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Canonical quantization of non-Lagrangian theories and its application to damped oscillator and radiated point-like charge

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Abstract

We present an approach to the canonical quantization of systems with non-Lagrangian equations of motion. We first construct an action principle for an equivalent first-order equations of motion. A hamiltonization and canonical quantization of theory with such an action is a non-trivial problem, since this theory involves time-dependent constraints. We adopt the general approach of hamiltonization and canonical quantization for such theories (Gitman, Tyutin, 1990) to the case under consideration. There exists an ambiguity (not reduced to a total time derivative) in associating a Lagrange function with the given set of equations. We give a complete description of this ambiguity. It is remarkable that the quantization scheme developed in the case under consideration provides arguments in favor of fixing this ambiguity. The proposed scheme is applied to quantization of a general quadratic theory. In addition, we consider the quantization of a damped oscillator and a radiating point-like charge.

1   Introduction

It is well-known that some physical systems like dissipative systems [1], Dirac monopole [2], radiating point-like charge, etc. are usually described in terms of second-order equations of motion which cannot be identified with Euler-Lagrange equations for an action principle. We call such equations of motion non-Lagrangian equations in what follows. Sometimes (but not always) non-Lagrangian equations can be reduced to Euler-Lagrange equations by multiplying by the so-called integrating multiplier [3]-[5]. The existence of an action principle for a given physical system, or what is the same, the existence of a Lagrange function for such a system, allows one to proceed with canonical quantization schemes. This, in particular, stresses the importance of formulating action principle for any physical system.

In the present work we discuss an approach to the canonical quantization of physical systems with non-Lagrangian equations of motion. On the classical level, we reduce non-Lagrangian equations of motion to an equivalent set of first-order differential equations. For such equations, one can always construct an action principle, the corresponding consideration is represented in section 2 and, partially, is based on results of works [6]-[10]. Thus, we obtain a Lagrangian formulation (in the first-order form) for initial theory with non-Lagrangian equations of motion. The hamiltonization of the constructed Lagrangian theory leads to a Hamiltonian theory with time-dependent constraints as it is demonstrated in section 3. The canonical quantization of such a theory is not a trivial problem (it follows to general consideration of [14]) and is represented in section 4. It is known that on the classical level, there exists an ambiguity in constructing Lagrange function (which is not reduced to a total time derivative) for a given set of equations [8]-[13]. Remarkable, that the proposed quantization scheme provides arguments in fixing the ambiguity in constructing the corresponding classical action. Following the general approach, we explicitly formulate the canonical quantization of theories with arbitrary linear inhomogenous equations of motion (general quadratic theories), see section 5. Then we consider the canonical quantization of a damped harmonic oscillator (sec. 6) and a radiating point-like charge (sec.7).

2 Action principle for non-Lagrangian systems

Let a system with n degrees of freedom be described by a set of n non-Lagrangian second-order differential equations of motion. To construct an action principle, we replace these equations (which is always possible) by an equivalent set of $2n$ first-order differential equations, solvable with respect to time derivatives. Suppose such a set has a form

$$\dot{x}^{\alpha} = f^{\alpha}(t, x), \quad \alpha = 1, .., 2n, \tag{1}$$

where $f^{\alpha}(t, x)$ are some functions of the indicated arguments. An action $S[x]$ that yields (1) as Euler–Lagrange equations, can be chosen in the following form

$$S[x] = \int dt \, (J_{\alpha}\dot{x}^{\alpha} - H), \tag{2}$$

where $J_{\alpha} = J_{\alpha}(t, x)$ and $H = H(t, x)$ are some functions of the indicated arguments. The Euler–Lagrange equations corresponding to (2) are given by

$$\Omega_{\alpha\beta}\dot{x}^{\beta} = \partial_{\alpha}H + \partial_{t}J_{\alpha}, \tag{3}$$

where

$$\Omega_{\alpha\beta} = \partial_{\alpha}J_{\beta} - \partial_{\beta}J_{\alpha} = \Omega_{\alpha\beta}(t, x). \tag{4}$$

Equations (3) can be identified with (1), provided

$$\det \Omega_{\alpha\beta} \neq 0, \tag{5}$$

$$\Omega_{\alpha\beta}f^{\beta} - \partial_{t}J_{\alpha} = \partial_{\alpha}H. \tag{6}$$

The functions J_{α} and H can be found from conditions (4)–(6). The consistency condition for equations (6) imply

$$\partial_{\beta}(\Omega_{\alpha\gamma}f^{\gamma}) - \partial_{\alpha}(\Omega_{\beta\gamma}f^{\gamma}) + \partial_{t}\Omega_{\alpha\beta} = 0. \tag{7}$$

In addition, the symplectic form $\Omega_{\alpha\beta}$ (4) must obey the Jacobi identity

$$\partial_{\alpha}\Omega_{\beta\gamma} + \partial_{\beta}\Omega_{\gamma\alpha} + \partial_{\gamma}\Omega_{\alpha\beta} = 0. \tag{8}$$

Namely, the consistency conditions for equations (4)–(6) provide a closed set of equations for the symplectic form $\Omega_{\alpha\beta}$. Having at our disposal solutions of these equations, we can easily find J_{α} from (4) and then H from (6).

It is known (see, for instance, [8], [9], [10]) that the general solution $\Omega_{\alpha\beta}$ of equation (7) can be constructed with the help of a solution of the Cauchy problem for equations (1). Let us suppose that

$$x^{\alpha} = \varphi^{\alpha}(t, x_{(0)}), \quad x_{(0)}^{\alpha} = \varphi^{\alpha}(0, x_{(0)}) \tag{9}$$

be a solution of equations (1) for any $x_{(0)} = \left(x_{(0)}^{\alpha}\right)$, and let $x_{(0)}^{\alpha} = \chi^{\alpha}(t, x)$ be a solution of (9), i.e., $x^{\alpha} \equiv \varphi^{\alpha}(t, \chi^{\alpha})$. Then we have

$$\Omega_{\alpha\beta}(t, x) = \partial_{\alpha}\chi^{\gamma}\,\Omega_{\gamma\delta}^{(0)}(\chi)\,\partial_{\beta}\chi^{\delta}. \tag{10}$$

From $\partial\chi^{\alpha}/\partial x^{\gamma}|_{t=0} = \delta_{\gamma}^{\alpha}$ we obtain

$$\Omega_{\alpha\beta}(t, x)|_{t=0} = \Omega_{\alpha\beta}^{(0)}(x),$$

which implies that the matrix $\Omega_{\alpha\beta}^{(0)}(x)$ yields the initial condition for $\Omega_{\alpha\beta}(t, x)$.

We can see that choosing the matrix $\Omega_{\alpha\beta}^{(0)}(x)$ subject to conditions (8) and (5) we provide the fulfilment of the same conditions for the complete matrix $\Omega_{\alpha\beta}(t, x)$ (10), since its components at the moment t are given by a simple change of coordinates.

Thus, there exists an ambiguity in constructing action (2) related to the choice of the matrix $\Omega_{\alpha\beta}^{(0)}$. Actions constructed from different $\Omega_{\alpha\beta}^{(0)}$ lead to the same equations of motion; however, they do not differ from each other by a total derivative.

It is convenient (as we can see below) to choose the matrix $\Omega_{\alpha\beta}^{(0)}$ as a canonical symplectic matrix:

$$\Omega_{\alpha\beta}^{(0)} = \begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix},$$ (11)

where $\mathbf{I}$ is an $n \times n$ unit matrix, and $\mathbf{0}$ denotes an $n \times n$ zero matrix.

Provided that the matrix Ω is known, we can find J and H from equations (4), (6). To this end, we remind that:

a) the general solution of the equation $\partial_k \alpha_i - \partial_i \alpha_k = \omega_{ki}$, provided that ω_{ki} is antisymmetric and obeys the Jacobi identity, is given by

$$\alpha_i(x) = \int_0^1 ds \, sx^k \omega_{ki}(sx) + \partial_i \varphi(x),$$

where $\varphi(x)$ is an arbitrary function;

b) the general solution of the equation $\partial_i f = g_i$, provided a vector g_i is a gradient, is given by

$$f(x) = \int_0^1 ds \, x^i g_i(sx) + c,$$

where c is a constant.

Taking the above into account, we obtain for J_α and H (we do not consider global problems which arise from a nontrivial topology of the x^α-space) the following representation:

$$J_\alpha(t,x) = \int_0^1 ds \, sx^\beta \Omega_{\beta\alpha}(t,sx) + \partial_\alpha \varphi(t,x),$$ (12)

$$H(t,x) = \int_0^1 ds \, x^\beta [\Omega_{\beta\alpha}(t,sx) f^\alpha(t,sx) - \partial_t J_\beta(t,sx)] + c(t),$$ (13)

where $\varphi(t,x)$ and $c(t)$ are arbitrary functions of the indicated arguments. It is easy to see that this arbitrariness is accumulated in the Lagrange function that corresponds to action (2) via the total time derivative of a function F,

$$F = \varphi(t,x) - \int c(t)dt.$$

Note that if solution (9) of the equation of motion (1) smoothly depends on some parameter, then because of (10) and (12), (13) action (2) will smoothly depend on this parameter as well. So, in this case any perturbation of the equations of motion can be represented as a corresponding perturbation of action.

The first-order action (2) can be regarded as a Lagrangian action, or as a Hamiltonian action with a noncanonical Poisson bracket. An equivalent second-order Lagrangian formulation is always possible; however, it may include additional variables [15].

One ought to say that it is always possible to construct a Lagrangian action for non-Lagrangian second-order equations in an extended configuration space following a simple idea first proposed by Bateman [24]. Such a Lagrangian has a form of a sum of initial equations of motion being multiplied by the corresponding Lagrangian multipliers, new variables. Euler-Lagrange equations for such an action contain besides the initial equations some new equations of motion for the Lagrange multipliers. In such an approach one has to think how to interprete the new variables already on the classical level. Additional difficulties (indefinite metric) can appear in course of the quantization.

As an example, we consider a theory with equations of motion of the form[1]

$$\dot{x} = A(t)x + j(t).$$ (14)

We call such a theory the general quadratic theory. Let us apply the above consideration to construct the action principle for such a theory.

Solution of the Cauchy problem for the equations (14) reads

$$x(t) = \Gamma(t)x_{(0)} + \gamma(t),$$ (15)

---

[1]Here we use matrix notation, $x = (x^\alpha)$, $A(t) = \left(A(t)_\beta^\alpha\right)$, $j(t) = (j(t)^\alpha)$, $\alpha, \beta = 1, .., 2n$.

where the matrix $\Gamma(t)$ is the fundamental solution of (14), i.e.,

$$\dot{\Gamma} = A\Gamma, \quad \Gamma(0) = 1, \tag{16}$$

and $\gamma(t)$ is a partial solution of (14). Then following (10), we construct the matrix Ω,

$$\Omega = \Lambda^T \Omega^{(0)} \Lambda, \quad \Lambda = \Gamma^{-1}. \tag{17}$$

and find the functions J and H according to (12) and (13),

$$J = \frac{1}{2}x\Omega, \quad H = \frac{1}{2}xBx - Cx, \tag{18}$$

where

$$B = \frac{1}{2}\left(\Omega A - A^T \Omega\right), \quad C = \Omega j. \tag{19}$$

Thus, the action functional for the general quadratic theory is

$$S[x] = \frac{1}{2}\int dt\,(x\Omega\dot{x} - xBx - 2Cx). \tag{20}$$

Another approach to constructing the action functional for the general quadratic theory was proposed in [1].

Note that Darboux coordinates x_0 can be written via a matrix Λ as follows:

$$x \to x_0 = R^{-1}(t)\Lambda(t)x. \tag{21}$$

Here, $R(t)$ is an arbitrary matrix of a linear (generally time-dependent) canonical transformation:

$$R^T(t)\Omega^{(0)}R(t) = \Omega^{(0)}. $$

In terms of the coordinates x_0, action (20) takes the form

$$S[x] = \frac{1}{2}\int dt\,\left(x_0\Omega^{(0)}\dot{x}_0 + x_0 R^T \Omega^{(0)} \dot{R} x_0 - 2C\Gamma R x_0\right). \tag{22}$$

The Darboux coordinates (21) can be divided into coordinates and corresponding momenta. The Euler–Lagrange equations for action (22) have the form of canonical Hamilton equations with the Hamiltonian

$$H_0 = -\frac{1}{2}x_0 R^T \Omega^{(0)} \dot{R} x_0 + C\Gamma R x_0. \tag{23}$$

Note that the choice $R = \text{const}$ yields a trivial Hamiltonian, which is consistent with the fact that in this case x_0 are the initial data without dynamics.

3 Hamiltonian formulation

We are now going to consider action (2) as a Lagrangian action with the Lagrange function

$$L = J_\alpha(t,x)\dot{x}^\alpha - H(t,x) \tag{24}$$

and construct a corresponding Hamiltonian formulation. To this end, we follow the general[2] scheme of [14]. We first construct the action $S^v[x,\pi,v]$, which, in this case, has the form

$$S^v[x,\pi,v] = \int \left[J_\alpha(t,x)v^\alpha - H(t,x) + \pi_\alpha(\dot{x}^\alpha - v^\alpha)\right]dt, \tag{25}$$

and depends on the momenta π_α conjugate to the coordinates x^α, as well as on the velocities v^α. The equations

$$\frac{\delta S^v}{\delta v^\alpha} = \Phi_\alpha(t,x,\pi) = \pi_\alpha - J_\alpha(t,x) = 0 \tag{26}$$

---

[2]Note that some of J_α can be equal to zero, for instance, if one deals with a canonical Hamiltonian action. In this case, one obtains the constraints $\Phi_\alpha = \pi_\alpha = 0$. Another way to examine this case is to use the method of hamiltonization for theories with degenerate coordinates [16].

do not allow one to express the velocities via x and π, which implies the appearance of primary constraints $\Phi_\alpha(t, x, \pi)$, and the velocities v^α become Lagrangian multipliers to these constraints, so that action (25) becomes a Hamiltonian action of a theory with the primary constraints (26),

$$S_H = \int dt \{\pi_\alpha \dot{x}^\alpha - H^{(1)}\}, \ H^{(1)} = H(t, x) + \lambda^\alpha \Phi_\alpha(t, x, \pi), \tag{27}$$

with the equations of motion

$$\dot{\eta} = \left\{\eta, H^{(1)}\right\}, \ \Phi = 0, \tag{28}$$

where $\eta = (x, \pi)$.

The primary constraints are second-class ones. Indeed, we have, in virtue of (5),

$$\{\Phi_\alpha, \Phi_\beta\} = \Omega_{\alpha\beta}(t, x) \Longrightarrow \det\{\Phi_\alpha, \Phi_\beta\} \neq 0. \tag{29}$$

Thus, secondary constraints do not appear, and all λ-s are determined from the consistency conditions for the primary constraints:

$$\dot{\Phi}_\alpha = \partial_t \Phi_\alpha + \{\Phi_\alpha, H^{(1)}\} = 0 \Longrightarrow -\partial_t J_\alpha - \partial_\alpha H + \lambda^\beta \{\Phi_\alpha, \Phi_\beta\} = 0 \Longrightarrow$$
$$\lambda^\beta = \omega^{\beta\alpha}(\partial_t J_\alpha + \partial_\alpha H), \ \omega^{\beta\alpha} = \Omega_{\beta\alpha}^{-1}. \tag{30}$$

Using the Lagrange multipliers (30) in equations (28), we can write these equations in the form

$$\dot{\eta} = \{\eta, H\}_{D(\Phi)} + \{\eta, \Phi_\alpha\}\omega^{\alpha\beta}\partial_t J_\beta, \ \Phi = 0, \tag{31}$$

where $\{\cdots, \cdots\}_{D(\Phi)}$ are the Dirac brackets with respect to the second-class constraints Φ. For the canonical variables the Dirac brackets are

$$\begin{aligned}
\{x^\alpha, x^\beta\}_{D(\Phi)} &= \omega^{\alpha\beta}, \\
\{\pi_\alpha, \pi_\beta\}_{D(\Phi)} &= \partial_\alpha J_\rho \omega^{\rho\gamma} \partial_\beta J_\gamma, \\
\{x^\alpha, \pi_\beta\}_{D(\Phi)} &= \delta^\alpha_\beta + \omega^{\alpha\gamma} \partial_\beta J_\gamma.
\end{aligned} \tag{32}$$

Formally introducing a momentum ϵ conjugate to the time t, and defining the Poisson brackets in an extended space of the canonical variables $(x, \pi; t, \epsilon) = (\eta; t, \epsilon)$, see [14], we rewrite (31) as follows:

$$\dot{\eta} = \{\eta, H + \epsilon\}_{D(\Phi)}, \ \Phi = 0. \tag{33}$$

Equations (33) present a Hamiltonian formulation of non-Lagrangian systems with first-order equations of motion (1). We note that the Hamiltonian constraints in this formulation are second-class ones and depend on time explicitly. The canonical quantization of theories with time-dependent second-class constraints can be carried out along the lines of [14]. Below, we present the details of such a quantization, and then adopt it to the system under consideration.

4 Canonical quantization

For a Hamiltonian theory with time-dependent second-class constraints, the quantization procedure in the "Schrödinger" picture is realized as follows. The phase-space variables η of a theory with time-dependent second-class constraints $\Phi_l(\eta, t)$ are assigned operators $\hat{\eta}(t)$ subject to the equal-time commutation relations and the constraints equations

$$[\hat{\eta}^A(t), \hat{\eta}^B(t)] = i\{\eta^A, \eta^B\}_{D(\Phi)}|_{\eta=\hat{\eta}}, \ \Phi_l(\hat{\eta}(t), t) = 0. \tag{34}$$

Their time evolution is postulated as (we neglect the problem of operator ordering [17])

$$\frac{d}{dt}\hat{\eta}(t) = \{\eta, \epsilon\}_{D(\Phi)}|_{\eta=\hat{\eta}} = -\{\eta, \Phi_l\}\{\Phi, \Phi\}_{ll'}^{-1}\partial_t \Phi_{l'}|_{\eta=\hat{\eta}}. \tag{35}$$

To each physical quantity F, given in the Hamiltonian formulation by a function $F(t, \eta)$, we assign a "Schrödinger" operator $\hat{F}(t)$, by the rule $\hat{F}(t) = F(t, \hat{\eta}(t))$. For arbitrary "Schrödinger" operators $\hat{F}(t)$ and $\hat{G}(t)$, the relation

$$[\hat{F}(t), \hat{G}(t)] = i\{F, G\}_{D(\Phi)}|_{\eta=\hat{\eta}} \tag{36}$$

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holds as a consequence of (34). The quantum states of the system are described by vectors Ψ of a Hilbert space with a scalar product (Ψ, Ψ'). Their time evolution is determined by the Schrödinger equation

$$i\frac{\partial \Psi(t)}{\partial t} = \hat{H}\Psi(t) , \tag{37}$$

where the quantum Hamiltonian $\hat{H}$ is constructed according to the classical function $H(t,\eta)$ as $\hat{H}(t) = H(t,\hat{\eta}(t))$. The mean values $\langle F \rangle_t$ of a physical quantity F are determined as the mean values of a corresponding "Schrödinger" operator $\hat{F}(t) = F(t,\hat{\eta}(t))$ with respect to state vectors $\Psi(t)$,

$$\langle F \rangle_t = \left(\Psi(t), \hat{F}(t)\Psi(t) \right) . \tag{38}$$

Provided that $\hat{H}$ is a self-adjoint operator, the time evolution of state vectors $\Psi(t)$ is unitary,

$$\Psi(t) = U(t)\Psi(0) , \quad U^+(t) = U^{-1}(t) , \tag{39}$$

where $U(t)$ is an evolution operator.

In the Heisenberg picture, where state vectors are "frozen" and the time evolution is governed by the Heisenberg operators $\check{\eta}(t) = U^{-1}(t)\hat{\eta}(t)U(t)$, one can see [14] that

$$\frac{d}{dt}\check{\eta} = \{\eta, H(t,\eta) + \epsilon\}_{D(\Phi)}|_{\eta=\check{\eta}} ,$$
$$[\check{\eta}^A(t), \check{\eta}^B(t)] = i\{\eta^A, \eta^B\}_{D(\Phi)}|_{\eta=\check{\eta}} , \quad \Phi(\check{\eta}(t),t) = 0 , \tag{40}$$

while for Heisenberg operators $\check{F}(t) = U^{-1}(t)\hat{F}(t)U(t) = F(t,\check{\eta}(t))$, we have

$$\frac{d}{dt}\check{F}(t) = \{F(t,\eta), H(t,\eta) + \epsilon\}_{D(\Phi)}|_{\eta=\check{\eta}} , \tag{41}$$

or

$$\frac{d}{dt}\check{F}(t) = -i\left[\check{F}(t), \check{H}(t)\right] + \{F(t,\eta), \epsilon\}_{D(\Phi)}|_{\eta=\check{\eta}} . \tag{42}$$

The above quantization provides the fulfilment of the correspondence principle since the quantum equations (40) have the same form as the classical ones (33).

Mean values $\langle F \rangle_t$ in the Heisenber picture in according to (38) and (39) are determined as

$$\langle F \rangle_t = \left(\Psi(0), \check{F}(t)\Psi(0) \right) . \tag{43}$$

The above quantization provides the fulfilment of the correspondence principle because quantum equations (40) has the same form as the classical one (33).

Note that the time-dependence of the Heisenberg operators in the theories under consideration is not unitary in the general case. In other words, there exists no such ("Hamiltonian") operator whose commutator with a physical quantity can produce its total time derivative. This is explained by the existence of two factors which determine the time evolution of a Heisenberg operator. The first one is the unitary evolution of a state vector in the "Schrödinger" picture, while the second one is the time variation of a "Schrödinger" operators $\hat{\eta}$, which in general has a non-unitary character. The existence of these two factors is related to the division of the right-hand side of (42) into two summands. Physically, this is explained by the fact that dynamics develops on a surface which changes with time – in the general case, in a nonunitary way.

Below, we apply the above quantization scheme to the system under consideration. Taking into account the Dirac brackets (32), we can write the equal-time commutation relations (34) for phase-space operators as

$$[\hat{x}^\alpha, \hat{x}^\beta] = i\omega^{\alpha\beta}|_{x=\hat{x}} ,$$
$$[\hat{\pi}_\alpha, \hat{\pi}_\beta] = i\,\partial_\alpha J_\rho \omega^{\rho\gamma} \partial_\beta J_\gamma|_{x=\hat{x}} , \tag{44}$$
$$[\hat{x}^\alpha, \hat{\pi}_\beta] = i\delta_\beta^\alpha + i\,\omega^{\alpha\gamma}\partial_\beta J_\gamma|_{x=\hat{x}} .$$

In this case, the classical Hamiltonian H does not depend on the momenta π_α, and therefore in order to determine the quantum Hamiltonian $\hat{H}$, we need to know only the time dependence of the operators $\hat{x}^\alpha$. From (35) it follows that

$$\frac{d}{dt}\hat{x}^\alpha = \omega^{\alpha\beta}(t,x)\partial_t J_\beta(t,x)|_{x=\hat{x}} . \tag{45}$$

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5 Quantization of general quadratic theory

The quantum-mechanical description of quadratic systems is a widely discussed physical problem which has a number of important applications (see, e.g., [18]–[23] and references therein). Almost all of these works deal with the case of "Hamiltonian" quadratic systems, i.e., systems described by canonical Hamiltonian equations of motion. On the other hand, we consider a general quadratic system, i.e., a system described by arbitrary linear inhomogeneous equations of motion (14). In this case conditions (44), (45) become

$$[\hat{x}^\alpha, \hat{x}^\beta] = i\omega^{\alpha\beta}(t),$$ (46)

$$\frac{d}{dt}\hat{x}^\alpha = -\frac{1}{2}\omega^{\alpha\beta}(t)\dot{\Omega}_{\beta\gamma}(t)\hat{x}^\gamma.$$ (47)

The time-dependence of the operators $\hat{x}$ can be easily found:

$$\hat{x}^\alpha(t) = \Phi^{\alpha\beta}(t)\hat{x}_0^\beta.$$ (48)

Here, the matrix Φ obeys the equation

$$\dot{\Phi} = -\frac{1}{2}\omega\dot{\Omega}\Phi, \quad \Phi(0) = E,$$ (49)

and the operators $\hat{x}_0$ obey the following commutation relations:

$$\left[\hat{x}_0^\alpha, \hat{x}_0^\beta\right] = i\left(\Omega_{\alpha\beta}^{(0)}\right)^{-1} = i\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix},$$ (50)

see (11). In what follows, it is useful to divide the operators $\hat{x}_0^\alpha$ into the operators of coordinates proper and corresponding momenta $\hat{x}_0^\alpha = (\hat{q}^i, \hat{p}_i)$, $\alpha = 1,..,2n$, $i = 1,..,n$. The operators $\hat{q}$ and $\hat{p}$ obey the canonical commutation relations

$$\left[\hat{q}^i, \hat{p}_j\right] = i\delta_j^i, \; \left[\hat{q}^i, \hat{q}^j\right] = [\hat{p}_i, \hat{p}_j] = 0.$$ (51)

The quantum Hamiltonian in eq. (37) takes the form

$$\hat{H} = \frac{1}{2}\hat{x}_0\Phi^T B\Phi\hat{x}_0 - C\Phi\hat{x}_0,$$ (52)

where the matrix B is determined by (19).

The above quantization is equivalent to quantization in Darboux coordinates, and the transformation $x \to \Phi(t)x_0$ provides, by itself, a passage to the Darboux coordinates x_0, because (49) implies

$$\Phi^t\Omega\Phi = \Omega_0.$$ (53)

Namely, in the coordinates x_0 the Poisson bracket is canonical. Therefore, $\Phi = \Gamma(t)R(t)$, where $\Gamma(t)$ is a fundamental solution of system (14). However, in contrast to the classical theory, now the matrix $R(t)$ is fixed, it must obey the conditions

$$\dot{R} = \Omega_0\Gamma^t B\Gamma R, \quad R(0) = E.$$ (54)

Thus, using (23) one can also rewrite the Hamiltonian in (52) as follows:

$$\hat{H} = -\frac{1}{2}\hat{x}_0 R^T \Omega^{(0)}\dot{R}\hat{x}_0 + C\Gamma R\hat{x}_0.$$ (55)

It is remarkable that if the matrix A that determines the set of equations (14) is constant, the matrix that determines the quadratic part of the Hamiltonian in (52) is constant as well, and equals to

$$\Phi^T B\Phi = B(0) = \frac{1}{2}\left(\Omega^{(0)}A - A^T\Omega^{(0)}\right).$$ (56)

This fact is easy to observe because the time derivative of this matrix, in view of (49), (17) and (19), is equal to zero:

$$\frac{d}{dt}\left(\Phi^T B\Phi\right) = 0.$$

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Thus, in this case, as distinct from the general case, the matrix Φ can be determined from the set of algebraic equations (53) and (56).

Note that if we start from a canonical Hamiltonian system the above quantization coincides with the usual canonical quantization, because in this case equation (47) becomes $d\hat{x}/dt = 0$, i.e., $\hat{x}(t) = \hat{x}_0$.

In the Heisenberg picture, equations (40) for the operators $\check{x}$ take the form

$$\frac{d}{dt}\check{x} = A(t)\check{x} + j(t),\tag{57}$$

$$\left[\check{x}^{\alpha}, \check{x}^{\beta}\right] = i\omega^{\alpha\beta}(t)\ .\tag{58}$$

Equations (57) coincide (the correspondence principle) with the classical equations of motion (14); however, the commutation relations (58) differ from the canonical ones. So, evolution of operators $\check{x}$ can be written as

$$\check{x}(t) = \Gamma(t)\check{x}_0 + \gamma(t)\,,\tag{59}$$

where operators $\check{x}_0$ as well as $\hat{x}_0$ obey the canonical commutation relations

$$\left[\check{x}_0^{\alpha}, \check{x}_0^{\beta}\right] = i\left(\Omega_{\alpha\beta}^{(0)}\right)^{-1} = i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.\tag{60}$$

Thus, mean values $\langle F \rangle_t$ of a physical quantity F according to (43) are determined as mean values of the corresponding operator $\check{F}(t) = F(t, \check{x}(t))$ with respect to initial states vectors $\Psi(0)$, i.e.

$$\langle F \rangle_t = \left(\Psi(0), F\left(t, \Gamma(t)\check{x}_0 + \gamma(t)\right)\Psi(0)\right)\ .\tag{61}$$

We see that the quantum evolution of physical quantities in general quadratic systems is completely determined by the classical one.

6 Quantization of damped harmonic oscillator

The above formulated quantization of non-Lagrangian theories and, in particular, of general quadratic theories can be immediately applied to quantizing a damped harmonic oscillator. The latter problem attracts attention for already more then 50 years , there exist different approaches to its solution, no one of them seems to be a final version which does not contain weak points, see e.g. [25]-[46], [1].

The classical equation of motion for a damped harmonic oscillator is non-Lagrangian, it has the form

$$\ddot{x} + 2\alpha\dot{x} + \omega^2 x = 0\,,\tag{62}$$

where ω is the angular frequency and $\alpha \geq 0$ is a friction coefficient. Introducing an auxiliary variable $y = \dot{x}$, we reduce (62) to the following equivalent pair of first-order equations:

$$\dot{x} = y\,,\ \dot{y} = -\omega^2 x - 2\alpha y\,.\tag{63}$$

Following the way proposed in sec. 2, we construct an action S that implies (63) as Euler-Lagrange equations,

$$S = \frac{1}{2}\int dt\left[y\dot{x} - x\dot{y} - (y^2 + 2\alpha xy + \omega^2 x^2)\right]e^{2\alpha t}\,.\tag{64}$$

Then we proceed with the canonical quantization described in the previous section. Equal-time commutations relations (46) and equations (47) determining time evolution of "Schrödinger" operators $\hat{x}$ and $\hat{y}$ are

$$[\hat{x}, \hat{y}] = ie^{-2\alpha t},\ [\hat{x}, \hat{x}] = [\hat{y}, \hat{y}] = 0,\tag{65}$$

$$\frac{d}{dt}\hat{x} = -\alpha\hat{x},\ \frac{d}{dt}\hat{y} = -\alpha\hat{y}\,.\tag{66}$$

A solution of these equations has the form

$$\hat{x} = e^{-\alpha t}\hat{q}\,,\ \hat{y} = e^{-\alpha t}\hat{p}\,,\tag{67}$$

where operators $\hat{q}$ and $\hat{p}$ obey canonical commutations relations

$$[\hat{q}, \hat{p}] = i \, , \ [\hat{q}, \hat{q}] = [\hat{p}, \hat{p}] = 0 \, .$$

According to (52), the corresponding quantum Hamiltonian reads

$$\hat{H} = \frac{1}{2} \left[\hat{p}^2 + \alpha \left(\hat{q}\hat{p} + \hat{p}\hat{q} \right) + \omega^2 \hat{q}^2 \right] \, .$$

It can be modified to the form

$$\hat{H} = \frac{1}{2} \left[\hat{P}^2 + \left(\omega^2 - \alpha^2 \right) \hat{Q}^2 \right] \tag{68}$$

by the help of the canonical transformation $(\hat{p}, \hat{q}) \to (\hat{P}, \hat{Q})$, where $\hat{P} = \hat{p} + \alpha\hat{q}$, and $\hat{Q} = \hat{q}$. The corresponding generating function is $W = qP - \alpha q^2/2$.

As usual we define the classical energy of the system by

$$E = \frac{1}{2} \left(\dot{x}^2 + \omega^2 x^2 \right) = \frac{1}{2} \left(y^2 + \omega^2 x^2 \right) \, .$$

One can easy see the energy depends of time as follows: $E = E_0 e^{-2\alpha t}$. Using (67), we obtain an expression for the operator $\hat{E}$ that corresponds to the classical quantity E,

$$\hat{E} = \frac{1}{2} e^{-2\alpha t} \left[\hat{P}^2 - \alpha(\hat{P}\hat{Q} + \hat{Q}\hat{P}) + (\omega^2 + \alpha^2)\hat{Q}^2 \right] \, . \tag{69}$$

Let us consider the underdumped case, $\alpha < \omega$. Then (68) is a Hamiltonian of a harmonic oscillator with an angular frequency $\tilde{\omega} = \sqrt{\omega^2 - \alpha^2}$. Stationary states of the corresponding Schrödinger equation have the form

$$\Psi_n = e^{-iE_n t} \psi_n(Q) \, , \ E_n = \tilde{\omega} \left(n + \frac{1}{2} \right) \, , \ \ n = 0, 1, \ldots,$$

$$\psi_n(Q) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\tilde{\omega}}{\pi} \right)^{1/4} e^{-\tilde{\omega} Q^2/2} H_n \left(Q\sqrt{\tilde{\omega}} \right) \, , \tag{70}$$

The mean values of energy (of the operator (69)) in such states can be easily calculated,

$$\langle E \rangle_n = \left(n + \frac{1}{2} \right) \frac{\omega^2}{\tilde{\omega}} e^{-2\alpha t} \, . \tag{71}$$

At each fixed time instant, the energy spectrum is discrete, however, it decreses with time exactly as in classical theory. The same conclusion was derived in [39]-[41] where a second-order action obtained by integrating-multiplier method was taken as a starting point for a quantization. A quantization of the damped oscillator following Bateman (see above) meets serious difficulties such as indefinite metric etc., [39, 42].

Overdumped cases, when $\alpha \geq \omega$ correspond to an aperiodic motion in classical theory [47]. Its quantum interpretation is not clear due to continuous character of the Hamiltonian spectrum.

7 Quantization of radiating point-like charge

Equations of motion for a nonrelativistic particle moving in electric **E** and magnetic **H** fields with account taken of the back reaction of the radiation emitted by the particle have the form [48]

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{f} \, , \ (\mathbf{r} = (x, y, z)) \, ,$$

$$\mathbf{F} = e\mathbf{E} + \frac{e}{c} [\dot{\mathbf{r}} \times \mathbf{H}] \, , \ \mathbf{f} = \frac{2e^2}{3c^3} \dddot{\mathbf{r}} \, . \tag{72}$$

Here **F** is the Lorentz force, **f** is the force of the back reaction of the radiation, e is the charge of the particle, and c is the light velocity. Derivatives with respect to time are denoted by dots above.

These equations are of third order, therefore a trajectory of a charged particle cannot by uniquely specified only by initial position and velocity of the particle. It was also pointed out that together with physically

meaningful solutions, equations (72) have a set of nonphysical solution [48]. However, in the case when the back reaction force f is small when compared to the Lorentz one **F**,

$$|\mathbf{f}| \ll |\mathbf{F}| \,, \tag{73}$$

these equations can be reduced to the second order equations by means of a reduction of order procedure. Then the above mentioned problem with nonphysical solutions does not appear. In the reduction procedure, equations (72) are replaced by second-order equations $\ddot{\mathbf{r}} = \mathbf{g}(\mathbf{r}, \dot{\mathbf{r}}, e)$ such that all the solutions of the latter equations would be solutions of (72). The last requirement implies a partial differential equation on the function $\mathbf{g}(\mathbf{r}, \dot{\mathbf{r}}, e)$ having a unique solution with the natural condition $\mathbf{g}(\mathbf{r}, \dot{\mathbf{r}}, 0) = 0$, see e.g. [48, 10].

Consider, for example, a particular case $\mathbf{E} = 0$, $\mathbf{H} = (0, 0, H = \text{const})$. In such a case, the reduced second-order equations have the form [10]

$$\ddot{x} = -\alpha\dot{x} - \beta\dot{y} \,, \ \ddot{y} = \beta\dot{x} - \alpha\dot{y} \,, \ \ddot{z} = 0 \,,$$

$$\alpha = \frac{\sqrt{6}\sqrt{3 + \sqrt{9 + 64e^6 H^2}} - 6}{8e^2} \approx \frac{2}{3}e^4 H^2 \,, \ \beta = \frac{eH\sqrt{6}}{\sqrt{3 + \sqrt{9 + 64e^6 H^2}}} \approx eH \,. \tag{74}$$

Here we have set $m = c = 1$ for simplicity. Since the evolution along the z-axis represents the free motion and decouples from the dynamics in the xy-plane, we restrict our consideration to the first two equations. At $\alpha = 0$, equations (74) are Lorentz equations with an "effective" magnetic field $\beta = (0, 0, \beta/e)$. In this case, the trajectories are concentric circles. If $\alpha \neq 0$, the particle spirals at the origin of xy-plane. So, it is natural to treat α as a friction coefficient.

In order to construct an action functional for the non-Lagrangian second-order equations (74), we introduce new variables as follows:

$$p = \dot{x} + \frac{\beta}{2}y \,, \ q = \dot{y} - \frac{\beta}{2}x \,.$$

In the new variables, we have a set of first-order equations,

$$\dot{x} = p - \frac{\beta}{2}y \,, \ \dot{y} = q + \frac{\beta}{2}x \,,$$

$$\dot{p} = -\frac{\beta}{2}q - \frac{\beta^2}{4}x - \alpha\left(p - \frac{\beta}{2}y\right) \,, \ \dot{q} = \frac{\beta}{2}p - \frac{\beta^2}{4}y - \alpha\left(q + \frac{\beta}{2}x\right) \,. \tag{75}$$

According to general formulas (18), (19), and (19), we construct the following action for the set (75):

$$S = \frac{1}{2(\alpha^2 + \beta^2)} \int e^{\alpha t} \left[a(p\dot{x} - x\dot{p} + q\dot{y} - y\dot{q}) + \beta(q\dot{x} - x\dot{q} + y\dot{p} - p\dot{y}) \right. \tag{76}$$
$$\left. + c(p\dot{q} - q\dot{p}) + d(x\dot{y} - y\dot{x}) - e(p^2 + q^2) - f(x^2 + y^2) - g(px + qy) - j(qx - py)\right] dt \,,$$

where time-dependent functions a, b, c, d, e, f, g, and j are

$$a = \alpha^2 \cos(\beta t) + \beta^2 \cosh(-\alpha t) \,, \ b = \alpha^2 \sin(\beta t) + \alpha\beta e^{-\alpha t} - \alpha\beta \cos(\beta t),$$

$$c = e^{-\alpha t}\beta - 2\alpha \sin(\beta t) \,, \ d = -\frac{\beta^3}{2}\sinh(-\alpha t) - \frac{1}{2}\alpha\beta \sin(\beta t) + \alpha^2\beta\left[\cos(\beta t) - e^{-\alpha t}\right] \,,$$

$$e = e^{-\alpha t}\beta^2 + \alpha^2 \cos(\beta t) - \alpha\beta \sin(\beta t) \,, \ g = \alpha\beta^2 \cos(\beta t) + \alpha^3 \cos(\beta t),$$

$$f = \frac{\beta}{4}\left\{\beta^3 e^{\alpha t} + \beta\alpha^2 \cos(\beta t) + \alpha[\beta^2 + 2\alpha^2]\sin(\beta t)\right\} \,, \ j = \alpha^3 \sin(\beta t) + e^{\alpha t}\beta^3 + \alpha^2\beta \cos(\beta t).$$

In the limit of zero friction $\alpha \rightarrow 0$, this action is reduced to the usual action for a charged particle in a homogeneous magnetic field β.

The set (75) is linear one with constant coefficients. For such a case, the corresponding quantum Hamiltonian can be constructed according to (56) as

$$\hat{H} = \frac{1}{2}\left[\hat{p}_1^2 + \hat{p}_2^2 + \frac{\alpha}{2}\left(\hat{p}_1\hat{q}_1 + \hat{q}_1\hat{p}_1 + \hat{p}_2\hat{q}_2 + \hat{q}_2\hat{p}_2\right) + \beta(\hat{p}_2\hat{q}_1 - \hat{p}_1\hat{q}_2) + \frac{\beta^2}{4}\left(\hat{q}_1^2 + \hat{q}_2^2\right)\right] \tag{77}$$

with operators $\hat{q}_i$ and $\hat{p}_j$ obeying canonical commutations relations,

$$[\hat{q}_i, \hat{p}_j] = i, \ [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, \ i, j = 1, 2 \,.$$

10

By help of a canonical transformation $(\hat{p}_1, \hat{q}_1; \hat{p}_2, \hat{q}_2) \rightarrow (\hat{p}, \hat{x}; \hat{q}, \hat{y})$, where

$$\hat{p} = \hat{p}_1 + \frac{\alpha}{2}\hat{q}_1, \ \hat{x} = \hat{q}_1, \ \hat{q} = \hat{p}_2 + \frac{\alpha}{2}\hat{q}_2, \ \hat{y} = \hat{q}_2 \ ,$$

we reduce (77) to the form

$$\hat{H} = \frac{1}{2}\left[\hat{p}^2 + \hat{q}^2 + \beta(\hat{q}\hat{x} - \hat{p}\hat{y}) + \frac{\beta^2 - \alpha^2}{4}\left(\hat{x}^2 + \hat{y}^2\right)\right]. \tag{78}$$

Condition (73) in the case under consideration implies $\alpha \ll \beta$, that is why α^2 will be omitted in (78) in what follows.

Consider eigenstates Ψ for two mutually commuting operators $\hat{H}$ and $\hat{L} = \hat{p}\hat{y} - \hat{q}\hat{x}$,

$$\hat{H}\Psi = E\Psi, \ \hat{L}\Psi = M\Psi. \tag{79}$$

It is convenient to perform the following canonical transformation $(\hat{p}, \hat{x}; \hat{q}, \hat{y}) \rightarrow (\hat{P}, \hat{X}; \hat{Q}, \hat{Y})$,

$$\hat{P} = \hat{p} - \frac{\beta}{2}\hat{y}, \ \hat{X} = \frac{1}{\beta}\left(\hat{q} + \frac{\beta}{2}\hat{x}\right) ,$$

$$\hat{Q} = \hat{q} - \frac{\beta}{2}\hat{x}, \ \hat{Y} = \frac{1}{\beta}\left(\hat{p} + \frac{\beta}{2}\hat{y}\right).$$

It is easy to see that

$$\hat{H} = \frac{1}{2}\left(\hat{P}^2 + \beta^2\hat{X}^2\right), \ \hat{L} = \beta^{-1}(\hat{H}_1 - \hat{H}), \ \hat{H}_1 = \frac{1}{2}\left(\hat{Q}^2 + \beta^2\hat{Y}^2\right).$$

Operators $\hat{H}$ and $\hat{H}_1$ are Hamiltonians of two independent harmonic oscillators. Then we can divide variables solving (79). Thus, we obtain solution of the eigenvalue problem (79),

$$\Psi = \Psi_{n,l}(X,Y) = \psi_n(X)\psi_l(Y), \ E_n = \beta\left(n + \frac{1}{2}\right), \ M_{nl} = l - n, \ n,l = 0,1,2,...$$

where ψ_n and ψ_l are eigenstates of the Hamiltonians $\hat{H}$ and $\hat{H}_1$ respectively (given e.g. by (70)). Finally, stationary states $\Psi(t)$ of the corresponding Schrödinger equation with the Hamiltonian $\hat{H}$ have the form

$$\Psi(X,Y,t) = e^{-iE_nt}\Psi_{n,l}(X,Y). \tag{80}$$

We define the classical energy E of the system under consideration according to [47] as the mechanical energy of the system without friction,

$$E = \frac{1}{2}\left[p^2 + q^2 + \beta(qx - py) + \frac{\beta^2}{4}\left(x^2 + y^2\right)\right] .$$

One can see that the energy depends of time as follows: $E = E_0 e^{-2\alpha t}$. An operator $\hat{E}$ that corresponds to the classical quantity E reads:

$$\hat{E} = \frac{1}{2}\left[\hat{P}^2 + \beta^2\hat{X}^2 + \alpha\left(\hat{X}\hat{Y} - \hat{P}\hat{Q}\right)\right]e^{-2\alpha t} - 2\alpha\left(\frac{\alpha}{\beta}\right)\left[\hat{P}\hat{Y} + \hat{Q}\hat{X}\right]e^{-2\alpha t} + o\left(\frac{\alpha}{\beta}\right) .$$

Mean values of this operator in stationary states (80) can be easily calculated, they are

$$\langle E \rangle_{nl} = \beta\left(n + \frac{1}{2}\right)e^{-2\alpha t}.$$

Similar to the damped oscillator case considered above, at each fixed time instant, the energy spectrum is discrete, however, it decreses with time exactly as in classical theory.

We would like to note that in the work [10] it was shown that although an action principle for the second-order equations (74) describing a radiating point-like charge does exist, none of the possible corresponding

Lagrangians in the limit of $\alpha \to 0$ reduces to the Lagrangian of a particle in a magnetic field modulo a total time derivative. That is, in the case of a radiating point-like charge a perturbation (in the friction parameter α) of a second-order action does not correspond to a perturbation of the equations of motion (74). For this reason, we expect some difficulties with the limit of $\alpha \to 0$ in the quantum theory of a radiating point-like charge resulting from quantization based on an action functional in the second-order form (such quantization for a damped harmonic oscillator was presented in [39]-[41]).

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