{dq^a} \left[z^\alpha \hat{K}_{\alpha\beta} z^\beta \right] .$$

Then, taking into account (43, 44), and the definition of the Euler–Lagrange derivative
get the following useful relation:

$$\frac{\delta \Delta S}{\delta q^a} = \hat{\Lambda}_a^\alpha \frac{\delta \Delta S}{\delta z^\alpha} , \quad \hat{\Lambda}_a^\alpha = \sum_{l=0} \left(-\frac{d}{dt} \right)^l z^\nu \frac{\partial \hat{K}_{\nu\beta}}{\partial q^{a[l]}} \left(\hat{U}^{-1}\right)^{\beta\alpha} ,$$

where $\hat{\Lambda}_a^\alpha$ is an LO.

IV. SYMMETRIES OF THE EXTENDED AND THE REDUCED ACTION

There exists a one-to-one correspondence (isomorphism) between the symmetry cl
of the extended action $\tilde{S}[\tilde{q}]$ and the reduced action $S[q]$. Below, we prove a set of assert
which justify, in fact, this correspondence.

i) If the transformation

$$\delta \tilde{q}^A = \begin{pmatrix} \delta' q^a \\ \delta z^\alpha \end{pmatrix} ,$$

is a symmetry of the extended action $\tilde{S}$, then the transformation

$$\delta q^a = \delta' q \big|_{z=0}$$

is a symmetry of the reduced action S.

Indeed, let (46) be a symmetry of the action $\tilde{S}$. Then

$$\hat{\delta}_{\delta\tilde{q}}\tilde{L} = \frac{d}{dt}\tilde{F}\,,$$

where $\tilde{F}$ is an LF. Considering (48) at $z = \delta z = 0$, we get

$$\hat{\delta}_{\delta q}L = \frac{d}{dt}F\,,\quad \delta q^a = \delta' q|_{z=0}\,,\quad F = \tilde{F}\Big|_{z=0}\,,$$

where L is given by (39). Thus, any symmetry of the action $\tilde{S}$ implies a symmetry of action S. The symmetry δq obtained in such a way can be called the symmetry reduc of the extended action.

ii) If the transformation δq is a symmetry of the reduced action S, then the transforma

$$\delta\tilde{q}^A = \begin{pmatrix} \delta q^a \\ \delta z^\alpha \end{pmatrix}\,,\ \delta z^\alpha = -\left(\hat{\Lambda}^T\right)^\alpha_a \delta q^a\,,$$

where the LO $\hat{\Lambda}$ defined by Eq. (45) is a symmetry of the extended action $\tilde{S}$.

To prove this assertion, let us consider the first variation $\hat{\delta}_{\delta\tilde{q}}\tilde{L}$ of the Lagrange func $\tilde{L}$. Since δq is a symmetry of the reduced action S, the relation $\hat{\delta}_{\delta q}L = dF/dt$, whe is an LF, holds true. Thus, with the help of the property (15), one may write the varia $\hat{\delta}_{\delta\tilde{q}}\tilde{L}$ in the form

$$\hat{\delta}_{\delta\tilde{q}}\tilde{L} = \left(\hat{\delta}_{\delta q} + \hat{\delta}_{\delta z}\right)\tilde{L} = \frac{d}{dt}F + \left(\hat{\delta}_{\delta q} + \hat{\delta}_{\delta z}\right)\Delta L\,.$$

Now, we present the variations $\hat{\delta}_{\delta q}\Delta L$ and $\hat{\delta}_{\delta z}\Delta L$ with the help of relation (18). Bes taking into account the expression (49) for the variation δz, we get

$$\hat{\delta}_{\delta\tilde{q}}\tilde{L} = \frac{d}{dt}\left(F + P_q + P_z\right) + \delta q^a\frac{\delta\Delta S}{\delta q^a} - \left[\left(\hat{\Lambda}^T\right)^\alpha_a \delta q^a\right]\frac{\delta\Delta S}{\delta \dot{z}^\alpha}\,,$$

where P_q and P_z are some LF. Using (45) and (5), we may write

$$\delta q^a\frac{\delta\Delta S}{\delta q^a} = \delta q^a\hat{\Lambda}^\alpha_a\frac{\delta\Delta S}{\delta z^\alpha} = \left[\left(\hat{\Lambda}^T\right)^\alpha_a \delta q^a\right]\frac{\delta\Delta S}{\delta \dot{z}^\alpha} + \frac{dG}{dt}\,,$$

where G is an LF. Thus, the variation $\hat{\delta}_{\delta\tilde{q}}\tilde{L}$ is reduced to the total derivative of an LF,

$$\hat{\delta}_{\delta\tilde{q}}\tilde{L} = \frac{d}{dt}\left(F + P_q + P_z + G\right)\,.$$

Thus, $\delta\tilde{q}$ is a symmetry of the extended action $\tilde{S}$.

iii) Any symmetry of the form

$$\delta\tilde{q} = \begin{pmatrix} 0 \\ \delta z \end{pmatrix}$$

of the extended action $\tilde{S}$ is trivial.

Since $\delta\tilde{q}$ is a symmetry of the action $\tilde{S}$, one can write

$$\hat{\delta}_{\delta\tilde{q}}\tilde{L} = \hat{\delta}_{\delta z}\tilde{L} = \frac{dF}{dt},$$

where F is an LF. Taking into account (18), we may rewrite Eq. (54) as

$$\delta z^{\alpha}\frac{\delta\tilde{S}}{\delta z^{\alpha}} = \frac{dF'}{dt},$$

where F' is an LF. The left-hand side of equation (55) can be transformed, with the he (42) and (5), to the form

$$\delta z^{\alpha}\frac{\delta\tilde{S}}{\delta z^{\alpha}} = \delta z^{\alpha}\hat{U}_{\alpha\beta}z^{\beta} = \left[\left(\hat{U}^{T}\right)_{\beta\alpha}\sigma^{\alpha}\right]z^{\beta} + \frac{dF''}{dt},$$

where F'' is an LF. Thus, the equation (55) may be reduced to

$$z^{\beta}f_{\beta} = \frac{d\Phi}{dt},\ f_{\beta} = \left(\hat{U}^{T}\right)_{\beta\alpha}\sigma^{\alpha},$$

where $f\left(Q^{[]}\right)$ and $\Phi\left(Q^{[]}\right)$ are some LF. Let us present the LF Φ as

$$\Phi\left(Q^{[]}\right) = \Phi_0\left(q^{[]}\right) + \Phi_1\left(Q^{[]}\right),$$

$$\Phi_0 = \Phi|_{z=0},\ \Phi_1|_{z=0} = \sum_{k=0}^{N}\Phi_{\alpha(k)}\left(Q^{[]}\right)z^{\alpha[k]},\ N < \infty.$$

It follows from equation (56) that $d\Phi_0/dt \equiv 0$. According to (6), the latter implies $\Phi_0 = \text{const}$. From (56), we get the equation

$$\sum_{k=0}^{N+1}\varphi_{\alpha(k)}z^{\alpha[k]} = 0,$$

where

$$\varphi_{\alpha(0)} = f_\alpha - \dot{\Phi}_{\alpha(0)}\,,\ \varphi_{\alpha(N+1)} = -\Phi_{\alpha(N)}\,,$$
$$\varphi_{\alpha(k)} = -\left[\Phi_{\alpha(k-1)} + \dot{\Phi}_{\alpha(k)}\right],\ k = 1, ..., N\,.$$

The general solution of Eq. (58) is

$$\varphi_{\alpha(k)} = \sum_{l=0}^{N+1} m_{\alpha(k)|\beta(s)l}\, z^{\beta[s]},\ \ m_{\alpha(k)|\beta(s)l} = -m_{\beta(s)|l\alpha(k)}\,,$$

where $m_{\alpha(k)|\beta(s)l}\left(Q^{[]}\right)$ are some LF. Then the LF $\Phi_{\alpha(k)}$ and f_α can be found from Eq. (

$$\Phi_{\alpha(k)} = -\sum_{m=0}^{N-k}\sum_{l=0}^{N+1}\left(-\frac{d}{dt}\right)^m \left[m_{\alpha(k+m+1)|\beta(l)} z^{\beta[l]}\right]\,,$$
$$f_\alpha = \sum_{m,l=0}^{N+1}\left(-\frac{d}{dt}\right)^m \left[m_{\alpha(m)|\beta(l)} z^{\beta[l]}\right] \equiv \hat{m}_{\alpha\beta} z^\beta\,,$$

where $\hat{m}_{\alpha\beta}$ is an antisymmetric LO. Thus, we get from (56)

$$\delta z^\alpha = \hat{M}^{\alpha\beta}\frac{\delta \tilde{S}}{\delta z^\beta},\ \hat{M}^{\alpha\beta} = \left[\left(\hat{U}^T\right)^{-1}\right]^{\alpha\gamma} \hat{m}_{\gamma\delta}\left(\hat{U}^{-1}\right)^{\delta\beta}\,,$$

where $\hat{M}^{\alpha\beta}$ is an antisymmetric LO. Therefore, the symmetry (53) is trivial.

iv) Suppose both transformations $\delta\tilde{q}_1$ and $\delta\tilde{q}_2$ to be symmetries of the extended acti such that their reductions coincide, that is

$$\delta' q_1|_{z=0} = \delta' q_2|_{z=0} = \delta q\,.$$

Then these symmetries are equivalent,

$$\delta\tilde{q}_1 \sim \delta\tilde{q}_2\,,$$

which means that $\delta\tilde{q}_1$ and $\delta\tilde{q}_2$ differ by a trivial symmetry.

Thus, we have to proved that the transformation

$$\Delta\tilde{q} = \delta\tilde{q}_1 - \delta\tilde{q}_2 = \begin{pmatrix} \Delta q' = \delta' q_1 - \delta' q_2 \\ \Delta z = \delta z_1 - \delta z_2 \end{pmatrix},\ \Delta q'|_{z=0} = 0\,,$$

is a trivial symmetry of the extended action $\tilde{S}$. In virtue of Eq. (63), the LF $\Delta q'$ ma presented as

$$\Delta q'^a = \hat{m}^a_\alpha z^\alpha ,$$

where $\hat{m}$ is an LO. With the help of (44), we get for $\Delta q'$ the following expression:

$$\Delta q'^a = \hat{M}^{a\beta} \frac{\delta \tilde{S}}{\delta z^\beta} ,$$

where $\hat{M} = \hat{m}\hat{U}^{-1}$ is an LO.

Let us present the transformation $\Delta\tilde{q}$ in the form $\Delta\tilde{q} = \Delta_1\tilde{q} + \Delta_2\tilde{q}$, where

$$\Delta_1\tilde{q} = \hat{M}^{AB}\frac{\delta\tilde{S}}{\delta\tilde{q}^B} , \ \hat{M}^{AB} = \begin{pmatrix} 0 & \hat{M}^{a\beta} \\ -\left(\hat{M}^T\right)^{\alpha b} & 0 \end{pmatrix} ,$$

and

$$\Delta_2\tilde{q} = \begin{pmatrix} 0 \\ \Delta\sigma'' \end{pmatrix} .$$

The transformations $\Delta_1\tilde{q}$ is a trivial symmetry since the LO $\hat{M}^{AB}$ is antisymmetric, th $\left(\hat{M}^T\right)^{AB} = -\hat{M}^{AB}$. Thus, $\Delta_2\tilde{q}$ is a symmetry of the extended action $\tilde{S}$. Besides, the l symmetry has a special form (68). It was proven in item c) that any symmetry of su form is trivial. Therefore, the symmetry $\Delta\tilde{q}$ is trivial as well.

v) Let a transformation $\delta\tilde{q}$ be a trivial symmetry of the extended action $\tilde{S}$. The reduction δq is a trivial symmetry of the reduced action S.

According to this assumption, we may write

$$\delta\tilde{q}^A = \begin{pmatrix} \delta' q^a = \hat{M}^{ab}\frac{\delta\tilde{S}}{\delta q^b} + \hat{M}^{a\beta}\frac{\delta\tilde{S}}{\delta z^\beta} \\ \delta z^\alpha = -\left(\hat{M}^T\right)^{b\alpha}\frac{\delta\tilde{S}}{\delta q^b} + \hat{M}^{\alpha\beta}\frac{\delta\tilde{S}}{\delta z^\beta} \end{pmatrix} ,$$

where the local operators $\hat{M}^{ab}$ and $\hat{M}^{\alpha\beta}$ are antisymmetric. Then the reduction $\delta q = \delta'$ of the transformation (69) reads

$$\delta q^a = \hat{m}^{ab}\frac{\delta S}{\delta q^b}, \ \hat{m}^{ab} = \hat{M}^{ab}\Big|_{z=0} .$$

The LO $\hat{m}^{ab}$ is antisymmetric. Thus, (70) is a trivial symmetry of the reduced action

vi) Let a symmetry δq of a reduced action S be trivial. Then any extension of symmetry to the symmetry $\delta\tilde{q}$ of the extended action $\tilde{S}$ is trivial as well.

Since δq is a trivial symmetry, one can write

$$\delta q^a = \hat{m}^{ab}\frac{\delta S}{\delta q^b},$$

where $\hat{m}^{ab}$ is an antisymmetric LO. Consider the following extension of the symmetry

$$\delta\tilde{q}_1 = \begin{pmatrix} \delta' q \\ 0 \end{pmatrix}, \ \delta' q^a = \hat{m}^{ab}\frac{\delta\tilde{S}}{\delta q^b},$$

which is a trivial symmetry of the extended action $\tilde{S}$. Any other extension of δq differs : $\delta\tilde{q}_1$ by a trivial symmetry, according to item (iv). Therefore, any extension of the tr symmetry is a trivial symmetry as well.

Concluding, we can see that there exists an isomorphism between classes of equiv symmetries of dynamically equivalent actions. Since the Lagrangian and Hamiltonia tions are dynamically equivalent, one can study the symmetry structure of any sin theory considering the first-order Hamiltonian action.

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REFERENCES

[1] P.G. Bergmann, Phys. Rev. **75** (1949) 680; P.M. Dirac, Can. J. Math. **2** (1950) *Lectures on Quantum Mechanics* (Yeshiva University, New York 1964); E.C.G. Su shan, N. Mukunda, *Classical Dynamics: A Modern Perspective* (Wiley, New York 1 D.M. Gitman and I.V. Tyutin, *Quantization of Fields with Constraints* (Springer-Ve Berlin 1990); M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Pri ton University Press, Princeton 1992)

[2] V.A. Borochov and I.V. Tyutin, Physics of Atomic Nuclei, **61** (1998) 1603; ibid **62** (1 1070

[3] D.M. Gitman, I.V. Tyutin, *Symmetries in Constrained Systems*, Talk of Internati Conference "Lie and Jordan algebras, their Representations and Applications, II, Bı Guaruja, 3-8 May 2004; hep-th/0409087; *General quadratic gauge theory. Const structure, symmetries, and physical functions*, hep-th/0409206, submitted to J. Phy

[4] D. M. Gitman, and I.V. Tyutin, Nucl. Phys. B630 (3) (2002) pp. 509-527

[5] M. Henneaux, Phys. Lett., **B238** (1990) 299; I.A. Batalin and I.V. Tyutin, Int. J. M Phys. **A11** (1996) 1352; D. M. Gitman, and I.V. Tyutin, Nucl. Phys. B630 (3) (2 509

[6] The functions F may depend on time explicitly, however, we do not include t in arguments of the functions.

[7] The functions L may depend on time explicitly, however, we do not include t in arguments of the functions.

[8] Inner variations vanish together with all their time derivatives at t_1 and t_2.

[9] Sometimes, we mark the transformation operator below by the corresponding varia