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FREE PARAMETERS IN QUANTUM FIELD THEORIES: AN ANALYSIS OF THE VARIATIONAL APPROACH

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Free parameters in quantum field theories: an analysis of the variational approach

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Abstract

The usual renormalization procedure for the variational approximation with a trial Gaussian ansatz for the $\lambda\phi^4$ model in 3+1 dimensions is re-analysed as a departing framework for the investigation of the parameters of the model. The so-called asymmetric phase of the model (where $< vac|\phi|vac > \neq 0$) is considered for the search of privileged values of these parameters (mass and coupling constant) and possible conditions they can be expected to satisfy. This also may yield a suitable approach for the investigation of the reliability and stability of the approximation. The extremization of the renormalized energy density with relation to the renormalized mass, coupling and $\bar{\phi}$ is done. The minimizations of the renormalized energy with relation to the mass and $\bar{\phi}$ provide different expressions from the ones obtained from the usual variational principle for the regularized theory. Sort of "energy scale" invariances are found in expressions for the renormalized mass and coupling constant. A different view on the restoration of symmetry issue is presented. The transcendental character of the GAP equation can be reduced or even eliminated by placing some variables in the complex plane. This procedure corresponds to move from one phase of the potential to the other by means of a sort of rotation of the mass parameters in the complex plane.

Key words: Mass, coupling constant, variational method, symmetry restoration, vacuum, non perturbative method, quantum field theory, spontaneous symmetry breaking, many body quantum theory, Gaussian wavefunctional, Bogoliubov transformation, Bose-Einstein condensation, renormalization, ground state.

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1 Introduction

The most developped approach to solve interacting field (and many body) theories is perturbation theory which only works well for very small coupling constants as it occurs in Quantum Electrodynamics. Also in this approach there is a systematic and direct way of dealing with ultraviolet (UV) divergences, i.e., one knows precisely how to renormalize the parameters and to make the theory finite [1, 2, 3]. There are many motivations for the development of non perturbative methods in Quantum Field and Many Body Theories such as for the description of strong interacting systems (with or without spontaneous symmetry breaking(s) (SSB)), bound states and phase transitions. One method which has been quite extensively investigated is the variational approximation which, with the use of Gaussian wave functional, has been showed to be useful in a wide variety of situations. It corresponds to a summation of "cactus" type loop diagrams [4, 5, 6, 10, 14]. It is equivalent to the Hartree Bogoliubov approach [11], in which a Bogoliubov transformation yields a non equivalent basis of Fock states [7, 15], and also to the leading order large N approximation [13, 16]. In this approach the ground state of the system is determined by (GAP) equations for the variational parameters, which are choosen to be a mass and the classical expected value of a field characteristic from a SSB state (for a scalar field $<\phi>\equiv\bar{\phi}$, which will be referred to as condensate [11]). These equations are derived from the minimization of the (regularized) averaged energy density with respect to variational parameters of the trial wavefunctional. As it is discussed below it is not evident whether the minimization of the renormalized energy density should provide the same relations among the parameters, i.e., the same ground state or not. The subtraction of the (equations of the) theory with $<\phi>=0$ from the ones of the theory in which $<\phi>\neq 0$ provides a consistent elimination of the ultraviolet divergences. The procedures for renormalization imply the introduction of a mass scale parameter leading to the definition of running coupling constant [10, 17] which permits to investigate the behavior of the model at different energy scales. Some earlier limitations of the approximations were pointed out and discussed in [19] which are nowadays rediscussed, with extensions and higher order calculations for static and time dependent formulations [20, 21, 22, 23, 24, 25, 26, 13, 27, 28, 29, 30, 31, 32, 33, 34, 16, 17, 18]. For the sake of conciseness extensions will not be discussed here although the main ideas of the present work apply to them. It is worth pointing out some general issues, related or not to the Gaussian approximation for the understanding of the present work. The renormalized coupling constant is directly proportional to the scattering amplitude and more strictly to the scattering length, as discussed for example in [17]. Therefore information of the existence of particular values of the coupling constant with its consequences on the structure of the model (for instance whether it corresponds to a energy minimum or maximum) has relevance for the scattering amplitude. The scattering length is, on its turn, of extreme relevance for Bose-Einstein-Condensation [42, 43]. On the other hand the emergence of a condensed state depends strongly on the values assumed by the coupling constant and also by the mass. In particular the masses (and eventually coupling constants) of particles in each of the phases of a theory which undergoes spontaneous symmetry breaking can vary. This is of relevance for *in medium* particle (hadrons) properties which are intensively investigated nowadays. The investigation of the model at different energy scales can yield relevant information and the Gaussian variational approximation (as well as large N, and the related ones) provides a suitable departing framework for investigating these effects.

It is usually highly desirable to predict the values of the free parameters of a physical theory, such as masses and couplings, from the theory itself before comparisons to experimental observations. For this it may be possible to predict values, either exact values or a range of them. These "privileged" values can be associated to the validity of the approximation method, to applicability of the model and also to points in which particular physical effects can be expected. The main aim of the present work is to suggest and investigate some reasonings according to which values (or range of values) for these parameters could be found with which the model can have particular behavior. Eventually this may suggest sort of "constraints" between the parameters inside the model. The basic ideas are: to search for renormalized couplings and masses which extremize (minimize/maximize) the renormalized energy density. This procedure can be considered as complementary to the renormalization group method [35]. Another procedure will be to consider some parameters in the complex plane to introduce auxiliar (imaginary) variables which can be eliminated afterwards.

The $\lambda\phi^4$ model has been extensively studied for different reasons among which to shed light on non perturbative effects in quantum field and many body theories (QFT, QMBT). It corresponds to one of the simplest self interacting model whose structure is expected to be (partially) present in several more elaborated theories and it presents interesting features [36, 37, 10, 35, 38, 39, 40, 41, 17]. It has also been considered for the study of cosmological models [44], some cases of Bose-Einstein-Condensation [42, 43, 11, 35] and of the Higgs particle in the standard model, for example in [39]. Besides that it shares several properties with the linear sigma model (LSM) which is an effective model for low energy QCD. Although it strongly seems to exhibit asymptotic freedom in the asymmetric phase [37, 10, 40], the model is "trivial" in the symmetric phase [5, 41, 35].

In the present work the usual renormalization scheme of the Gaussian approach as carried out, for example, in [8] for the $\lambda\phi^4$ model is used as starting point for further investigation. It is proposed the extremization of the renormalized energy density with relation to the renormalized parameters (coupling constant and mass). Besides that some variables are placed in the complex plane to search suitable (physical) values and eventual conditions for these parameters. The work is organized as follows. In the next section the Gaussian approximation is summarized: the GAP equation (transcendental) is derived, obtained from the regularized theory (with a cutoff). The renormalization procedure of the mass and coupling constant as proposed in [8] is considered. In sections 3, 4 and 5 values of the renormalized mass, condensate and coupling constant which extremize the energy density are searched and analysed. They also could yield privileged values of the parameters with which the approximation can be more appropriated, which can be associated to the validity of the approximation method, to the applicability of the model and also to points of the phase space in which particular physical effects can be expected. In some cases instabilities are found for values of the parameters. In section 6 a mathematical trick is used to search non transcendetal solutions or/and an expression which constrains further the parameters. This is done by allowing some parameters to be complex such that the imaginary part disappears in the end of the calculation to keep real values of mass parameters and coupling constant. In the last section the results are summarized.

2 Gaussian approximation for the $\lambda\phi^4$ model

The Lagrangian density for the scalar field $\phi(\mathbf{x})$ with bare mass m_0^2 and coupling constant λ is given by:

$$\mathcal{L}(\mathbf{x}) = \frac{1}{2}\left\{\partial_\mu\phi(\mathbf{x})\partial^\mu\phi(\mathbf{x}) - m_0^2\phi^2(\mathbf{x}) - \frac{\lambda}{12}\phi^4(\mathbf{x})\right\} \tag{1}$$

The theory is quantized in the Schrodinger picture [9] being the action of the field and momentum operators over a state $|\Psi[\phi]>$ given respectively by:

$$\hat{\phi}|\Psi>=\phi|\Psi> \qquad \hat{\pi} = -i\hbar\frac{\delta}{\delta\phi}|\Psi> \tag{2}$$

In the static Gaussian approximation at zero temperature the trial ground state wave functional Ψ is parametrized by the Gaussian:

$$\Psi\left[\phi(\mathbf{x})\right] = Nexp\left\{-\frac{1}{4}\int d\mathbf{x}d\mathbf{y}\delta\phi(\mathbf{x})G^{-1}(\mathbf{x},\mathbf{y})\delta\phi(\mathbf{y})\right\}, \tag{3}$$

Where $\delta\phi(\mathbf{x}) = \phi(\mathbf{x}) - \bar{\phi}(\mathbf{x})$ is the field shifted by the condensate, the point where the wave function is centered; the normalization factor is N, the variational parameters are the (classical) expected value of the field, $\bar{\phi}(\mathbf{x}) =< \Psi|\phi|\Psi >$, and the quantum fluctuations represented by the two point function, i.e., the width of the Gaussian: $G(\mathbf{x},\mathbf{y}) =< \Psi|\phi(\mathbf{x})\phi(\mathbf{y})|\Psi >$. In variational calculations the averaged energy calculated with $\Psi[\phi(\mathbf{x})]$ is to be minimized to obtain the GAP equations. In principle it would yield a maximum bound for the ground state (averaged) energy, although ultraviolet divergences make this not necessarily reliable. The minimization of the renormalized theory is useful for this theoretical bound of the variational principle. Each of these variational parameters represents one component of the scalar field: the expected value in the ground state ("classical" part) and the two-point Green's function with the mass of the quantum which is decomposed into creation and annihilation operators [11].

The average value of the Hamiltonian is calculated and expressed in terms of the variational parameters by means of expressions (2) and (3). It is given by:

$$\begin{aligned}\mathcal{H} \; &= \tfrac{1}{2}\Big[\tfrac{1}{4}G^{-1}(\mathbf{x},\mathbf{x}) - \Delta G(\mathbf{x},\mathbf{x}) + m_0^2 G(\mathbf{x},\mathbf{x}) + \tfrac{\lambda}{4}G^2(\mathbf{x},\mathbf{x}) + \\ &+ m_0^2\bar{\phi}^2(\mathbf{x}) + (\nabla\bar{\phi}(\mathbf{x}))^2 + \tfrac{\lambda}{12}\bar{\phi}^4(\mathbf{x}) + \tfrac{\lambda}{2}\bar{\phi}^2(\mathbf{x})G(\mathbf{x},\mathbf{x})\Big]\,.\end{aligned} \tag{4}$$

Although in this expression the variational parameters were allowed to have spatial dependence they will be assumed to be constant. Variations of the averaged energy density with respect to the variational parameters yield the following GAP and condensate equations which define the ground state of the model:

$$\begin{aligned}&\frac{\delta\mathcal{H}}{\delta G(\mathbf{x},\mathbf{y})} \rightarrow \; 0 = -\frac{1}{8}G^{-2}(\mathbf{x},\mathbf{y}) + \frac{\Gamma(\mathbf{x},\mathbf{y})}{2} + \frac{\lambda}{2}\bar{\phi}(\mathbf{x})^2 \qquad (i) \\ &\frac{\delta\mathcal{H}}{\delta\bar{\phi}(\mathbf{x})} \rightarrow \; 0 = \Gamma(\mathbf{x},\mathbf{y})\bar{\phi}(\mathbf{y}) + \frac{\lambda}{6}\bar{\phi}^2(\mathbf{x}), \qquad (ii)\end{aligned} \tag{5}$$

Where $\Gamma(\mathbf{x},\mathbf{y}) = -\Delta + \left(m_0^2 + \frac{\lambda}{2}G(\mathbf{x},\mathbf{x})\right)\delta(\mathbf{x}-\mathbf{y})$. The Green's function G can be written from expressions above as:

$$G_0(\mathbf{x},\mathbf{y}) =< \mathbf{x}|\frac{1}{\sqrt{-\Delta + m^2}}|\mathbf{y} > \tag{6}$$

where m^2 is given by the self consistent (transcendental) GAP equation (expression (5)):

$$m^2 = m_0^2 + \frac{\lambda}{2}TraceG(x,x,m^2) + \frac{\lambda}{2}\bar{\phi}^2. \tag{7}$$

An analogous expression holds for the case in which $\bar{\phi} = 0$, i.e.,

$$\mu^2 = m^2(\bar{\phi} = 0) = m_0^2 + \frac{\lambda}{2}TraceG(x,x,\mu^2).$$

Expression (6) is equivalent to the Feynman Green's function with time integrated and with sign changed in the imaginary part by replacing the self consistent mass by the bare mass m_0^2. The physical masses in the different phases can assume different values from each other. The condition of minimum for this procedure and its stability was partially investigated in [10] and it corresponds to analysing the second order variation of the energy density with respect to the variational parameters.

From the above expressions it is seen that the non zero solutions for the condensate, $\bar{\phi}$, can be written as:

$$\bar{\phi}^2 = -6\frac{m_0^2}{\lambda} - 3G(m^2) = \frac{3m^2}{\lambda}. \tag{8}$$

For $G = 0$ the tree level value for $\bar{\phi}$ is obtained in terms of the bare mass.

The above expression for the Gaussian width (6) (and its inverse G_0^{-1}) can be calculated in the momentum space with a regulator Λ (cutoff) yielding (for $\Lambda >> m^2$):

$$\begin{aligned} G(m^2) &= \frac{1}{8\pi^2}\left(\Lambda^2 - m^2 Ln\left(\frac{2\Lambda}{\sqrt{e}m}\right)\right), \\ G^{-1}(m^2) &= \frac{1}{8\pi^2}\left(2\Lambda^4 + 2m^2\Lambda^2 - \frac{m^4}{4} - m^4 Ln\left(\frac{2\Lambda}{\sqrt{e}m}\right)\right), \end{aligned} \tag{9}$$

where $d = 2/\sqrt{e}$. In the (local) limit of infinite cutoff the average energy and observables diverge and the divergences must be eliminated. The renormalization procedure has been performed in three dimensions for example in [8, 5, 37, 10, 17]. According to the procedure shown below the non equivalence of the Fock basis for each of the phases (explicited through the Gaussian covariances or equivalently m^2 and μ^2) is exhibitted as discussed in [7].

2.1 Renormalized parameters

The renormalization procedure of the parameters of the model is done as follows [8, 10]. The energy density of the symmetric phase, as well as its GAP equation (7), is subtracted from the corresponding expression of the asymmetric phase. The GAP equation as defined in expression (7) can be rewritten as:

$$\mu^2 = m^2 + g_R\left(\bar{\phi}^2 + \frac{m^2}{8\pi^2} Ln\left(\frac{m}{\mu}\right)\right), \tag{10}$$

where the renormalized parameters were defined as:

$$\begin{aligned} \mu^2 = m_R^2 &\equiv \frac{m_0^2 + \frac{\lambda\Lambda^2}{16\pi^2}}{1 + \frac{\lambda}{16\pi^2} log\left(\frac{d\Lambda}{\mu}\right)}, \\ g_R &= \frac{-\frac{\lambda}{2}}{1 + \frac{\lambda}{16\pi^2} log\left(\frac{d\Lambda}{\mu}\right)}. \end{aligned} \tag{11}$$

In the first of these expressions $m_R^2 \equiv \mu^2$ was chosen to produce the usual effective potential [5, 8]. It is seen from the second of these expressions that in the limit of $\Lambda \to \infty$ the bare coupling constant would go to zero in order to keep g_R finite if μ is kept constant. This is the "triviality" problem.

The resulting subtracted energy density, $\mathcal{H}_{sub} = \mathcal{H}(\bar{\phi}) - \mathcal{H}(\bar{\phi} = 0)$, is re-written in terms of the renormalized mass, coupling constant and the mass scale eliminating the cutoff. It is given by:

$$\mathcal{H}_{sub} = \frac{m^2}{2}\bar{\phi}^2 + \frac{1}{4g_R}\left(m^2 - \mu^2\right)^2 + \frac{1}{128\pi^2}\left(m^4 Ln\left(\frac{m^4}{\mu^4}\right) - m^4 + \mu^4\right). \tag{12}$$

The mass scale μ^2 is not a free parameter for the ground state in fact, it can be considered to be a function of the mass m^2 and the coupling g_R by the GAP expression (10). Other approaches can be of interest for investigating the variational method in the Schrodinger picture [3]. In the ground state the parameters $\bar{\phi}, m^2, \mu^2$ (for a given g_R) are related by the GAP and condensate expressions shown above. Any deviation of these values fixed in the GAP equation induce temporal evolution or can correspond to excited states (stable or not).

It is possible to verify whether the renormalized GAP equation obtained from the regularized energy density, given by expression (10), still is an equation defining the minimum of the energy density given by expression (12) or not in two ways. The minimization of expression (12) with relation to m^2 is done in the next section. However the integration of the GAP equation with relation to m^2 should also yield an expression equal to (12) in the case the order of performing renormalization and extracting the ground state does not change results. The integral of the GAP equation is given by:

$$\begin{aligned} &\int \left(-\mu^2 + m^2 + g_R\left(\bar{\phi}^2 + \frac{m^2}{8\pi^2}Ln\left(\frac{m}{\mu}\right)\right)\right) dm^2 = \\ &= -\mu^2 m^2 + \frac{m^4}{2} + g_R\bar{\phi}^2 m^2 + \frac{g_R}{16\pi^2}\left(\frac{m^4}{2}Ln\left(\frac{m^2}{\mu^2}\right) - \frac{m^6}{6\mu^2}\right) + C(\bar{\phi}, \mu^2), \end{aligned} \tag{13}$$

where $C(\bar{\phi}, \mu^2)$ does not depend on m^2. This expression contains terms different from the renormalized expression (12). This means either that the minimization of the regularized energy is not equivalent to the minimization of the renormalized one or/and that the renormalization procedure has to be improved to make both procedures coincident - if this is possible or desirable. This will be discussed below with the minimization of the energy density with respect to $\bar{\phi}$.

3 Energy density and renormalized mass

In this section the minimum of the renormalized energy density $\mathcal{H}_{sub}$ with relation to the renormalized (physical) mass is searched:

$$\frac{\partial \mathcal{H}_{sub}}{\partial m} = 0. \tag{14}$$

Considering that $\bar{\phi}^2$ is in fact dependent on m_R^2 by expression (10) the resulting expression is given by:

$$0 = m^3 \left[Ln^2 \left(\frac{m}{\mu} \right) a_1 + Ln \left(\frac{m}{\mu} \right) a_2 + a_3 \right], \tag{15}$$

where a_i can be given in terms of

$$J = 1 - \frac{g_R}{(8\pi)^2} = 1 - G_R,$$

by:

$$\begin{aligned} a_1 &= \frac{1}{g_R} J^2 + \frac{1}{32\pi^2}, \\ a_2 &= \frac{2}{g_R} \left(-1 + J + \frac{J^2}{(32\pi^2)} \right) + \frac{1}{128\pi^2} \left(1 + \frac{2J^2}{(8\pi)^2} \right), \\ a_3 &= \frac{1}{32\pi^2} \left(1 + \frac{g_R}{32\pi^2} \right). \end{aligned} \tag{16}$$

Expression (15) is not equal to the GAP (15) obtained from the minimization of the regularized energy density with relation to $G(m^2)$, i.e. the variational parameter. There are therefore five solutions for the renormalized mass m^2 which can be written in the following form:

$$\begin{aligned} m^3 &= 0, \\ m^{\pm} &= \mu \; exp(H^{\pm}), \end{aligned} \tag{17}$$

where:

$$H^{\pm} = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}. \tag{18}$$

These solutions for $m^{\pm}$ can be viewed as having corrections for the value of μ due to the self interaction through the parameters $H^{\pm}$ due to the appearance of $\bar{\phi} \neq 0$. It is noted that there is a sort of "energy scale" invariance in these expressions for $m^{\pm}$ with simultaneous changes in the mass renormalization parameter μ.

The particular case of $m^2 = \mu^2$, for which the GAP equation is trivially satisfied with $\bar{\phi} \to 0$, is found for

$$a_3 = 0, \quad \to \quad g_R = -32\pi^2. \tag{19}$$

This point can correspond to a restoration of the symmetry broken by the scalar condensate. However according to expression (18) it also yields $m^+ = 0$, which is not consistent as it will be discussed below, and the following solution:

$$\frac{m^-}{\mu} = exp \left(-\frac{a_2}{a_1} \right). \tag{20}$$

These resulting values can be compared to the limit for the stability of a two-body bound state, when $m = \mu$, found by Kerman and Lin: $g < -8\pi^2$ [17].

A similar case is obtained when $a_1 = 0$. In this point $g_R = 64\pi^2$ yielding undefined ratio m^+/μ (i.e., this ratio can be defined by the GAP equation as usually for the coupling above).

The zero mass solutions correspond to a saddle point, they are not minima neither maxima of the energy density in agreement with [10, 17]. If the others solutions are minima is checked via the positiveness of the second derivative:

$$\frac{\partial^2 \mathcal{H}_{sub}}{\partial m^2} = \frac{m^2}{(8\pi)^2} \left(2Ln \left(\frac{m}{\mu} \right) a_1 + a_2 \right) > 0. \tag{21}$$

For the derivation of these expressions the complete self consistency of the Gaussian equations was not completely considered. There has been used a truncation on the dependence on μ, i.e., the dependence of $Ln(\mu/m)$ on μ (self consistency) was considered only for μ not very different from m, i.e. $\mu^2 = m^2 + \delta$ where $\delta << m^2$. Out of this range the above solutions are not expected to be valid. Considering that in each of the phases the corresponding particle can be expected to have different masses, eventually in a Higgs-like picture, this means that these masses in the different phases are not very different.

In Figures 1a and 1b the solutions of the above equations ($m^\pm/\mu$ from (17)) are shown as a function of $G_R = g_R/(32\pi^2)$. All the solutions of figure 1a, for m^+, correspond to stable solutions of minima ($d^2\mathcal{H}/dm^2 > 0$). The solutions of figure 1b, for m^-, are stable minima for G_R nearly equal or smaller than -1.45 or equal or greater than nearly 1.25. The point $g_R = 0$ is not plotted. Values between $-1 < G_R < 0$ do not correspond to stable minima of the energy for the corresponding values of the condensate shown below, in expression (25).

In the limits of $g_R \to \pm\infty$ it is obtained analytically that either $m = \mu$ or $m = 0$. For $\mu \to \infty$ the renormalized coupling constant $G_R \to 0$. While the solution m_R^- in the weak coupling regime can be identified to the renormalization point usually considered (for $\mu >> m$ and/or the cutoff going to infinite), although in the present analysis $\delta << \mu^2$, there is a consistent stable solution m^+ for which $\mu \simeq m^+$. Therefore the ground state can impose several restrictions on the values that mass and coupling can assume. It is worth to remind that the masses in the different phases (m^2 and μ^2) can be expected to be different, even though very close, giving rise to $\bar{\phi} \neq 0$ according to the GAP equation. This is actually intensively investigated in strong interacting systems at finite density and temperature where the QCD scalar condensate, that varies considerably with energy density and temperature, is expected to provide a very important part of the hadron masses (if not all of them) which vary accordingly, for example in [24] and references therein.

4 The condensate: $\bar{\phi}$

The variational equation for the condensate (expression (5 (ii))) is obtained from the regularized energy density $\mathcal{H}_{reg}$. The minimization of the renormalized energy density is done in the following:

$$\frac{\partial \mathcal{H}_{sub}}{\partial \bar{\phi}} = 0. \tag{22}$$

For this derivation the GAP equation provides the dependence of the mass on the condensate, i.e., $m^2(\bar{\phi})$ and $\mu^2 \equiv m^2(\bar{\phi} = 0)$ is kept constant. It yields the following expressions:

$$\begin{aligned} &\bar{\phi} = 0, \\ &\bar{\phi}^2 = -\frac{m^2}{g_R}\left(1 + \frac{1}{8\pi^2} Ln\left(\frac{m}{\mu}\right)\right). \end{aligned} \tag{23}$$

This last expression can be imposed to be equal to the expression of $\bar{\phi}_0$ obtained from the minimization of the regularized energy density (expression (8)) depending on the relation between λ and mass scale μ as it will be shown below. However it is not completely consistent with the GAP equation (10) which is obtained from the minimization of the reguralized energy density with respect to the mass m in the asymmetric phase and then renormalized. To make these expressions compatible it would be necessary to consider the following alternatives for these expressions:

$$\mu^2 \neq m_R^2, \qquad \text{or} \qquad \mu^2 = (g_R - 1)\frac{m^2}{8\pi^2} Ln\left(\frac{m}{\mu}\right), \tag{24}$$

where m_R^2 is the one of expression (11). It is not clear whether these identifications are reasonable or if they imply a meaningful loss of generality. The limit of $g_R = 1$ does not seem to be reasonable, being a very particular point. Furthermore to be meaningful it requires $\mu^2 = 0$ (and thus $m^2 = 0$) or $m^2 \to \infty$ according to these expressions (24). Therefore the two minimization procedures (of the regularized and the renormalized energy densities with respect to the regularized and renormalized parameters respectively) do not seem to yield necessarily the same expressions for the parameters in the ground state. Nevertheless it is worth to remember that renormalization is performed essencially from the regularized GAP equation. These points are discussed further latter.

From the expression (23) the following conditions to obtain non zero real values of $\bar{\phi}$ can be considered:

$$\begin{aligned} &if: g_R > 0 \to Ln\left(\frac{m}{\mu}\right) < -8\pi^2, \\ &if: g_R < 0 \to Ln\left(\frac{m}{\mu}\right) > -8\pi^2. \end{aligned} \tag{25}$$

The energy density is expected be stable for the condensate values found in expression (23). This minimum is verified by calculating the second derivative of the energy density with relation to $\bar{\phi}$, i.e.:

$\partial^2 \mathcal{H}_{sub}/\partial \bar{\phi}^2 > 0$. Its positiveness corresponds to the condition:

$$g_R \left(1 + \frac{g_R}{32\pi^2} \right) > 0. \tag{26}$$

From this it is seen that for positive coupling constant g_R, it can assume any value (from this stability criterium) whereas if $g_R < 0$ one would have to consider $g_R < -32\pi^2$. Again, this value can be compared to the value obtained in [17] for the threshold of the two particle bound state given by: $g < -8\pi^2$. Expressions (25) and (26) can correspond to constraints for the values that the renormalized coupling assumes in order to yield stable real ground states.

Expression (23) can be written as:

$$g_R \bar{\phi}^2 = -m^2 \left(1 + \frac{1}{8\pi^2} Ln \left(\frac{m}{\mu} \right) \right). \tag{27}$$

When $\mu = m\ exp(8\pi^2)$ it follows that either $\bar{\phi} = 0$ or $g_R = 0$ in the asymmetric phase of the potential. This can correspond to the so called symmetry restoration when the condensate disappears at a particularly high excitation energy, i.e., the symmetry is restored. A different solution for the particular limit of $\bar{\phi} = 0$ was found in expression (19) where the energy density is minimum with relation to the mass for $m^2 = \mu^2$.

The above expression for the condensate (23) can be equated to the previous (regularized) one (8). Taking into account the expression of the renormalized coupling constant in terms of the bare one (expression (11)) this can be written as:

$$\lambda = \frac{16\pi^2}{Ln\left(\frac{\Lambda d}{\mu}\right)} \left(-1 + \frac{3}{2\left(1 + \frac{1}{8\pi^2} Ln\left(\frac{m}{\mu}\right)\right)} \right). \tag{28}$$

If the cutoff is sent to infinite the bare coupling constant assumes different values depending on the ratio of μ/m. For example, there is a case in which $\lambda = 0$ if either $\Lambda \to \infty$ for finite μ or:

$$\frac{m}{\mu} = exp(4\pi^2), \tag{29}$$

being therefore $m^2 >> \mu^2$. Varying μ together with Λ it can yield solutions with non zero λ. For Λ/μ finite, the coupling λ can even diverge when:

$$\frac{m}{\mu} = exp(-8\pi^2). \tag{30}$$

This is the same point found above (for expression (27)) for the possible restoration of the symmetry.

It is worth emphasizing that it has been assumed, according to the variational principle, that the minimum of the effective potential with relation to the condensate necessarily defines the ground state together with its minimum in respect to the (physical) mass m^2 in the regularized theory. The different results found in this work from the minimization of the regularized and renormalized theory may indicate that these assumptions are not (complete) correct. Furthermore in the renormalized theory the bare mass and coupling which determine the effective potential at the tree level are eliminated in favor of the renormalized (physical) ones.

5 Analysis of the renormalized coupling constant

Analogously to what was done for the renormalized mass in the preceeding section the extremization of the renormalized energy density with respect to the renormalized coupling constant is done in this section. Moreover one relevant subject for any approximation method is the understanding of the range of values of the parameters of the model (as mass and mainly coupling constants) for which the approximation is more appropriated. The extremization is found from:

$$\frac{\partial \mathcal{H}_{sub}}{\partial g_R} = 0.$$

It is considered, in the following, a truncation of the self consistency of the GAP equations. This is done by taking the scale parameter to be close to the mass $\mu^2 = m^2 + \delta$, where $\delta << m^2$ is determined from the GAP equation self consistently. From the renormalized GAP equation (expression (10)) it follows that:

$$\delta = \frac{g_R \bar{\phi}^2}{1 + \frac{g_R}{16\pi^2}}. \tag{31}$$

Since $\delta << m^2$ either $\bar{\phi}$ is large or g_R is very large for positive coupling g_R. For $\bar{\phi} \neq 0$ it is also required that g_R is very different from $-16\pi^2$. The minimization of the renormalized energy yields the following third order algebraic expression:

$$(G'_R)^3 + (G'_R)^2 \left(3 + (1+H)\frac{1}{16\pi^2}\right) + G'_R \left(3 + (1+H)\frac{3}{2} + (1+H)\frac{1}{32\pi^2}\right) + 1 + \frac{(1+H)^2}{2} = 0, \tag{32}$$

where

$$H = \frac{Ln\left(\frac{m}{\mu}\right)}{(8\pi^2)}, \qquad G'_R = \frac{g_R}{(16\pi^2)}.$$

In figures 2a, 2b and 2c the solutions of expression (32) are showed as function of a limited range of H, i.e., $Ln(m/\mu)$. Figures 2b and 2c exhibit the same behavior. It is plotted only the region in which the above truncation scheme of the self consistency can be expected to be reliable. The values for g_R are large and obtained for $m^2 \sim \mu^2$. These values cannot be simultaneously compatible with the results of

Figures 1a and 1b (from the minimization of the energy with relation to the mass m^2). For instance, the point $H = -0.001$ corresponds to $m/\mu = 0.985$ which is not obtained for the larger values of g_R from figures 1a and 1b.

These solutions (32) correspond to maxima of the solutions of the above equation. This is seen by the non positiveness of the second derivative: $\frac{\partial^2 \mathcal{H}_{sub}}{\partial g_R^2} < 0$. All the solutions have a negative second derivative corresponding to maxima of $\mathcal{H}_{sub}(g_R)$, instead of minima. At this level of approximation, these values of coupling constant lead to a maximum amount of energy involved in scattering process of two scalars at the threshold, whose amplitude reduces to the scattering length. However these couplings which maximize the energy density are positive and do not give rise to bound states. Conversely the two-scalar scattering is favored in energy ranges which are associated to a different range of values of the coupling constant.

The solutions for the coupling constant of expression (32) depend only on the ratio m/μ and not on the absolute values of these parameters. This also can be seen as a sort of energy scale invariance for different physical processes (eventually in different systems) at different energy scales with different physical masses. These maxima of the energy density yield renormalized coupling constants which are directly proportional to the scattering length of T matrix [17]. Therefore in the respective ground state as provided by the corresponding energy density, for each set of values of the parameters, the coupling constant depends only on the ratio m/μ. They also determine the scattering lengths being relevant for the more general scattering amplitude. Conversely the scattering length can fix the mass scale, μ, for the respective physical situation yielding information about the ranges of values for the free parameters which should assume physical values.

5.1 Fixing the energy density

For the analysis of the system with an energy density given by $\mathcal{H}_{sub}$ and a given mass scale, μ, (acceptable) values for the renormalized coupling constant and mass can be suggested such that the values remain in the physically allowed part of the phase space of the model [34]. This corresponds to fix the renormalized mass and energy density ($\mathcal{H}_{sub}$) and to calculate the resulting physical coupling constants for the process involved at a scale μ. A third degree algebraic equation is obtained, it can be written as:

$$g_R^3 \frac{H^2}{128\pi^2} + g_R^2 \left(\frac{H^2}{4} - \frac{2H^2}{128\pi^2} \right) + g_R \left(-\frac{H(H+1)}{2} + \frac{1}{128\pi^2} \left(\frac{H}{4} - 1 + H^2 \right) - \frac{\mathcal{H}_{sub}}{m^4} \right) - \frac{1}{4} + \frac{H^2}{4} = 0. \tag{33}$$

where $H = Ln(m/\mu)/(8\pi^2)$. This expression also presents a sort of "energy scale" invariance for the parameters m/μ unless for the term which depends on the total energy density, if $\mathcal{H}_{sub}$ scales differently from μ^4 with similar features to those pointed out in [17].

In figures 3a, 3b and 3c the solutions of this algebraic equation are shown as functions of H for a fixed energy density $\mathcal{H} = (100MeV)^{-4}$ and $m_R = 100MeV$. The coupling g_R can be strong in the region of $\mu \simeq m$. In particular in the limit of $\mu = m$ (which can correspond to the limit of $\bar{\phi} \to 0$) an unique value is found, it is given by:

$$g_R = -\frac{1}{4\left(\frac{\mathcal{H}^{sub}}{m^4} + \frac{1}{128\pi^2}\right)}. \tag{34}$$

For $H \to -\infty$, which is equivalent to $\mu/m \to \infty$, it follows $g_R \to 0$ as seen in figure 3a. Similar conclusion was obtained in [17]. However solutions of figures 3b and 3c do not correspond to $g_R \to 0$, but to a finite (quite strong) value, g_R close to 10. This seems to be related to asymptotic freedom for the energy and mass scale involved [37, 10].

6 The transcendental character of the GAP equation

In this section an heuristic trick is used to extract analytical non transcendental solutions from the GAP equation or to provide possible further relations among the parameters. Firstly it is considered that the the mass scale and renormalized mass develop imaginary parts:

$$\mu^2 \to \nu^2 = re^{i\theta}, \qquad m^2 \to \tau^2 = te^{i\omega}, \tag{35}$$

where r, s, θ, ω are respectively modulus and phases. Depending on the relative values of these parameters this parametrization corresponds simply to a rotation. With these parametrizations the GAP equation (10) can be written as:

$$\begin{aligned} &\Big(rcos\theta - tcos\omega - g_R\bar{\phi}^2 - D(t\,cos\omega\,Ln(t/r) - (\omega-\theta)tsin\omega)\Big) + \\ &+i\left[t\ sin\omega - r\ sin\theta + D(t\ cos\omega\ (\omega-\theta) + t\ sin\omega\ Ln(t/r))\right] = 0, \end{aligned} \tag{36}$$

where $D = \frac{g_R}{8\pi^2}$. Both the real and the imaginary parts in this expression have basically the same structure of the usual GAP equation. It is worth to emphasize that requiring the GAP equation to be real is a requirement to keep mass parameters real numbers. The parametrization in the complex plane can be just a trick to reduce the transcendental character of the GAP equation (10). However an imaginary part for the covariance of the Gaussian (G or correspondingly m^2, μ^2) can introduce time dependence or instability of the system if considered in the wavefunctional, although in this case other considerations are needed to keep unitarity [12]. Therefore the phases θ, ω can have different roles, eventually corresponding to (dynamical) corrections to the calculation of the ground state from virtual unstable states. This will not be discussed further here.

Several cases can be analysed separatedly.

1a) Firstly for $\theta = \omega$ the GAP equation is given by:

$$\begin{aligned} &\Big((r-t)cos\omega - g_R\bar{\phi}^2 - D(t\, cos\omega\, Ln(t/r))\Big) + \\ &+i\left[(t-r)\; sin\omega + Dt\; sin\omega\; Ln(t/r)\right] = 0. \end{aligned} \tag{37}$$

The real part of the GAP equation is the usual one (apart from the $cos\theta$ term which corresponds to a normalization of the mass parameters) and the imaginary part, like a rotation in this complex plane, is the GAP equation in the symmetric phase.

2a) For $r = t$ the GAP equation is rewritten as:

$$\begin{aligned} &r(cos\theta - cos\omega) - g_R\bar{\phi}^2 - D(\theta-\omega)r sin\omega + \\ &+i\left[r\; (sin\omega - sin\theta) + Dr\; cos\omega\; (\omega-\theta)\right] = 0. \end{aligned} \tag{38}$$

In this case, these two expressions yield values for ω and θ and they can be solved as function of r and $D \propto g_R$. The resulting (real) values for m^2 and μ^2 still can be different. However the GAP expression has not the logarithmic term anymore. It is decomposed in two trigonometric equations.

Other limits yield interesting features as well.

3a) Requiring the GAP equation (36) to have only real component (this is considered to be a stable system) the imaginary part is set to zero. The expression still is quite complicated but the analysis of some particular cases will be very useful. For $\omega = 0$ it follows that:

$$r sin\theta = -D\, t\, \theta, \tag{39}$$

This can be written as:

$$cos\theta = \sqrt{1 - \frac{B^2\theta^2}{r^2}}. \tag{40}$$

In this case the self consistent character of the GAP equation remains strong. The real part of expression (36) keeps the same form of expression (10) basically with the mass parameters m^2, μ^2 replaced by r, t.

4a) For $\theta = 0$ (and $\omega \neq 0$) the resulting expressions for the real part of the GAP equation and its imaginary part (to be equated to zero) can be obtained from expression (36). They can be written as:

$$\begin{aligned} &r - t\, cos\omega - g_R\bar{\phi}^2 - D(t\, cos\omega\, Ln(t/r) - \omega t\, sin\omega) = 0, \\ &+i\left[t\, sin\omega + D(\omega t\, cos\omega + tLn(t/r)\; sin\omega)\right] = 0. \end{aligned} \tag{41}$$

It does not provide simpler solutions and therefore they are not shown. The resulting number of free parameters is not reduced because although there is one more expression ($\Im m(GAP)$) there also is one extra variable (ω).

Since the phases are auxiliar parameters it is reasonable to assume they are very small without (great) loss of generality for the results. The expression for the imaginary part of the GAP equation in the limit when $sin(\theta) \sim \theta$ and $sin(\omega) \sim \omega$ is given by:

$$\omega t \left(1 + D + DLn\left(\frac{t}{r}\right)\right) = \theta\left(r + Dt\right). \tag{42}$$

This expression can be regarded as fixing the ratio θ/ω.

Several particular cases are analyzed below although the more interesting case is obtained for ω, θ non zero and very small.

(1b) Assuming the phases are equal $\theta = \omega$ expression (42) reduces to:

$$r - t = D\, t\, Ln\left(\frac{t}{r}\right), \tag{43}$$

which fixes the ratio r/t or correspondently m^2/μ^2. This expression is only consistent with the renormalized GAP equation 10 for $\bar{\phi} = 0$ (which is obtained from the minimization of the regularized energy density). Besides that it was mentioned above that, since $\omega \neq 0$ and $\theta \neq 0$, it is not clear whether μ^2 and m^2 remain real although the GAP equation is necessarily real. This happens because, in this case, the imaginary part of both parameters can cancel with each other to result a real GAP equation instead of allowing for independent cancelation. On the other hand each angle (ω or θ) can be set to zero separatedly as done below.

(2b) For $\omega = 0$ it follows from expression (42):

$$r \simeq -D\, t, \tag{44}$$

which also fixes the ratio m^2/μ^2 being a real number only for $g_R < 0$.

(3b) For $\theta = 0$, expression (42) is re-computed up to the order of $O(\omega^2)$ and it reduces to:

$$\omega^2 = \frac{6 + 6D + 6DLn\left(\frac{t}{r}\right)}{1 + 3D + Ln\left(\frac{t}{r}\right),}, \tag{45}$$

where it has been assumed that $sin\omega \sim \omega$. In this case it is reasonable to consider $\omega^2 \sim 0$ leading to the expression:

$$\frac{t}{r} = exp\left(\frac{1+D}{D}\right). \tag{46}$$

For $g_R = -8\pi^2$ it follows that $t = r$, and therefore $m^2 \simeq \mu^2$.

If $\omega \neq 0$ it will appear in the real part of the GAP equation and therefore the number of free parameters in the renormalized equation does not diminish with the new parametrization. Therefore $\omega = 0$ would be the only possibly interesting case. This does not happens because of expression (44) which imposes negative coupling $g_R < 0$.

The real part of the GAP equation for small angles keeps nearly the form of the original GAP, it can be written as:

$$t - r + g_R \bar{\phi}^2 + D\left[tLn\left(\frac{t}{r}\right) - t\,\omega\,(\omega - \theta)\right] = 0, \tag{47}$$

where either r or t can be written as a function of the other by means of the constraints of the imaginary parts from the expressions (43), (44) or (45). In this third case the auxiliar parameter ω was not eliminated (although $\theta = 0$). However for very small phases the expression (47) reduces to the usual real GAP equation (10). In this case the real part of the GAP equation is the same as expression (10) written as:

$$t - r + g_R \bar{\phi}^2 + D\,t\,Ln\left(\frac{t}{r}\right) = 0. \tag{48}$$

Simultaneously the renormalized energy density must be a real number. The imaginary part due to the introduction of parametrization (35) has to disappear. However it is easy to notice from expression (42) that the resulting expression for the imaginary part of $\mathcal{H}_{sub}$ will be quite complicated. Below it will be assumed that the phases have small values. This should not impose great limitations in the results because they are auxiliar parameters. With this assumption several simplifications occurs because: $sin(\theta) \sim \theta$ and $sin(\omega) \sim \omega$. The result for the imaginary part of the energy density, up to first order in the phases, will be given by:

$$\Im m(\mathcal{H}_{sub}) = \omega\left(\frac{t\bar{\phi}^2}{2} + 2t^2 A_- + \frac{tr}{2g_R} + \frac{r^2}{64\pi^2}\right) + \theta\left(2A_+ r^2 - \frac{rt}{2g_R} + \frac{r^2}{32\pi^2} Ln\left(\frac{t}{r}\right) - r^2\right) \to 0, \tag{49}$$

Where

$$A_{\pm} = \frac{1}{4g_R} \pm \frac{1}{128\pi^2}.$$

One of these variables (A_+) can be identified with the solution of fixed $\mathcal{H}_{sub}(\omega = \theta = 0)$ for $\mu = m$ given by expression (34):

$$A_+ = -\left.\frac{\mathcal{H}_{sub}}{m^4}\right|_{\mu=m}. \tag{50}$$

Expression (49) still is very complicated and it can also be used to fix the ratio θ/ω which can be equated to the same ratio obtained from expression (42). However this has been written for $\mathcal{H}_{sub}$ in the form given by expression (12), which can be written differently by means of the GAP equation for $m^2 = m^2(\mu^2)$. This allows to re-arrange an equation of r as a function of t and to eliminate one of these variables. The resulting identity reads:

$$\frac{\omega}{\theta} = -\frac{2A_+r^2 - \frac{rt}{2g_R} + \frac{r^2}{32\pi^2}Ln\left(\frac{t}{r}\right) - r^2}{\frac{t\bar{\phi}^2}{2} + 2t^2A_- + \frac{tr}{2g_R} + \frac{r^2}{64\pi^2}} = -\frac{\frac{r}{t} + D}{\left(1 + D + DLn\left(\frac{t}{r}\right)\right)}. \tag{51}$$

In this expression the same parameter is used: $D = g_R/(8\pi^2)$. This (highly non transcendental) expression appears in addition to the usual real part of the GAP equation, expression (48), making a system of two algebraic expressions with two variables (r, t). g_R is a remaining input/free parameter.

7 Summary

An analysis of the variational approximation with Gaussian wavefunctionals was done as a non perturbative method with its renormalized parameters departing from the renormalized energy density. This is a different approach from the usual one within the regularized theory. The renormalized energy density was extremized with respect to the renormalized mass, renormalized coupling and to the condensate. Some physical situations in which these ranges of the space of parameters of the model can be of relevance were proposed. Other aspects were raised including the eventual equivalence to the Bogoliubov transformation for describing superfluid systems and the possibility of modification in the physical mass of the particle due to the presence of the condensate $m^2(\bar{\phi}) \neq \mu(\bar{\phi} = 0)$.

Concerning the extremization with respect to the mass, five solutions were found, two of which which can correspond to stable vacua in specific ranges of the renormalized coupling constant. For this it was considered that the mass scale μ is close to the physical mass. Specific values of the ratio m/μ were found for which effects were found, like the restoration of the symmetry, stability of the model, existence of ground state. A sort of "energy scale" invariant algebraic expression was found in this calculation. In other words, changes in the renormalized (physical) mass m^2 with corresponding change in the renormalization mass scale parameter μ^2 yield the same solutions and effects in different energy scales.

Values for $\bar{\phi}$, for the ground state, were also found by minimizing the renormalized energy density with relation to it. The resulting expression is not completely consistent with the renormalized GAP equation unless the expression (11) is modified such that $\mu^2 \neq m_R^2 \to 0$. From this expression it was pointed out that either the "condensate" or g_R disappears when the mass scale (introduced in the renormalization procedure) assumes the value

$$\bar{\phi}\left(\mu = m\; exp(8\pi^2)\right) = 0.$$

This can be seen as a restoration of the spontaneous symmetry breaking. Considering the comparison between the minimization of the regularized and renormalized energy densities of expressions (24) it would follow that $g_R = 1 - exp(16\pi^2) << -1$. Other values for the ratio m/μ were found to yield $\bar{\phi} = 0$, such as that of expression (20). With this value for μ, the bare coupling λ can also diverge for Λ/μ finite, as shown in expression (28).

Particular values of the renormalized coupling constant which extremize the energy density were also found. The coupling constant can constraint the values of the renormalized mass which yield maxima of the energy density. This extremization with respect to the coupling constant was also performed in the limiting case that the mass scale μ is close to the physical mass m. This is a way of truncating the self-consistency of the approximation. Another kind of "energy scale" invariant expression was also obtained. Only maxima with relation to g_R were found (considered without the whole renormalization group equations) within the truncation scheme which was adopted. This can be of relevance for the scattering amplitude structure [17], in particular for the scattering length for which particular values of the renormalized coupling constant can maximize the energy involved in the process eventually difficulting it. The renormalized energy was also fixed to provide specific values for the coupling constant as a function of the energy density and of the mass.

The masses were allowed to assume complex values to search non-transcendental solutions for the GAP equation and other relations among the parameters reducing the number of free parameters. The renormalized theory when considered with complex parameters, as it was shown in section 6, exhibits an structure akin to that required by the Bogoliubov transformation. According to this, a rotation in the complex mass parameters plane makes the system to move from one phase of the potential ($\bar{\phi} = 0$) to the other ($\bar{\phi} = 0$). The imaginary part of the mass parameters were required to be zero at the end of the calculation for keeping static and stable solutions. This produced another expression which relates the mass, coupling and the renormalization scale parameter. Consequently this complex parametrization can lead to new relations between the parameters reducing the number of free variables. These imaginary parameters were also required to be very small ($sin(\omega) \sim \omega$ or $sin(\theta) \sim \theta$). The same parametrization is applied to the energy density which also must be a real number. The number of free parameters (m^2 or μ^2, and g_R) is reduced and non transcendental solutions can be obtained such as that of expression (44) or eliminated such as expressions (44) or (45).

The ground state in the framework of the variational approximation is found by the minimization with respect to the two point function $G(\mathbf{x}, \mathbf{y}, m^2)$ (which is a function of the physical mass m^2 or $\mu^2 = m^2(\bar{\phi} = 0)$) and to the condensate $\bar{\phi}$ - they are the variational parameters (given in expressions (5)). Although they are regarded initially as independent variables, the GAP equation (for ground state) relates them. While the GAP equation is used for the renormalization of parameters in the vacuum, expression (23) was calculated from a renormalized expression to find the ground state. It was shown with sections 2.1, 3 and 4 that the minimizations of the renormalized energy with relation to the mass and $\bar{\phi}$ yield different ground state (GAP) expressions from the ones obtained by the usual variational procedure for the regularized theory. This can have several meanings. It is not evident whether these variational parameters are really or completely suitable as independent parameters for the Gaussian approximation and extensions (or leading order large N, Hartree Bogoliubov), or in the *exact ground state*, i.e., the energy must be minimum with respect to particular combination(s) of these (or other) (physical?) variables. Notwithstanding the minimization of the regularized energy can be eventually required to be equivalent to the minimization of the renormalized one because in the regularized theory there still are other (bare) parameters which are eliminated in the renormalization procedure corresponding to constraints among them. It can be asked to what extent this is an indication that the renormalization procedure has to be improved such as to make both ways of obtaining the ground state expressions equivalent or it is natural to expect different ground states. This last conclusion should be more reliable since quantum loops always are expected to re-arrange the system. However in the first case the renormalization procedure would be allowed to be done at any moment independently of the order of the the variation, renormalization and extraction of observables within a given (non)perturbative approach.

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Figure captions

Figure 1a - First solution of expression (15) - a new GAP equation - for the ratio of the renormalized mass to the mass scale μ as a function of $G_R = g_R/(8\pi^2)$.

Figure 1b - Second solution of expression (15) - a new GAP equation - for the ratio of the renormalized mass to the mass scale μ as a function of $G_R = g_R/(8\pi^2)$.

Figure 2a - First solution of expression (32) for the renormalized coupling constant - from the minimization of the energy density with relation to the renormalized coupling constant - as a function of $H = Ln(m_R/\mu)/(8\pi^2)$.

Figure 2b - Second solution of expression (32) for the renormalized coupling constant - from the minimization of the energy density with relation to the renormalized coupling constant - as a function of $H = Ln(m_R/\mu)/(8\pi^2)$.

Figure 2c - Third solution of expression (32) for the renormalized coupling constant - from the minimization of the energy density with relation to the renormalized coupling constant - as a function of

$H = Ln(m_R/\mu)/(8\pi^2)$.

Figure 3a - First solution for the renormalized coupling constant of expression (33) - fixing $\mathcal{H} = (100MeV)^4$ and $m_R = 100MeV$ - as a function of $H = Ln(m_R/\mu)/(8\pi^2)$.

Figure 3b - Second solution for the renormalized coupling constant of expression (33) - fixing $\mathcal{H} = (100MeV)^4$ and $m_R = 100MeV$ - as a function of $H = Ln(m_R/\mu)/(8\pi^2)$.

Figure 3c - Third solution for the renormalized coupling constant of expression (33) - fixing $\mathcal{H} = (100MeV)^4$ and $m_R = 100MeV$ - as a function of $H = Ln(m_R/\mu)/(8\pi^2)$.

Figure 1a

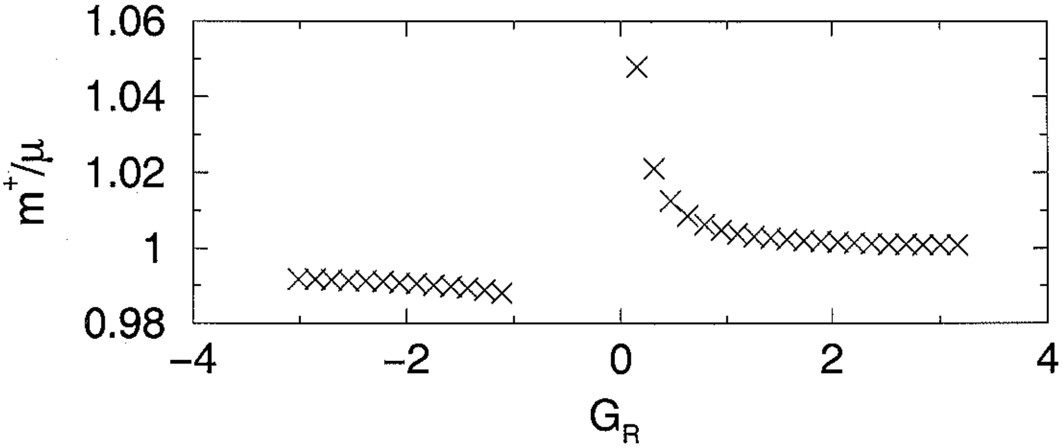

Figure 1b

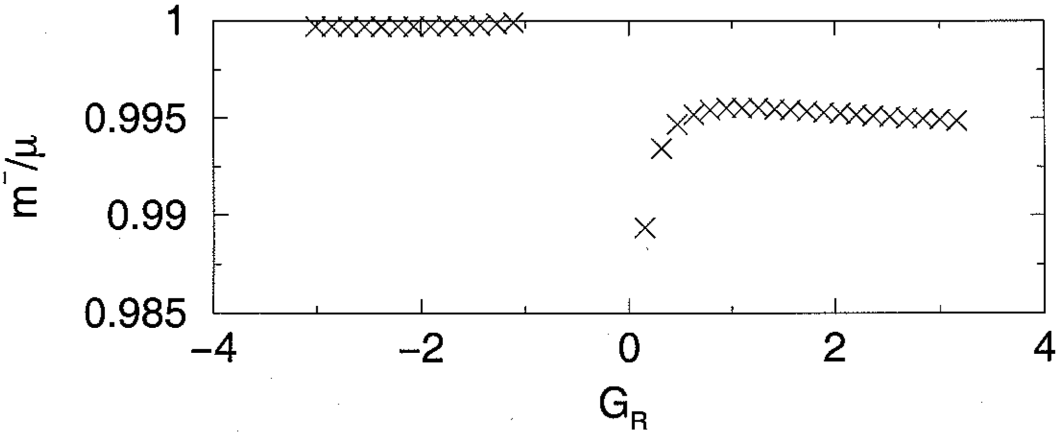

Figure 2a

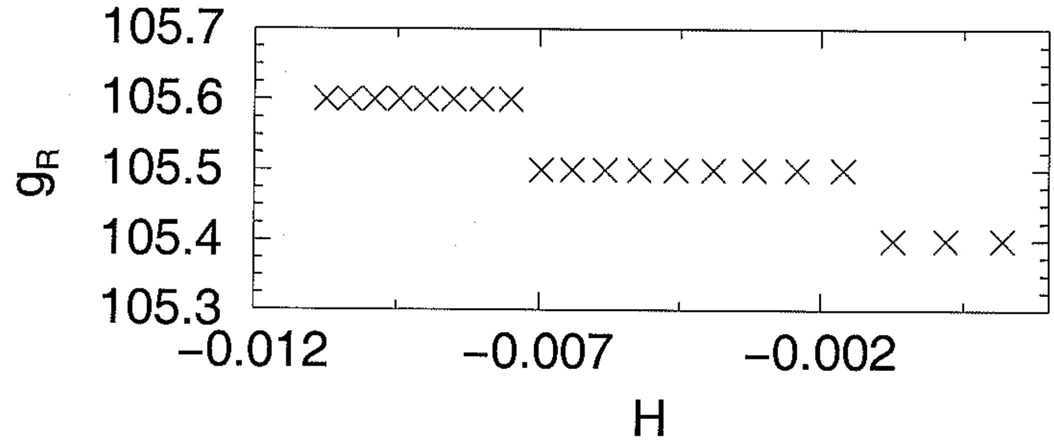

Figure 2b

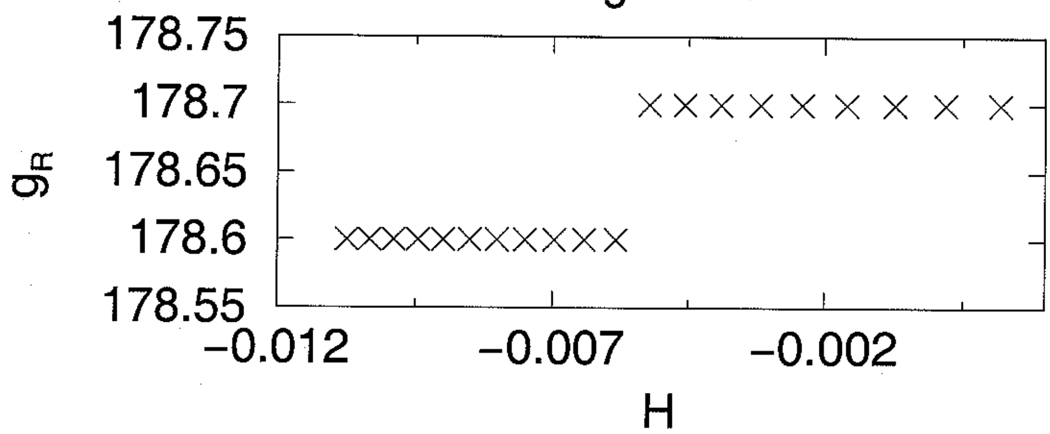

Figure 2c

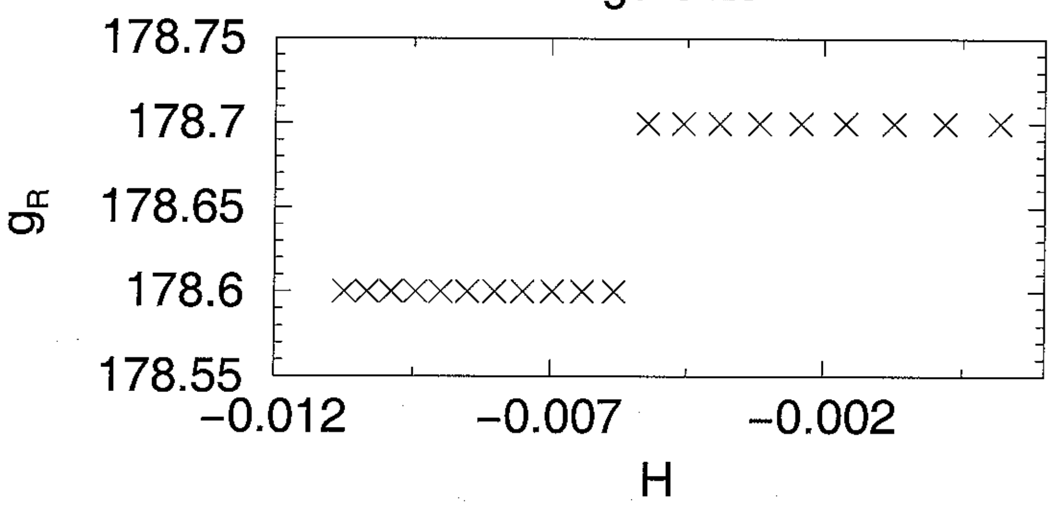

Figure 3a

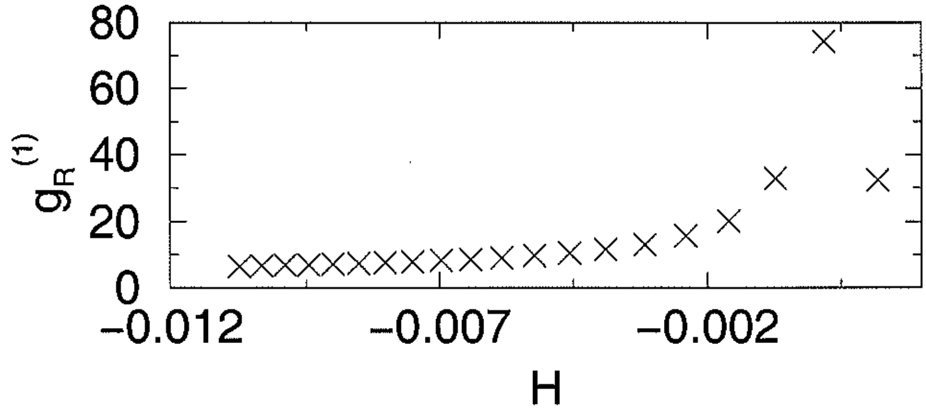

Figure 3b

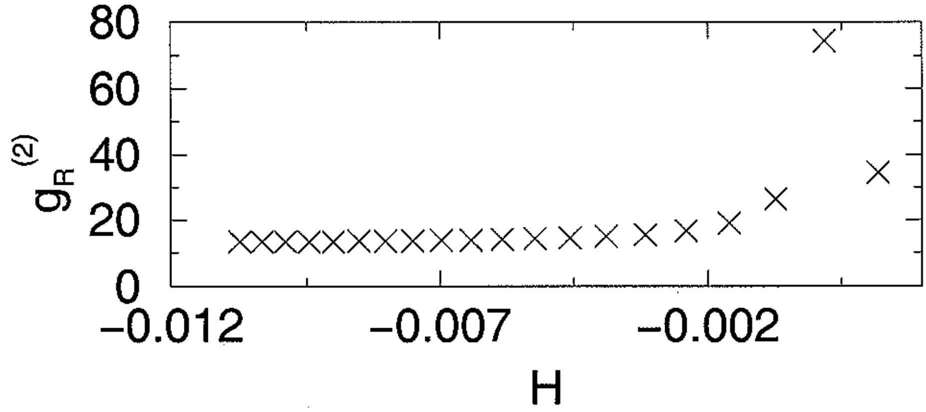

Figure 3c