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Density matrix of a quantum field in a particle-creating background

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Abstract

We consider the time evolution of a quantized field in backgrounds that violate the stability of vacuum (particle-creating backgrounds). Our aim is to study the exact form of the final quantum state (the density operator at a final instant of time) that has emerged from a given arbitrary initial state (from a given arbitrary density operator at the initial time instant) in the course of evolution. We find the generating functional for an ensemble average at a final instant of time for any possible initial state. Averaging over states of a subsystem of antiparticles (particles), we obtain reduced density matrices for subsystems of particles (antiparticles). Studying one-particle correlation functions, we establish a one-to-one correspondence between these functions and the reduced density matrices. It is shown that in the general case a presence of bosons (e.g. gluons) in an initial state increases the creation of the same kind of bosons. We discuss in detail the question (and its relation to the initial stage of quark-gluon plasma formation) whether the thermal form of one-particle distribution can appear even if the final state of the complete system is not a thermal equilibrium. In this respect, we discuss some cases when a pair creation by an electric-like field can mimic a one-particle thermal distribution. We apply our technics to some QFT problems in slowly varying electric-like backgrounds: electric, SU(3) chromoelectric, and metric. In particular, we study the time and temperature behavior of mean numbers of created particles when the effects of switching on and off are negligible and the particle creation in a slowly varying electric external field at high temperatures.

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1 Introduction

The effect of particle creation from vacuum by an external background (vacuum instability in external fields) ranks among the most intriguing nonlinear phenomena in quantum theory. Its theoretical consideration must be nonperturbative and its experimental observation would verify the validity of a theory in the domain of superstrong fields. The study of this effect began in connection with the so-called Klein [1] paradox, and was continued by Schwinger [2], who calculated the vacuum-to-vacuum transition probability in a constant electric field. A complete study of particle creation from vacuum by a constant electric field is represented in [3, 4]. It should be mentioned that the effect can actually be observed as soon as the external field strength approaches the characteristic value (critical field) $E_c = m^2c^3/|e|\hbar \simeq 1,3\cdot 10^{16}\ V/cm$. Although a real possibility of creating such fields under laboratory conditions does not exist at present, e^+e^--pair production by a slowly varying external electric field from vacuum is probably relevant to phenomenology with the advent of new laser technology which may access the truly strong-field domain. It is widely discussed nowadays [5] at SLAC and TESLA X-ray laser facilities. Such strong fields may be relevant in astrophysics, where characteristic values of electromagnetic fields and gravitational fields near black holes are enormous. The electric field near a cosmic string can become extremely strong [6]. In this respect, one has to mention that the Coulomb field of superheavy

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nuclei may create electron-positron pairs, see [7]. Aside from the pure QED problems, there are problems in QFT where the vacuum instability in various external backgrounds play an important role, for example, phase transitions in non-Abelian theories, the problem of boundary conditions or the influence of topology on the vacuum, the problem of a consistent vacuum construction in QCD and GUT, multiple particle creation in the context of heavy-ion collisions, and so on. The particle creation by background metrics are important in black hole physics [8, 9] and in the study of early Universe dynamics [10]. Recently, it has also been recognized that the presence of background electric field must be taken into account in string theory constructions, see, e.g., [11] and references therein.

Considerable attention has been recently focused on a non-perturbative parton production from vacuum by a classical chromoelectric field of SU(3) [12, 13] and SU(2) [14] in the framework of a modern version of the known chromoelectric flux tube model [15], the latter being an effective model for the confinement of quarks in QCD (previously the pair creation by a constant field has been calculated for SU(2) in [16] and for SU(3) in [17]). The model ensures a very good description of the phenomenology of hadron jets in high-energy $e^{+}-e^{-}$ and $p-\bar{p}$ collision experiments (for further development of the basic model and phenomenological applications, see, e.g., review [18]). Probably, this model describes reasonably well the initial stage of quark-gluon plasma formation (in particular, the transversal spectrum of produced soft partons). Such a state may be produced at high-energy large-hadron colliders such as RHIC (Au-Au collisions at $\sqrt{s}=200$ GeV) [19] and LHC (Pb-Pb collisions at $\sqrt{s}=5,5$ TeV) [20]. At present, this initial stage is related to an effective theory, the color glass condensate [21] (see also the review papers [22]), which is the coherent limit of the quark-gluon plasma at high energies. In such a picture, after a nuclei collision, a strong classical chromo-electric-magnetic field is created due to relatively slow fluctuations of the color density. Such a field is sufficiently uniform in the direction that is transversal to the beam direction and has a longitudinal chromoelectric component [14, 23]. This component is much more intensive than the transversal component, see [14]. Thus, the color glass condensate picture provides strong arguments in favor of the chromoelectric flux tube model and allows one to calculate configurations of the field in a tube. In particular, such a physical picture allows one to accept a quasiconstant chromoelectric field as a good approximation at the above-mentioned initial stage. One ought to say that experimental data of a heavy-ion collisions that exist at present can be interpreted as a quantum parton production by an external chromoelectric field both from vacuum and from many-particle states.

There exists a considerable interest to particle creation at finite temperatures and a finite particle density which is basically motivated by a heavy-ion collisions, cosmological QCD phase transitions and dark matter formation. For example, thermally influenced pair production in a constant electric field has been searched for at the one-loop level [24, 25, 26, 27].

All the above calculations were made in the framework of the theory of a quantized field placed in an external background. A consistent formulation of complete QED (interacting quantum electromagnetic and matter fields in particle creation backgrounds) with unstable vacuum that treats interaction with an external backgrounds nonperturbatively was elaborated in [28]. Possible generalizations of the formalism to an external gravitational field and non-Abelian gauge fields were presented in [29] and [30] (see also [31]), respectively. An attempt to extend this technics to the thermal case was given in [32]. Calculating particle creation by black-hole metric, Hawking has discovered that the density matrix of created particles at the spatial infinity has a thermal character. Is such a character related to the particle creation mechanism in general or to the gravitational origin of the background? A way to answer this question is to elaborate an adequate technics which would allow one to include arbitrary mixed initial states, in particular thermal initial states, in the corresponding particle creation formalism [28].

In this article, we present a development of the particle creation formalism [28] that could answer some of the above questions. We consider the time evolution of a quantized field (bosonic or fermionic) in backgrounds that violate the stability of vacuum. Our aim is to study the exact form of the final quantum state (the density operator at a final instant of time) that has emerged from a given arbitrary initial state (from a given arbitrary density operator at the initial time instant) in the course of evolution. The article is organized as follows. Section 2 has an original but rather technical character. Here, we derive exact expressions for density operators (more appropriate for the generating density operator) by applying a path integration method. Some necessary relevant formulas are placed in Appendix. In section 3, having an exact expression for the generating density operator, we derive reduced density operators for subsystems of particles and antiparticles. We introduce and calculate one-particle correlation functions and establish a one-to-one correspondence between these functions and the reduced density operators. In particular, this allows one to restore the reduced density operator of a complete system from one-particle distributions (of course, this is possible only in the model under consideration, being a quadratic theory). It is shown that in the general case a presence of bosons (e.g. gluons) in an initial state increases the creation of the same kind of bosons. We discuss in detail the question (and its relation to the initial stage of quark-gluon plasma formation) whether the thermal form of one-particle distribution can appear even if the final state of the complete system is not a thermal equilibrium. In section 4, we discuss the obtained expressions for density operators and one-particle distributions in electric-like backgrounds: electric, SU(3) chromoelectric, and metric. In particular, we analyze density operators and one-particle distributions in the so-called T-constant electric background (a field exists during a finite period of time T) and demonstrate how such a problem is related to particle creation in a gravitational field (Hawking's effect). We present some examples when a pair creation by an electric-like field can mimic a one-particle thermal distribution. Then, we analyze the time and temperature behavior of particle creation when effects of switching on and off are negligible. In particular, we show that at high temperatures the production rate is non-trivially time-dependent. This result has to be taken into account at high temperatures.

2 Density operator in pair-creating backgrounds

We consider a quantum field $\psi(x)$ in an external background. The quantum field can be scalar, spinor, etc. field, and the background can be a classical external electromagnetic, Yang-Mills, or gravitational field. In the general case, the background is intense, time-dependent, and violates the vacuum stability. Such a background must be treated nonperturbatively. We are going to follow the formulation proposed in [28].

2.1 Some relevant relations

It is supposed that there is a set of creation and annihilation operators $a_n^\dagger(t_{in})$, $a_n(t_{in})$ of particles $a_n^\dagger(t_{in})$, $a_n(t_{in})$, and antiparticles $b_n^\dagger(t_{in})$, $b_n(t_{in})$ respectively at an initial time instant t_{in} ($t_{in} \to -\infty$), and a set of creation and annihilation operators $a_n^\dagger(t_{out})$, $a_n(t_{out})$, of particles, and $b_n^\dagger(t_{out})$, $b_n(t_{out})$ of antiparticles at an final time instant t_{out} ($t_{in} \to \infty$). By n we denote a complete set of possible quantum numbers. The total Hamiltonian of the quantized field under consideration is diagonalized (and has a canonical form) in terms of the first set at the initial time instant, and is diagonalized (and has a canonical form) in terms of the second set at the final time instant. Nonzero commutators[1] are

$$\begin{aligned}&[a_n(t_{in}), a_m^\dagger(t_{in})\} = [a_n(t_{out}), a_m^\dagger(t_{out})\}\\&= [b_n(t_{in}), b_m^\dagger(t_{in})\} = [b_n(t_{out}), b_m^\dagger(t_{out})\} = \delta_{nm}\,.\end{aligned} \tag{1}$$

Vacuum states $|0, t_{in}\rangle$ at t_{in} and $|0, t_{out}\rangle$ at t_{out} are defines as usual,

$$a(t_{in})|0, t_{in}\rangle = b(t_{in})|0, t_{in}\rangle = 0\,,\; a(t_{out})|0, t_{out}\rangle = b(t_{out})|0, t_{out}\rangle = 0\,.$$

To pass to the Heisenberg picture, we introduce finite-time evolution operators $\Omega_{(\pm)}$,

$$\Omega_{(+)} = U(t, t_{in})\;,\; \Omega_{(-)} = U(t, t_{out})\;,\; U(t_{out}, t_{in}) = \Omega_{(-)}^\dagger \Omega_{(+)}\,,$$

where $U(t, t')$ is an unitary evolution operator of the system. Then, we define a set of creation and annihilation operators $a_n^\dagger(in)$, $a_n(in)$ of *in*-particles, similar operators $b_n^\dagger(in)$, $b_n(in)$ of *in*-antiparticles, the corresponding *in*-vacuum $|0, in\rangle$, and a set of creation and annihilation operators $a_n^\dagger$, a_n, of *out*-particles and similar operators $b_n^\dagger$, b_n of *out*-antiparticles, and corresponding *out*-vacuum $|0\rangle$,

$$\begin{aligned}&\left(a^\dagger(in), a(in), b^\dagger(in), b(in)\right) = \Omega_{(+)}\left(a^\dagger(t_{in}), a(t_{in}), b^\dagger(t_{in}), b(t_{in})\right)\Omega_{(+)}^\dagger\,,\\&\left(a^\dagger, a, b^\dagger, b\right) = \Omega_{(-)}\left(a^\dagger(t_{out}), a(t_{out}), b^\dagger(t_{out}), b(t_{out})\right)\Omega_{(-)}^\dagger\,,\\&|0, in\rangle = \Omega_{(+)}|0, t_{in}\rangle\,,\; |0\rangle = \Omega_{(-)}|0, t_{out}\rangle\,.\end{aligned} \tag{2}$$

The *in*- and *out*-operators obey the canonical commutation relations (1).

[1]It is the usual commutator in the Bose case and the anticommutator in the Fermi case.

All the information about processes of particle creation, annihilation, and scattering is contained in elementary probability amplitudes,

$$\begin{aligned}
&w\left(+|+\right)_{mn}=c_v^{-1}\langle 0\left|a_m a_n^{\dagger}(in)\right|0,in\rangle,\\
&w\left(-|-\right)_{nm}=c_v^{-1}\langle 0\left|b_m b_n^{\dagger}(in)\right|0,in\rangle,\\
&w\left(0|-+\right)_{nm}=c_v^{-1}\langle 0\left|b_n^{\dagger}(in)a_m^{\dagger}(in)\right|0,in\rangle,\\
&w\left(+-|0\right)_{mn}=c_v^{-1}\langle 0\left|a_m b_n\right|0,in\rangle,
\end{aligned}\tag{3}$$

where c_v is the vacuum-to-vacuum transition amplitude

$$c_v=\langle 0|0,in\rangle\,. \tag{4}$$

The amplitudes (3) can be calculated via appropriate sets of solutions of corresponding relativistic wave equation (Klein-Gordon, Dirac, linearized Yang-Mills), see [28, 30].

The sets of *in* and *out*-operators are related to each other by a linear canonical transformation (it is called sometimes the Bogolubov transformation). As was demonstrated, in the general case such a relation has the form (see [28])

$$\begin{aligned}
&V\left(a^{\dagger},a,b^{\dagger},b\right)V^{\dagger}=\left(a^{\dagger}(in),a(in),b^{\dagger}(in),b(in)\right)\,,\\
&|0,in\rangle=V|0\rangle\quad(c_v=\langle 0|V|0\rangle)\,,
\end{aligned}\tag{5}$$

where an unitary operator V reads

$$V=v_4v_3v_2v_1\,, \tag{6}$$

and[2]

$$\begin{aligned}
&v_1=\exp\left\{-bw\left(0|-+\right)a\right\}\,,\ v_2=\exp\left\{a^{\dagger}\ln w\left(+|+\right)a\right\}\,,\\
&v_3=\exp\left\{-\kappa b\ln w\left(-|-\right)b^{\dagger}\right\}\,,\ v_4=\exp\left\{-\kappa a^{\dagger}w\left(+-|0\right)b^{\dagger}\right\}\,,\\
&\kappa=\left\{\begin{array}{l}1\ \ \text{Fermi particles}\\-1\ \ \text{Bose particles}\end{array}\right.
\end{aligned}\tag{7}$$

Using this explicit expression for V, one can easily find

$$c_v=\langle 0|V|0\rangle=\exp\left\{-\kappa\operatorname{tr}\ln w\left(-|-\right)\right\}\,. \tag{8}$$

Let $\hat{\rho}\left(t_{in}\right)=\rho\left(a^{\dagger}(t_{in}),a(t_{in}),b^{\dagger}(t_{in}),b(t_{in})\right)$, $\operatorname{tr}\hat{\rho}\left(t_{in}\right)=1$, be a density operator of the system under consideration at the initial time instant. Evolving in time, this density operator becomes $\hat{\rho}\left(t_{out}\right)$ at the final time instant,

$$\hat{\rho}\left(t_{out}\right)=U\left(t_{out},t_{in}\right)\hat{\rho}\left(t_{in}\right)U^{\dagger}\left(t_{out},t_{in}\right)\,. \tag{9}$$

The density operator $\check{\rho}$ of the system under consideration in the Heisenberg representation is defined as

$$\begin{aligned}
\check{\rho}&=\Omega_{(+)}\hat{\rho}\left(t_{in}\right)\Omega_{(+)}^{\dagger}=\rho\left(a^{\dagger}(in),a(in),b^{\dagger}(in),b(in)\right)\,,\\
\check{\rho}&=\Omega_{(-)}\hat{\rho}\left(t_{out}\right)\Omega_{(-)}^{\dagger}\,,\ \operatorname{tr}\check{\rho}=1\,.
\end{aligned}\tag{10}$$

Suppose a physical quantity is given by an operator $\hat{F}\left(t_{out}\right)$ at the final time instant as

$$\hat{F}\left(t_{out}\right)=F\left(a^{\dagger}(t_{out}),a(t_{out}),b^{\dagger}(t_{out}),b(t_{out})\right)\,. \tag{11}$$

Then its mean value at the final time instant is

$$\langle F\rangle=\operatorname{tr}\left[\hat{F}\left(t_{out}\right)\hat{\rho}\left(t_{out}\right)\right]=\operatorname{tr}\left[\check{F}\check{\rho}\right]\,, \tag{12}$$

where

$$\check{F}=\Omega_{(-)}\hat{F}\left(t_{out}\right)\Omega_{(-)}^{\dagger}=F\left(a^{\dagger},a,b^{\dagger},b\right) \tag{13}$$

is an operator of the physical quantity in the Heisenberg representation[3].

[2] We use condenced notations here and in what follows. For example,

$$bw\left(0|-+\right)a=\sum_{n,m}b_n w_{nm}\left(0|-+\right)a_m\,.$$

[3] All operators in the Heisenberg representation are denoted by the turned over hat in what follows, e.g. $\check{A}$.

2.2 Generating density operator

We introduce the following generating operator $\check{R}(J)$:

$$\check{R}(J)=\frac{1}{Z}\underline{\check{R}}(J)\ ,\quad \mathrm{tr}\check{R}(J)=1\,, \tag{14}$$
$$\underline{\check{R}}(J)=\mathcal{N}_{in}\exp\left[a^{\dagger}(in)\left(\mathbb{J}^{(+)}-1\right)a(in)+b^{\dagger}(in)\left(\mathbb{J}^{(-)}-1\right)b(in)\right]\,,$$

where Grassmann-even variables $J=\left(J_n^{(\zeta)}\right)$ are sources, $\mathbb{J}_{mn}^{(\zeta)}=\delta_{mn}J_n^{(\zeta)}$, $\mathcal{N}_{in}$ is the sign of the normal form with respect to in-vacuum, and $Z=\mathrm{tr}\underline{\check{R}}(J)$ is a normalization factor (statistical sum). Here and in what follows, $\zeta=\pm$, being $(+)$ for particles and $(-)$ for antiparticles.

In order to fulfil the calculations it is effective to use a path integral representation. For the fermion case we use a path integral over anticommuting (Grassmann) variables which is understood as Berezin's integral [33] at $\kappa=1$,

$$:e^{-\kappa a^{\dagger}Ka}:=\det K^{\kappa}:\int\exp\left\{\kappa\lambda^{*}K^{-1}\lambda+a^{\dagger}\lambda+\lambda^{*}a\right\}\Pi d\lambda^{*}d\lambda:, \tag{15}$$

where $a^{\dagger}$, a are some creation and annihilation operators and $:\ldots:$ realizes the normal form of the operators $a^{\dagger}$, a. All the operators $a^{\dagger}$ and a can be considered as Grassmann-odd variables under the normal form sign, therefore, we can calculate the complete path integral (15) as a Gaussian one over Grassmann-odd variables. For the boson case we use a path integral (15) over commuting variables at $\kappa=-1$. In this case, we can consider all the operators $a^{\dagger}$ and a as bosonic (ordinary) variables under the normal form sign, such that path integral in (15) is an usual Gaussian path integral, where $\lambda^{*}K^{-1}\lambda>0$.

Let us note that the trace of a normal product of creation and annihilation operators can be calculated by using the path integral representation, as presented in (105) from the Appendix. For example, calculating Z we obtain

$$Z=\exp\left\{\kappa\sum_{n}\left[\ln\left(1+\kappa\mathbb{J}^{(+)}\right)\right]_{nn}+\kappa\sum_{m}\left[\ln\left(1+\kappa\mathbb{J}^{(-)}\right)\right]_{mm}\right\}. \tag{16}$$

Knowing the generating operator (14), we can obtain different density operators (in the Heisenberg representation), corresponding to different initial states of the system. We represent some examples below:

a) Setting all $J=0$, we obtain a density operator $\check{\rho}_v$ of the system that was in a pure vacuum state at the initial time instant,

$$\check{\rho}_v=\check{R}(0)\,.$$

Indeed, using relation (103) from the Appendix, we have

$$\check{\rho}_v=\mathcal{N}_{in}\exp\left\{-\left[a^{\dagger}(in)a(in)+b^{\dagger}(in)b(in)\right]\right\}=|0,in\rangle\langle 0,in|\,. \tag{17}$$

In addition, we define the following generating functional of moments,

$$\Phi^{v}(J)=\langle 0,in|\exp\left[a^{\dagger}\mathbb{J}^{(+)}a+b^{\dagger}\mathbb{J}^{(-)}b\right]|0,in\rangle=\mathrm{tr}\,\check{\phi}(J)\,,$$
$$\check{\phi}(J)=\exp\left[a^{\dagger}\mathbb{J}^{(+)}a+b^{\dagger}\mathbb{J}^{(-)}b\right]\check{\rho}_v\,, \tag{18}$$

which is useful to study a final state evolved from a vacuum at the initial time instant.

b) A density operator $\check{\rho}_{\{m\}_M;\{n\}_N}$ of the system that was in a pure state with M particles and N antiparticles (with the quantum numbers $\{m_1,\ldots,m_M\}=\{m\}_M$ and $\{n_1,\ldots,n_N\}=\{n\}_N$ respectively) at the initial time instant can be obtained from the generating operator $\underline{\check{R}}(J)$ as follows:

$$\check{\rho}_{\{m\}_M;\{n\}_N}=\left.\frac{\partial^{M+N}\underline{\check{R}}(J)}{\partial(J_{m_1}^{(+)}\ldots J_{m_M}^{(+)}J_{n_1}^{(-)}\ldots J_{n_N}^{(-)})}\right|_{J=0}=|\Psi_{\{m\}_M;\{n\}_N}(in)\rangle\langle\Psi_{\{m\}_M;\{n\}_N}(in)|\,, \tag{19}$$

where

$$|\Psi_{\{m\}_M;\{n\}_N}(in)\rangle=\prod_{i=1}^{M}a_{m_i}^{\dagger}(in)\prod_{j=1}^{N}b_{n_j}^{\dagger}(in)|0,in\rangle\,,$$
$$\langle\Psi_{\{m\}_M;\{n\}_N}(in)|=\langle 0,in|\prod_{j=1}^{N}b_{n_j}(in)\prod_{i=1}^{M}a_{m_i}(in)\,.$$

c) Let us set

$$J_n^{(\zeta)} = e^{-E_n^{(\zeta)}}, \; E_n^{(\zeta)} = \beta\left(\varepsilon_n^{(\zeta)} - \mu^{(\zeta)}\right), \; \beta^{-1} = \Theta, \tag{20}$$

where $\varepsilon_n^{(\zeta)}$ are energies of particles or antiparticles with quantum numbers n; $\mu^{(\zeta)}$ are the corresponding chemical potentials, and Θ is the absolute temperature. One can see that with such choice of sources, the generating operator (14) becomes the density operator $\check{\rho}_\beta$ of the system that was in thermal equilibrium at the initial time instant. Using relation (100) from the Appendix, we obtain an explicit expression for $\check{\rho}_\beta$,

$$\check{\rho}_\beta = \check{R}\left(e^{-E_n^{(\zeta)}}\right) = \frac{1}{Z}\exp\left\{-\left[a^\dagger(in)E^{(+)}a(in) + b^\dagger(in)E^{(-)}b(in)\right]\right\},$$

$$Z = \exp\left\{\kappa\sum_n \ln\left(1+\kappa e^{-E_n^{(+)}}\right) + \kappa\sum_m \ln\left(1+\kappa e^{-E_n^{(-)}}\right)\right\},$$

or

$$\check{\rho}_\beta = Z^{-1}\exp\left\{-\beta\left[\check{H} - \sum_{\zeta=\pm}\mu^{(\zeta)}\check{N}^{(\zeta)}\right]\right\}, \tag{21}$$

where $\check{H}$ is the system Hamiltonian (written in terms of *in*-operators), $\check{N}^{(\zeta)}$ are operators of *in*-particle or *in*-antiparticle numbers,

$$\check{H} = a^\dagger(in)\varepsilon^{(+)}a(in) + b^\dagger(in)\varepsilon^{(-)}b(in),$$
$$\check{N}^{(+)} = a^\dagger(in)a(in), \; \check{N}^{(-)} = b^\dagger(in)b(in),$$

and the matrices $E^{(\zeta)}$ and $\varepsilon^{(\zeta)}$ are defines as: $E_{mn}^{(\zeta)} = \delta_{mn}E_n^{(\zeta)}$ and $\varepsilon_{mn}^{(\zeta)} = \delta_{mn}\varepsilon_n^{(\zeta)}$.

One can see that the problem of calculating mean values of an operator $\hat{F}(t_{out})$ in a system state in the final time instant is related to the problem of calculating the quantity tr $\left[\check{F}\underline{\check{R}}(J)\right]$, where $\check{F}$ is a Heisenberg operator corresponding to $\hat{F}(t_{out})$. Such a quantity can be represented as follows:

$$\mathrm{tr}\left[\check{F}\underline{\check{R}}(J)\right] = \sum_{M,N=0}^{\infty}\sum_{\{m\}\{n\}}\frac{1}{M!N!}\langle\Psi(\{m\}_M,\{n\}_N)|\check{F}(J)\underline{\check{R}}|\Psi(\{m\}_M,\{n\}_N)\rangle,$$
$$|\Psi(\{m\}_M,\{n\}_N)\rangle = a_{m_1}^\dagger\ldots a_{m_M}^\dagger b_{n_1}^\dagger\ldots b_{n_N}^\dagger|0\rangle,$$
$$\langle\Psi(\{m\}_M,\{n\}_N)| = \langle 0|\, b_{n_N}\ldots b_{n_1}a_{m_M}\ldots a_{m_1} \tag{22}$$

Calculating tr $\left[\check{F}\underline{\check{R}}(J)\right]$ according to (22), it is convenient to have an expression for the operator $\underline{\check{R}}(J)$ in terms of *out*-operators. One can easily see that such an expression has the form

$$\underline{\check{R}}(J) = VU(J)V^\dagger, \; U(J) =: \exp\left[a^\dagger\left(\mathbb{J}^{(+)}-1\right)a + b^\dagger\left(\mathbb{J}^{(-)}-1\right)b\right]:, \tag{23}$$

where $:\cdots:$ is the sign of the normal form with respect to *out*-vacuum, and the operator V is defined by (6). A normal form of the operator $\underline{\check{R}}(J)$ with respect to *out*-vacuum is calculated below.

2.3 Normal form of the generating operator

First, we rewrite the operator expression (23) as follows

$$\underline{\check{R}}(J) = v_4 v_3 v_2 \tilde{Y}(J)\, v_2^\dagger v_3^\dagger v_4^\dagger, \; \tilde{Y} = v_1 U(J)\, v_1^\dagger, \tag{24}$$

where the operators ν_i, $i = 1,...,4$, are given by (7). Using formula (99) from the Appendix, we represent the operator $\tilde{Y}(J)$ in the form

$$\tilde{Y}(J) = Y(J)\,U(J), \; Y(J) = \exp(-bBa)\exp\left(-a^\dagger A(J)\,b^\dagger\right),$$
$$A(J) = \mathbb{J}^{(+)}B^\dagger\mathbb{J}^{(-)}, \; B = w(0|-+). \tag{25}$$

Both operator exponents in the expression for $Y(J)$ can be written in terms of Gaussian path integrals.

Consider first the fermi-particle case. In such a case, we can treat anticommuting operators a and b (or a^+ and b^+) as Grassmann-odd variables. Then according to the representation (15) at $\kappa = 1$ we have

$$Y = \det A \det B \int \exp\left(\tilde{\lambda}^* B^{-1}\tilde{\lambda} + \lambda^* A^{-1}\lambda\right) \Phi \Pi d\tilde{\lambda}^* d\tilde{\lambda} d\lambda^* d\lambda,$$
$$\Phi = \exp\left(b\tilde{\lambda} + \tilde{\lambda}^* a\right) \exp\left(a^\dagger\lambda + \lambda^* b^\dagger\right). \tag{26}$$

By the help of the relation (101) from the Appendix, we represent the operator Φ in the normal form,

$$\Phi =: \exp\left(a^\dagger\lambda + \lambda^* b^\dagger + b\tilde{\lambda} + \tilde{\lambda}^* a + \tilde{\lambda}^*\lambda + \tilde{\lambda}\lambda^*\right) : .$$

Then, using the formula (15) we can calculate the complete path integral (26). In the Bose case, we can consider all the operators $a^\dagger$, $b^\dagger$, a, and b as bosonic (ordinary) variables under the normal form sign, such that the operator $Y(J)$ can be represented as an usual Gaussian path integral by application of the representation (15) at $\kappa = -1$. Calculating such Gaussian integrals, we obtain the normal form of the operator Y,

$$Y = \det(1 + \kappa AB)^\kappa : \exp\left\{-a^\dagger A_{++} a - b^\dagger A_{--} b - a^\dagger A_{+-} b^\dagger - b A_{-+} a\right\} :,$$
$$A_{++} = \kappa AB (1 + \kappa AB)^{-1},\; A_{--}^T = \kappa BA (1 + \kappa BA)^{-1},$$
$$A_{+-} = (1 + \kappa AB)^{-1} A,\; A_{-+} = B (1 + \kappa AB)^{-1}. \tag{27}$$

By the help of relation (102) from Appendix, we represent now the operator $\tilde{Y}$ in the normal form,

$$\tilde{Y} = \det(1 + \kappa AB)^\kappa : \exp\left\{-a^\dagger \widetilde{A}_{++} a - b^\dagger \widetilde{A}_{--} b - a^\dagger \widetilde{A}_{+-} b^\dagger - b \widetilde{A}_{-+} a\right\} :,$$
$$\widetilde{A}_{++} = 1 - (1 - A_{++})\mathbb{J}^{(+)},\; \widetilde{A}_{--} = 1 - (1 - A_{--})\mathbb{J}^{(-)},$$
$$\widetilde{A}_{+-} = \widetilde{A}_{-+}^\dagger,\; \widetilde{A}_{-+} = \mathbb{J}^{(-)} A_{-+} \mathbb{J}^{(+)}. \tag{28}$$

Using relation (8), we rewrite the operator v_3 as

$$v_3 = \exp\left[-\kappa b \ln w(-|-) b^\dagger\right] = c_v \exp\left[b^\dagger \ln w(-|-)^T b\right].$$

Then using formulas (100) derived in the Appendix, we represent the operators $v_3 v_2$ and $v_2^\dagger v_3^\dagger$ from (24) in the normal forms as follows:

$$v_3 v_2 = c_v : \exp\left[b^\dagger\left(w(-|-)^T - 1\right) b\right] \exp\left[a^\dagger\left(w(+|+) - 1\right) a\right] :,$$
$$v_2^\dagger v_3^\dagger = c_v^* : \exp\left[a^\dagger\left(w(+|+)^\dagger - 1\right) a\right] \exp\left[b^\dagger\left(w(-|-)^{T\dagger} - 1\right) b\right] : .$$

Finally, applying relation (102) in tandem, we obtain the normal form of the operator $\underline{\breve{R}}(J)$,

$$\underline{\breve{R}}(J) = |c_v|^2 \det(1 + \kappa AB)^\kappa : \exp\left[-a^\dagger(1 - D_+) a - b^\dagger(1 - D_-) b - a^\dagger C^\dagger b^\dagger - b C a\right] :,$$
$$D_+ = w(+|+)(1 + \kappa AB)^{-1} \mathbb{J}^{(+)} w(+|+)^\dagger,$$
$$D_-^T = w(-|-)^\dagger \mathbb{J}^{(-)} (1 + \kappa BA)^{-1} w(-|-),$$
$$C = w(-|-)^\dagger \mathbb{J}^{(-)} B (1 + \kappa AB)^{-1} \mathbb{J}^{(+)} w(+|+)^\dagger + \kappa w(+-|0)^\dagger. \tag{29}$$

The representation (29) is useful since it allows one to calculate the trace (22) using a path integral techniques described in Appendix (see, eq. (105)).

As an example, let us consider again the density operator $\breve{\rho}_v$ defined by (17). Using (29), we represent this operator in terms of *out*-operators and in the normal form

$$\breve{\rho}_v = \underline{\breve{R}}(0) = |c_v|^2 : \exp\left[-a^\dagger a - b^\dagger b - \kappa a^\dagger w(+-|0) b^\dagger - \kappa b w(+-|0)^\dagger a\right] : . \tag{30}$$

In a similar manner the operator $\breve{\phi}(J)$ in the expression of generating functional (18) can be transformed to the normal form, which is

$$\breve{\phi}(J) = |c_v|^2 : \exp\left[-a^\dagger a - b^\dagger b - \kappa a^\dagger e^{\mathbb{J}^{(+)}} w(+-|0) e^{\mathbb{J}^{(-)}} b^\dagger - \kappa b w(+-|0)^\dagger a\right] :, \tag{31}$$

where the representation (30) for $\check{\rho}_v$ is used. Then using the path integral representation for traces (105) and applying in tandem the formula (102), we represent the generating functional of moments as

$$\Phi^v(J) = |c_v|^2 \exp\left\{\kappa \mathrm{tr} \ln \left[1 + \kappa w(+-|0)^\dagger e^{\mathbb{J}^{(+)}} w(+-|0) e^{\mathbb{J}^{(-)}}\right]\right\}. \quad (32)$$

3 Reduced density operators and correlation functions

3.1 Reduced density operators

In the general case, states of the system under consideration at the final time instant contain both particles and antiparticles due to the pair creation by the external field and the structure of the initial state. On the other hand, a very often we are interested in physical quantities $F_\pm$ that describe only particles (+) or antiparticle (−) at the final time instant. The corresponding operators $\check{F}_\pm$ are functions of either operators $a^\dagger$,a or $b^\dagger$,b,

$$\check{F}_+ = F_+(a^\dagger, a), \; \check{F}_- = F_-(b^\dagger, b). \quad (33)$$

Mean values of the operators $\check{F}_\pm$ and all the information about subsystems of particles and antiparticles, can be obtained from the so-called reduced density operators, which we are going to define below.

We present the basis vectors from (22) as follows

$$\begin{aligned} &|\Psi(\{m\}_M, \{n\}_N)\rangle = |\Psi_a(\{m\}_M)\rangle \otimes |\Psi_b(\{n\}_N)\rangle, \; |0\rangle = |0\rangle_a \otimes |0\rangle_b, \\ &|\Psi_a(\{m\}_M)\rangle = a^\dagger_{m_1} \ldots a^\dagger_{m_M}|0\rangle_a, \; |\Psi_b(\{n\}_N)\rangle = b^\dagger_{n_1} \ldots b^\dagger_{n_N}|0\rangle_b, \end{aligned} \quad (34)$$

where $|0\rangle_a$ and $|0\rangle_b$ are vacuum vectors of particle and antiparticle subsystems. The mean values of the operators $\check{F}_\pm$ are

$$\langle F_\pm \rangle = \mathrm{tr}_+ \mathrm{tr}_- \left(\check{F}_\pm \check{\rho}\right), \quad (35)$$

where $\check{\rho}$ is a density operator of a system and reduced traces $\mathrm{tr}_\pm$ of an operator $\check{A}$ are defined as

$$\begin{aligned} \mathrm{tr}_+ \check{A} &= \sum_{M=0}^{\infty} \sum_{\{m\}} (M!)^{-1} \langle \Psi_a(\{m\}_M) | \check{A} | \Psi_a(\{m\}_M) \rangle, \\ \mathrm{tr}_- \check{A} &= \sum_{M=0}^{\infty} \sum_{\{m\}} (M!)^{-1} \langle \Psi_b(\{m\}_M) | \check{A} | \Psi_b(\{m\}_M) \rangle. \end{aligned} \quad (36)$$

We define the reduced density operators (in the Heisenberg picture) $\check{\rho}_\pm$ of particle and antiparticle subsystems respectively as

$$\check{\rho}_\pm = \mathrm{tr}_\mp \check{\rho}. \quad (37)$$

Then mean values (35) can be calculated by the help of the reduced density operators $\check{\rho}_\pm$ as

$$\langle F_\pm \rangle = \mathrm{tr}_\pm \left(\check{F}_\pm \check{\rho}_\pm\right). \quad (38)$$

Even if an initial state of the system is a pure state, the reduced density operators $\check{\rho}_\pm$ describes mixed states. In some physical problems the use of the reduced density operators is inevitable. For example, considering the particle creation by a gravitation field of a black hole, we have only reduced operator of particles created outside the black hole, since we do not have any information about particles behind the horizon, [8, 9].

In the similar manner, we introduce reduced generating operators $\check{R}_\pm(J)$ as follows

$$\check{R}_\pm(J) = \mathrm{tr}_\mp \check{R}(J).$$

Using a path integral representation for traces (105), the representation (29), as well as applying in tandem (102), we get[4]

$$\begin{aligned} &\check{R}_+(J) = Z_+^{-1} : \exp\left\{-a^\dagger (1 - K_+(J)) a\right\} :, \\ &\check{R}_-(J) = Z_-^{-1} : \exp\left\{-b^\dagger (1 - K_-(J)) b\right\} :, \\ &K_\pm(J) = D_\pm + C^\dagger \left(1 + \kappa D_\mp^T\right)^{-\kappa} C, \\ &Z_\pm^{-1}(J) = Z^{-1} |c_v|^2 \det(1 + \kappa AB)^\kappa \det(1 + \kappa D_\mp)^\kappa. \end{aligned} \quad (39)$$

[4]One ought to say that symbols of the normal form of the operator $\hat{R}_\zeta$ were represented in [24] via some path integrals. The explicit form of the operator was written there for $J_n^{(\zeta)} = e^{-E_n^{(\zeta)}}$. It contains, unfortunately, essential misprints.

The reduced generating operators $\check{R}_\pm(J)$ allow one to obtain the reduced density operators $\check{\rho}_\pm$ for different initial states of the system. Consider below some examples:

a) Selecting all $J=0$ in (39), we obtain reduced density operators $\check{\rho}_{v\pm} = \check{R}_\pm(0)$ of the system that was in a pure vacuum state at the initial time instant. Explicit expressions for $\check{R}_\pm(0)$ follow from (39) with account taken of

$$K_\pm(0) = w(+-|0)\, w(+-|0)^\dagger\ ,\ Z_\pm^{-1}(0) = |c_v|^2\ . \tag{40}$$

The same result was obtained in [24, 34] by a straightforward calculation.

b) Reduced density operators $\check{\rho}_{0;n\pm}$ and $\check{\rho}_{m;0\pm}$ of the system that was in a pure state with one particles or one antiparticles respectively at the initial time instant can be obtained from the generating operator $\check{\underline{R}}_\pm(J) = Z\check{R}_\pm(J)$ as follows:

$$\begin{aligned}
\check{\rho}_{m;0+} &= \left.\frac{\partial \check{\underline{R}}_+(J)}{\partial J_m^{(+)}}\right|_{J=0} = \left[a^\dagger w(+|+)\right]_m \check{\rho}_{v+}\left[w(+|+)^\dagger a\right]_m\ ,\\
\check{\rho}_{0;m-} &= \left.\frac{\partial \check{\underline{R}}_-(J)}{\partial J_m^{(-)}}\right|_{J=0} = \left[w(-|-)\,b^\dagger\right]_m \check{\rho}_{v-}\left[bw(-|-)^\dagger\right]_m\ ,\\
\check{\rho}_{0;m+} &= \left.\frac{\partial \check{\underline{R}}_+(J)}{\partial J_m^{(-)}}\right|_{J=0} = \check{\rho}_{v+}\left[w(-|-)\,w(-|-)^\dagger\right]_{mm}\\
&- \left[a^\dagger w(+-|0)\,w(-|-)^\dagger\right]_m \check{\rho}_{v+}\left[w(-|-)\,w(+-|0)^\dagger a\right]_m\ ,\\
\check{\rho}_{m;0-} &= \left.\frac{\partial \check{\underline{R}}_-(J)}{\partial J_m^{(+)}}\right|_{J=0} = \check{\rho}_{v-}\left[w(+|+)^\dagger w(+|+)\right]_{mm}\\
&- \left[b^\dagger w(+-|0)\,w(+|+)^*\right]_m \check{\rho}_{v-}\left[w(+|+)^T w(+-|0)^\dagger b\right]_m\ .
\end{aligned}$$

c) Let us set sources in (39) as in (20). One can see that with such choice of sources, the reduced generating operators (14) become the reduced density operators $\check{\rho}_{\beta\pm}$ of the system that was in thermal equilibrium at the initial time instant.

3.2 One-particle correlation functions

Let us consider the following generating functions

$$\begin{aligned}
\mathbb{N}_{nm}^{(+)} &= \mathrm{tr}\left(a_n^\dagger a_m \check{R}\right) = \mathrm{tr}_+\left(a_n^\dagger a_m \check{R}_+\right)\ ,\\
\mathbb{N}_{nm}^{(-)} &= \mathrm{tr}\left(b_n^\dagger b_m \check{R}\right) = \mathrm{tr}_-\left(b_n^\dagger b_m \check{R}_+\right)\ .
\end{aligned} \tag{41}$$

They generate one-particle correlation functions for different initial states of the system. Setting sources (taking corresponding derivatives if necessary) in (41) as was demonstrated in sec.2, we chose needed initial states. Diagonal elements $\mathbb{N}_{mm}^{(\zeta)}$ are generating functionals for mean numbers $N_m^{(\zeta)}$ of particles/antiparticles with quantum numbers m at the final time instant (further differential mean numbers). In what follows, we call the quantities (41) simply correlation functions.

The correlation functions $\mathbb{N}_{nm}^{(\zeta)}$ can be expressed via matrices K_ζ (39) and vice-versa as follows

$$\mathbb{N}^{(\zeta)} = \left(\frac{K_\zeta}{1+\kappa K_\zeta}\right)^T\ ,\ K_\zeta = \frac{\mathbb{N}^{(\zeta)T}}{1-\kappa\mathbb{N}^{(\zeta)T}}\ . \tag{42}$$

We note that the quantities K_ζ are functions of elementary probability amplitudes (3).

Relations (42) can be proved in the following way: First, using commutation relations (1), we represent (41) as traces of operators in the normal form,

$$\mathbb{N}_{nm}^{(+)} = \mathrm{tr}_+\left[a_n^\dagger \check{R}_+ (K_+ a)_m\right]\ ,\ \mathbb{N}_{nm}^{(-)} = \mathrm{tr}_-\left[b_n^\dagger \check{R}_- (K_- b)_m\right]\ . \tag{43}$$

The quantities $\mathbb{N}_{nm}^{(\zeta)}$ can be obtained from generating functions $\mathcal{Z}_\zeta(\bar{\jmath}, \jmath)$ as follows:

$$\mathbb{N}_{nm}^{(\zeta)} = \left.\frac{\partial^2 \mathcal{Z}_\zeta(\bar{\jmath},\jmath)}{\partial\bar{\jmath}_n \partial\jmath_m}\right|_{\bar{\jmath}=\jmath=0}\ , \tag{44}$$

where

$$\mathcal{Z}_+(\bar{j},j) = Z_+^{-1}\mathrm{tr}_+ : \exp\left\{-a^\dagger\left[1-\mathbb{I}K_+\right]a\right\}:,$$
$$\mathcal{Z}_-(\bar{j},j) = Z_-^{-1}\mathrm{tr}_- : \exp\left\{-b^\dagger\left[1-\mathbb{I}K_-\right]b\right\}:,$$
$$\mathcal{Z}_\zeta(0,0) = 1\,,\ \mathbb{I}_{mn} = \delta_{mn} + \bar{j}_m j_n\,,\ \zeta = \pm\,, \quad (45)$$

and $\bar{j}$ and j are some new sources. Traces in (45) can be calculated using formula (105) from the Appendix. Thus, we get

$$\mathcal{Z}_\zeta(\bar{j},j) = Z_\zeta^{-1}\exp\left\{\kappa\sum_n\left[\ln\left(1+\kappa\mathbb{I}K_\zeta\right)\right]_{nn}\right\}. \quad (46)$$

Then the relations (42) follow from (44) and (46).

Normalization conditions and second relation (42) imply that the quantities Z_ζ can be expressed in terms of $\mathbb{N}^{(\zeta)}$ as

$$Z_\zeta = \exp\left\{\kappa\sum_n\left[\ln\left(1+\kappa K_\zeta\right)\right]_{nn}\right\} = \exp\left\{-\kappa\sum_n\left[\ln\left(1-\kappa\mathbb{N}^{(\zeta)T}\right)\right]_{nn}\right\}. \quad (47)$$

Now we are going to relate the quantities $\mathbb{N}^{(\zeta)}_{nm}$ with correlation functions $\mathbb{N}^{(\zeta)}_{nm}(in)$ of *in*-operators,

$$\mathbb{N}^{(+)}_{nm}(in) = \mathrm{tr}\left[a_n^\dagger(in)a_m(in)\check{R}\right]\,,\ \mathbb{N}^{(-)}_{nm}(in) = \mathrm{tr}\left[b_n^\dagger(in)b_m(in)\check{R}\right]. \quad (48)$$

Using the representation (14) for $\check{R}$, one can easily see that

$$\mathbb{N}^{(\zeta)}_{nm}(in) = \delta_{nm}N_m^{(\zeta)}(in),\quad N_m^{(\zeta)}(in) = \frac{J_m^{(\zeta)}}{1+\kappa J_m^{(\zeta)}}\,, \quad (49)$$

where $N_m^{(\zeta)}(in)$ are differential mean numbers (generating functions for differential mean numbers). Indeed, let us take expressions (41) for $\mathbb{N}^{(\zeta)}$ via traces in the complete Fock space. These traces can be written in the *in*-basis $|\Psi(\{m\}_M,\{n\}_N;in)\rangle = V|\Psi(\{m\}_M,\{n\}_N)\rangle$. Using canonical transformation (5), we express the operators $a^\dagger, a, b^\dagger, b$ via the operators $a^\dagger(in), a(in), b^\dagger(in), b(in)$ and calculate the traces explicitly. Thus, we obtain

$$\mathbb{N}^{(+)T} = G\left({}^+|_+\right)\mathbb{N}^{(+)}(in)G\left({}_+|^+\right) + G\left({}^+|_-\right)\left[1-\kappa\mathbb{N}^{(-)}(in)\right]G\left({}_-|^+\right),$$
$$\mathbb{N}^{(-)} = G\left({}^-|_-\right)\mathbb{N}^{(-)}(in)G\left({}_-|^-\right) + G\left({}^-|_+\right)\left[1-\kappa\mathbb{N}^{(+)}(in)\right]G\left({}_+|^-\right), \quad (50)$$

where

$$G\left({}_+|^+\right) = w\left(+|+\right)^{-1}\,,\ \ G\left({}^-|_-\right) = w\left(-|-\right)^{-1}\,,$$
$$G\left({}^-|_+\right) = -w\left(-|-\right)^{-1}w\left(0|-+\right) = \kappa\left[w\left(+|+\right)^{-1}w\left(+-|0\right)\right]^\dagger\,,$$
$$G\left({}_-|^+\right) = \kappa w\left(0|-+\right)w\left(+|+\right)^{-1} = -\left[w\left(+-|0\right)w\left(-|-\right)^{-1}\right]^\dagger\,, \quad (51)$$

and the property $G\left({}_\zeta|^{\zeta'}\right) = G\left({}^{\zeta'}|_\zeta\right)^\dagger$ is used[5].

Thus, due to relations (42), (50), we have explicit expressions of the complete generating density operator (29) and reduced generating density operators (39) via both correlation functions of *in*-particles and *out*-particles, and via elementary probability amplitudes (3) as well.

We stress that there is one-to one correspondence between one-particle correlation functions and the form of the reduced density operator of the total system is related to choice of the model, which is a quantized field placed in an external background. In fact, we deal with a quadratic system of noninteracting (between themselves) particles. Of course, such a fact is well-known for free particle systems. Our consideration generalizes it to the presence of a particle-creating background. For systems of interacting particles, there remains an

[5]One can express the matrices G as an inner product of special solutions of associated relativistic wave equation, see [28]. In fact, element of these matrices are matrix elementa of the evolution operator of the relativistic wave equation in a special basis.

important question: Suppose the one-particle distribution at the final time instant is thermal one. Can one assert that the complete system is in a thermal state with a given temperature (the same that determines the one-particle distribution)? Such a question seems to be relevant to the problem of particle creation in a black-hole gravitational field (Hawking radiation), where one-particle distributions of particles created have a thermal form.

Below, we consider some illustrations of general formulas derived above.

Let the initial state of the system is a vacuum state ($J=0$), then (50) reproduces the well-known formulas for the differential mean numbers $\aleph_m^{(\zeta)} = N_m^{(\zeta)}\Big|_{J=0}$ of particles/antiparticles created from vacuum by an external field,

$$\aleph_m^{(+)} = \left[G\left({}^{+}|_{-}\right)G\left({}_{-}|^{+}\right)\right]_{mm},\ \aleph_m^{(-)} = \left[G\left({}^{-}|_{+}\right)G\left({}_{+}|^{-}\right)\right]_{mm}, \tag{52}$$

see [28].

Let us consider a commonly encountered case (for example, an uniform external field) when particle/antiparticle states are specified by quantum numbers (the same for particles and antiparticles) that are integrals of motion. In such a case, all the matrices $G\left({}_{\zeta}|^{\zeta'}\right)$ are diagonal and differential mean numbers (52) of the particles/antiparticles created from the vacuum coincide, $\aleph_m^{(+)} = \aleph_m^{(-)} = \aleph_m$. Using formulas (50), (52), and unitarity relations

$$\begin{aligned}
&G\left({}_{\zeta}|^{+}\right)G\left({}_{\zeta}|^{+}\right)^{\dagger} + \kappa G\left({}_{\zeta}|^{-}\right)G\left({}_{\zeta}|^{-}\right)^{\dagger} = \zeta^{\frac{1-\kappa}{2}},\\
&G\left({}_{+}|^{+}\right)G\left({}_{-}|^{+}\right)^{\dagger} + \kappa G\left({}_{+}|^{-}\right)G\left({}_{-}|^{-}\right)^{\dagger} = 0,\\
&G\left({}_{+}|^{+}\right)^{\dagger}G\left({}_{+}|^{-}\right) + \kappa G\left({}_{-}|^{+}\right)^{\dagger}G\left({}_{-}|^{-}\right) = 0,
\end{aligned} \tag{53}$$

one can find the following expressions for differential mean numbers of particles/antiparticles

$$N_m^{(\zeta)} = \left(1-\kappa\aleph_m\right)N_m^{(\zeta)}(in) + \aleph_m\left[1-\kappa N_m^{(-\zeta)}(in)\right]. \tag{54}$$

If an initial system state is different from the vacuum, differential mean numbers of particles/antiparticles created by an external field is given by the difference $\Delta N_m^{(\zeta)} = N_m^{(\zeta)} - N_m^{(\zeta)}(in)$. One can see that

$$\begin{aligned}
&\Delta N_m^{(+)} = \Delta N_m^{(-)} = \Delta N_m,\\
&\Delta N_m = \aleph_m\left[1-\kappa\left(N_m^{(+)}(in) + N_m^{(-)}(in)\right)\right].
\end{aligned} \tag{55}$$

Even if $\aleph_m \neq 0$, no particle creation of fermions with quantum numbers m occurs if $N_m^{(+)}(in) + N_m^{(-)}(in) = 1$. Since $\kappa = -1$ for bosons, ΔN_m is always positive and more than $\aleph_m$. That is, the presence of a matter in the initial state increases the mean number of the bosons created.

In some articles devoted to the chromoelectric flux tube model (see, e.g., [12, 14, 35]) one can meet an inexact interpretation of the well-known Schwinger formulas describing pair-creation from vacuum by a constant electric field [2]. Such an interpretation may lead to incorrect results for some field strengths, as noted in [36]. Below we discuss this problem and present correct relations that will be used in the subsequent section. We recall that using the proper time method Schwinger calculated the one-loop effective Lagrangian L in the electric field and assumed that the probability P^v of no actual pair-creation occurring within the history of the field for the time T in the volume V can be written as $P^v = |c_v|^2 = \exp\{-VT2\,\mathrm{Im}\,L\}$ (for subsequent development, see review [37]). Schwinger interpreted $2\,\mathrm{Im}\,L$ as a probability, per time unit and per volume unit, of creating a pair by a constant electric field. Some arguments in favour of such an interpretation can be found, for example, in the book [38] and in the article [15]. The interpretation remains approximately true as long as the WKB calculation is applicable, that is, $VT2\,\mathrm{Im}\,L \ll 1$. Then the total probability of pair-creation reads $1-P^v \approx VT2\,\mathrm{Im}\,L$. To calculate differential probabilities of pair-creation with quantum numbers m (for example, momentum and spin polarization) one can represent the probability P^v as an infinite product,

$$P^v = \prod_m e^{-2\,\mathrm{Im}\,S_m}, \tag{56}$$

where a certain discretization scheme is used such that the effective action $S = VTL$ is written as $S = \sum_m S_m$. All this is possible only if m are selected to be integrals of motion. Then $e^{-2\,\mathrm{Im}\,S_m}$ is the vacuum-persistence probability in a cell of the space of quantum numbers m. Using the WKB approximation in the case $2\,\mathrm{Im}\,S_m \ll 1$, one obtains for the probability P_m of single pair-production with quantum numbers m and for the corresponding mean values $\aleph_m$ of created pairs the following relation:

$$\aleph_m \approx P_m \approx 2\,\mathrm{Im}\,S_m\,. \tag{57}$$

By analogy with one-particle quantum mechanics, one usually rewrites (57) for fermions as

$$\aleph_m \approx -\ln(1-P_m) \approx 2\,\mathrm{Im}\,S_m\,. \tag{58}$$

It is clear that (57) and (58) coincide in the first order with respect to P_m . Then it follows from (56)

$$P^v \approx \prod_m (1-P_m)\ . \tag{59}$$

Using the same analogy for bosons and rewriting (57) as

$$\aleph_m \approx \ln(1+P_m) \approx 2\,\mathrm{Im}\,S_m\,, \tag{60}$$

one obtains the following approximate relation:

$$P^v \approx \prod_m (1+P_m)^{-1}\ . \tag{61}$$

It turns out that for the field under consideration, using WKB calculations and relations (58)–(61) one can reproduce the Schwinger's result for P^v. This fact causes the temptation to interpret the latter formulas as exact ones, replacing there "$\approx$" by "$=$". One ought, however, to say that such an interpretation is, in particular, equivalent to the assumption that $\aleph_m = 2\,\mathrm{Im}\,S_m$. However, as we demonstrate below, the latter relation is not exact and it holds only in the approximation $2\,\mathrm{Im}\,S_m \ll 1$.

Exact treatment in the framework of QFT with unstable vacuum (see, for example, [4, 28, 30]) yields the following expressions for the scattering $P(-|-)_m$ of a particle (and an antiparticle) and a pair-creation $P(+-|0)_m$ probabilities, respectively (see subsection 2.1 for notation):

$$P(-|-)_m = \left|w\left(-|-\right)_{mm}\right|^2 P^v\,,\quad P(+-|0)_m = \left|w\left(+-|0\right)_{mm}\right|^2 P^v\,, \tag{62}$$

where due to the relations (51), (52) and (53) the corresponding relative probabilities are

$$\left|w\left(-|-\right)_{mm}\right|^2 = \frac{1}{1-\kappa\aleph_m}\,,\quad \left|w\left(+-|0\right)_{mm}\right|^2 = \frac{\aleph_m}{1-\kappa\aleph_m}\,. \tag{63}$$

As long as the semiclassical approximation is valid ($P^v \approx 1$, $\aleph_m \ll 1$), we have

$$P(+-|0)_m \approx \left|w\left(+-|0\right)_{mm}\right|^2 \approx \aleph_m\,.$$

Thus, we can see that the quantities $P(+-|0)_m$, $\left|w\left(+-|0\right)_{mm}\right|^2$ and $\aleph_m$ can be identified only in the approximation under consideration. The exact expression for P^v in terms of mean values $\aleph_m$ follows from (8), (63) and reads

$$P^v = \exp\left\{\kappa \sum_m \ln(1-\kappa\aleph_m)\right\}\ . \tag{64}$$

Formulas (56) and (64) imply the following exact relation between $\mathrm{Im}\,S_m$ and $\aleph_m$,

$$2\,\mathrm{Im}\,S_m = -\kappa\ln(1-\kappa\aleph_m)\ . \tag{65}$$

It has to be used in the general case when the WKB approximation is not applicable.

3.3 Is it really a thermal distribution?

Considerable attention has been recently focused on a mechanism of fast thermalization in heavy-ion collisions (see [14, 39] and references therein). A possibility is discussed of a thermal one-particle distribution due to quantum creation of particles from vacuum by strong electric-like fields. Some of such distributions are already known in QED, and their relation to thermal spectrum of the Hawking radiation has been discussed (see references in the next section). We present some examples of such distributions in subsection 4.3. However, thermal one-particle distribution of created particles does not guarantee the character of thermal equilibrium for the corresponding complete quantum state of the system and only mimics, in a certain sense, the latter state.

One ought to say that in contrast to the case of Hawking's radiation, in which we do not have any information about another member of each created pair behind the horizon, both particles and antiparticles created from vacuum by the chromoelectric field can in principle be observed. Because of the one-to-one correspondence between one-particle correlation functions and the reduced density operator of the total system, all the moments of the particle (or antiparticle) distribution coincide. However, the higher moments of the simultaneous distributions of particles and antiparticles are different.

In what follows, we formally examine the above questions.

Suppose that the differential mean numbers N_m of particles/antiparticles in a final state of a system subject to an external field have the form of a one-particle thermal distribution. There arises the question if one can be sure that in such a case the final state of the complete system is a thermal equilibrium or the thermal form of one-particle distribution can appear even if the final state of the complete system is not a thermal equilibrium. To answer these questions, we are going to examine two different possibilities of having the same one-particle thermal distribution for two distinct states of a complete system, one of them being a thermal equilibrium and the other one a pure state. Let the first state of the complete system be described by the thermal density operator

$$\breve{\rho}_\beta^{out} = \frac{1}{Z}\exp\left\{-\left[a^\dagger E^{(+)}a + b^\dagger E^{(-)}b\right]\right\},$$
$$Z = \exp\left\{\kappa\sum_n \ln\left(1+\kappa e^{-E_n^{(+)}}\right) + \kappa\sum_m \ln\left(1+\kappa e^{-E_n^{(-)}}\right)\right\}, \tag{66}$$

where $E^{(\pm)}$ are given by (20). It is obvious that in such a state the differential mean numbers N_m have the form

$$N_m = \left(e^{E_m}+\kappa\right)^{-1}. \tag{67}$$

On the other hand, if we have a causal evolution from vacuum, the density operator of the corresponding pure state has the form $\breve{\rho}_v$ (17) (see the normal form in (30)). Such a state provides the differential mean numbers (67) if (63) holds true. We can see that measuring the one-particle distribution cannot distinguish between both cases. However, they can be distinguished by measuring the next moments, as demonstrated below. Let us calculate the variances Var_m in the states described by the density matrices (66) and (17), respectively,

$$\begin{aligned}\mathrm{Var}_m^{th} &= \mathrm{tr}\left[\left(a_m^\dagger a_m + b_m^\dagger b_m - 2N_m\right)^2 \breve{\rho}_\beta^{out}\right],\\ \mathrm{Var}_m^{v} &= \mathrm{tr}\left[\left(a_m^\dagger a_m + b_m^\dagger b_m - 2N_m\right)^2 \breve{\rho}_v\right].\end{aligned}$$

Since the differential mean values coincide in both states, one can see that

$$\mathrm{Var}_m^{th} - \mathrm{Var}_m^{v} = 2\left(Q_m^{th} - Q_m^{v}\right),$$
$$Q_m^{th} = \mathrm{tr}\left[a_m^\dagger a_m b_m^\dagger b_m \breve{\rho}_\beta^{out}\right],\ Q_m^{v} = \mathrm{tr}\left[a_m^\dagger a_m b_m^\dagger b_m \breve{\rho}_v\right]. \tag{68}$$

To calculate the quantities Q_m^{th} and Q_m^{v} and demonstrate that they are different, we are going to use the generating functional $\Phi^v(J)$ (32) and the generating functional of moments for the thermal distribution,

$$\Phi^{th}(J) = \mathrm{tr}\left\{\exp\left[a^\dagger \mathbb{J}^{(+)}a + b^\dagger \mathbb{J}^{(-)}b\right]\breve{\rho}_\beta^{out}\right\}.$$

Then

$$Q_m^{th} = \left.\frac{\partial^2\Phi^{th}}{\partial J_m^{(+)}\partial J_m^{(-)}}\right|_{J=0} = N_m^2, \tag{69}$$

where the expression

$$\Phi^{th}(J) = \frac{1}{Z}\exp\left\{\kappa\sum_n \ln\left(1+\kappa e^{-E_n^{(+)}+J_n^{(+)}}\right) + \kappa\sum_m \ln\left(1+\kappa e^{-E_n^{(-)}+J_n^{(-)}}\right)\right\}$$

is used. On the other hand

$$Q_m^v = \left.\frac{\partial^2\Phi^v}{\partial J_m^{(+)}\partial J_m^{(-)}}\right|_{J=0} = N_m\left[1+(1-\kappa)N_m\right]. \tag{70}$$

4 Particle creation in an electric-like background

4.1 Quasiconstant electric field

Below, we consider some applications of the above-elaborated formalism in QED with quasiconstant (slowly varying) uniform electric fields violating the stability of vacuum. We emphasize that our consideration can be relevant in QCD with an electric-like color field and in some QFT models with a curved space-time as was demonstrated, for example, in [40]. It was shown [4] that the distribution of pairs created from vacuum by a quasiconstant electric field has a thermal-like form. It appears that such a form has an universal character, i.e., it emerges in any theory with quasiconstant external fields, and when applied to particle creation in external constant gravitational field it exactly reproduces the Hawking temperature. Thus, our consideration in QED with a quasiconstant electric field allows us to reveal the typical properties of any strong-field QFT.

We note that in the case under consideration, particle states are specified by continuous quantum numbers of momentum $\mathbf{p}$ and spin projections $r = \pm 1$ (we formally set $r = 0$, for scalar particles). From now on, we assume that the standard volume regularization is used, so that $\delta(\mathbf{p}-\mathbf{p}')$ is replaced by $\delta_{\mathbf{p},\mathbf{p}'}$ in normalization conditions. Thus, our particles are labeled by a set of discrete quantum numbers $m = (\mathbf{p}, r)$.

As usual, we are going to describe the electric field by time-dependent vector potentials. The states of the quantum system in question are far-from-equilibrium due to the field influence. We study in detail the time dependence of various mean values, in particular, mean values of created particles. In a physically correct setting of the problem, we consider a model of a quasiconstant electric field $E(x^0)$ which effectively acts only for a finite period of time T, and is zero out of the interval (we further call it the T-constant field). In our model $E(x^0) = E$ for $t_1 \le x^0 \le t_2$, $t_2 = -t_1 = T/2$. Thus, the field produces a finite work in a finite space volume. We accept the initial vacuum to be a free particle vacuum. A relevant calculations in QED with T-constant field can be found in [4]. Below, we use these results for evaluating the leading terms in particle creation phenomena at large T, when the effects of switching on and off are negligible.

Let us describe the T-constant field. It is nonstationary, but with a constant space direction. We chose the latter along the x^3-axis. We denote by q the charge of a particle (by $-q$ of an antiparticle) and by M its mass. The corresponding potentials can be chosen in the form: $A_0 = A_1 = A_2 = 0$, and

$$A_3(x^0) = \begin{cases} Et_1, & x^0 \in I \\ Ex^0, & x^0 \in II \\ Et_2, & x^0 \in III\,. \end{cases}$$

where the time intervals are $I = (-\infty, t_1)$, $II = [t_1, t_2]$, $III = (t_2, +\infty)$.

If the time T is sufficiently large,

$$T >> T_0 = (1+\lambda)/\sqrt{|qE|}\,,$$

the differential mean numbers $\aleph_m$ read

$$\aleph_m = \begin{cases} e^{-\pi\lambda}\left[1+O\left(\left[\frac{1+\lambda}{K}\right]^3\right)\right], & -\sqrt{|qE|}\frac{T}{2} \le \xi \le -K\,, \\ O(1), & -K < \xi \le +K\,, \\ O\left(\left[\frac{1+\lambda}{\xi^2}\right]^3\right), & \xi > K\,, \end{cases} \tag{71}$$

where K is a sufficiently large arbitrary constant, $K >> 1+\lambda$,

$$\lambda = \frac{M^2+\mathbf{p}_\perp^2}{|qE|},\quad \mathbf{p}_\perp = (p^1, p^2, 0),\quad \xi = \frac{|p_3| - |qE|\,T/2}{\sqrt{|qE|}}\,, \tag{72}$$

and p_3 is a longitudinal momentum of a particle [4]. One can consider the limit $T \to \infty$ at any given $\mathbf{p}$ in the above expression. In such a limit, the differential mean numbers take a simple form

$$\aleph_m = e^{-\pi\lambda} \quad (73)$$

which coincides with one obtained in the constant electric field by Nikishov [3]. One can see that the stabilization of the differential mean numbers to the asymptotic form (73) for finite longitudinal momenta is reached at $T >> T_0$. The characteristic time T_0 is called the stabilization time.

In order to study the effects of switching on and off for $T >> T_0$, let us consider another example of a quasiconstant electric field,

$$E(x^0) = E\cosh^{-2}\left(\frac{x^0}{\alpha}\right). \quad (74)$$

This field switches on and off adiabatically as $x^0 \to \pm\infty$ and is quasiconstant at finite times. It is called an adiabatic field. The differential mean numbers of particles created by such a field were found in [41]. For further discussion, we need these numbers for a large α. As was demonstrated in [4], the differential mean numbers in the field (74) take the asymptotic form (73) for $\alpha >> \alpha_0 = (1+\sqrt{\lambda})/\sqrt{|qE|}$ and for $|p_3| << |qE|\alpha$. Thus, α_0 can be interpreted as stabilization time for the adiabatic field. At the same time, the latter fact means that the effects of switching on and off are not essential at large times and finite longitudinal momenta for both fields. Extrapolating such a conclusion, one may suppose that particle creation effects in any electric field, that is quasiconstant $\approx E$ at least for a time period $T >> T_0$ and switches on and off out of this period arbitrary, do not depend on the details of switching on and off. Thus, our calculations in a T-constant field are typical for a large class of quasiconstant electric fields.

It is of interest for phenomenological application to calculate the distribution of particles created with all possible p_3 values and a given $p_\perp$ (it is called $p_\perp$ distribution and is denoted by $n_{p_\perp}$ in what follows). Studying the total mean number of particles created in the T-constant field, we go over from the summation to an integration, $\sum_{\mathbf{p}} \to \frac{V}{(2\pi)^3}\int d\mathbf{p}$. Then the total mean number (we denote it by $\aleph$) can be presented as

$$\aleph = V\int d^2p_\perp n_{p_\perp}, \quad (75)$$

where

$$n_{p_\perp} = \frac{1}{(2\pi)^3}\sum_r\int dp_3\aleph_m \quad (76)$$

is the $p_\perp$ distribution density of particles created per unit volume. $\aleph_m$ is constant for $|p_3| \le \sqrt{|qE|}\left(\sqrt{|qE|}T/2 - K\right)$ and at $T >> T_0$, and decreases rapidly for $|p_3| > \sqrt{|qE|}\left(\sqrt{|qE|}T/2 + K\right)$. The contribution to the integral (76) from the intermediate region can be estimated as $2\sqrt{|qE|}K$. This implies

$$n_{p_\perp} = \frac{J\sqrt{|qE|}}{(2\pi)^3}\left[\sqrt{|qE|}Te^{-\pi\lambda} + O(K)\right], \quad (77)$$

where J is the number of spin degrees of freedom ($J=1$ for scalar particles and $J=2$ for fermions). Thus, the $p_\perp$ distribution density of the particle production rate is

$$\frac{dn_{p_\perp}}{dT} = \frac{J\,|qE|}{(2\pi)^3}e^{-\pi\lambda}. \quad (78)$$

The total number of particles created per unit volume is

$$\frac{\aleph}{V} = J\frac{(qE)^2T}{(2\pi)^3}\exp\left\{-\pi\frac{M^2}{|qE|}\right\}. \quad (79)$$

For a strong electric field, $M^2/|qE| \lesssim 1$, and large T, the energy density of created pairs reads $\mathcal{E} = |qE|T\aleph/V$, see [42]. We can neglect the back-reaction of particles created by the electric field in case their energy density is essentially smaller than the energy density of the electric field, $\mathcal{E} \ll E^2/8\pi$. Consequently, the concept of a strong constant electric field is consistent only if the following condition holds true:

$$1 \ll |qE|T^2 \ll \frac{\pi^2}{Jq^2}\exp\left\{\pi\frac{M^2}{|qE|}\right\}. \quad (80)$$

Following [4], we represent the asymptotic formula (73) in the universal form

$$\aleph_m = \exp\left\{-2\pi\frac{\omega_m}{g}\right\}, \tag{81}$$

where ω_m is the work of an external field creating a particle from a pair in a given state m,

$$\omega_m = \frac{1}{2}\left[p_0(t_f) + p_0(t_i) + \Delta\epsilon_{vac}\right],$$

where $p_0(t_f)$ and $p_0(t_i)$ are particle energies at a final time instant t_f and at an initial time instant t_i, respectively, and $\Delta\epsilon_{vac}$ is a shift of the vacuum energy due to the time evolution. The quantity g is the classical acceleration of a particle in the final time instant. In the case of the T-constant field, one can calculate

$$\omega_m = \frac{M^2 + \mathbf{p}_\perp^2}{2p_0(t_f)} = \frac{\lambda}{T}, \quad g = \frac{|qE|}{p_0(t_f)} = \frac{2}{T}.$$

Thus, we can see that the differential mean values (81) are given, in fact, by the Boltzmann formula with the temperature $\theta = \frac{g}{2\pi k_B}$ (where k_B is the Boltzmann constant), the latter having literally the Hawking form [8], see below.

We recall that the Hawking result for bosons created by a static gravitational field of a black hole in a specific thermal environment has the Planck form

$$\aleph_m = \left[\exp\left\{2\pi\frac{\omega_m}{g_{(H)}}\right\} - 1\right]^{-1}. \tag{82}$$

Here ω_m is the energy of a created particle and the Hawking temperature reads $\theta_{(H)} = \frac{g_{(H)}}{2\pi k_B}$, where $g_{(H)} = \frac{GM}{r_g^2}$ is the free-fall acceleration at the gravitational radius r_g of a black hole with mass M. In this case of a quasistatic gravitation field, the evolution shift of the vacuum energy is $\Delta\epsilon_{vac} = 0$, so that one identifies the work ω_m (we have introduced) with the energy of a particle in formula (82). It is also known [43] that an observer that moves with a constant acceleration $g_{(R)}$ (with respect to its proper time) will probably register in the Minkowski vacuum some particles (Rindler particles). The distribution of Rindler bosons have the same Planck form (82), where one has to replace $g_{(H)}$ by $g_{(R)}$, so that the correspondent temperature is $\theta_{(R)} = \frac{g_{(R)}}{2\pi k_B}$.

It is a direct consequence of the equivalence principle that the effective temperature θ of the distribution (81) has literally the Hawking form. The different form of distributions can be caused by essentially different structures of the Fock space in both cases. We believe that the Planck distribution arises necessarily due to the formation of an event horizon (there is a boundary of the domain of the Hamiltonian), that is, due to the condition for which the space domains of the particle and antiparticle vacua are not the same. On the other hand, the final state can be treated as an equilibrium state. In contrast to this, in a uniform electric field we deal, in fact, with both the particle vacuum and the antiparticle vacuum defined over the entire space, that is, these space domains coincide. In this case, the mixed state of particles (antiparticles) described by the $\tilde{\rho}_{v+}$ ($\tilde{\rho}_{v-}$) density matrix of Sec. 3 can be represented as a pure state in an extended phase space where the space domains of both the particle vacuum and the antiparticle vacuum are the same, being a state of a far-from-equilibrium system. Let us note that in framework of a semiclassical description at $\omega_m/g << 1$ the Boltzmann spectrum closely approaches the Planck spectrum.

4.2 Soft parton production by SU(3) chromoelectric field

As mentioned in Introduction, in QCD there exist physical situations that are quite efficiently described by the chromoelectric flux tube model. In this model, the back-reaction of created pairs induces a gluon mean field and plasma oscillations (see [44] and references therein). It appears that for calculating particle creation in this model one needs to apply the general formalism of QFT for pair production at a finite temperature and at zero temperature both from vacuum and from many-particle states (see, physical reasons for that in [26, 36, 45]). The consideration of various time scales in heavy-ion collisions shows that the stabilization time T_0 is far smaller than the period of plasma and mean-field oscillations. Then, according to the condition (80), the approximation of a strong T-constant chromoelectric field can be used in treating such collisions during a period when the produced partons can be considered as weakly coupled due to the property of asymptotic freedom in QCD. It may be also reasonable to neglect dynamical back-reaction effects and to consider only pair-production from vacuum by a constant $SU(3)$ chromoelectric field.

Here, we would like to turn our attention to results obtained in QCD with a constant $SU(3)$ chromoelectric field E^a $(a = 1, \dots, 8)$ along the x^3-axis, see [12]. In this work, the imaginary parts of one-loop effective actions for quarks S^{quark} and gluons S^{gluon} were calculated via gauge-invariant $p_\perp$ distributions $S^{quark}_{p_\perp}$ and $S^{gluon}_{p_\perp}$ respectively. They have the form

$$\begin{aligned}
&\operatorname{Im} S^{quark} = \int d^2p_\perp \operatorname{Im} S^{quark}_{p_\perp}, \quad \operatorname{Im} S^{gluon} = \int d^2p_\perp \operatorname{Im} S^{gluon}_{p_\perp}, \\
&\operatorname{Im} S^{quark}_{p_\perp} = -\frac{VT}{8\pi^3} \sum_{j=1}^{3} |qE_{(j)}| \ln\left(1 - e^{-\pi\lambda_{(j)}}\right), \\
&\operatorname{Im} S^{gluon}_{p_\perp} = \frac{VT}{8\pi^3} \sum_{j=1}^{3} \left|q\tilde{E}_{(j)}\right| \ln\left(1 + e^{-\pi\tilde{\lambda}_{(j)}}\right), \\
&\lambda_{(j)} = \frac{M^2 + \mathbf{p}_\perp^2}{|qE_{(j)}|}, \quad \tilde{\lambda}_{(j)} = \frac{\mathbf{p}_\perp^2}{\left|q\tilde{E}_{(j)}\right|},
\end{aligned} \tag{83}$$

where $E_{(j)}$ are eigenvalues of the matrix iT^aE^a for the fundamental representation of $SU(3)$; $\tilde{E}_{(j)}$ are all the positive eigenvalues of the matrix $if^{abc}E^c$ for the adjoint representation of $SU(3)$; and q is the coupling constant. These eigenvalues are the following gauge invariant quantities:

$$\begin{aligned}
E_{(1)} &= \sqrt{C_1/3}\cos\theta\,, \; E_{(2)} = \sqrt{C_1/3}\cos(2\pi/3 - \theta)\,, \\
E_{(3)} &= \sqrt{C_1/3}\cos(2\pi/3 + \theta)\,,
\end{aligned}$$

where θ is given by $\cos^2 3\theta = 3C_2/C_1^3$, and

$$\begin{aligned}
\tilde{E}_{(1)} &= \left[\frac{C_1}{2}\left(1 - \cos\tilde{\theta}\right)\right]^{1/2}, \; \tilde{E}_{(2)} = \left[\frac{C_1}{2}\left(1 + \cos\left(\frac{\pi}{3} - \tilde{\theta}\right)\right)\right]^{1/2}, \\
\tilde{E}_{(3)} &= \left[\frac{C_1}{2}\left(1 + \cos\left(\frac{\pi}{3} + \tilde{\theta}\right)\right)\right]^{1/2},
\end{aligned}$$

where $\tilde{\theta}$ is given by $\cos^3\tilde{\theta} = -1 + 6C_2/C_1^3$. Here, C_1 and C_2 are Casimir invariants for $SU(3)$,

$$C_1 = E^aE^a, \quad C_2 = \left(d_{abc}E^aE^bE^c\right)^2,$$

where d_{abc} is a symmetric invariant tensor in the adjoint representation of $SU(3)$. Then the probabilities P^v for vacuum to remain vacuum are found, for both quarks and gluons, from relation (56). However, formulas for parton production rates derived in [12] hold only in the approximation $2\operatorname{Im} S^{quark}_{p_\perp} \ll 1$ and $2\operatorname{Im} S^{gluon}_{p_\perp} \ll 1$ by virtue of the arguments we presented at the end of subsection 3.2. To obtain exact results, we can use the following line of reasoning. The results (83) can be treated as ones obtained in the case of a T-constant chromoelectric field when the integration over the longitudinal momentum and the summation over the spin and color degrees of freedom have been carried out. Then, using the relation (65), we can extract from the representation (83) an exact expression for $p_\perp$ distribution densities of quarks $n^{quark}_{p_\perp}$ and gluons $n^{gluon}_{p_\perp}$ produced per unit volume. Those are

$$n^{quark}_{p_\perp} = \frac{T}{4\pi^3} \sum_{j=1}^{3} |qE_{(j)}|\, e^{-\pi\lambda_{(j)}}, \quad n^{gluon}_{p_\perp} = \frac{T}{4\pi^3} \sum_{j=1}^{3} \left|q\tilde{E}_{(j)}\right| e^{-\pi\tilde{\lambda}_{(j)}}, \tag{84}$$

where T is a sufficiently large period of constant field action. The $p_\perp$ distribution densities of particle production rates can be found as $dn^{quark}_{p_\perp}/dT$ and $dn^{gluon}_{p_\perp}/dT$, respectively. The total numbers of quarks and gluons created per unit volume can be obtained from (84) as

$$\frac{\aleph^{quark}}{V} = \frac{T}{4\pi^3} \sum_{j=1}^{3} \left(qE_{(j)}\right)^2 \exp\left\{-\pi\frac{M^2}{|qE_{(j)}|}\right\}, \quad \frac{\aleph^{gluon}}{V} = \frac{3Tq^2C_1}{8\pi^3}, \tag{85}$$

where the relation $\sum_{j=1}^{3}\tilde{E}_{(j)}^2 = 3C_1/2$ is used. Taking into account the relation $\sum_{j=1}^{3}E_{(j)}^2 = C_1/2$, one can see from (85) that in a sufficiently strong field $E_{(j)}$, $M^2/\left|qE_{(j)}\right| \ll 1$, the densities of created quarks and gluons are related as $\aleph^{quark}/V = \aleph^{gluon}/3V$.

We can see that the (j)- terms in expressions (83) and (85) can be interpreted as those which are obtained for an Abelian-like electric fields $E_{(j)}$ and $\tilde{E}_{(j)}$, respectively. The maxima of the fields are restricted by the relations $\left|E_{(j)}\right| \leq \sqrt{C_1/3}$ and $\left|\tilde{E}_{(j)}\right| \leq \sqrt{C_1}$. Thus, to study the validity of the constant $SU(3)$ chromoelectric field approximation we need to take into account only the energy density of gluons created by the field $\tilde{E}_{(j)}$. We know from the previous subsection that this energy per a single pair is $\left|q\tilde{E}_{(j)}\right|T$. Then the total energy density of the created gluons reads

$$\mathcal{E} = \frac{T^2}{4\pi^3}\sum_{j=1}^{3}\left|q\tilde{E}_{(j)}\right|^3 \lesssim |q|\sqrt{C_1}T\frac{\aleph^{gluon}}{V}.$$

One can neglect a back-reaction of these gluons created by the chromoelectric field only if $\mathcal{E} \ll C_1/8\pi$. Finally, the validity condition of the T-constant $SU(3)$ chromoelectric field approximation can be written as

$$1 \ll |q|\sqrt{C_1}T^2 \ll \frac{\pi^2}{3q^2}. \tag{86}$$

Thus, we can see that the T-constant $SU(3)$ chromoelectric field approximation is consistent during the period when the produced partons can be treated as weakly coupled.

In the following sections, we turn once again to particle creation by an electric field in QED. The above discussion shows that it can be useful for understanding the effects of quark and gluon creation in QCD.

4.3 Thermal-like distributions

As has been said above, the thermalization stage of multiparticle production in ion-ion collisions at high energy is very important. On the other hand, as we know from section 3, it is sometimes difficult to distinguish a real thermal equilibrium from a state where we have a one-particle thermal distribution. In this connection, we consider below some simple examples when pair creation by an electric field can mimic a one-particle thermal distribution.

We recall that due to the screen of created pairs, the original electric field may have an exponential fall-off,

$$E(x^0) = Ee^{-x^0/\alpha}. \tag{87}$$

The differential mean number of particles created from vacuum by this field was calculated in [46]. The result is

$$\aleph_m = \begin{cases} \left[\frac{\cosh\left[\pi\alpha\left(\varepsilon+p_3'\right)\right]}{\cosh\left[\pi\alpha\left(\varepsilon-p_3'\right)\right]}e^{2\pi\alpha\varepsilon} - 1\right]^{-1} & \text{for bosons,} \\ \left[\frac{\sinh\left[\pi\alpha\left(\varepsilon+p_3'\right)\right]}{\sinh\left[\pi\alpha\left(\varepsilon-p_3'\right)\right]}e^{2\pi\alpha\varepsilon} + 1\right]^{-1} & \text{for fermions,} \end{cases} \tag{88}$$

where $\varepsilon = \sqrt{M^2 + \mathbf{p}^2}$ and $p_3' = p_3\mathrm{sgn}(qE)$. At $\pi\alpha|p_3| \ll 1$ these expressions coincide with Bose and Fermi distributions at the temperature $\theta = (2\pi k_B\alpha)^{-1}$ respectively.

Another example is the pulse of electric field (74). The differential mean number of particles created from vacuum by the sharp field pulse (74) at $\alpha|qE|/\varepsilon \ll 1$ can be extracted from the result [41] and has the following form:

$$\aleph_m = \begin{cases} \left(\pi qE\alpha^2\right)^2\left[\left(qE\alpha^2\right)^2 + (p_3/\varepsilon)^2\right]\sinh^{-2}(\pi\alpha\varepsilon) & \text{for bosons,} \\ \left(\pi qE\alpha^2\right)^2\left[1-(p_3/\varepsilon)^2\right]\sinh^{-2}(\pi\alpha\varepsilon) & \text{for fermions,} \end{cases} \tag{89}$$

When the ratio $|p_3|/\varepsilon$ is sufficiently small and the external field is not strong, $\varepsilon\Big/\sqrt{|qE|} \gg 1$, there is a range of values α, $\pi\alpha\varepsilon \gg 1$ in which the distributions (89) take the Boltzmann form with the temperature $\theta = (2\pi k_B\alpha)^{-1}$.

As we noted in the previous subsection, depending on the details of the chromoelectric flux tube model, its stage and field strength, pair production both from vacuum and from many-particle states by a T-constant electric field may be relevant. The well-known asymptotic form (73) of the differential mean numbers for pairs created from vacuum by a constant electric field can lead to a thermal-like distribution of created pairs if, for example, the chromoelectric string tension undergoes Gaussian fluctuations [47]. It means a modification of the original flux tube model by introducing a fluctuating string tension. In case the original flux tube model still holds, nevertheless, the differential mean numbers of pairs created by a T-constant electric field can be represent as the Boltzmann distribution (81) with the temperature $\theta = (\pi k_B T)^{-1}$, as we have seen in subsection 4.1. Then it may be reasonable to examine the phenomenological model with slowly oscillated mean electric field and suppose that the pair creation by the mean field during semiperiod of oscillation can be effectively approximated by a T-constant electric field (we recall that the time scale of stabilization T_0 is far smaller than the period of oscillations). In this case, the electric field produces pairs for a semiperiod of oscillation in the presence of pairs created at previous stages. This is the way to take into account the effects of back-reaction in such a model. In other words, we are going to consider pair creation from the initial state given by a distribution of previously created particles. Formula (55) is relevant in this analysis.

Starting from the initial vacuum state, one has $\Delta N_m = \aleph_m$, where $\aleph_m$ belongs to the asymptotic form (73). Then, at the end of the first stage, when the mean field is depleted for the first time, the distribution of particles (it is equal for antiparticles) is $N_m^{(1)} = \aleph_m$. During the second stage, the direction of the mean field is opposite to the field direction at the first stage. Due to the condition of stabilization, it is of no importance since the $\aleph_m$ is an even function of qE. Thus, when the mean field is depleted for the second time, using (55) one has the relation

$$N_m^{(2)} = \aleph_m + (1 - 2\kappa\aleph_m)\, N_m^{(1)},$$

and at the end of the n stage

$$N_m^{(n)} = \aleph_m + (1 - 2\kappa\aleph_m)\, N_m^{(n-1)}.$$

Consequently, the total number of the particles created at the end of the n stage is

$$N_m^{(n)} = \aleph_m \sum_{l=0}^{n-1} (1 - 2\kappa\aleph_m)^l .$$

We have this result if the created particles do not leave the region of the active field. To take into account a possible loss of particles due to interaction, movement, etc. We also assume that the total number of particles in the initial state of the n stage is less than the number $N^{(n-1)}$ of particles created at the end of the $n-1$ stage and is $\gamma N^{(n-1)}$, where $\gamma < 1$ is the factor of loss. Then the modified relation is

$$N_m^{(n)} = \aleph_m + (1 - 2\kappa\aleph_m)\, \gamma N_m^{(n-1)}, \tag{90}$$

and we finally have

$$N_m^{(n)} = \aleph_m \sum_{l=0}^{n-1} \gamma^l (1 - 2\kappa\aleph_m)^l . \tag{91}$$

Supposing that γ is a constant, one can calculate the sum in (91),

$$N_m^{(n)} = \aleph_m \frac{1 - r^n}{1 - r},\; r = \gamma (1 - 2\kappa\aleph_m) . \tag{92}$$

For fermions, $\kappa = +1$, then $N_m^{(n)} \leq 1$. Energy dissipation after a period of oscillation is estimated (for real parameters of heavy-ion collisions) not to be large so that damping is small and the number of oscillations can be quite large; damping decreases with an increasing field strength. If the number of cycles is sufficiently large, we get the limiting thermal-like distribution

$$N_m^{\Sigma} = \frac{\aleph_m}{1 - \gamma (1 - 2\kappa\aleph_m)} = \frac{1}{2\gamma} \cdot \frac{1}{e^{\pi\lambda} (1 - \gamma)/2\gamma + \kappa} . \tag{93}$$

In other words, the system reaches a quasiequilibrium state. For bosons, $\kappa = -1$, then $N_m^{(n)}$ grows. This is the phenomenon of resonance, and the increase can be either limited or unlimited depending on the factor γ. The increase is limited as long as $r < 1$. In this case, formula (93) is valid for bosons as well. We can see that the back-reaction induced plasma oscillations can reach a quasistationary form specified by the thermal-like distribution for both fermions and bosons.

4.4 Particle creation at finite temperatures

We are now ready to present explicitly, for example, the mean number of (anti)particles in the mode m (with the finite longitudinal momenta, $|p_3| \leq \sqrt{|qE|}\left(\sqrt{|qE|}T/2 - K\right)$) for the final state of evolution in a quasiconstant field from the initial thermodynamic equilibrium, $N_m^{(\zeta)}(in) = \left(e^{E_m} + \kappa\right)^{-1}$, at equal chemical potentials $\mu^{(+)} = \mu^{(-)} = \mu$, ($\mu < M$ for bosons),

$$N_m^{(\zeta)} = \left(e^{E_m} + \kappa\right)^{-1} + e^{-\pi\lambda} \left(\tanh\left(E_m/2\right)\right)^{\kappa}, \tag{94}$$

where $E_m = \beta\left(\varepsilon_m - \mu\right)$, $\varepsilon_m = \sqrt{M^2 + \mathbf{p}_\perp^2 + \left(\pi_3\right)^2}$, $\pi_3 = p_3 + qET/2$, and it is implied that λ is given by (72). This result for the electric field coincides with the one obtained in [24]. Due to the effect of stabilization, it seems that the time dependence of the final distributions in question is absent. However, the integral mean numbers vary as long as a quasiconstant field is active.

It is of interest to establish some general behavior of the integral mean numbers of created particles when the effects of switching on and off are negligible. As shown above, we can satisfy this condition by selecting the action time T of the T-constant field ($T >> T_0$) as an effective period of pair creation. It is implied that, in general, the final time instant, t_f, and the initial time instant, t_i, are so selected that a quasiconstant field is closely approximated by the T-constant field for a period from t_i to t_f, and $t_f - t_i = T$.

Let us estimate the sum over the longitudinal momentum p_3 of ΔN_m in (55), which is the mean number of particles created with all possible values p_3. As above, $\sum_{\mathbf{p}} \rightarrow \frac{V}{(2\pi)^3}\int d\mathbf{p}$ and the distribution $\aleph_m$ plays the role of the cut-off factor for the integral over p_3. Then, one can conclude that the $p_\perp, r$ distribution density of particles produced per unit volume is finite and can be presented as

$$\begin{aligned} n_{\mathbf{p}_\perp,r}^{cr} &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \Delta N_m dp_3 \\ &= \frac{1}{(2\pi)^3}\left[e^{-\pi\lambda}\int_{-|qE|T/2}^{|qE|T/2} n_m(\beta) dp_3 + \sqrt{|qE|}O(K)\right], \\ n_m(\beta) &= \left(\tanh\left(E_m/2\right)\right)^{\kappa}. \end{aligned} \tag{95}$$

From (95), one can estimate the $p_\perp, r$ distribution density of the particle production rate,

$$\frac{dn_{\mathbf{p}_\perp,r}^{cr}}{dT} = \frac{|qE|}{(2\pi)^3}\, n_m(\beta)\big|_{\pi_3 = |qE|T}\, e^{-\pi\lambda}. \tag{96}$$

We have $(qET)^2 \gg M^2 + \mathbf{p}_\perp^2$ according to the condition of stabilization. Then, the low and high temperature limits for the production rate are only defined by the final longitudinal kinetic momentum $|qE|T$ and the temperature Θ relation: $\beta|qE|T \gg 1$ and $\beta|qE|T \ll 1$, respectively. For simplicity, we assume that $|qE|T \gg \mu$. Considering these limits, one obtains for the temperature dependent term in (96)

$$\begin{aligned} n_m(\beta)\big|_{\pi_3=|qE|T} &= 1 - 2\kappa e^{-\beta|qE|T}, \ \beta|qE|T \gg 1, \\ n_m(\beta)\big|_{\pi_3=|qE|T} &= \left[\beta|qE|T/2\right]^{\kappa}, \ \beta|qE|T \ll 1. \end{aligned}$$

We can see that at high temperatures the rate $\frac{dn_{\mathbf{p}_\perp,r}^{cr}}{dT}$ is time-dependent: it is much lower than the zero temperature value but increasing for fermions and considerably higher than the zero temperature value but decreasing for bosons. Consequently, the frequently used notion of a number of particles created per unit of time makes sense only at low temperatures and in this limit it coincides with the zero temperature value of the production rate. We consider two temperature limits for the $p_\perp, r$ distribution density (95): low temperatures at $\beta\left(\varepsilon_\perp - \mu\right) \gg 1$, $\varepsilon_\perp = \sqrt{M^2 + \mathbf{p}_\perp^2}$, when all the energies of the particles created in the modes with a given $\mathbf{p}_\perp$ are considerably higher than the temperature Θ, and high temperatures at $\beta|qE|T \ll 1$, when all the energies of the created particles are much lower than the temperature Θ,

$$
\begin{aligned}
n^{cr}_{\mathbf{p}_\perp,r} &= \frac{\sqrt{|qE|}}{(2\pi)^3}\left[\sqrt{|qE|}Te^{-\pi\lambda}+O(K)\right],\ \kappa=\pm1,\ \beta(\varepsilon_\perp-\mu)\gg1,\\
n^{cr}_{\mathbf{p}_\perp,r} &= \frac{\beta|qE|}{(2\pi)^3}\left[|qE|T^2/2+O\left(\sqrt{|qE|T}\right)\right]e^{-\pi\lambda},\ \kappa=+1,\ \beta|qE|T\ll1,\\
n^{cr}_{\mathbf{p}_\perp,r} &= \frac{\sqrt{|qE|}}{(2\pi)^3}\left[\frac{4}{\beta\sqrt{|qE|}}\ln\left(\sqrt{|qE|}T/K\right)e^{-\pi\lambda}+O(K)\right],\ \kappa=-1,\ \beta|qE|T\ll1.
\end{aligned}
\tag{97}
$$

The result at low temperatures is not different from the zero temperature result [4] within the accuracy of the analysis. Integrating expressions (97) over $\mathbf{p}_\perp$, one finds the total number of particles created per unit volume at low temperature and high temperature limits, respectively:

$$
\begin{aligned}
\frac{N^{cr}}{V} &= J\frac{(qE)^2T}{(2\pi)^3}e^{-\pi M^2/|qE|},\ \beta(M-\mu)\gg1,\\
\frac{N^{cr}}{V} &= \frac{\beta|qE|^3T^2}{(2\pi)^3}e^{-\pi M^2/|qE|},\ \kappa=+1,\ \beta|qE|T\ll1,\\
\frac{N^{cr}}{V} &= \frac{|qE|\ln\left(\sqrt{|qE|}T\right)}{2\pi^3\beta}e^{-\pi M^2/qE},\ \kappa=-1,\ \beta|qE|T\ll1,
\end{aligned}
\tag{98}
$$

where the summation over $r=\pm1$ is carried out for the fermions, and only the leading T-dependent terms are shown. From (97),(98), we can see that the values of the integral mean numbers for fermions at high temperatures are much lower than the corresponding values at low temperatures. For bosons, the integral mean numbers at high temperatures are considerably higher than the corresponding values at low temperatures.

As mentioned in the introduction, thermally influenced pair production in a constant electric field has been investigated in several approaches [26, 24, 25, 27]. The results obtained are rather contradictory, varying from the absence of creation to values of the rate of fermion production higher than the rate at the zero temperature. Now, we are ready to discuss such contradictions. As shown above, the initial thermal distribution affects the number of states in which pairs are created by the quasiconstant field. Hence, the pair production exists at any temperatures, and, in particular, the fermion production rate cannot be higher than the rate at the zero temperature, by any means. Note that our calculations are based on the generalized Furry representation elaborated especially for the case of vacuum instability in accordance with the basic principles of quantum field theory. On the other hand, all conclusions in [26, 27] concerning the pair production rate and/or the mean numbers of pairs created at non-zero temperatures are based on either the standard real-time or imaginary-time one-loop effective actions. However, such formalisms do not work in the presence of unstable modes. The real part of the standard effective action describes effects of vacuum polarization and has nothing to do with the time-dependent conduction current of created pairs. For example, it can be seen at the zero temperature (see [42]). In this case, the information about pair creation comes from the imaginary part of the standard effective action. The extension of real-time techniques for finite-temperature quantum electrodynamics with unstable vacuum was presented in [32]. In this article, one can see that the relevant Green functions in a constant electric field are quite different from the standard proper-time representation given by Schwinger. Then, the relevant real-time one-loop effective action must be different from the standard one[6]. The standard imaginary-time formalism was derived under the assumption of thermal equilibrium and the appearance of a contradiction with the Pauli exclusion principle shows that the attempts of generalization to a far-from-equilibrium system have failed. The functional Schrödinger picture used in [25] to calculate the N^{cr} at high temperatures appears relevant. Its asymptotic expressions for N^{cr} at high temperatures are in agreement with ours in (98).

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[6] We will present the relevant real-time one-loop effective action elsewhere.

Appendix

I. For both the Bose and Fermi cases, the following relations hold:

$$ae^{a^\dagger Da} = e^{a^\dagger Da}e^{D}a\,,\;\; a^\dagger e^{a^\dagger Da} = e^{a^\dagger Da}a^\dagger e^{-D}\,, \tag{99}$$

$$e^{a^\dagger Da} =: \exp\left\{a^\dagger\left(e^{D}-1\right)a\right\}:, \tag{100}$$

where D is a matrix. To prove (100), let us consider the operator function $F(s) = e^{sa^\dagger Da}$, where s is a parameter. The function is a solution of the following equation:

$$\frac{dF(s)}{ds} = a^\dagger DaF(s)\,,\;\; F(0) = 1\,.$$

Using relation (99), we can rewrite the right-hand side of the equation as follows:

$$\frac{dF(s)}{ds} = a^\dagger F(s)De^{sD}a\,,\;\; F(0) = 1\,.$$

Now, we can verify that a solution of such an equation reads

$$F(s) =: \exp\left\{a^\dagger\left(e^{sD}-1\right)a\right\}:\,.$$

Setting $s = 1$, we justify (100).

II. We often use the well-known relation

$$e^{\lambda a}e^{a^\dagger\tilde{\lambda}} = e^{a^\dagger\tilde{\lambda}}e^{\lambda a}e^{\lambda\tilde{\lambda}}\,, \tag{101}$$

where λ and $\tilde{\lambda}$ are Grassmann-odd or Grassmann-even variables depending on statistics. For a product of two normal forms there is a generalization of (101),

$$:e^{a^\dagger Da}::e^{a^\dagger\tilde{D}a}:=:e^{a^\dagger(D+\tilde{D}+D\tilde{D})a}:, \tag{102}$$

where D and $\tilde{D}$ are some matrices.

III. The projection operator on the vacuum state can be written as follows:

$$P_0 = |0\rangle\langle 0| =: e^{-a^\dagger a}:\,. \tag{103}$$

Such a representation was first used by Berezin [33]. One can see that the operator P_0 obeys the equations

$$aP_0 = 0\,,\;\; P_0a^\dagger = 0\,,\;\; P_0|0\rangle = |0\rangle\,.$$

Using the Wick theorem, one can see that $:e^{-a^\dagger a}:$ is a solution of these equations.

IV. The trace of a normal product of creation and annihilation operators can be calculated by using the following path integral representation. Let $X\left(a^\dagger, a\right)$ be an operator expression of creation and annihilation operators a and $a^\dagger$. Then the trace of its normal form

$$\mathrm{tr}\left\{:X\left(a^\dagger,a\right):\right\} = \sum_{M=0}^{\infty}\sum_{\{m\}}(M!)^{-1}\langle 0|a_{m_M}\ldots a_{m_1}:X\left(a^\dagger,a\right):a^\dagger_{m_1}\ldots a^\dagger_{m_M}|0\rangle\,,$$

can be represented as the following vacuum mean value:

$$\mathrm{tr}\left\{:X\left(a^\dagger,a\right):\right\} = \langle 0|T:X\left(a^\dagger,a\right):e^{a(t_f)a^\dagger(t_i)}|0\rangle \tag{104}$$

where the notation $a = a\left(t_f\right)$, $a^\dagger = a^\dagger\left(t_i\right)$ is used for operators a to the left of $:X\left(a^\dagger,a\right):$ and $a^\dagger$ to the right of $:X\left(a^\dagger,a\right):$, and T is ordering operator putting $a\left(t_f\right)$ to the left of $:X\left(a^\dagger,a\right):$ and $a^\dagger\left(t_i\right)$ to the right of $:X\left(a^\dagger,a\right):$. Using either the Berezin path integral or the Gaussian integral over ordinary variables, depending on statistics, one can rewrite (104) as

$$\mathrm{tr}\left\{:X\left(a^\dagger,a\right):\right\} = \langle 0|\int\exp\left\{\kappa\lambda^*\lambda + \lambda^* a\right\}:X\left(a^\dagger,a\right):\exp\left\{a^\dagger\lambda\right\}\Pi d\lambda^* d\lambda|0\rangle, \tag{105}$$

where $a\left(t_f\right) = a$ and $a^\dagger\left(t_i\right) = a^\dagger$ are used after rewriting.

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