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Four- level systems.

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Two interacting spins in external fields.
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Abstract

In the present article, we consider the so-called two-spin equation that describes four-level quantum systems. Recently, these systems attract attention due to their relation to the problem of quantum computation. We study general properties of the two-spin equation and show that the problem for certain external backgrounds can be identified with the problem of one spin in an appropriate background. This allows one to generate a number of exact solutions for two-spin equations on the basis of already known exact solutions of the one-spin equation. Besides, we present some exact solutions for the two-spin equation with an external background different for each spin but having the same direction. We study the eigenvalue problem for a time-independent spin interaction and a time-independent external background. A possible analogue of the Rabi problem for the two-spin equation is defined. We present its exact solution and demonstrate the existence of magnetic resonances in two specific frequencies, one of them coinciding with the Rabi frequency, and the other depending on the rotating field magnitude. The resonance that corresponds to the second frequency is suppressed with respect to the first one.

1 Introduction

1.1 Overview

Finite-level systems have always played an important role in quantum physics. In particular, two-level systems possess a wide range of applications, for example, in the semi-classical theory of laser beams [1], optical resonance [2],

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absorption resonance, and nuclear induction experiments [3], in the explanation of the behavior of a molecule in a cavity immersed in electric or magnetic fields [4], and so on. The best known physical systems that could be identified with two-level systems are, for example, two-level atoms (atoms in which interaction with specific electromagnetic fields naturally selects only two energy levels important for the consideration) and spin-one-half objects interacting with magnetic field. Four-level systems describe two interacting one-half spins, e.g., those of an electron and a proton in an atom, or those of two electrons frozen in space, and so on, see, e.g., [5, 6]. Recently, two- and four-level systems attract even more attention due to their relation to the problem of quantum computation, see, for example, [7]. In this problem, the state of each bit of conventional computation is permitted to be any quantum-mechanical state of a *qubit* (quantum bit), which can be regarded as a two-level system. Computation is performed by manipulating these qubits with the help of the so-called *quantum gates*. Although these gates depend on the number of involved qubits, it is possible to demonstrate that all manipulations can be efficiently accomplished by using gates with just one and two qubits, where two-qubit gates can be identified with a four-level system, see [7]. For these reasons, two- and four-level systems are crucial elements of possible quantum computers. For physical applications, it is very important to have explicit exact solutions of two- and four-level system equations. One can mention, in this respect, the Rabi solution of a two-level system equations (*the spin equation* according to our terminology), which has a great importance in the treatment of numerous physical phenomena. In our opinion, the most recent and complete study of the spin equation and its exact solutions is presented in our work [8]. In the present article, we turn our attention to four-level systems. We study the general properties of the corresponding Schrödinger equation (called *the two-spin equation* in what follows) and present some exact solutions of this equation. In sec. 2, we define a two-spin equation with an external field and describe its general properties. In particular, we show that the problem for an external background equal for both spins is reduced to the problem of one spin in a certain background. This allows one to generate a number of exact solutions for the two-spin equation on the basis of already known exact solutions of the spin equation. In sec. 3, we present some exact solutions for the two-spin equation with external backgrounds different for each spin but having the same direction. In sec. 4, we study the eigenvalue problem for a time-independent spin interaction and a time-independent external background. In sec. 5, we define a possible analogue of the Rabi problem for the two-spin equation. We present its exact solution and show the existence of magnetic resonances in two specific frequencies, one of them coinciding with the Rabi frequency, and the other depending on the rotating field magnitude. The resonance that corresponds to the second frequency is suppressed with respect to the first one.

1.2 Spin equation

The nonrelativistic spin operator $\hat{\mathbf{s}} = \frac{\hbar}{2}\boldsymbol{\sigma}$ is a particular case of the momentum operators[1] and describes particles with spin s one-half ($\hat{\mathbf{s}}^2 = 3/4 = s(s+1)$, $s = 1/2$). Consider the ϑ_λ-basis

$$s_z\vartheta_\lambda = (-1)^{\lambda-1} s\vartheta_\lambda\,,\ \lambda = 1,2;\ \vartheta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \vartheta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1}$$

In the ϑ_λ-basis, state vectors $\psi(t)$ can be associated with a time-dependent two-columns:

$$\psi(t) = \sum_{\lambda=1,2} v_\lambda(t)\,\vartheta_\lambda \Longleftrightarrow \psi(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}. \tag{2}$$

The dynamics of a spin-one-half particle subject to a time-dependent external field is described by the Schrödinger equation with a Hamiltonian $\hat{h}$. In the ϑ_λ-basis, the most general form of the Hamiltonian is $\hat{h} = 2(\hat{\mathbf{s}}\cdot\mathbf{F}) = (\boldsymbol{\sigma}\cdot\mathbf{F})$, where $\mathbf{F} = (F_1(t), F_2(t), F_3(t))$ is an arbitrary time-dependent vector (external field). Then the corresponding Schrödinger equation reads

$$i\frac{d\psi}{dt} = \hat{h}\psi\,,\ \psi = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix},\ \hat{h} = \begin{pmatrix} F_3 & F_1 - iF_2 \\ F_1 + iF_2 & -F_3 \end{pmatrix}. \tag{3}$$

Equation (3) implies the following coupled equations for the components $v_i(t)$, $i = 1, 2$:

$$i\dot{v}_1 = F_3 v_1 + (F_1 - iF_2)\,v_2\,,\ i\dot{v}_2 = -F_3 v_2 + (F_1 + iF_2)\,v_1\,. \tag{4}$$

Equation (3) is called *the spin equation*. The spin equation with a real external field can be regarded as a reduction of the Pauli equation [9] to the $(0+1)$-dimensional case. Such an equation is used to describe a (frozen in space) spin-1/2 particle of magnetic momentum μ, immersed in a magnetic field $\mathbf{B}$ (in this case, $\mathbf{F} = -\mu\mathbf{B}$), and has been intensely studied in connection with the problem of magnetic resonances [3, 10]. The spin equation with complex external fields describes a possible damping of two-level systems [8]. There exist various equations that are equivalent, or (in a sense) related, to the spin equation. For example, the well-known top equation, which appears in the gyroscope theory, in the theory of precession of a classical gyromagnet in a magnetic field (see [4]), and so on. The spin equation with an external field in which $F_s(t)$, $s = 1, 2$ are purely imaginary and F_3 is constant is a degenerate case of the Zakharov–Shabad equation, which plays an important role in the soliton theory [11]. The first exact solution of the spin equation was found by Rabi [12] for an external field of the form $\mathbf{F} = (f_1\cos\omega t,\ f_2\sin\omega t,\ F_3)$, where $f_{1,2}$, ω, and F_3 are real constant. A number of exact solutions of the spin equation were found in [8, 13, 14]. For periodic, or quasiperiodic, external fields, the equations of a two-level system have been studied by many authors using different approximation methods, e.g., perturbative expansions [15], see also [16].

[1] Here, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. In what follows, we set $\hbar = 1$.

2 Two-spin equation and its properties

2.1 Two-spin equation

Let us consider two interacting spins one-half. We choose the state space for such a system as the direct-product space of the state spaces of individual spins. In this space, we choose basis states Θ_μ , $\mu = 1, 2, 3, 4$, as the direct product of individual bases:

$$\Theta_1 = \vartheta_1 \otimes \vartheta_1 \, , \; \Theta_2 = \vartheta_1 \otimes \vartheta_2 \, , \; \Theta_3 = \vartheta_2 \otimes \vartheta_1 \, , \; \Theta_4 = \vartheta_2 \otimes \vartheta_2 \, . \tag{5}$$

The spin operators for the first and second subsystems are $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$,

$$\begin{aligned}
&\hat{\mathbf{s}}_1 = \frac{\boldsymbol{\sigma}}{2} \otimes I \, , \; \hat{\mathbf{s}}_2 = I \otimes \frac{\boldsymbol{\sigma}}{2} \, , \\
&\hat{s}_{1z}\Theta_{1,2} = \frac{1}{2}\Theta_{1,2} \, , \; \hat{s}_{1z}\Theta_{3,4} = -\frac{1}{2}\Theta_{3,4} \, , \\
&\hat{s}_{2z}\Theta_{1,3} = \frac{1}{2}\Theta_{1,3} \, , \; \hat{s}_{2z}\Theta_{2,4} = -\frac{1}{2}\Theta_{2,4} \, ,
\end{aligned}$$

where I is a 2×2 unity matrix, and the total spin operator is $\hat{\mathbf{S}} = \hat{\mathbf{s}}_1 + \hat{\mathbf{s}}_2$.

The Hamiltonian of two interacting spins subject to the external fields $\mathbf{G}$ and $\mathbf{F}$, respectively, is chosen as[2]

$$\begin{aligned}
&\hat{H}\left(\mathbf{G}, \mathbf{F}, J\right) = 2\left[\hat{h}_1 \otimes I + I \otimes \hat{h}_2 + J\hat{\mathbf{s}}_1 \otimes \hat{\mathbf{s}}_2\right] \, , \\
&\hat{h}_1 = \left(\hat{\mathbf{s}} \cdot \mathbf{G}\right) = \frac{1}{2}\left(\boldsymbol{\sigma} \cdot \mathbf{G}\right) \, , \; \hat{h}_2 = \left(\hat{\mathbf{s}} \cdot \mathbf{F}\right) = \frac{1}{2}\left(\boldsymbol{\sigma} \cdot \mathbf{F}\right) \, , \\
&\hat{\mathbf{s}}_1 \otimes \hat{\mathbf{s}}_2 = \frac{1}{4}\left[\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3\right] \, ,
\end{aligned} \tag{6}$$

where $J = J(t)$, in general, is a function of time, and $\mathbf{G} = (G_1(t), G_2(t), G_3(t))$ and $\mathbf{F} = (F_1(t), F_2(t), F_3(t))$ are, in general, time-dependent vectors (external fields for each particle) [5, 6].

In the representation generated by the basis (5), the evolution of the system is describes by the Schrödinger equation

$$i\frac{d\Psi}{dt} = \hat{H}\left(\mathbf{G}, \mathbf{F}, J\right)\Psi \, , \; \Psi = \begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \end{pmatrix} , \tag{7}$$

where the Hamiltonian $\hat{H}(\mathbf{G}, \mathbf{F}, J)$ is given by a 4×4 matrix:

$$\hat{H} = \begin{pmatrix} F_3 + G_3 + \frac{J}{2} & F_1 - iF_2 & G_1 - iG_2 & 0 \\ F_1 + iF_2 & G_3 - F_3 - \frac{J}{2} & J & G_1 - iG_2 \\ G_1 + iG_2 & J & F_3 - G_3 - \frac{J}{2} & F_1 - iF_2 \\ 0 & G_1 + iG_2 & F_1 + iF_2 & \frac{J}{2} - G_3 - F_3 \end{pmatrix} . \tag{8}$$

[2] When restoring the Plank constant, we need to replace J by $j/\hbar^2$. The factor $\hbar^2$ in the denominator of the interaction term ensures that J has the dimension of energy.

Equation (7) implies for the components $v_\mu(t)$ the coupled equations

$$
\begin{aligned}
i\dot{v}_1 &= \left(F_3 + G_3 + \frac{J}{2}\right) v_1 + (F_1 - iF_2)\, v_2 + (G_1 - iG_2)\, v_3\,, \\
i\dot{v}_2 &= (F_1 + iF_2)\, v_1 + \left(G_3 - F_3 - \frac{J}{2}\right) v_2 + Jv_3 + (G_1 - iG_2)\, v_4\,, \\
i\dot{v}_3 &= (G_1 + iG_2)\, v_1 + Jv_2 + \left(F_3 - G_3 - \frac{J}{2}\right) v_3 + (F_1 - iF_2)\, v_4\,, \\
i\dot{v}_4 &= (G_1 + iG_2)\, v_2 + (F_1 + iF_2)\, v_3 + \left(\frac{J}{2} - G_3 - F_3\right) v_4\,. \qquad (9)
\end{aligned}
$$

We call Eqs. (7) or (9) *the two-spin equation.*

Let us introduce the matrices $\boldsymbol{\Sigma}$ =diag $(\boldsymbol{\sigma},\boldsymbol{\sigma})$ and $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3)$,

$$
\rho_1 = -\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},\ \rho_2 = i\gamma^0\gamma^5 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix},\ \rho_3 = \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},
$$

where the gamma-matrices are in the standard representation. Then

$$
\boldsymbol{\Sigma} = I \otimes \boldsymbol{\sigma}\,,\ \boldsymbol{\rho} = \boldsymbol{\sigma} \otimes I\,,\ (\boldsymbol{\Sigma} \cdot \boldsymbol{\rho}) = \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \otimes \sigma_i\,, \qquad (10)
$$

so that the Hamiltonian (8) can be written via these matrices as follows:

$$
\hat{H}\,(\mathbf{G},\mathbf{F},J) = (\boldsymbol{\rho}\cdot\mathbf{G}) + (\boldsymbol{\Sigma}\cdot\mathbf{F}) + \frac{J}{2}\,(\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})\,. \qquad (11)
$$

Let us now consider a nonsingular 4×4 orthogonal matrix A,

$$
\begin{aligned}
&A = \frac{1}{2}\left[\mathbb{I} + (\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \det A = -1\,, \\
&A = A^{+} = A^{-1}\,, \quad A^2 = \mathbb{I}\,, \qquad (12)
\end{aligned}
$$

where $\mathbb{I}$ is a 4×4 unity matrix. Using the properties of the σ-matrices,

$$
[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k\,, \quad [\sigma_i, \sigma_j]_+ = 2\delta_{ij}\,, \qquad (13)
$$

where ε_{ijk} is the Levi-Civita symbol ($\varepsilon_{123} = 1$), one can easily verify that

$$
\begin{aligned}
&A\,\boldsymbol{\Sigma}\,A = \boldsymbol{\rho}\,, \quad A\,\boldsymbol{\rho}\,A = \boldsymbol{\Sigma}\,, \quad A\,(\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})\,A = (\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})\,, \qquad (14) \\
&A\,(\boldsymbol{\Sigma} + \boldsymbol{\rho})\,A = \boldsymbol{\Sigma} + \boldsymbol{\rho}\,, \quad A\,(\boldsymbol{\Sigma} - \boldsymbol{\rho})\,A = -(\boldsymbol{\Sigma} - \boldsymbol{\rho})\,. \qquad (15)
\end{aligned}
$$

By the use of the above properties, one can prove that

$$
A\hat{H}\,(\mathbf{G},\mathbf{F},J)\,A = \hat{H}\,(\mathbf{F},\mathbf{G},J)\,. \qquad (16)
$$

The latter relation implies that a solution Ψ_1 with external fields $\mathbf{G}$ and $\mathbf{F}$ is related to a solution Ψ_2 with external fields $\mathbf{F}$ and $\mathbf{G}$ by the matrix A, i.e., $\Psi_2 = A\Psi_1$.

Let us introduce the evolution operator $R_t(\mathbf{G},\mathbf{F},J)$ for the two-spin equation:

$$\Psi(t) = R_t(\mathbf{G},\mathbf{F},J)\,\Psi_0\,,\ R_0 = \mathbb{I}\,,$$

$$i\frac{d}{dt}R_t(\mathbf{G},\mathbf{F},J) = \hat{H}\,(\mathbf{G},\mathbf{F},J)\,R_t(\mathbf{G},\mathbf{F},J)\,.$$

Taking (14) and (15) into account, one can easily see that

$$AR_t(\mathbf{G},\mathbf{F},J)A = R_t(\mathbf{F},\mathbf{G},J)\,.$$

2.2 Some general properties

1. Let $\Psi(t)$ be a solution that corresponds to the Hamiltonian $\hat{H}\,(\mathbf{G},\mathbf{F},J)$, then $\Psi\,(T\,(t))$, with $T\,(t)$ being a differentiable function of t, is a solution that corresponds to the Hamiltonian

$$\hat{H}\left(\dot{T}\,(t)\,\mathbf{G}\,(T\,(t))\,,\dot{T}\,(t)\,\mathbf{F}\,(T\,(t))\,,\dot{T}\,(t)\,J\,(T\,(t))\right)\,.$$

   Thus, each solution of equation (7) generates a set of solutions with explicitly indicated arbitrariness.

2. Let the external fields be zero, $\mathbf{G} = \mathbf{F} = 0$, and $J\,(t)$ be an arbitrary function (two interacting spins without external backgrounds). In this case, the evolution operator has the form

$$R_t\,(0,0,J) = \exp\left[i\Phi\,(t)\,/2\right]\left[\mathbb{I}\cos\Phi\,(t) - i\,A\sin\Phi\,(t)\right]\,, \tag{17}$$

   where

$$\Phi\,(t) = \int_{t_0}^{t} J\,(\tau)\;d\tau\,. \tag{18}$$

3. Let the spins do not interact, $J = 0$, and the fields $\mathbf{G}\,(t)$ and $\mathbf{F}\,(t)$ be arbitrary. Then one can write the expression for the evolution operator $R_t\,(\mathbf{G},\mathbf{F},0)$ as follows:

$$R_t\,(\mathbf{G},\mathbf{F},0) = R_t\,(\mathbf{G},0,0)\,R_t\,(0,\mathbf{F},0)\,. \tag{19}$$

   In this case, the general solution of equation (7) reads

$$\Psi\,(t) = R_t\,(\mathbf{G},0,0)\,R_t\,(0,\mathbf{F},0)\,\Psi_0 = AR_t\,(0,\mathbf{G},0)\,AR_t\,(0,\mathbf{F},0)\,\Psi_0\,. \tag{20}$$

   The second form of the general solution in (20) seems to be more convenient since it is expressed via solutions of the two-spin equation with a free first spin. This solution, in fact, is reduced to a solution of the evolution operator for the spin equation (3) because the corresponding Hamiltonians $(\mathbf{\Sigma}{\cdot}\mathbf{G})$ and $(\mathbf{\Sigma}{\cdot}\mathbf{F})$ are diagonal in this case.

4. Let the external fields be the same for both spins, $\mathbf{G}(t)=\mathbf{F}(t)$, and the interaction $J(t)$ be arbitrary. Taking into account the property (15), one can find the form of the evolution operator in this case:

$$
\begin{aligned}
&R_t(\mathbf{G},\mathbf{G},J)=R_t(0,0,J)\,R_t(\mathbf{G},\mathbf{G},0)\\
&=R_t(0,0,J)\;R_t(\mathbf{G},0,0)\,R_t(0,\mathbf{G},0)=R_t(0,0,J)\;AR_t(0,\mathbf{G},0)\,AR_t(0,\mathbf{G},0),
\end{aligned}
\tag{21}
$$

where $R_t(0,0,J)$ is given by (17). Therefore, in the case under consideration, we have reduced the two-spin problem to the one-spin problem in the external field $\mathbf{G}(t)$.

5. Due to the spherical symmetry, rotations commute with the interaction operator $(\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})=\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}$ in (11). Therefore, as in the case of a single spin, the method of a *rotating coordinate system* [3] can be applied to the case of two interacting spins. Let us apply, to a spinor ψ that is a solution of the spin equation (3), $i\dot{\psi}=\hat{h}\psi$, the transformation $\psi'=r^{-1}\psi$. The spinor ψ' obeys the spin equations with a Hamiltonian $\hat{h}'$ given by

$$
\hat{h}'=r^{-1}\hat{h}r-ir^{-1}\dot{r}\,. \tag{22}
$$

Suppose now that r is a rotation, then the operator $\mathcal{R}=r\otimes r$ commutes with $(\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})$. Therefore, if Ψ is a solution of the two-spin equation (7), with Hamiltonian $\hat{H}$ (6), the rotated spinor $\Psi'=\mathcal{R}^{-1}\Psi$ is a solution of this equation for a Hamiltonian $\hat{H}'$ given by

$$
\begin{aligned}
\hat{H}'&=\mathcal{R}^{-1}\hat{H}\mathcal{R}-i\mathcal{R}^{-1}\dot{\mathcal{R}}\\
&=\hat{h}_1'\otimes I+I\otimes\hat{h}_2'+\frac{J}{2}(\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})\,,
\end{aligned}
\tag{23}
$$

with the same original interaction J and $\hat{h}_i'$ given by (22). An example of the use of this property will be given later, namely in Eq. (43), when we study a possible generalization of the Rabi problem for two interacting spins.

3 Some exact solutions of two-spin equation

3.1 Parallel external fields

Let the two spins be subject to different time-dependent external fields having the same fixed direction,

$$
\mathbf{G}=\mathbf{n}B_1(t)\ ,\ \mathbf{F}=\mathbf{n}B_2(t)\ ,
$$

where $\mathbf{n}$ is a constant unity vector. Choosing an appropriated coordinate system, or using a constant rotation, which due to (23) does not change the problem, we can set $\mathbf{n}=(0,0,1)$, so that

$$
\mathbf{G}=(0,0,B_1)\ ,\ \mathbf{F}=(0,0,B_2)\ B_{1,2}=B_{1,2}(t)\ .
$$

Then the Hamiltonian (11) takes the form

$$i\dot{\Psi} = \hat{H}\Psi\,,\ \hat{H} = \frac{1}{2}\left[(\Sigma_3 + \rho_3)\,B_+ - (\Sigma_3 - \rho_3)\,B_- - J\right] + AJ,$$
$$B_\pm(t) = B_1(t) \pm B_2(t)\,.$$

For the components v_1 and v_4 of the four-spinor Ψ, we get from the Schrödinger equation (7):

$$v_1 = C_1 \exp\left[-i\int_0^t \left(\frac{J}{2} + B_+\right) d\tau\right]\,,\ v_4 = C_4 \exp\left[-i\int_0^t \left(\frac{J}{2} - B_+\right) d\tau\right]\,, \tag{24}$$

with $C_{1,4}$ being complex constants. And for the components $v_{2,3}$ we obtain the equation

$$i\dot{\psi}' = \left[(\boldsymbol{\sigma}\cdot\mathbf{K}) - \frac{J}{2}\right]\psi'\,,\ \psi' = \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}\,, \tag{25}$$

$$\mathbf{K}(t) = (J(t)\,, 0, B_-(t))\,. \tag{26}$$

Doing the transformation

$$\psi'(t) = \exp\left[\frac{i}{2}\int_0^t J(\tau)\,d\tau\right]\psi(t)\,, \tag{27}$$

we can see that the spinor ψ obeys the spin equation (3) with the external field $\mathbf{K}(t)$.

Exact solutions of the spin equation (3) for 26 different types of external fields of the form (26) are described in [8]. Respectively, they generate 26 sets of exact solutions of the two-spin-equation.

Below we consider two specific cases:

1. Constant spin interaction
   Let $J = \varepsilon = \text{const}$. Then equations (24) imply

$$v_1 = C_1 e^{-i\varepsilon t/2} \exp\left(-i\int_0^t B_+\,d\tau\right)\,,\ v_4 = C_4 e^{-i\varepsilon t/2} \exp\left(i\int_0^t B_+\,d\tau\right)\,.$$

In this case, a solution ψ' of the problem (25) can be written as

$$\psi'(t) = \exp\left(\frac{i\varepsilon}{2}t\right)\psi(t)\,,$$

where ψ is a solution of the spin equation (3) with the field

$$\mathbf{K}(t) = (\varepsilon, 0, f(t))\,,\ f(t) = B_-(t)\,. \tag{28}$$

In [14], one can find several functions $f(t)$ for which exact solutions of the spin equation can be found.

2. Fields with a constant difference
Let the difference $B_{-}=B_1-B_2=\varepsilon$ does not depend on time, $\varepsilon=\text{const}$, while the spin interaction J can depend on time. It happens sometimes in case of two interacting quantum dots [17]. For fields with a constant difference, a solution ψ' of the problem (25) can be written as (27), where ψ is a solution of the spin equation with the field

$$\mathbf{M}(t)=(f(t),0,\varepsilon)\ ,\ f(t)=J(t)\ . \tag{29}$$

If φ is a solution of the spin equation (3) with the external field $\mathbf{K}$ (28), one can construct a solution ψ for this equation with the external field $\mathbf{M}$ (29) by the transformation

$$\psi=(2)^{-1/2}(\sigma_1+\sigma_3)\varphi\,.$$

Then, we can use solutions from [14] in order to construct exact solutions of the two-spin equation in the fields with a constant difference.

4 Time-independent spin interaction and constant external fields

Let the interaction J and the fields $\mathbf{F}$ and $\mathbf{G}$ be time-independent,

$$J(t)=2\gamma,\quad \mathbf{F}(t)=\mathbf{a},\quad \mathbf{G}(t)=\mathbf{b}, \tag{30}$$

where γ, $\mathbf{a}$, $\mathbf{b}$ are constant. In this case, we search for solutions of two-spin equation of the form

$$\Psi(t)=\exp(-i\lambda t)\,C, \tag{31}$$

where C is a constant spinor. Substituting (31) into two-spin equation, we obtain the following equation for C:

$$D(\lambda)\,C=0,\ D(\lambda)=\gamma(\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})+(\boldsymbol{\Sigma}\cdot\mathbf{a})+(\boldsymbol{\rho}\cdot\mathbf{b})-\lambda\mathbb{I}\,. \tag{32}$$

Its solutions exist under the following condition:

$$d(\lambda)=\det D(\lambda)=0. \tag{33}$$

Direct calculation yields the following expression for $d(\lambda)$:

$$\begin{aligned}d(\lambda)&=\lambda^4-2\lambda^2\left(a^2+b^2+3\gamma^2\right)+8\lambda\gamma\left[\gamma^2-(\mathbf{ab})\right]-3\gamma^4\\&+2\gamma^2\left[a^2+b^2+4(\mathbf{ab})\right]+\left(a^2-b^2\right)^2,\quad a^2=(\mathbf{aa}),\quad b^2=(\mathbf{bb})\ .\end{aligned} \tag{34}$$

The determinant $d(\lambda)$ can also be presented as

$$\begin{aligned}d(\lambda)&=(\lambda-\gamma)^3(\lambda+3\gamma)-2\left(\lambda^2-\gamma^2\right)\left(a^2+b^2\right)-8\gamma(\mathbf{ab})(\lambda-\gamma)+\left(a^2-b^2\right)^2\\&=\left[(\lambda+\gamma)^2-4\gamma^2-q^2\right]\left[(\lambda-\gamma)^2-p^2\right]-p^2q^2+(\mathbf{pq})^2,\ \mathbf{p}=\mathbf{a}+\mathbf{b},\ \mathbf{q}=\mathbf{a}-\mathbf{b}.\end{aligned} \tag{35}$$

In principle, equation (33) can be solved explicitly (four roots can be found), then equation (32) allows one to find the spinor C.

In the particular case $\mathbf{a} = (0,0,a), \mathbf{b} = (0,0,b)$, $\mathbf{p} = (0,0,p = a+b)$, $\mathbf{q} = (0,0,q = a-b)$, the general solution of the two-spin equation has the form

$$\Psi(t) = \hat{U}(t - t_0)\,\Psi_0\,, \tag{36}$$

where the evolution operator reads

$$\begin{aligned}
&2\hat{U}(\tau) = [(I + \rho_3\Sigma_3)\cos p\tau - i(\rho_3 + \Sigma_3)\sin p\tau]\exp(-i\gamma\tau) \\
&+ \{(I - \rho_3\Sigma_3)\cos\omega\tau - (i/\omega)[q(\rho_3 - \Sigma_3) + 2\gamma(\rho_1\Sigma_1 + \rho_2\Sigma_2)]\sin\omega\tau\}\exp(i\gamma\tau)\,, \\
&\omega^2 = 4\gamma^2 + q^2.
\end{aligned} \tag{37}$$

5 Analogue of the Rabi problem for two-spin system

5.1 Rabi problem for one spin

We recall that Rabi considered one spin placed in a constant magnetic field and perpendicular to the latter a rotating field [3]. In fact, the Rabi problem is reduced to solving spin equation (3) with the external field of the form $\mathbf{F} = (A\cos\omega t, A\sin\omega t, A_0)$, where A and A_0 are real constants. The evolution operator for the spin equation with the Rabi field has the form

$$\begin{aligned}
&\hat{u}_F = \hat{r}_z(\omega t)\left[I\cos\omega_R t + \frac{(\alpha - ia'\sigma_2)^2}{\alpha^2 + a'^2} i\sigma_3\sin\omega_R t\right], \\
&\hat{r}_z(\omega t) = \exp\left(-i\sigma_3\frac{\omega}{2}t\right),\ \omega_R^2 = A^2 + \left(A_0 - \frac{\omega}{2}\right)^2, \\
&a' = \frac{A}{\omega},\ a_0 = \frac{A_0}{\omega} - \frac{1}{2},\ \alpha = a_0 - \frac{\omega_R}{\omega},
\end{aligned} \tag{38}$$

where ω_R is the *Rabi frequency*. Calculating the transition probability between two orthogonal one-spin states, we get following Rabi

$$|\langle 2|\,\hat{u}_F(t)\,|1\rangle|^2 = \frac{\left(\frac{\omega_R}{\omega}\right)^2 - a_0^2}{\left(\frac{\omega_R}{\omega}\right)^2}\sin^2\omega_R t,\ |1\rangle = \begin{pmatrix}1\\0\end{pmatrix},\ |2\rangle = \begin{pmatrix}0\\1\end{pmatrix}. \tag{39}$$

Since $(\omega_R/\omega)^2 \geq a_0^2$, the amplitude of the probability has a maximum in the resonance frequency $\omega = 2A_0$ of external field.

If we consider two noninteracting spins, one of them free and another one placed in the Rabi field, we can describe such a system by the two-spin equation with $\mathbf{G} = J = 0$ and $\mathbf{F} = (A\cos\omega t, A\sin\omega t, A_0)$, and the Hamiltonian

$$\hat{H} = (\mathbf{\Sigma}\cdot\mathbf{F})\,. \tag{40}$$

The evolution operator for the two-spin equation (40) in such a case can be written, in virtue of relations (10), as follows:

$$
\begin{aligned}
R_t(0,\mathbf{F},0) &= I\otimes\hat{u}_F = \exp\left(-i\Sigma_3\frac{\omega}{2}t\right)R_\Sigma(\omega_R t)\ ,\\
R_\Sigma(\omega_R t) &= \left[\mathbb{I}\cos\omega_R t + \frac{(\alpha - ia'\Sigma_2)^2}{\alpha^2+a'^2} i\Sigma_3\sin\omega_R t\right]\ .
\end{aligned} \tag{41}
$$

5.2 Possible generalizations of the Rabi problem for two interacting spins

5.2.1 Different Rabi fields for each spin

Let us considered two interacting spins each of them placed in a Rabi field and with a spin interaction that does not depend on time. Such a situation is described by the two-spin equation (7) with

$$
\begin{aligned}
&J(t) = J,\ \mathbf{F} = (A\cos(\omega t+\varphi_1), A\sin(\omega t+\varphi_1), A_0)\ ,\\
&\mathbf{G} = (B\cos(\omega t+\varphi_2), B\sin(\omega t+\varphi_2), B_0)\ ,
\end{aligned} \tag{42}
$$

where J, A, B, A_0, B_0, φ_1, φ_2, ω are constant. Its solution can be chosen in the form

$$
\begin{aligned}
&\Psi(t) = \exp(-i\lambda\omega t)\,\mathcal{R}_z(\omega t)\,C\,,\\
&\mathcal{R}_z(\omega t) = \exp\left[-i\frac{\omega t}{2}(\Sigma_3+\rho_3)\right]\ .
\end{aligned} \tag{43}
$$

where C, with components $C_k\,(k=1,2,3,4)$, is a constant bispinor. Note that $\mathcal{R}_z$ is a rotation in the z-direction by the angle ωt, such that we use a *rotating coordinate system* that rotates with the field, similar to ordinary Rabi problem. Taking the Hamiltonian $\hat{H}$ with the fields (42), the rotation $\mathcal{R}_z$ obeying (23), and setting $D(\lambda) = \hat{H}' - \lambda\mathbb{I}$, we see that the bispinor C obeys a linear set of equations

$$
D(\lambda)\,C = 0, \tag{44}
$$

where the *constant* matrix $D(\lambda)$ has the form (32) with

$$
\begin{aligned}
&\gamma = \frac{J}{2\omega},\ \mathbf{a} = (a'\cos\varphi_1, a'\sin\varphi_1, a_0)\ ,\ a' = \frac{A}{\omega},\ a_0 = \frac{A_0}{\omega} - \frac{1}{2}\,,\\
&\mathbf{b} = (b'\cos\varphi_2, b'\sin\varphi_2, b_0)\ ,\ b' = \frac{B}{\omega},\ b_0 = \frac{B_0}{\omega} - \frac{1}{2}\,.
\end{aligned} \tag{45}
$$

Thus, the problem under consideration is reduced to a problem for two spins in time-independent fields and with a constant interaction.

5.2.2 Equal Rabi fields for both spins

Let us considered two interacting spins each of them placed in the same Rabi field and with a time-dependent spin interaction $J(t)$. Such a situation is described by two-spin equation (7) with

$$\mathbf{G}=\mathbf{F}=(A\cos\omega t, A\sin\omega t, A_0)\ ,\ \gamma(t)=\frac{1}{2\omega}\int_{t_0}^{t}J(\tau)\,d\tau.$$

In the case under consideration, the evolution operator has the form (21)

$$\begin{aligned}R_t(\mathbf{F},\mathbf{F},J) &= R_t(0,0,\gamma)\,R_t(\mathbf{F},0,0)\,R_t(0,\mathbf{F},0)\ ,\\ R_t(0,0,\gamma) &= \exp\left[-i\left(\boldsymbol{\Sigma}\cdot\boldsymbol{\rho}\right)\omega\gamma(t)\right],\end{aligned}$$

with $R_t(0,\mathbf{F},0)$ given by (41) and

$$\begin{aligned}R_t(\mathbf{F},0,0) &= \exp\left(-i\rho_3\frac{\omega}{2}t\right)R_\rho(\omega_R t)\ ,\\ R_\rho(\omega_R t) &= \left[\mathbb{I}\cos\omega_R t+\frac{(\alpha-i\alpha'\rho_2)^2}{\alpha^2+\alpha'^2}i\rho_3\sin\omega_R t\right].\end{aligned}$$

Using commutation relations between rotations and the operator $(\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})$, and the fact that $[\Sigma_i,\rho_j]=0$, we find

$$\begin{aligned}R_t(\mathbf{F},\mathbf{F},J) &= \mathcal{R}_z(\omega t)\exp\left[-i\left(\boldsymbol{\Sigma}\cdot\rho\right)\omega\gamma(t)\right]R_\rho(t)\,R_\Sigma(t)\ ,\\ \mathcal{R}_z(\omega t) &= \hat{r}_z\otimes\hat{r}_z=\exp\left[-i\left(\rho_3+\Sigma_3\right)\frac{\omega}{2}t\right].\end{aligned}$$

In the absence of the circular field ($A=0$), the corresponding stationary states $|\Psi_i\rangle$ and energy eigenvalues λ_i of the problem are:

$$\begin{aligned}|\Psi_1\rangle &= |1\rangle\otimes|1\rangle\ ,\ \lambda_1=\frac{J}{2}+2A_0\,,\\ |\Psi_2\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle\otimes|2\rangle+|2\rangle\otimes|1\rangle\right)\ ,\ \lambda_2=\frac{J}{2}\,,\\ |\Psi_3\rangle &= \frac{1}{\sqrt{2}}\left(|2\rangle\otimes|1\rangle-|1\rangle\otimes|2\rangle\right)\ ,\ \lambda_3=-\frac{3}{2}J\,,\\ |\Psi_4\rangle &= |2\rangle\otimes|2\rangle\ ,\ \lambda_4=\frac{J}{2}-2A_0\,.\end{aligned}\tag{46}$$

Using the fact that $|\Psi_i\rangle$ are eigenvectors of $(\boldsymbol{\Sigma}\cdot\boldsymbol{\rho})$ with the eigenvalues $\lambda_{1,2,4}=1/2$ and $\lambda_3=-3/2$, we obtain:

a) $\langle\Psi_3|\,R_t(\mathbf{F},\mathbf{F},J)\,|\Psi_k\rangle=0\,,k=1,2,4$, which is, in fact, a consequence of the conservation of the total angular momentum.

b)

$$\begin{aligned}\langle\Psi_4|\,R_t(\mathbf{F},\mathbf{F},J)\,|\Psi_1\rangle &= \exp\left(-i\omega\gamma\right)\langle 2|\,\hat{u}_F\,|1\rangle^2\ ,\\ |\langle\Psi_4|\,R_t(\mathbf{F},\mathbf{F},J)\,|\Psi_1\rangle|^2 &= \left[\frac{\left(\frac{\omega_R}{\omega}\right)^2-a_0^2}{\left(\frac{\omega_R}{\omega}\right)^2}\sin^2\omega_R t\right]^2,\end{aligned}$$

where $|\langle 2|\,\hat{u}_F\,|1\rangle|^2$ is given by (39).

c)

$$\langle \Psi_2|\, R_t\,(\mathbf{F},\mathbf{F},J)\,|\Psi_1\rangle = \frac{2}{\sqrt{2}}\exp\left(-i\omega\gamma\right)\langle 1|\,\hat{u}_F\,|1\rangle\,\langle 2|\,\hat{u}_F\,|1\rangle\ ,$$
$$\langle \Psi_2|\, R_t\,(\mathbf{F},\mathbf{F},J)\,|\Psi_4\rangle = \frac{2}{\sqrt{2}}\exp\left(-i\omega\gamma\right)\langle 1|\,\hat{u}_F\,|2\rangle\,\langle 2|\,\hat{u}_F\,|2\rangle\ . \qquad (47)$$

The one-spin amplitudes involved in (47) can be obtained from (38),

$$\langle 2|\,\hat{u}_F\,(t)\,|1\rangle = e^{i\frac{\omega}{2}t}\frac{2a'\alpha}{\alpha^2+a'^2}\sin\omega_R t\ ,$$
$$\langle 1|\,\hat{u}_F\,(t)\,|1\rangle = e^{i\frac{\omega}{2}t}\left(\cos\omega_R t - i\omega\frac{a_0}{\omega_R}\sin\omega_R t\right)\ ,$$
$$\langle 2|\,\hat{u}_F\,(t)\,|2\rangle = e^{-i\frac{\omega}{2}t}\left(\cos\omega_R t + i\omega\frac{a_0}{\omega_R}\sin\omega_R t\right)\ .$$

We see that the transition from the state $|\Psi_1\rangle$ to the one $|\Psi_4\rangle$ has a resonance at $\omega = \omega_1 = 2A_0$,

$$\max\left(|\langle\Psi_1|\,R_t\,|\Psi_4\rangle|^2\right) = 1\,.$$

Yet, for an external field in the frequency $\omega = \omega_1$, transitions from the state $|\Psi_2\rangle$ to $|\Psi_1\rangle$ and $|\Psi_4\rangle$ have resonances,

$$\max\left(|\langle\Psi_2|\,R_t\,|\Psi_1\rangle|^2\right) = \max\left(|\langle\Psi_2|\,R_t\,|\Psi_4\rangle|^2\right) = 1/2\,,$$

In addition, these transitions have resonances at $\omega = \omega_2 = 2\left(A_0 - A\right)$,

$$\max\left(|\langle\Psi_2|\,R_t\,|\Psi_1\rangle|^2\right) = \max\left(|\langle\Psi_2|\,R_t\,|\Psi_4\rangle|^2\right) = 1/2\,.$$

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