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One-loop energy-momentum tensor in QED with electric-like background

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Abstract

We have obtained nonperturbative one-loop expressions for the mean energy-momentum tensor and current density of Dirac's field on a constant electric-like background. One of the goals of this calculation is to give a consistent description of back-reaction in such a theory. Two cases of initial states are considered: the vacuum state and the thermal equilibrium state. First, we perform calculations for the vacuum initial state. In the obtained expressions, we separate the contributions due to particle creation and vacuum polarization. The latter contributions are related to the Heisenberg-Euler Lagrangian. Then, we study the case of the thermal initial state. Here, we separate the contributions due to particle creation, vacuum polarization, and the contributions due to the work of the external field on the particles at the initial state. All these contributions are studied in detail, in different regimes of weak and strong fields and low and high temperatures. The obtained results allow us to establish restrictions on the electric field and its duration under which QED with a strong constant electric field is consistent. Under such restrictions, one can neglect the back-reaction of particles created by the electric field. Some of the obtained results generalize the calculations of Heisenberg-Euler for energy density to the case of arbitrary strong electric fields.

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1 Introduction

It is well-known that quantum field theory (QFT) in external backgrounds is a good model for the study of quantum processes in cases when a part of a quantized field is strong enough to be treated as a classical one. Here, it is supposed that quantum processes under consideration do not affect significantly such a classical field (back-reaction is supposed to be small) and the field is treated as an external one. Of course, it is well-understood that, in principle, back-reaction must be calculated, at least, to evaluate the limits of applicability of the obtained results. However, the latter problem often remains open, being quite involved. From physical considerations, it is clear that back-reaction may be very strong, so it must be taken into account for external backgrounds that can create particles from vacuum (for theories with an unstable vacuum under the action of an external field) and produce actual work on particles. The effect of particle-creation from vacuum by an external field (vacuum instability in external fields) ranks among the most intriguing nonlinear phenomena in quantum theory. Its consideration is theoretically important, since it requires one to go beyond the scope of perturbation theory, and its experimental observation would verify the validity of a theory in the superstrong field domain. For the history of the subject, the reader may refer to the books [1, 2, 3, 4] and articles [5] for recent computational developments. Typical problems examined in the models with an unstable vacuum are related to the calculation of the density of particles created from vacuum. In some cases, this allows one to make phenomenological conclusions about back-reaction, see, e.g., [6]. However, a complete description of back-reaction is related to the calculation of mean values for the current density and energy-momentum tensor (EMT) of the matter field. This problem, in fact, is one of the aims of the present article. In this connection, we recall that Heisenberg and Euler calculated the mean value of energy density of Dirac field in constant electromagnetic background at zero temperature under the condition of weakness of electric field, $|E| \ll E_c = M^2c^3/|e|\hbar \simeq 1,3 \cdot 10^{16}\ V/cm$, see [7] Having this energy density in hands, they constructed a one-loop effective Lagrangian $\mathcal{L}$. It turns out that this Lagrangian was re-introduced by

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Schwinger [8] (for subsequent developments, see the review [9]) to define the vacuum-to-vacuum transition amplitude c_v as

$$c_v = \exp\left(i\int dx\mathcal{L}\right), \quad (1)$$

without any restrictions on the external field. He demonstrated that the probability P^v of a vacuum to remain a vacuum in a constant electric field is related to the imaginary part of $\mathcal{L}$ as follows:

$$P^v = |c_v|^2 = \exp\{-VT2\,\mathrm{Im}\,\mathcal{L}\}. \quad (2)$$

Here, T is the field duration, and V is the volume of observation. Obviously, the effect can actually be observed as soon as the external field strength approaches the critical value E_c. Given that, there remains the open question of what is the form of energy density, as well as the form of the entire EMT, in arbitrary constant fields. In the present article, we study the mean current density and EMT of a spinor field subject to the action of an electric-like external background, while paying a special attention to the initial state of thermal equilibrium. A more exact and detailed setting of the problem is given in the next section.

Aside from the principal importance of describing back-reaction for an uderstanding of the limits of applicability of QED in external fields, the calculations of the article are closely related to a wide class of problems (in the framework of quantum-field models in external fields) that recently attract attention. Below, we present a brief review of these problems and of their importance, which may be considered as an additional motivation for this research.

Even though an actual possibility of creating superstrong fields under laboratory conditions does not exist at present, e^+e^--pair production from vacuum by a slowly varying external electric field is relevant to phenomenology with the advent of a new laser technology capable of accessing the true strong-field domain. It is widely discussed nowadays [10] at the SLAC and TESLA X-ray laser facilities. Such strong fields may be relevant in astrophysics, where characteristic values of electromagnetic and gravitational fields near black holes are enormous. The Coulomb barrier at the quark surface of a hot strange star may be a powerful source of e^+e^--pairs, which are created in extremely strong constant electric fields (dosens of times higher that the critical field E_c) of the barrier and flow away from the star [11]. Such emission could be the main observational signature of quark stars. Electric field near a cosmic string can also become extremely strong [12].

Aside from purely QED problems, there exist closely related QFT problems in which the vacuum instability in various electric-like external backgrounds plays an important role, for example, phase transitions in non-Abelian theories, the problem of boundary conditions, or the influence of topology on the vacuum, the problem of a consistent vacuum construction in QCD and GUT, multiple particle-creation in the context of heavy-ion collisions, and so on. Recently, it has also been recognized that the presence of a background electric field must be taken into account in string theory constructions; see, e.g., [13] and references therein.

In the early 1970s, finite-temperature effects were recognized as very important for QFT, and then Dittrich [14] computed the one-loop effective potential of finite-temperature QED in the presence of a constant magnetic field, starting from the Schwinger relation (1). This was followed by an intense study of finite-temperature and finite-chemical-potential one-loop effects for fermions in a constant uniform magnetic field; for a review, see [15, 16]. This study was motivated by possible applications to various areas of physics, including astrophysics, condensed matter, particle physics, etc., where the existence of strong electromagnetic fields is possible. The presence of an electric-like field leads to a qualitative change (in comparison with a magnetic-like external field alone) in the character of processes involved, due to vacuum instability and the work of the field on the particles at the initial state.

In some of the above-menitoned examples with an electric-like external field, an initial state is in thermal equilibrium. In the chromoelectric flux-tube model [17], the back-reaction of created pairs induces a gluon mean field and plasma oscillations (see [6] and references therein). Various problems of cosmological QCD phase transitions and dark matter formation are discussed on the basis of the chromoelectric flux tube model (see, for example, [18] and references therein). It appears that for a calculation of particle creation in this model one needs to apply the general formalism of QFT for pair-production at finite temperatures and at zero temperature, both from vacuum and from many-particle states (see the corresponding physical reasons in [19, 20, 21]). The consideration of various time scales in heavy-ion collisions shows that the stabilization time (a time interval during which the characteristic asymptotic form for differential mean numbers of particles created by a constant chromoelectric field from vacuum is achieved) is far less than the period of plasma and mean-field oscillations. Then, the approximation of a strong quasiconstant chromoelectric field can be used in the treatment of such collisions during a period when the partons produced can be regarded as weakly coupled, due to the property of asymptotic freedom in QCD. It may also be reasonable to neglect dynamical back-reaction effects [22]. In the case of a strange star [11], it has been argued that there is a macroscopic time interval during which local thermal equilibrium is achieved, but the strong electric field has not yet been depleted due to e^+e^--pair production. By this reason, there has been a considerable interest in establishing a relevant formalism for finite-temperature QED with an unstable vacuum. An adequate techniques for such a case was proposed in [23]. The special case of thermally-influenced pair production in a constant electric field has been studied at the one-loop level in [19, 22, 24, 25, 26]. The results of [22, 24, 25], obtained by various methods in the framework of the generalized Furry representation, are in mutual agreement. The authors of [19, 26] came to different results, being, at the same time, in contradiction with themselves. We believe that the difficulties [19, 26] are related with attempts to generalize relation (1) to nonzero temperatures; see Discussion and Summary.

The paper is organized as follows. In Section 2, we describe the setting of the problem and recall some necessary details of the nonperturbative techniques that we use in our calculations. In Section 3, we calculate the current density and EMT for the vacuum initial state. In the obtained expressions, we separate the contributions due to particle creation and vacuum polarization. The latter contributions are related to the Heisenberg–Euler Lagrangian. In Section 4, we calculate the current density and EMT for the thermal initial state. In the obtained expressions, we separate the contributions due to particle creation, vacuum polarization, and the contributions that appear due to the work of the external field on the particles at the initial state. All these contributions are studied in detail in different regimes, limits of weak and strong field and low and high temperatures. We have established restrictions on the electric field and its duration under which QED with a strong constant electric field is consistent. Under such restrictions, one can neglect the back-reaction of particles created by the electric field. In the last section, we summarize and discuss the main results.

2 Setting the problem

We consider a quantized spinor (Dirac) field $\psi(x)$, $x = \left(x^0 = t, \mathbf{x}\right)$, in an external electromagnetic background specified by a quasiconstant electric field and a constant magnetic field. The stress tensor $F_{\mu\nu}$ of the field has nonzero invariants. In this case, we can always choose a reference frame in which the electric and magnetic fields are parallel and directed along the x^3-axis:

$$F_{\mu\nu} = F^E_{\mu\nu} + F^B_{\mu\nu}\,, \;\; F^E_{\mu\nu} = E(\delta^0_\mu\delta^3_\nu - \delta^3_\mu\delta^0_\nu)\,, \;\; F^B_{\mu\nu} = B(\delta^2_\mu\delta^1_\nu - \delta^1_\mu\delta^2_\nu)\,. \tag{3}$$

Our aim is to study the mean values of electric current and EMT in states that have evolved from vacuum and thermal equilibrium. It is well-known that a uniform constant electric field acting during an infinite time creates an infinite number of pairs from vacuum. There appear another divergences in the course of QED calculations related to this fact. That is why one needs a regularization to deal with these divergences. One type of such a regularization is the following: from the beginning, we consider a uniform electric field which efficiently acts only during a sufficiently large but finite time T. Such a regulirization was used in our works [3, 27, 28]. In these works, we studied particle creation effects in the so-called T-constant field, being a uniform electric field, constant (and nonzero) within a time interval T and zero outside this interval[1]. A similar regularization is used in the present article: we choose the T-constant field to switch on at the instant $t_1 = -T/2$ and to switch off at the instant $t_2 = T/2$. The results of our calculations will be presented in a form in which contributions of different nature are separated. These are contributions due to vacuum polarization, particle-creation from vacuum, and the contributions due to the work of the external field on the particles at the initial state. Some of these contributions have constant parts, as well as parts depending on the time interval $t - t_1$ that passes since the instant the electric field turns on. Among the contributions depending on $t - t_1$, there exist contributions both increasing and decreasing as the interval $t - t_1$ increases. Having the general expressions in hands, we study their dependence on the magnitude E of electric field and the time interval $t - t_1$, as well as their behavior at low and high temperatures. In the case of a strong electric field, when among the parts depending on $t - t_1$ the leading ones are the contributions due to particle creation, we are interested in the divergent parts related to large $t - t_1$. In the latter parts, we retain only the contributions leading in $t - t_1$. In particular, we study the dynamics of mean-energy growth due to particle creation from vacuum and due to the work of electric field on real particles. Making a comparison of this dynamics with the energy density of the external electric field, one can establish the limits of correct applicability of the concepts of a constant external field. For calculations, we use the technics and general results shortly outlined below in this section with the corresponding citations.

We suppose the initial state to be a state of free *in*-particles in termal equilibrium at a given temperature θ. In such a model, we are going to calculate the mean values of electric current and the EMT

[1]The physical interpretation in the case of a field that violates the stability of vacuum (for instance, a constant electric field) is extremely involved when the field is given by a space-dependent potential which does not disappear asymptotically. This happens because the state space of strong-field QFT is quite different from the state space of standard scattering theory, in which such a gauge for an external field is appropriate. In the general case of a space-dependent potential, one cannot use the standard definition of vacuum in QFT because such a state is well-defined only on a space-like hypersurface. It is then generally unclear when a strong electric field is given by a space-dependent potential, what particles and antiparticles are, what the vacuum state is, and what is the relation of the applied definitions to the standard QFT approach. In that case, instead of the standard evolution in time, one has to use the rather obscure "evolution in space".

of the Dirac field. In the general case, the background under consideration is intense, time-dependent, and violates the vacuum stability. Such a background must be treated nonperturbatively. Doing this, we follow the formulation proposed in [29]. Our calculations can be treated as the one-loop approximation within the method of complete QED, developed in [3, 30] and [23, 22]. Namely, this method provides a tehnique of calculating the mean values for systems with unstable vacuum, which is based on the generalized Furry picture for systems in strong external backgrounds (see [29, 3]); this technique may be interpreted as a generalization of the Schwinger–Keldysh technique ([31]; see also the review [32]) on systems with unstable vacuum.

It is supposed that in the Heisenberg picture (see [3, 22] for notation) there exists a set of creation and annihilation operators $a_n^\dagger(in)$, $a_n(in)$ of *in*-particles (electrons), and similar operators $b_n^\dagger(in)$, $b_n(in)$ of *in*-antiparticles (positrons), with the corresponding *in*-vacuum $|0,in\rangle$, and a set of creation and annihilation operators $a_n^\dagger(out)$, $a_n(out)$, of *out*-electrons and similar operators $b_n^\dagger(out)$, $b_n(out)$ of *out*-positrons, with the corresponding *out*-vacuum $|0,out\rangle$. By n we denote a complete set of possible quantum numbers. The *in*- and *out*-operators obey the canonical anticommutation relations

$$[a_n(in),a_m^\dagger(in)]_+ = [a_n(out),a_m^\dagger(out)]_+ = [b_n(in),b_m^\dagger(in)]_+ = [b_n(out),b_m^\dagger(out)]_+ = \delta_{nm}\,. \tag{4}$$

The *in*-particles ($\zeta = +$ for electrons and $\zeta = -$ for positrons) are associated with a complete set of solutions of the Dirac equation with the external electromagnetic field, given by a vector potential $\mathbf{A}(t,\mathbf{x})$ only; *in*-set $\{{}_\zeta\psi_n(x)\}$ with asymptotics ${}_\zeta\psi_n(t_1,\mathbf{x})$ at the initial time-instant t_1 that are eigenvectors of the one-particle Dirac Hamiltonian $\mathcal{H}(t) = \gamma^0([M + \boldsymbol{\gamma}(i\boldsymbol{\nabla} - q\mathbf{A}(t,\mathbf{x}))])$,

$$\mathcal{H}(t_1)\,{}_\zeta\psi_n(t_1,\mathbf{x}) = \zeta\varepsilon_n^{(\zeta)}\,{}_\zeta\psi_n(t_1,\mathbf{x})\,, \tag{5}$$

where $\varepsilon_n^{(\zeta)}$ are the energies of *in*-particles in a state specified by a complete set of quantum numbers n, and $\varepsilon_n^{(\pm)} > 0$. The *out*-particles are associated with a complete *out*-set of solutions $\{{}^\zeta\psi_n(x)\}$ of the Dirac equation with asymptotics ${}^\zeta\psi_n(t_2,\mathbf{x})$ at t_2 that are eigenvectors of the one-particle Dirac Hamiltonian at t_2, namely,

$$\mathcal{H}(t_2)\,{}^\zeta\psi_n(t_2,\mathbf{x}) = \zeta\tilde{\varepsilon}_n^{(\zeta)}\,{}^\zeta\psi_n(t_2,\mathbf{x})\,, \tag{6}$$

where $\tilde{\varepsilon}_n^{(\pm)}$ are the energies of *out*-particles in a state specified by a complete set of quantum numbers n, and $\tilde{\varepsilon}_n^{(\pm)} > 0$.

The *out*-set can be decomposed in the *in*-set as follows:

$${}^\zeta\psi(x) = {}_+\psi(x)G\left({}_+|{}^\zeta\right) + {}_-\psi(x)G\left({}_-|{}^\zeta\right)\,, \tag{7}$$

where the decomposition coefficients $G\left({}_\zeta|{}^{\zeta'}\right)$ are expressed via inner products of these sets. These coefficients obey unitary conditions that follows from normalization conditions for the solutions.

The Hamiltonian $H(t)$ of the quantized Dirac field is time-dependent due to the external field. It is diagonalized (and has a canonical form) in terms of the first set at the initial time instant, and is diagonalized (and has a canonical form) in terms of the second set at the final time instant. For example,

$$H(t_1) = \sum_n \left[\varepsilon_n^{(+)}a_n^\dagger(in)\,a_n(in) + \varepsilon_n^{(-)}b_n^\dagger(in)\,b_n(in)\right]\,. \tag{8}$$

Correspondingly, the initial vacuum is defined by $a_n(in)|0,in\rangle = b_n(in)|0,in\rangle = 0$ for every n.

All the information about the processes of particle creation, annihilation, and scattering is contained in the elementary probability amplitudes

$$\begin{aligned}
&w(+|+)_{mn} = c_v^{-1} < 0,out\left|a_m(out)a_n^\dagger(in)\right|0,in >= G^{-1}\left({}_+|{}^+\right)_{mn}\,,\\
&w(-|-)_{nm} = c_v^{-1} < 0,out\left|b_m(out)b_n^\dagger(in)\right|0,in >= \left[G^{-1}\left({}_-|{}^-\right)\right]_{nm}^\dagger\,,\\
&w(0|-+)_{nm} = c_v^{-1} < 0,out\left|b_n^\dagger(in)a_m^\dagger(in)\right|0,in >= -\left[G\left({}_+|{}^-\right)G^{-1}\left({}_-|{}^-\right)\right]_{nm}^\dagger\,,\\
&w(+-|0)_{mn} = c_v^{-1} < 0,out\left|a_m(out)b_n(out)\right|0,in >= \left[G^{-1}\left({}_+|{}^+\right)G\left({}_+|{}^-\right)\right]_{mn}\,,\\
&c_v = \langle 0,out|0,in\rangle\,,
\end{aligned} \tag{9}$$

where c_v is the vacuum-to-vacuum transition amplitude.

The sets of *in* and *out*-operators are related to each other by a linear canonical transformation (sometimes called the Bogolubov transformation). It has been demonstrated that in the general case such a relation has the form (see [29])

$$V\left(a^\dagger(out),a(out),b^\dagger(out),b(out)\right)V^\dagger = \left(a^\dagger(in),a(in),b^\dagger(in),b(in)\right)\,,\ |0,in\rangle = V|0,out\rangle\,, \tag{10}$$

where the unitary operator V reads[2]

$$
\begin{aligned}
V &= \exp\left\{-a^{\dagger}(out)w\,(+-|0)\,b^{\dagger}(out)\right\}\exp\left\{-b(out)\ln w\,(-|-)\,b^{\dagger}(out)\right\} \\
&\times\exp\left\{a^{\dagger}(out)\ln w\,(+|+)\,a(out)\right\}\exp\left\{-b(out)w\,(0|-+)\,a(out)\right\},
\end{aligned} \tag{11}
$$

and

$$
c_v = \langle 0, out|V|0, out\rangle = \exp\left\{-\mathrm{Tr}\ln w\,(-|-)\right\}. \tag{12}
$$

Note that exact formula (12) holds true for any external field. It was demonstrated in [33] that in the case of a constant electric-like field the Schwinger formula (1) leads to the same result.

As the final purpose, we are going to calculate and analyze the following mean values:

$$
\langle j_\mu(t)\rangle = \mathrm{Tr}\left[\rho_{in} j_\mu\right], \quad \langle T_{\mu\nu}(t)\rangle = \mathrm{Tr}\left[\rho_{in} T_{\mu\nu}\right], \tag{13}
$$

where ρ_{in} is a density operator of the initial state in the Heisenberg picture, the operators of current density j_μ and energy-momentum tensor $T_{\mu\nu}$ have the form

$$
\begin{aligned}
j_\mu &= \frac{q}{2}\left[\bar\psi(x), \gamma_\mu\,\psi(x)\right], \quad T_{\mu\nu} = \frac{1}{2}\left(T^{can}_{\mu\nu} + T^{can}_{\nu\mu}\right), \\
T^{can}_{\mu\nu} &= \frac{1}{4}\left\{\left[\bar\psi(x), \gamma_\mu P_\nu\,\psi(x)\right] + \left[P^{*}_{\nu}\bar\psi(x), \gamma_\mu\,\psi(x)\right]\right\},
\end{aligned}
$$

$P_\mu = i\partial_\mu - qA_\mu(x)$ and $\psi(x)$ are are Dirac field operators in the Heisenberg representation, that obey the Dirac equation with the external field $A_\mu(x)$.

In the next section, we calculate and analize the mean values (13) for the initial vacuum state. We denote them as

$$
\langle j_\mu(t)\rangle^0 = \langle 0, in|j_\mu|0, in\rangle, \quad \langle T_{\mu\nu}(t)\rangle^0 = \langle 0, in|T_{\mu\nu}|0, in\rangle. \tag{14}
$$

Then, we analyze the more complicated case (13), in which the initial state of the system under consideration is prepared as an equilibrium state of noninteracting *in*-particles at temperature θ with the chemical potentials $\mu^{(\zeta)}$ and is characterized by the density operator ρ_{in} in the Heisenberg picture,

$$
\rho_{in} = Z^{-1}\exp\left\{\beta\left(\sum_{\zeta=\pm}\mu^{(\zeta)}N^{(\zeta)} - H(t_1)\right)\right\}, \quad \mathrm{Tr}\,\rho_{in} = 1, \tag{15}
$$

where Z is a normalization constant, $\beta = \theta^{-1}$, and $N^{(\zeta)}$ are operators of *in*-paricles numbers,

$$
N^{(+)} = \sum_n a^{\dagger}_n(in)\,a_n(in), \quad N^{(+)} = \sum_n b^{\dagger}_n(in)\,b_n(in).
$$

In our calculations, we also need the following matrix elements:

$$
\begin{aligned}
\langle j_\mu(t)\rangle^c &= \langle 0, out|j_\mu|0, in\rangle c_v^{-1}, \quad \langle T_{\mu\nu}(t)\rangle^c = \langle 0, out|T_{\mu\nu}|0, in\rangle c_v^{-1}, \\
\langle j_\mu(t)\rangle^0_{out} &= \langle 0, out|j_\mu|0, out\rangle, \quad \langle T_{\mu\nu}(t)\rangle^0_{out} = \langle 0, out|T_{\mu\nu}|0, out\rangle.
\end{aligned} \tag{16}
$$

The matrix elements (13), (14), (16) can be expressed via relevant singular functions of the Dirac fields, as follows:

$$
\begin{aligned}
&i\mathrm{Tr}\left\{\rho_{in}T\psi(x)\,\bar\psi(x')\right\} = S^c_{in}(x,x') + S^\theta(x,x'), \\
&i\mathrm{Tr}\left\{\rho_{in}\bar\psi(x')\,\psi(x)\right\} = S^+_{in}(x,x') - S^\theta(x,x'), \\
&i\mathrm{Tr}\left\{\rho_{in}\psi(x)\,\bar\psi(x')\right\} = S^-_{in}(x,x') + S^\theta(x,x'), \\
&S^c_{in}(x,x') = \theta(x_0 - x'_0)\,S^-_{in}(x,x') - \theta(x'_0 - x_0)\,S^+_{in}(x,x'),
\end{aligned} \tag{17}
$$

where $S^{(\ldots)}_{in}$ are singular functions at zero temperature,

$$
\begin{aligned}
S^c_{in}(x,x') &= i\langle 0, in|T\psi(x)\bar\psi(x')|0, in\rangle, \\
S^-_{in}(x,x') &= i\langle 0, in|\psi(x)\bar\psi(x')|0, in\rangle, \\
S^+_{in}(x,x') &= i\langle 0, in|\bar\psi(x')\psi(x)|0, in\rangle.
\end{aligned} \tag{18}
$$

[2]Here and elsewhere, we use condenced notations, for example,

$$
bw\,(0|-+)\,a = \sum_{n,m} b_n w\,(0|-+)_{nm}\,a_m\,.
$$

All the singular functions can be expressed in terms of a complete and orthonormal set of solutions $\{_\pm\psi_n(x)\}$ of the Dirac equation with an external field, that have beene defined above,

$$
\begin{aligned}
&S^{\mp}_{in}(x,x') = i\sum_n {}_\pm\psi_n(x)_\pm\bar{\psi}_n(x')\,, \\
&S^{\theta}(x,x') = \tilde{S}^{-}_{\theta}(x,x') - \tilde{S}^{+}_{\theta}(x,x')\,, \\
&\tilde{S}^{\mp}_{\theta}(x,x') = -i\sum_n {}_\pm\psi_n(x)_\pm\bar{\psi}_n(x')\,N^{(\pm)}_n(in)\,,
\end{aligned} \tag{19}
$$

where

$$
N^{(\zeta)}_n(in) = \left[\exp\left\{\beta\left(\varepsilon^{(\zeta)}_n - \mu^{(\zeta)}\right)\right\} + 1\right]^{-1}. \tag{20}
$$

It is important to stress that in the general case S^c_{in} differs from the causal Green function,

$$
S^c(x,x') = i\langle 0, out|T\psi(x)\bar{\psi}(x')|0, in\rangle c_v^{-1}. \tag{21}
$$

From (21), it follows that

$$
\begin{aligned}
&S^c(x,x') = \theta(x_0 - x_0')\,S^-(x,x') - \theta(x_0' - x_0)\,S^+(x,x')\,, \\
&S^-(x,x') = i\sum_{n,m} {}^+\psi_n(x)\,G\left({}_+|^+\right)^{-1}_{nm}\,{}_+\bar{\psi}_m(x')\,, \\
&S^+(x,x') = i\sum_{n,m} {}_-\psi_n(x)\left[G\left({}_-|^-\right)^{-1}\right]^{\dagger}_{nm}\,{}^-\bar{\psi}_m(x')\,,
\end{aligned} \tag{22}
$$

see [29]. Then the difference $S^p(x,x') = S^c_{in}(x,x') - S^c(x,x')$ has the form

$$
S^p(x,x') = i\sum_{nm} {}_-\psi_n(x)\,\left[G({}_+|^-)G({}_-|^-)^{-1}\right]^{\dagger}_{nm}\,{}_+\bar{\psi}_m(x'). \tag{23}
$$

This function vanishes for the case of a stable vacuum, since it contains the coefficients $G({}_+|^-)$ related to the mean number of created particles.

Another kind of Green's function that will be used in the following is given by

$$
S^c_{out}(x,x') = i\langle 0, out|T\psi(x)\bar{\psi}(x')|0, out\rangle. \tag{24}
$$

It is related to $S^c(x,x')$ as follows:

$$
\begin{aligned}
&S^c_{out}(x,x') = S^c(x,x') + S^{\bar{p}}(x,x')\,, \\
&S^{\bar{p}}(x,x') = -i\sum_{nm} {}^+\psi_n(x)\,\left[G({}_+|^+)^{-1}G({}_+|^-)\right]_{nm}\,{}^-\bar{\psi}_m(x').
\end{aligned} \tag{25}
$$

We can see that there exist relations between the mean values (13), (14) and matrix elements (16), namely,

$$
\begin{aligned}
\langle j_\mu(t)\rangle = {}& \langle j_\mu(t)\rangle^0 + \langle j_\mu(t)\rangle^\theta, \;\; \langle T_{\mu\nu}(t)\rangle = \langle T_{\mu\nu}(t)\rangle^0 + \langle T_{\mu\nu}(t)\rangle^\theta, \\
\langle j_\mu(t)\rangle^0 = {}& \langle j_\mu(t)\rangle^c + \langle j_\mu(t)\rangle^p, \;\; \langle T_{\mu\nu}(t)\rangle^0 = \langle T_{\mu\nu}(t)\rangle^c + \langle T_{\mu\nu}(t)\rangle^p, \\
\langle j_\mu(t)\rangle^0_{out} = {}& \langle j_\mu(t)\rangle^c + \langle j_\mu(t)\rangle^{\bar{p}}, \;\; \langle T_{\mu\nu}(t)\rangle^0_{out} = \langle T_{\mu\nu}(t)\rangle^c + \langle T_{\mu\nu}(t)\rangle^{\bar{p}}.
\end{aligned} \tag{26}
$$

These relations involve the quantities

$$
\begin{aligned}
&\langle j_\mu(t)\rangle^{c,p,\bar{p},\theta} = iq\,\mathrm{tr}\left[\gamma_\mu S^{c,p,\bar{p},\theta}(x,x')\right]\Big|_{x=x'}\,, \\
&\langle T_{\mu\nu}(t)\rangle^{c,p,\bar{p},\theta} = i\,\mathrm{tr}\left[A_{\mu\nu}S^{c,p,\bar{p},\theta}(x,x')\right]\Big|_{x=x'}\,, \\
&A_{\mu\nu} = 1/4\left[\gamma_\mu\left(P_\nu + P'^{*}_\nu\right) + \gamma_\nu\left(P_\mu + P'^{*}_\mu\right)\right]\,, \\
&P'^{*}_\mu = -i\frac{\partial}{\partial x'^\mu} - q\,A_\mu(x')\,,
\end{aligned} \tag{27}
$$

where $\mathrm{tr}\,[\cdots]$ is the trace in the space of 4×4 matrices, and the equality $x = x'$ is understood as

$$
\mathrm{tr}\,[\cdots(x,x')]|_{x=x'} = \frac{1}{2}\left[\lim_{t\to t'-0}\mathrm{tr}\,[\cdots(x,x')] + \lim_{t\to t'+0}\mathrm{tr}\,[\cdots(x,x')]\right]\Bigg|_{\mathbf{x}=\mathbf{x}'}.
$$

The quantities (27) are expressed in terms of $S^c(x,x')$, which is the causal Green function (propagator) of the Dirac equation with external field, and in terms of $S^{p,\bar{p},\theta}$, which are solutions of the same equation. The quantities $< j_\mu(t) >^0$, $< T_{\mu\nu}(t) >^0$, $< j_\mu(t) >^0_{out}$, and $< T_{\mu\nu}(t) >^0_{out}$ are contributions to the corresponding mean values at zero temperature and density. The quantities $< j_\mu(t) >^\theta$and $< T_{\mu\nu}(t) >^\theta$ present the contributions due to the existence of the initial thermal distribution.

The quantities $\langle j_\mu(t)\rangle^0$, $\langle T_{\mu\nu}(t)\rangle^0$, $\langle j_\mu(t)\rangle^0_{out}$, $\langle T_{\mu\nu}(t)\rangle^0_{out}$, $\langle j_\mu(t)\rangle^\theta$, and $\langle T_{\mu\nu}(t)\rangle^\theta$ are real-valued by construction, due to the properties of the singular functions S^c_{in}, S^c_{out} and S^θ. On the contrary, the quantities $\langle j_\mu(t)\rangle^{c,p,\bar{p}}$ and $\langle T_{\mu\nu}(t)\rangle^{c,p,\bar{p}}$ are not necessarily real. For the purpose of the following consideration, it is useful to rewrite $\langle j_\mu(t)\rangle^0$ and $\langle T_{\mu\nu}(t)\rangle^0$ in the form

$$\begin{aligned}
\langle j_\mu(t)\rangle^0 &= \operatorname{Re}\langle j_\mu(t)\rangle^c + \operatorname{Re}\langle j_\mu(t)\rangle^p\,,\\
\langle T_{\mu\nu}(t)\rangle^0 &= \operatorname{Re}\langle T_{\mu\nu}(t)\rangle^c + \operatorname{Re}\langle T_{\mu\nu}(t)\rangle^p\,,\\
\langle j_\mu(t)\rangle^0_{out} &= \operatorname{Re}\langle j_\mu(t)\rangle^c + \operatorname{Re}\langle j_\mu(t)\rangle^{\bar{p}}\,,\\
\langle T_{\mu\nu}(t)\rangle^0_{out} &= \operatorname{Re}\langle T_{\mu\nu}(t)\rangle^c + \operatorname{Re}\langle T_{\mu\nu}(t)\rangle^{\bar{p}}\,.
\end{aligned} \tag{28}$$

3 Initial state as a vacuum

3.1 T-constant field regularization

In this section, we study the quantities $< j_\mu(t) >^0$ and $< T_{\mu\nu}(t) >^0$ that represent contributions to the corresponding mean values at zero temperature. To calculate such quantities, the T-constant field regularization is necessary. In particular, we are going to study the leading contributions to the divergent parts of the quantities at $T \to \infty$. To make the consideration complete, and to provide the reader with some formulas necessary for the further consideration, we start by reproducing some of the results on the mean numbers of created particle that were obtained in [28].

We chose the T-constant field potentials A^E_μ in the following form, $A^E_0 = A^E_1 = A^E_2 = 0$:

$$A^E_3(t) = \begin{cases} Et_1\,, & t\in(-\infty,t_1)\,,\\ Et\,, & t\in[t_1,t_2],\\ Et_2\,, & t\in(t_2,+\infty)\,,\end{cases}$$

where $t_2 = -t_1 = T/2$. The corresponding electric field $E(t)$ is given by

$$E(t) = \begin{cases} 0\,, & t\in(-\infty,t_1)\,,\\ E\,, & t\in[t_1,t_2],\\ 0\,, & t\in(t_2,+\infty)\,,\end{cases} \tag{29}$$

First, let us suppose that magnetic field is absent.

For the purpose of T-regulirization, it is sufficient to choose, for $\{{}_\zeta\psi_n(x)\}$ and $\{{}^\zeta\psi_n(x)\}$, defined above, the following orthonormalized sets of solutions of the Dirac equation with a constant electric field:

$$\begin{aligned}
&{}_\pm\psi_{\mathbf{p},r}(x) = (\gamma P+M)\,{}_\pm\phi_{\mathbf{p},\pm1,r}(x)\,,\quad {}^\pm\psi_{\mathbf{p},r}(x) = (\gamma P+M)\,{}^\pm\phi_{\mathbf{p},\mp1,r}(x)\,,\\
&{}_\pm\phi_{\mathbf{p},s,r}(x) = {}_\pm\phi_{\mathbf{p},s}(t)\exp\{i\mathbf{px}\}v_{s,r}\,,\quad {}^\pm\phi_{\mathbf{p},s,r}(x) = {}^\pm\phi_{\mathbf{p},s}(t)\exp\{i\mathbf{px}\}v_{s,r}\,,\\
&{}^-_+\phi_{\mathbf{p},s}(t) = CD_{\nu-\frac{1+s}{2}}(\pm(1-i)\xi)\,,\quad {}^+_-\phi_{\mathbf{p},s}(t) = CD_{-\nu-\frac{1-s}{2}}(\pm(1+i)\xi)\,,\ s=\pm1\,,
\end{aligned} \tag{30}$$

where $\mathbf{p}$ is the momentum and $r = \pm1$ is the spin projection; $D_\nu(z)$ is the Weber parabolic cylinder (WPC) function [34], and

$$\begin{aligned}
&\nu = \frac{i\lambda}{2},\ \ \lambda = \frac{M^2+\mathbf{p}_\perp^2}{|qE|},\ \mathbf{p}_\perp = (p^1,p^2,0),\ \ \xi = \xi(t) = \frac{qEt-p_3}{\sqrt{|qE|}},\\
&C = (2\pi)^{-3/2}(2|qE|)^{-1/2}\exp(-\pi\lambda/8)\,,
\end{aligned}$$

$v_{s,r}$ are constant orthonormal spinors, $v^\dagger_{s,r}v_{s,r'} = \delta_{r,r'}$, subject to the supplementary condition $(1\pm\gamma^0\gamma^3)v_{\mp1,r} = 0$. Using an asymptotic expansion of the WPC-function [34],

$$D_\nu(z) = z^\nu\exp\left(-\frac{z^2}{4}\right)\left(\sum_{n=0}^N\frac{(-\frac{1}{2}\nu)_n(\frac{1}{2}-\frac{1}{2}\nu)_n}{n!(-\frac{1}{2}z^2)^n} + O(|z|^{-2(N+1)})\right),\ |\arg z| < 3\pi/4, \tag{31}$$

we can obtain the asymptotics of particle energies as $T\to\infty$,

$$\varepsilon^{(\zeta)}_{\mathbf{p}} = |\frac{qET}{2}+p_3|\,,\ \ \tilde{\varepsilon}^{(\zeta)}_{\mathbf{p}} = |\frac{qET}{2}-p_3|\,. \tag{32}$$

One can see that the matrices $G\left({}_{\zeta}|{}^{\zeta'}\right)$ (7) are diagonal:

$$G\left({}_{\zeta}|{}^{\zeta'}\right)_{\mathbf{p},r,\mathbf{p}',r'} = \delta_{r,r'}\delta(\mathbf{p}-\mathbf{p}')g\left({}_{\zeta}|{}^{\zeta'}\right). \tag{33}$$

The differential mean numbers of electrons (equal to the corresponding differential mean number of pairs) with a given momentum $\mathbf{p}$ and spin projections r created from vacuum are

$$\aleph_{\mathbf{p},r} = \langle 0, in|a^{\dagger}_{\mathbf{p},r}(out)a_{\mathbf{p},r}(out)|0, in\rangle = \left|g\left({}_{-}|{}^{+}\right)\right|^2, \tag{34}$$

were the standard volume regularization is used, so that $\delta(\mathbf{p}-\mathbf{p}') \to \delta_{\mathbf{p},\mathbf{p}'}$. Note that $\aleph_{\mathbf{p},r}$ is even with respect to the sign of qE.

If the time T is sufficiently large, $T >> T_0 = (1+\lambda)/\sqrt{|qE|}$, the differential mean numbers $\aleph_{\mathbf{p},r}$ have the form

$$\aleph_{\mathbf{p},r} = \begin{cases} e^{-\pi\lambda}\left[1 + O\left(\left[\frac{1+\lambda}{K_p}\right]^3\right)\right], & -\sqrt{|qE|}\frac{T}{2} \le \bar{\xi} \le -K_p, \\ O(1), \ -K_p < \bar{\xi} \le K_p, \\ O\left(\left[\frac{1+\lambda}{\bar{\xi}^2}\right]^3\right), & \bar{\xi} > K_p, \end{cases} \tag{35}$$

where

$$\bar{\xi} = \frac{|p_3| - |qE|\,T/2}{\sqrt{|qE|}},$$

and K_p is a sufficiently large arbitrary constant, $K_p >> 1+\lambda$, see [28]. In the limit $T \to \infty$, the differential mean numbers have a simple form:

$$\aleph_{\mathbf{p},r} = e^{-\pi\lambda}, \tag{36}$$

which coincides with the one obtained in a constant electric field by Nikishov [33]. One can see that the stabilization of the differential mean numbers to the asymptotic form (36) for finite longitudinal momenta is reached at $T >> T_0$. The characteristic time T_0 is called the stabilization time.

In order to study the effects of switching on and off (of the electric field) at $T >> T_0$, one can consider another example of a quasiconstant electric field:

$$E(t) = E\cosh^{-2}\left(\frac{t}{\alpha}\right). \tag{37}$$

This field switches on and off adiabatically as $t \to \pm\infty$ and is quasiconstant at finite times. The differential mean numbers of particles created by such a field were found in [35]. As shown in [28], the differential mean numbers in the field (37) have the asymptotic form (36) for sufficiently large α, $\alpha >> \alpha_0 = (1+\sqrt{\lambda})/\sqrt{|qE|}$ and for $|qE|\,\alpha \gg |p_3|$. Thus, α_0 can be interpreted as the stabilization time for such a field. At the same time, the latter implies that the effects of switching on and off are not essential at large times and finite longitudinal momenta for both fields. Extrapolating this conclusion, one may suppose that in any electric field being quasiconstant $\approx E$ at least for a time period $T >> T_0$ and switching on and off outside this period particle-creation effects have no dependence whatsoever on the details of switching on and off. Thus, calculations in a T-constant field are representative for a large class of quasiconstant electric fields.

3.2 Contributions due to particle creation

Let us apply these results to a calculation of the leading terms in $\mathrm{Re}\langle j_\mu(t)\rangle^{p,\bar{p}}$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^{p,\bar{p}}$ from (28) at $T \to \infty$, or, more specifically, to the case of a time interval that passes since the instant of switching-on of the electric field, $t - t_1 = t + T/2$, being sufficiently large. In fact, we use the following dimensionless combination $\sqrt{|qE|}\,(t+T/2)$ to determine large time intervals. Then, our consideration holds for

$$\begin{aligned} &\sqrt{|qE|}\,(t+T/2) \gg 1 + m^2/\,|qE|\,, \ \ t \le T/2\,, \\ &\sqrt{|qE|}T \gg 1 + m^2/\,|qE|\,, \ \ t > T/2\,. \end{aligned} \tag{38}$$

One can see that this is a stabilization condition when the leading terms do not depend on the details of switching on and off. As follows from (27), we need the singular functions S^p (23) and $S^{\bar{p}}$ (25) at $x \approx x'$ in such an approximation, which provides the required contribution to $\mathrm{Re}\langle j_\mu(t)\rangle^{p,\bar{p}}$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^{p,\bar{p}}$ when (38) is valid. Time-dependence arises due to integration over p_3 under the condition $-\sqrt{|qE|}\frac{T}{2} \le \bar{\xi} \le -K$

with the subsidiary condition that $|\xi(t)|$ must be sufficiently large, $|\xi(t)| \geq K$, where K is subject to the condition $K \gg 1 + m^2/|qE|$. The distribution $\aleph_{\mathbf{p},r}$ plays the role of a cut-off factor in the integral over $p_\perp$, the T-dependent contribution of $S^{p,\bar{p}}$ being thus convergent.

The range of integration over momenta in S^p that determines the leading contributions can be defined as

$$D : \begin{cases} |\mathbf{p}_\perp| \leq \sqrt{|qE|}\left[\sqrt{|qE|}\,(t + T/2) - K\right]^{1/2} \\ -T/2 - K/\sqrt{|qE|} \leq p_3/qE \leq t - K/\sqrt{|qE|} \end{cases} .$$

Using (7) to express the solutions ${}_\pm\psi_n$ via ${}^\pm\psi_n$ and taking into account the asymptotic expansion (31), we can calculate the leading contributions to the function $S^p(x,x')$ (23),

$$S^p(x,x') = -i\int_D d\mathbf{p} \sum_{r=\pm 1} \aleph_{\mathbf{p},r}\left[{}^+\psi_{\mathbf{p},r}(x){}^+\bar{\psi}_{\mathbf{p},r}(x') - {}^-\psi_{\mathbf{p},r}(x){}^-\bar{\psi}_{\mathbf{p},r}(x')\right] . \quad (39)$$

Making summation over r, we represent (39) as

$$\begin{aligned} S^p(x,x') &= (\gamma P + M)\Delta^p(x,x') , \\ \Delta^p(x,x') &= i\int_D \aleph_{\mathbf{p},0}\exp\{i\mathbf{p}\,(\mathbf{x}-\mathbf{x}')\}\left[{}^+\phi_{\mathbf{p},0}(t){}^+\phi^*_{\mathbf{p},0}(t') +{}^- \phi_{\mathbf{p},0}(t){}^-\phi^*_{\mathbf{p},0}(t')\right] d\mathbf{p} . \end{aligned} \quad (40)$$

Then, integrating over $\mathbf{p}_\perp$ in (40), we obtain

$$\Delta^p(x,x') = i\int_{-T/2+K/\sqrt{|qE|}}^{t-K/\sqrt{|qE|}} h_\parallel\left(x_\parallel, x'_\parallel\right) h_\perp\left(\mathbf{x}_\perp, \mathbf{x}'_\perp\right) d\left(\frac{p_3}{qE}\right) , \quad (41)$$

where

$$\begin{aligned} h_\parallel\left(x_\parallel, x'_\parallel\right) &= \frac{qE}{2\pi\,|qEt - p_3|}\exp\left\{\frac{i}{2}\left[\xi(t')^2 - \xi(t)^2\right] + ip_3\,(x_3 - x'_3)\right\} , \\ h\left(\mathbf{x}_\perp, \mathbf{x}'_\perp\right) &= (2\pi)^{-2}\,|qE|\exp\left(-\frac{\pi m^2}{|qE|} - \frac{(\mathbf{x}_\perp - \mathbf{x}'_\perp)^2\,|qE|}{4\pi}\right) , \end{aligned} \quad (42)$$

and the notation $x^\mu_\parallel = \left(x^0, 0, 0, x^3\right)$ and $\mathbf{x}_\perp = \left(x^1, x^2, 0\right)$ is used.

In the same manner, we calculate the leading contributions to the function $S^{\bar{p}}(x,x')$ (25),

$$\begin{aligned} S^{\bar{p}}(x,x') &= (\gamma P + M)\Delta^{\bar{p}}(x,x') , \\ \Delta^{\bar{p}}(x,x') &= i\int_{\bar{D}} \aleph_{\mathbf{p},0}\exp\{i\mathbf{p}\,(\mathbf{x}-\mathbf{x}')\}\left[{}_+\phi_{\mathbf{p},0}(t){}_+\phi^*_{\mathbf{p},0}(t') +{}_- \phi_{\mathbf{p},0}(t){}_-\phi^*_{\mathbf{p},0}(t')\right] d\mathbf{p} . \end{aligned} \quad (43)$$

The range of integration $\bar{D}$ reads

$$\bar{D} : \begin{cases} |\mathbf{p}_\perp| \leq \sqrt{|qE|}\left[\sqrt{|qE|}\,(t + T/2) - K\right]^{1/2} \\ t + K/\sqrt{|qE|} \leq p_3/qE \leq T/2 - K/\sqrt{|qE|} \end{cases} .$$

Integrating over $\mathbf{p}_\perp$ in (43), we find

$$\Delta^{\bar{p}}(x,x') = i\int_{t+K/\sqrt{|qE|}}^{T/2-K/\sqrt{|qE|}} h_\parallel\left(x_\parallel, x'_\parallel\right) h_\perp\left(\mathbf{x}_\perp, \mathbf{x}'_\perp\right) d\left(\frac{p_3}{qE}\right) , \quad (44)$$

The following results will be outlined in the presence of a constant magnetic field, $B \neq 0$. For such a field, we choose the nonzero potential as $A^B_1 = Bx^2$. Now, the complete set of quantum numbers that discribes particles is (p_1, n_B, p_3, r), $n_B = 0, 1, \ldots$, $\lambda = \left(M^2 + |qB|(2n_B + 1 - r)\right)|qE|^{-1}$. The space-time dependent part of the function $\phi_{\mathbf{p},s,r}(x)$ (see (30)) is modified as

$$\begin{aligned} &(2\pi)^{-3/2}\exp\{i\mathbf{p}\mathbf{x}\} \to (2\pi)^{-1/2}\exp\{-ip_3x^3\}\phi_{p_1,n_B,r}(\mathbf{x}_\perp) , \\ &\phi_{p_1,n_B,r}(\mathbf{x}_\perp) = \left(\frac{\sqrt{|qB|}}{2^{n_B+1}\pi^{3/2}n_B!}\right)^{1/2}\exp\left\{-ip_1x^1 - \frac{|qB|}{2}\left(x^2 - \frac{p_1}{qB}\right)^2\right\} \\ &\times\mathcal{H}_{n_B}\left[\sqrt{|qB|}\left(x^2 - \frac{p_1}{qB}\right)\right] . \end{aligned} \quad (45)$$

Here, $\mathcal{H}_{n_B}(x)$ are the Hermite polynomials. Then, integration over $\mathbf{p}_\perp$ in all of the above formulas must be replaced by integration over p_1 from $-\infty$ to $+\infty$ with summation over the integer quantum numbers n_B in the interval

$$n_B \le \left[\sqrt{|qE|}\,(t+T/2)-K\right]|E/2B| \; .$$

After that, one can see that formula (41) needs only one modification. Namely, the new (in the presence of magnetic field) value of $h_\perp(\mathbf{x}_\perp,\mathbf{x}'_\perp)$ reads

$$h_\perp(\mathbf{x}_\perp,\mathbf{x}'_\perp)=\frac{qB\exp\left(\pi\Sigma^3 B/E\right)}{4\pi\sinh(\pi B/E)}\exp\left\{-\frac{\pi M^2}{|qE|}-(\mathbf{x}_\perp-\mathbf{x}'_\perp)^2\frac{qB}{4}\coth(\pi B/E)\right\}. \tag{46}$$

Using (41) and (44), one can represent $\langle j_\mu(t)\rangle^{p,\bar p}$ and $\langle T_{\mu\nu}(t)\rangle^{p,\bar p}$ in (27) as follows:

$$\begin{aligned}\langle j_\mu(t)\rangle^{p,\bar p} &= iq\,\mathrm{tr}\left[\gamma_\mu\gamma P\Delta^{p,\bar p}(x,x')\right]\big|_{x=x'} \; ,\\ \langle T_{\mu\nu}(t)\rangle^{p,\bar p} &= i\,\mathrm{tr}\left[A_{\mu\nu}\gamma P\Delta^{p,\bar p}(x,x')\right]\big|_{x=x'} \; .\end{aligned}$$

Then, using (41), (42), (44), and (46), we find

$$\begin{aligned}\langle j_\mu(t)\rangle^{p,\bar p} &= -iq\,P_\mu \mathrm{tr}\,\Delta^{p,\bar p}(x,x')\big|_{x=x'} \; ,\\ \langle T_{\mu\mu}(t)\rangle^{p,\bar p} &= -i\,P_\mu^2 \mathrm{tr}\,\Delta^{p,\bar p}(x,x')\big|_{x=x'} \; .\end{aligned} \tag{47}$$

The off-diagonal matrix elements of $\langle T_{\mu\nu}\rangle^p$ and $\langle T_{\mu\nu}\rangle^{\bar p}$ are all equal to zero.

Taking derivatives, calculating traces, and integrating over p_3, we obtain the leading contributions at large T . First of all,

$$\begin{aligned}&\langle j_\mu(t)\rangle^p = -2\delta^3_\mu q\,\mathrm{sgn}\,(qE)\,(1/2+t/T)\,n^{cr},\\ &\langle j_\mu(t)\rangle^{\bar p} = -2\delta^3_\mu q\,\mathrm{sgn}\,(qE)\,(1/2-t/T)\,n^{cr},\\ &\langle T_{00}(t)\rangle^p = \langle T_{33}(t)\rangle^p = |qE|\,T\,(1/2+t/T)^2\,n^{cr},\\ &\langle T_{00}(t)\rangle^{\bar p} = \langle T_{33}(t)\rangle^{\bar p} = |qE|\,T\,(1/2-t/T)^2\,n^{cr},\end{aligned} \tag{48}$$

where

$$n^{cr}=\frac{q^2}{4\pi^2}EBT\,\coth(\pi B/E)\left[\exp\left(-\pi\frac{M^2}{|qE|}\right)+O\left(\frac{K}{\sqrt{|qE|}T}\right)\right]. \tag{49}$$

Second,

$$\begin{aligned}&\langle T_{11}(t)\rangle^p = \langle T_{22}(t)\rangle^p\,,\;\langle T_{11}(t)\rangle^{\bar p} = \langle T_{22}(t)\rangle^{\bar p}\,,\\ &\langle T_{11}(t)\rangle^p = \widetilde{n}\left\{\begin{array}{l}\ln\left[\sqrt{|qE|}\,(T/2+t)\right]+O(\ln K)\,,\;\sqrt{|qE|}\,(T/2+t)>K\\ O(\ln K)\;,\;\;\sqrt{|qE|}\,(T/2+t)\le K\end{array}\right. ,\\ &\langle T_{11}(t)\rangle^{\bar p} = \widetilde{n}\left\{\begin{array}{l}-\ln\left[\sqrt{|qE|}\,(T/2-t)\right]+O(\ln K)\,,\;\sqrt{|qE|}\,(T/2-t)>K\\ -O(\ln K)\;,\;\;\sqrt{|qE|}\,(T/2-t)\le K\end{array}\right. ,\end{aligned} \tag{50}$$

where

$$\widetilde{n}=\frac{(qB)^2}{4\pi^2\sinh^2(\pi B/E)}\exp\left(-\pi\frac{M^2}{|qE|}\right). \tag{51}$$

We can see that all of the leading contributions are real-valued. Note that the current density $\langle j_\mu(t)\rangle^p$ and the component of EMT $\langle T_{\mu\nu}(t)\rangle^p$ are zero for $t\le t_1$ and change with time until t_2, $\langle j_3(t)\rangle^p$ being linear, $\langle T_{00}(t)\rangle^p=\langle T_{33}(t)\rangle^p$ quadratic, and $\langle T_{11}(t)\rangle^p=\langle T_{22}(t)\rangle^p$ logarithmic, respectively. As will be demonstrated, the rate n^{cr} in (48) is the total number-density of pairs created by the T-constant electric field.

After switching the electric field (at $t>t_2$), all the mean values (48), (50) are constant and retain their values at t_2.

One ougth to say that formulas (48), (50) are obtained for the first time.

At large times $t-t_1=t+T/2\gg K/\sqrt{|qE|}$, it is useful to consider the quanties

$$j^{cr}_\mu(t)=\langle j_\mu(t)\rangle^0-\langle j_\mu(t)\rangle^0_{out}\,,\;\;T^{cr}_{\mu\nu}(t)=\langle T_{\mu\nu}(t)\rangle^0-\langle T_{\mu\nu}(t)\rangle^0_{out}\,, \tag{52}$$

which can be interpreted as those corresponding to *out*-partilces at any large t. At the final time instant t_2, they coincide with the physical quantities $\langle j_\mu(t_2)\rangle^p$ and $\langle T_{\mu\nu}(t_2)\rangle^p$. Then it follows from (26) and (48), (50) that

$$j_\mu^{cr}(t) = \langle j_\mu(t)\rangle^p - \langle j_\mu(t)\rangle^{\bar{p}} = -2\delta_\mu^3 q \,\mathrm{sgn}(qE)(2t/T)\, n^{cr}, \quad T_{\mu\nu}^{cr}(t) = \langle T_{\mu\nu}(t)\rangle^p - \langle T_{\mu\nu}(t)\rangle^{\bar{p}}, \tag{53}$$

and

$$\begin{aligned} \langle T_{00}(t)\rangle^{cr} &= \langle T_{33}(t)\rangle^{cr} = 2|qE|\, t n^{cr}, \quad \langle T_{11}(t)\rangle^{cr} = \langle T_{22}(t)\rangle^{cr}, \\ \langle T_{11}(t)\rangle^{cr} &= \tilde{n} \begin{cases} \ln\left[|qE|\left((T/2)^2 - t^2\right)\right] + O(\ln K), & \sqrt{|qE|}(T/2 - t) > K \\ \ln\left[\sqrt{|qE|}(T/2 + t)\right] + O(\ln K), & \sqrt{|qE|}(T/2 - t) \le K \end{cases}. \end{aligned} \tag{54}$$

At $t \ge t_2$, the production of pairs terminates, and the quantities (53) and (54) mantain their values (53) and (54) at $t = t_2$, and present the current density and EMT of created particles. For example, from j_μ^{cr} in (53) we can see that n^{cr} is the total number-density of pairs created during the entire time of action of the electric field.

We stress that expressions (48), (50) for $\langle j_\mu(t)\rangle^p$ and $\langle T_{\mu\nu}(t)\rangle^p$ represent contributions to $\langle j_\mu(t)\rangle^0$ and $\langle T_{\mu\nu}(t)\rangle^0$, respectively, at any time instant t if the stabilization condition (38) is valid. On the contrary, expressions (53), (54) are valued only for time instants $\bar{t}$ that are sufficiently close to t_2,

$$\bar{t}: \ (T - 2\bar{t})/T \ll 1,$$

when the intrepretation in terms of final particles already makes sense.

3.3 Vacuum polirization contributions

Now, we are going to calculate the real-valued parts of the quantities $\langle j_\mu(t)\rangle^c$ and $\langle T_{\mu\nu}(t)\rangle^c$ defined in (27), which do not have any T-divergences. In these expressions, we can use the causal Green function $S^c(x,x')$ (22) in the constant field, since the above quantities do not need any regularization.

For such a Green function, the so-called Fock–Schwinger proper time representation holds true:

$$S^c(x,x') = (\gamma P + M)\Delta^c(x,x'), \quad \Delta^c(x,x') = \int_0^\infty f(x,x',s)ds, \tag{55}$$

see [33] and [36], where the Fock–Schwinger kernel $f(x,x',s)$ reads

$$\begin{aligned} f(x,x',s) &= \exp\left(-i\frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu}s\right) f^{(0)}(x,x',s), \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu], \\ f^{(0)}(x,x',s) &= \frac{q^2 EB\exp(iq\Lambda)}{(4\pi)^2\sinh(qEs)\sin(qBs)}\exp\left[-iM^2 s - i\frac{(x-x')\,qF\coth(qFs)\,(x-x')}{4}\right], \end{aligned} \tag{56}$$

see [8, 37]. Singularities of the kernel are situated at the origin and on the imaginary axis, $|qE|\,s = -i\pi n$, $n = 1, 2, 3, \ldots$.

One ought to say that in this expression the only term

$$\Lambda = \Lambda_\parallel + \Lambda_\perp, \quad \Lambda_\parallel = (x_0 + x_0')(x_3 - x_3')E/2, \quad \Lambda_\perp = -\int_{x'}^{x} A_\mu^B dx^\mu, \tag{57}$$

is potential-dependent. Here, A_μ^B is the potential of magnetic field $F_{\mu\nu}^B$, and the integral is taken along the line connecting the points x and x'.

Using this representation, we calculate $\mathrm{Re}\langle j_\mu(t)\rangle^c$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^c$. It easy to see that $\langle j_\mu(t)\rangle^c = 0$, as should be expected due to translational symmetry, and $\langle T_{\mu\nu}(t)\rangle^c = 0$, $\mu \ne \nu$.

Calculating the diagonal terms $\langle T_{\mu\mu}(t)\rangle^c$, we discower that it can be derived from the Heisenberg–Euler Lagrangian $\mathcal{L}$,

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}\int_0^\infty s^{-1} f(x,x,s)\, ds.$$

Subtracting the zero external field contribution from $\mathcal{L}$ and performing standard renormalizations of $\mathcal{L} - \mathcal{L}|_{F=0}$, leaving $qF_{\mu\nu}$ invariant, i.e., the standard renormalizations of the coupling constant q^2 and potentials A_μ, we obtain a finite expression:

$$\mathcal{L}_{ren} = \int_0^\infty \frac{ds\exp(-iM^2 s)}{8\pi^2 s}\left[q^2 EB\coth(qEs)\cot(qBs) - \frac{1}{s^2} - \frac{q^2}{3}(E^2 - B^2)\right], \tag{58}$$

see [7]. Making the same renormalization for $\langle T_{\mu\mu}(t)\rangle^c$, we can see that there holds the relation

$$\langle T_{00}(t)\rangle^c_{ren} = -\langle T_{33}(t)\rangle^c_{ren} = E\frac{\partial\mathcal{L}(t)_{ren}}{\partial E} - \mathcal{L}(t)_{ren}\,,$$
$$\langle T_{11}(t)\rangle^c_{ren} = \langle T_{22}(t)\rangle^c_{ren} = \mathcal{L}(t)_{ren} - B\frac{\partial\mathcal{L}(t)_{ren}}{\partial B}\,, \quad (59)$$

where $\mathcal{L}(t)_{ren} = \mathcal{L}_{ren}|_{E\to E(t)}$, and $E(t)$ is defined by (29)[3].

Finally, the vacuum mean values of the current density and EMT have the form

$$\langle j_\mu(t)\rangle^0 = \mathrm{Re}\langle j_\mu(t)\rangle^p,\ \langle T_{\mu\nu}(t)\rangle^0_{ren} = \mathrm{Re}\langle T_{\mu\nu}(t)\rangle^c_{ren} + \mathrm{Re}\langle T_{\mu\nu}(t)\rangle^p, \quad (60)$$

where $\mathrm{Re}\langle j_\mu(t)\rangle^p$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^p$ are presented by (48) and (50). We can see that the T-dependent contributions $\mathrm{Re}\langle j_\mu(t)\rangle^p$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^p$ arise due to vacuum instability; they are global physical quantities, and they have the factor $\exp\left\{-\pi M^2/\left|qE\right|\right\}$. This factor is exponentially small for a weak electric field, $M^2/\left|qE\right| \gg 1$, and these quantities can actually be observed as soon as the external field strength approaches the critical value E_c. On the other hand, the term $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^c_{ren}$ is t-dependent and T-independent; it is therefore local and does exist in arbitrary electric fields. When the T-constant electric field switches off, the local contribution from the electric field to $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^c_{ren}$ vanishes, but the global one, given by $\mathrm{Re}\langle T_{\mu\nu}(t_2)\rangle^p$, does remain. Thus, in the general case, both kinds of contributions are important. We stress that in order to make $\langle T_{\mu\nu}(t)\rangle^0_{ren}$ finite the only term $\langle T_{\mu\nu}(t)\rangle^c$ has to be regularized and renormalized due to the standard ultraviolet divergences, which is consistent with the fact that the ultraviolet divergences have a local nature.

Using expression (60), we can find a condition which gives restrictions for the strength of classical constant electric field. Note that for a strong electric field ($B = 0$), $M^2/\left|qE\right| \ll 1$, and for a large T, there is a well-known asymptotic expression for the vacuum energy density $\mathrm{Re}\langle T_{00}(t)\rangle^c_{ren}$,

$$\mathrm{Re}\,\langle T_{00}(t)\rangle^c_{ren} = -\frac{q^2}{24\pi^2}E^2\ln\frac{|qE|}{M^2}\,. \quad (61)$$

It does not depend on T. The energy density of a classical electric field is $E^2/8\pi$. Then (see [38]) one makes the conclusion that the notion of a strong constant electric field is physically meaningful when (61) is less than $E^2/8\pi$, which implies

$$\frac{q^2}{3\pi}\ln\frac{|qE|}{M^2} \ll 1\,. \quad (62)$$

However, if T is large, one also has to take into account the T-dependent term $\mathrm{Re}\langle T_{00}(t)\rangle^p$ in (60). At $t = t_2$, we have

$$\mathrm{Re}\,\langle T_{00}(t_2)\rangle^p = \frac{q^2E^2}{4\pi^3}\,|qE|\,T^2\,. \quad (63)$$

One can neglect the back-reaction of created pairs in electric field only in case (63) is far less than $E^2/8\pi$, which implies

$$|qE|\,T^2 \ll \frac{\pi^2}{2q^2}\,. \quad (64)$$

The restriction (64) is independent and, in a sence, additional to (62). It connects the maximal strengh of electric field and its duration. It is much more restrictive than (62), and is related to the back-reaction of created particles.

4 Initial state as thermal equilibrium

We are now going to calculate the quantities $< j_\mu(t) >$ and $< T_{\mu\nu}(t) >$ (13) with the density operator (15), by using relations (26). In fact, the contributions $< j_\mu(t) >^0$ and $< T_{\mu\nu}(t) >^0$ to these qiantities have been calculated in the previous section. Here, we study the remaining temperature-dependent contributions $\langle j_\mu(t)\rangle^\theta$ and $\langle T_{\mu\nu}(t)\rangle^\theta$ to these quantities.

[3] In the absence of electric field, the Green function $S^c(x,x')$, $S^c_{in}(x,x')$, and $S^c_{out}(x,x')$ coincide. Therefore, in case $t < t_1$ we have $\langle T_{\mu\nu}(t)\rangle^0 = .\langle T_{\mu\nu}(t)\rangle^0\big|_{E=0} = \langle T_{\mu\nu}(t)\rangle^c\big|_{E=0}$. For $t > t_2$, relation (52) implies

$$\langle T_{\mu\nu}(t)\rangle^0 = \langle T_{\mu\nu}(t)\rangle^0_{out} + T^{cr}_{\mu\nu}(t)\ ,$$

where $T^{cr}_{\mu\nu}(t)$ is given by (53),(54) and $\langle T_{\mu\nu}(t)\rangle^0_{out} = .\langle T_{\mu\nu}(t)\rangle^0\big|_{E=0} = \langle T_{\mu\nu}(t)\rangle^c\big|_{E=0}$, i.e., this is a vacuum polarization contribution.

4.1 Contributions due to the particle creation

Let us examine the function $S^{\theta}(x,x')$ in (19). The form of the function $\tilde{S}_{\theta}^{\mp}$ in (19) until the moment t_1 the electric field turns on is known. Therefore, we have to investigate the case of $t > t_1, t' > t_1$. First of all, we separate the contributions from particle creation in the same manner as it has been done for the function $S_{in}^{c}(x,x')$ in Sec.2. Using relations (7), we find

$$\begin{aligned}
\pm\tilde{S}_{\theta}^{\mp}(x,x') &= S_{\theta,\pm}(x,x') + S_{\theta,\pm}^{p}(x,x') , \\
S_{\theta,+}(x,x') &= -i\sum_{n,m} N_m^{(+)}(in)\ {}^{+}\psi_n(x)\, G\left({}_{+}|{}^{+}\right)^{-1}_{nm}\ {}^{+}\bar{\psi}_m(x') , \\
S_{\theta,-}(x,x') &= i\sum_{n,m} N_n^{(-)}(in)\ {}_{-}\psi_n(x)\left[G\left({}_{-}|{}^{-}\right)^{-1}\right]^{\dagger}_{nm}\ {}^{-}\bar{\psi}_m(x') , \\
S_{\theta,+}^{p}(x,x') &= -i\sum_{nm} N_m^{(+)}(in)\ {}_{-}\psi_n(x)\ \left[G({}_{+}|{}^{-})G({}_{-}|{}^{-})^{-1}\right]^{\dagger}_{nm}\ {}_{+}\bar{\psi}_m(x'), \\
S_{\theta,-}^{p}(x,x') &= -i\sum_{nm} N_n^{(-)}(in)\ {}_{-}\psi_n(x)\ \left[G({}_{+}|{}^{-})G({}_{-}|{}^{-})^{-1}\right]^{\dagger}_{nm}\ {}_{+}\bar{\psi}_m(x') .
\end{aligned} \tag{65}$$

Taking into account (9), we can see that $S_{\theta,\pm}^{p}$ describe the contributions from particle creation.

Then, we represent the real-valued terms $\langle j_{\mu}(t)\rangle^{\theta}$ and $\langle T_{\mu\nu}(t)\rangle^{\theta}$ in (27) as follows:

$$\begin{aligned}
\langle j_{\mu}(t)\rangle^{\theta} &= \mathrm{Re}\langle j_{\mu}(t)\rangle_{\theta}^{c} + \mathrm{Re}\langle j_{\mu}(t)\rangle_{\theta}^{p} , \\
\langle T_{\mu\nu}(t)\rangle^{\theta} &= \mathrm{Re}\langle T_{\mu\nu}(t)\rangle_{\theta}^{c} + \mathrm{Re}\langle T_{\mu\nu}(t)\rangle_{\theta}^{p} ,
\end{aligned} \tag{66}$$

where

$$\begin{aligned}
\langle j_{\mu}(t)\rangle_{\theta}^{c} &= iq\sum_{\zeta=\pm}\mathrm{tr}\left[\gamma_{\mu}S_{\theta,\zeta}(x,x')\right]\big|_{x=x'} ,\ \langle T_{\mu\nu}(t)\rangle_{\theta}^{c} = i\sum_{\zeta=\pm}\mathrm{tr}\left[A_{\mu\nu}S_{\theta,\zeta}(x,x')\right]\big|_{x=x'} , \\
\langle j_{\mu}(t)\rangle_{\theta}^{p} &= iq\sum_{\zeta=\pm}\mathrm{tr}\left[\gamma_{\mu}S_{\theta,\zeta}^{p}(x,x')\right]\Big|_{x=x'} ,\ \langle T_{\mu\nu}(t)\rangle_{\theta}^{p} = i\sum_{\zeta=\pm}\mathrm{tr}\left[A_{\mu\nu}S_{\theta,\zeta}^{p}(x,x')\right]\Big|_{x=x'} .
\end{aligned} \tag{67}$$

Due to the symmetry $\gamma^{0}\tilde{S}_{\theta}^{\mp}(x,x')^{\dagger}\gamma^{0} = \tilde{S}_{\theta}^{\mp}(x',x)$, which can be observed from representation (19), we can calculate the real-valued parts of the right-hand sides of (67) at $x = x'$ as follows:

$$\begin{aligned}
\mathrm{tr}\left[\gamma_{\mu}S_{\theta,+}(x,x')\right]\big|_{x=x'} &= \lim_{x_0\to x_0'+0}\mathrm{tr}\left[\gamma_{\mu}S_{\theta,+}(x,x')\right]\Big|_{\mathbf{x}=\mathbf{x}'} , \\
\mathrm{tr}\left[\gamma_{\mu}S_{\theta,-}(x,x')\right]\big|_{x=x'} &= \lim_{x_0\to x_0'-0}\mathrm{tr}\left[\gamma_{\mu}S_{\theta,-}(x,x')\right]\Big|_{\mathbf{x}=\mathbf{x}'} ,
\end{aligned} \tag{68}$$

and so on.

Using (26), (60), and (66), we present the mean values $\langle j_{\mu}(t)\rangle$ and $\langle T_{\mu\nu}(t)\rangle_{ren}$:

$$\begin{aligned}
&\langle j_{\mu}(t)\rangle = \mathrm{Re}\langle j_{\mu}(t)\rangle_{\theta}^{c} + J_{\mu}^{p}(t) , \\
&J_{\mu}^{p}(t) = \mathrm{Re}\langle j_{\mu}(t)\rangle^{p} + \mathrm{Re}\langle j_{\mu}(t)\rangle_{\theta}^{p} , \\
&\langle T_{\mu\nu}(t)\rangle_{ren} = \mathrm{Re}\langle T_{\mu\nu}(t)\rangle_{ren}^{c} + \mathrm{Re}\langle T_{\mu\nu}(t)\rangle_{\theta}^{c} + \tau_{\mu\nu}^{p}(t) , \\
&\tau_{\mu\nu}^{p}(t) = \mathrm{Re}\langle T_{\mu\nu}(t)\rangle^{p} + \mathrm{Re}\langle T_{\mu\nu}(t)\rangle_{\theta}^{p} .
\end{aligned} \tag{69}$$

Now, $\langle T_{\mu\nu}(t)\rangle$ is replaced by the renormalized (with respect to the ultraviolet divergences) quantity $\langle T_{\mu\nu}(t)\rangle_{ren}$ by inserting $\langle T_{\mu\nu}(t)\rangle_{ren}^{c}$ in the r.h.s. instead of $\langle T_{\mu\nu}(t)\rangle^{c}$. The terms $J_{\mu}^{p}(t)$ and $\tau_{\mu\nu}^{p}(t)$ with the upper script p represent the contributions from particle creation, the terms with the upperscript c represent the remaining contributions. One ought to recall that the vacuum polarization contribution $\langle T_{\mu\nu}(t)\rangle_{ren}^{c}$ has been already calculated and has the form (59). In the absence of electric field (for $t < t_1$ and $t > t_2$), the quantity (69) does not depend on time, in particular, $J_{\mu}^{p}(t) = \tau_{\mu\nu}^{p}(t) = 0$ for $t < t_1$. As already mentioned, the intrepretation in terms of *out*-particles makes sense only for $t > \bar{t}$, where $\bar{t}$ is sufficiently close to t_2, $(T - 2\bar{t})/T \ll 1$. At the same time, for $t > \bar{t}$ the quantities $J_{\mu}^{p}(t)$ and $\tau_{\mu\nu}^{p}(t)$ are the current density and EMT of *out*-particles created by the electric field. The quantities $\mathrm{Re}\langle j_{\mu}(t)\rangle_{\theta}^{c}$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle_{\theta}^{c}$ describe the corresponding contributions related to the existence of real *in*-particles at the initial state.

The quantities $\mathrm{Re}\langle j_\mu(t)\rangle_\theta^p$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^p$ present the contributions from particle creation at finite temperature. They are T-divergent. The leading contributions to these quantities at large T can be found by analogy with the case of the zero temperature, which has been considered above. We examine the leading contributions to $S_{\theta,\zeta}^p$, which are given by the expression

$$S_{\theta,\zeta}^p(x,x') = i\int_D d\mathbf{p}\sum_{r=\pm1} N_{p_3}^{(\zeta)}(in)\,\aleph_{\mathbf{p},r}\left({}^+\psi_{\mathbf{p},r}(x){}^+\bar\psi_{\mathbf{p},r}(x') - {}^-\psi_{\mathbf{p},r}(x){}^-\bar\psi_{\mathbf{p},r}(x')\right). \tag{70}$$

The integral $\int_D d\mathbf{p}$ is defined in (39); the differential numbers $\aleph_{\mathbf{p},r}$ are defined in (36), and

$$N_{p_3}^{(\zeta)}(in) = \left\{\exp\left[\beta\left(\varepsilon_{\mathbf{p}}^{(\zeta)} - \mu^{(\zeta)}\right)\right] + 1\right\}^{-1}, \tag{71}$$

where $\varepsilon_{\mathbf{p}}^{(\zeta)}$ in (71) are quasi-energies at t_1, see (32).

To obtain a generalization of (70) to the presence of a constant magnetic field, $B \neq 0$, we have to follow the way that has been already described in Subsec. 3.2. Then, making summation and integration, we obtain the leading contribution to the function $S_{\theta,\zeta}^p(x,x')$ in the case under consideration:

$$\begin{aligned} S_{\theta,\zeta}^p(x,x') &= (\gamma P + M)\Delta_{\theta,\zeta}^p(x,x'), \\ \Delta_{\theta,\zeta}^p(x,x') &= -i\int_{-T/2+K/\sqrt{|qE|}}^{t-K/\sqrt{|qE|}} N_{p_3}^{(\zeta)}(in)\,h_\parallel\left(x_\parallel,x'_\parallel\right)h_\perp(\mathbf{x}_\perp,\mathbf{x}'_\perp)\,d\left(\frac{p_3}{qE}\right). \end{aligned} \tag{72}$$

The functions $h_\parallel\left(x_\parallel,x'_\parallel\right)$ and $h_\perp(\mathbf{x}_\perp,\mathbf{x}'_\perp)$ are given by expressions (42) and (46), respectively.

Thus, we obtain the leading contributions to $\langle j_\mu(t)\rangle_\theta^p$ and $\langle T_{\mu\nu}(t)\rangle_\theta^p$:

$$\begin{aligned} \langle j_\mu(t)\rangle_\theta^p &= -iq\sum_{\zeta=\pm} P_\mu \mathrm{tr}\,\Delta_{\theta,\zeta}^p(x,x')\Big|_{x=x'}, \\ \langle T_{\mu\nu}(t)\rangle_\theta^p &= 0,\ \mu\neq\nu;\ \langle T_{\mu\mu}(t)\rangle_\theta^p = -i\sum_{\zeta=\pm} P_\mu^2 \mathrm{tr}\,\Delta_{\theta,\zeta}^p(x,x')\Big|_{x=x'}. \end{aligned} \tag{73}$$

To examine the temperature-dependent contributions to $J_\mu^p(t)$ and $\tau_{\mu\nu}^p(t)$ in (69), we have to analyze the quantities (73). At low temperatures, $\beta\left|\left(\varepsilon_{\mathbf{p}}^{(\zeta)} - \mu^{(\zeta)}\right)\right| \gg 1$, and assuming $\varepsilon_{\mathbf{p}}^{(\zeta)} > \left|\mu^{(\zeta)}\right|$, one can see that contributions from $\mathrm{Re}\langle j_\mu(t)\rangle_\theta^p$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^p$ are very small in comparison with the vacuum contributions $\mathrm{Re}\langle j_\mu(t)\rangle^p$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^p$. Therefore, in the case under consideration the leading contributions to $J_\mu^p(t)$ and $\tau_{\mu\nu}^p(t)$ are

$$J_\mu^p(t) = \mathrm{Re}\langle j_\mu(t)\rangle^p,\ \tau_{\mu\nu}^p(t) = \mathrm{Re}\langle T_{\mu\nu}(t)\rangle^p. \tag{74}$$

At $\varepsilon_{\mathbf{p}}^{(\zeta)} < \left|\mu^{(\zeta)}\right|$, with sufficiently large $\left|\mu^{(\zeta)}\right| \gg \sqrt{|qE|}K$, contributions from $\mathrm{Re}\langle j_\mu(t)\rangle_\theta^p$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^p$ are comparable with contributions from $\mathrm{Re}\langle j_\mu(t)\rangle^p$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^p$, respectively. For example, at $\left|\mu^{(\zeta)}\right| \gtrsim |qE|(t+T/2)$ we have

$$\mathrm{Re}\langle j_\mu(t)\rangle_\theta^p = -2\,\mathrm{Re}\langle j_\mu(t)\rangle^p,\ \ \mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^p = -2\,\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^p.$$

Then

$$J_\mu^p(t) = -\,\mathrm{Re}\langle j_\mu(t)\rangle^p,\ \tau_{\mu\nu}^p(t) = -\,\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^p. \tag{75}$$

The negative sign in these expressions means that the total number-density of particles is decreasing due to electron-postron annihilation in electric field, and, as a consequence, $\langle j_\mu(t)\rangle$ and $\langle T_{\mu\nu}(t)\rangle_{ren}$ in (69) are less than $\mathrm{Re}\langle j_\mu(t)\rangle_\theta^c$ and $\langle T_{\mu\nu}(t)\rangle_{ren}^c + \mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c$, respectively. This is in agreement with the fact that due to particle creation there holds the relation

$$N_m^{(\zeta)}(out) = (1-\aleph_m)\,N_m^{(\zeta)}(in) + \aleph_m\left[1 - N_m^{(-\zeta)}(in)\right].$$

between the initial $N_m^{(\zeta)}(in)$ and final $N_m^{(\zeta)}(out)$ differential mean numbers; see [22]. Thus, the differential mean numbers of created particles are given by the difference $\Delta N_m^{(\zeta)} = N_m^{(\zeta)}(out) - N_m^{(\zeta)}(in)$. As a result, $\Delta N_m^{(\zeta)}$ is negative in case $N_m^{(+)}(in) + N_m^{(-)}(in) > 1$.

At high temperatures, $\beta\left[|qE|\left(t+T/2\right)-\mu^{(\pm)}\right]\ll 1$, we find

$$
\begin{aligned}
J_{\mu}^{p}(t) &= \frac{\beta}{2}\left[\frac{1}{2}\left\{|qE|\left(t+T/2\right)-\mu^{(+)}-\mu^{(-)}\right\}+\sqrt{|qE|}O\left(K\right)\right]\operatorname{Re}\langle j_{\mu}(t)\rangle^{p}\,,\\
\tau_{00}^{p}(t) &= \tau_{33}^{p}(t)=\frac{\beta}{2}\left[\frac{1}{3}|qE|\left(t+T/2\right)-\frac{1}{2}\left(\mu^{(+)}+\mu^{(-)}\right)+\sqrt{|qE|}O\left(K\right)\right]\operatorname{Re}\langle T_{00}(t)\rangle^{p}\,,\\
\tau_{11}^{p}(t) &= \tau_{22}^{p}(t)=\frac{\beta}{2}\left[|qE|\left(t+T/2\right)-\frac{1}{2}\left(\mu^{(+)}+\mu^{(-)}\right)\right]\\
&\times\left[1+\sqrt{|qE|}O\left(\frac{\ln K}{\ln\left[\sqrt{|qE|}\left(t+T/2\right)\right]}\right)\right]\operatorname{Re}\langle T_{11}(t)\rangle^{p}\,. \qquad (76)
\end{aligned}
$$

We can see that all the component of $J_{\mu}^{p}(t)$ and $\tau_{\mu\nu}^{p}(t)$ at high temperature are far less than the corresponding components of the vacuum contributions $\langle j_{\mu}(t)\rangle^{p}$ and $\langle T_{\mu\nu}(t)\rangle^{p}$, which is natural for Fermi particles.

The resulting expressions for the mean energy density of created particles make it possible to conclude that the restriction (64) for the external constant electric field, established for vacuum, is also valid at the initial low-temperature state. In case the temperature of the initial state is suffciently high in comparison with the mean kinetic energy of created particles, $\beta|qE|T\ll 1$, this restriction changes considerably. In this case, the mean energy density of created particles after the electric field turns off,

$$\tau_{00}^{p}(t_{2})=\frac{1}{6}\beta|qE|T\operatorname{Re}\langle T_{00}(t_{2})\rangle^{p}\,, \qquad (77)$$

in accordance with (76), is much smaller than the coresponding vacuum expression $\operatorname{Re}\langle T_{00}(t_{2})\rangle^{p}$ in (63): Repeating the argumentis of Subsec. 3.3, we find: one can neglect the back-reaction of created pairs in electric field only in case (77) is far less than $E^{2}/8\pi$, which implies

$$\beta|qE|^{2}T^{3}\ll\frac{3\pi^{2}}{q^{2}}\,. \qquad (78)$$

It is much less restrictive than (64).

4.2 Conventional contribution

Here, we are going to calculate the temperature-dependent contributions to the mean current density and EMT, which are labelled by an upperscript, c. We refer to these contribitions as conventional contributions. Such contributions are not related to either particle creation or vacuum polarization. Thus, it is natural to say that they are determined only by the quantum statistical behavior of the initial thermal gas of charged particles in the constant external field under consideration.

4.2.1 Proper-time representation

Let us represent $\langle j_{\mu}(t)\rangle_{\theta}^{c}$ and $\langle T_{\mu\nu}(t)\rangle_{\theta}^{c}$ in (67) as integrals over the proper time. To this end, we relate $S_{\theta,\zeta}$ in (65) to the proper-time representation (55) of S^{c}. For simplicity, we assume that the chemical potentials are not very large, $\left|\mu^{(\zeta)}\right|<M$.

As always, we consider large T, namely,

$$\sqrt{|qE|}T\gg 1+m^{2}/|qE|\,. \qquad (79)$$

As has been mentioned, the form of the function $\tilde{S}_{\theta}^{\mp}$ in (19) until the moment t_{1} the electric field turns on is known. Therefore, we continue their investigation in the case of $t>t_{1}$, $t'>t_{1}$, which we have begun in the previous section. We are now interested in the components of this functions, which are denoted in (65) as $S_{\theta,\zeta}$.

Due to the cutting-off factor, the integrals $S_{\theta,\zeta}$ are finite. However, they have derivatives proportional to the time interval, $t-t_{1}$, which may be large. Therefore, in the study of conventional contributions we have to pay attention to the contributions growing as $t-t_{1}$ increases. Note that, as distincts from the contributions $J_{\mu}^{p}(t)$ and $\tau_{\mu\nu}^{p}(t)$, which we have extimated only for suffcintly large intervals $t-t_{1}$, conventional contributions are calculated for any t, $0\leq t-t_{1}\leq T$.

We remind that the functions ${}_{\zeta}\psi_{n}(x)$ and ${}^{\zeta}\psi_{n}(x)$ are solutions of the Dirac equation in a constant field, described in Subsec. 3.1. In relation (35) of that subsection, we have presented the expressions for the coefficientes $G\left({}_{-}|^{+}\right)$, obtained for the T-constant field, which determine all the coefficientes $G\left({}_{\zeta}|^{\zeta}\right)$

via the unitarity condition. These expressions for $G\left({}_{-}|^{+}\right)$ at longitudinal momenta p_3 being large and comparable with $|qE|T$ are different from the asimptotic form at $T\to\infty$. Accordingly, for such large values p_3 the coefficients $G\left({}_{\zeta}|^{\zeta}\right)$ are different from the corresponding asimptotic forms. However, it can be shown that under condition (79) these differences in expressions for $S_{\theta,\zeta}$ can be neglected, and one can use, for every p_3, the asimptotic form of the coefficients $G\left({}_{\zeta}|^{\zeta}\right)$, which holds true for a constant field. On the other hand, we need to attract attention to the above-mentioned fact, which will be used to obtain the necessary representaion, whereas a manifest form of these coefficients in a constant field will be unnecessary.

The energy spectrum of *in*-particles at $t<t_1$ has the form

$$\varepsilon_n^{(\zeta)}=\varepsilon_n=\sqrt{M^2+\omega+(\pi_3)^2},\ ,\ \pi_3=\left(\frac{qET}{2}+p_3\right),$$

where

$$n=(p_1,n_B,p_3,r),\ \omega=|qB|(2n_B+1-r)\,,\ n_B=0,1,\ldots,\ B\neq 0;$$
$$n=(\mathbf{p},r),\ \omega=p_1^2+p_2^2\,,\ \ B=0\,.$$

We expand $N_n^{(\zeta)}(in)$ (20) as the power series of $\exp\left\{-\beta\left(\varepsilon_n^{(\zeta)}-\mu^{(\zeta)}\right)\right\}$ and make use of the formula (see [39], Eqs. 3.471.9 and 8.469.3)

$$\exp\{-\beta l\varepsilon_n\}=\frac{1}{2\sqrt{\pi}}\int_0^\infty\frac{du}{u^{1/2}}\exp\left(-\frac{u}{4}-\frac{\beta^2l^2\varepsilon_n^2}{u}\right). \tag{80}$$

Since the operator $(\mathcal{H}(t_1))^2=M^2-(\gamma\mathbf{P}_\perp)^2+(i\partial_3+qET/2)^2$ is an integral of motion in the T-constant field, the relation

$$(\mathcal{H}(t_1))^2\ {}_{\pm}\psi_n(x)=\varepsilon_n^2\ {}_{\pm}\psi_n(x)$$

holds true at any time instant t. Then, $S_{\theta,\zeta}$ (65) can be represented as follows:

$$\begin{aligned}
S_{\theta,+}(x,x') &= i\sum_{n,m}\Upsilon^{(+)}{}_{+}\psi_n(x)\,G\left({}_{+}|^{+}\right)^{-1}_{nm}\ {}_{+}\bar\psi_m(x')\,,\\
S_{\theta,-}(x,x') &= -i\sum_{n,m}\Upsilon^{(-)}{}_{-}\psi_n(x)\left[G\left({}_{-}|^{-}\right)^{-1}\right]^{\dagger}_{nm}\ {}^{-}\bar\psi_m(x')\,,
\end{aligned}\tag{81}$$

where

$$\begin{aligned}
\Upsilon^{(\zeta)} &= \sum_{l=1}^{\infty}(-1)^l e^{\beta l\mu^{(\zeta)}}\frac{1}{2\sqrt{\pi}}\int_0^\infty\frac{du}{u^{1/2}}\exp\left(-\frac{u}{4}-\frac{\beta^2l^2\mathcal{E}_\zeta^2}{u}\right),\\
\mathcal{E}_+^2 &= M^2-(\gamma\mathbf{P}_\perp'^{*})^2+(-i\partial_3'+qET/2)^2\,,\\
\mathcal{E}_-^2 &= M^2-(\gamma\mathbf{P}_\perp)^2+(i\partial_3+qET/2)^2\,.
\end{aligned}$$

The representations (81) and (22) are formaly related as follows:

$$S_{\theta,\pm}(x,x')=\pm\Upsilon^{(\pm)}S^{\mp}(x,x')\,. \tag{82}$$

To calculate $\langle j_\mu(t)\rangle_\theta^c$ and $\langle T_{\mu\nu}(t)\rangle_\theta^c$ in (67). according to the prescription (68), we need the expressions for $S_{\theta,+}(x,x')$ at $x_0>x_0'$ and $S_{\theta,-}(x,x')$ at $x_0<x_0'$ only. Under this condition, the proper-time representation for $S^{\mp}(x,x')$ has the form

$$\begin{aligned}
S^{\mp}(x,x') &= (\gamma P+M)\Delta^{\mp}(x,x')\,,\\
\pm\Delta^{\mp}(x,x') &= \Delta^c(x,x')\,,\ x_0-x_0'\gtrless 0\,,
\end{aligned}\tag{83}$$

where $\Delta^c(x,x')$ is defined by the proper-time representations (55) and (56); see [1, 33, 36]. For the function $S_{\theta,\zeta}(x,x')$ we find

$$\begin{aligned}
S_{\theta,\zeta}(x,x') &= (\gamma P+M)\Delta_{\theta,\zeta}^c(x,x')\,,\\
\Delta_{\theta,\zeta}^c(x,x') &= \int_0^\infty f_{\theta,\zeta}(x,x',s)ds\,,\ \ f_{\theta,\zeta}(x,x',s)=\Upsilon^{(\zeta)}f(x,x',s)\,,
\end{aligned}\tag{84}$$

where $f(x,x',s)$ is given by (56).

Thus, we also have proper-time representations for the quantities $\langle j_\mu(t)\rangle_\theta^c$ and $\langle T_{ik}(t)\rangle_\theta^c$. The latter quantities are finite at finite $t-t_1$. This implies that the representation (84) can be used in calculating the quantities $\langle j_\mu(t)\rangle_\theta^c$ and $\langle T_{ik}(t)\rangle_\theta^c$. Then, the quantity $\langle T_{00}(t)\rangle_\theta^c$ must be calculated via $\langle T_{ik}(t)\rangle_\theta^c$ as follows:

$$\langle T_{00}(t)\rangle_\theta^c=\sum_{k=1,2,3}\langle T_{kk}(t)\rangle_\theta^c+iM\sum_{\zeta=\pm}\operatorname{tr}S_{\theta,\zeta}(x,x')\big|_{x=x'}\ , \tag{85}$$

The latter follows from (67) with allowance for the fact that $S_{\theta,\zeta}$ are solutions of the Dirac equation. Then, we represent expressions in (67) and (85) as

$$\begin{aligned}\langle j_\mu(t)\rangle_\theta^c=&\quad iq\sum_{\zeta=\pm}\operatorname{tr}\left[\gamma_\mu\gamma P\Delta_{\theta,\zeta}^c(x,x')\right]\big|_{x=x'}\ ,\\ \langle T_{\mu\nu}(t)\rangle_\theta^c=&\quad i\sum_{\zeta=\pm}\operatorname{tr}\left[A_{\mu\nu}\gamma P\Delta_{\theta,\zeta}^c(x,x')\right]\big|_{x=x'}\ ,\quad \mu,\nu\neq0\,,\\ \langle T_{00}(t)\rangle_\theta^c=&\quad\sum_{k=1,2,3}\langle T_{kk}(t)\rangle_\theta^c+iM^2\sum_{\zeta=\pm}\operatorname{tr}\left[\Delta_{\theta,\zeta}^c(x,x')\right]\big|_{x=x'}\ .\end{aligned} \tag{86}$$

Following Schwinger (see [8]), the kernel $f(x,x',s)$ can be treated as a matrix element of an operator in x-representation,

$$f(x,x',s)=i\langle x\left|\exp\left\{-is\left[M^2-\left(\gamma\hat{P}\right)^2\right]\right\}\right|x'\rangle\,. \tag{87}$$

Here, $|x'\rangle$ is a complete set of eigenvectors of some commuting operators $\hat{X}^\mu$, such that $\hat{X}^\mu|x\rangle=x^\mu|x\rangle$, $\langle x|x'\rangle=\delta^{(4)}(x-x')$. There exist canonically conjugated operators $\hat{P}_\mu$, and the commutation relations $\left[\hat{X}^\mu,\hat{P}_\nu\right]=i\delta_\nu^\mu$, $\left[\hat{P}_\nu,\hat{P}_\nu\right]=-iqF_{\mu\nu}$, where $F_{\mu\nu}$ is the stress tensor of the external field under consideration. Using the representation (87), we rewrite $f_{\theta,\zeta}(x,x',s)$ (84) as follows:

$$f_{\theta,\zeta}(x,x',s)=\sum_{l=1}^{\infty}(-1)^l e^{\beta l\mu^{(\zeta)}}\frac{1}{2\sqrt{\pi}}\int_0^\infty\frac{du}{u^{1/2}}e^{-u/4}g(x,x',s,\tau), \tag{88}$$

where

$$\begin{aligned}g(x,x',s,\tau)&=\exp\left[-\tau\left(i\partial_3+qET/2\right)^2\right]f(x,x',s,\tau),\quad \tau=(\beta l)^2/u\,,\\ f(x,x',s,\tau)&=i\exp\left\{-\tau\left[M^2-\left(\gamma\hat{P}_\perp\right)^2\right]\right\}\langle x\left|\exp\left\{-is\left[M^2-\left(\gamma\hat{P}\right)^2\right]\right\}\right|x'\rangle\,,\end{aligned} \tag{89}$$

and

$$\left(\gamma\hat{P}\right)^2=\left(\gamma\hat{P}_\parallel\right)^2+\left(\gamma\hat{P}_\perp\right)^2,\quad \hat{P}_\perp=\left(0,\hat{P}_1,\hat{P}_2,0\right),\ \hat{P}_\parallel=\left(\hat{P}_0,0,0,\hat{P}_3\right).$$

Since the operators $\left(\gamma\hat{P}_\parallel\right)^2$ and $\left(\gamma\hat{P}_\perp\right)^2$ commute, we represent $f(x,x',s,\tau)$ as

$$f(x,x',s,\tau)=i\langle x\left|\exp\left\{is\left(\gamma\hat{P}_\parallel\right)^2-i\left(s-i\tau\right)\left[M^2-\left(\gamma\hat{P}_\perp\right)^2\right]\right\}\right|x'\rangle\,.$$

Applying the Schwinger operator techniques [8] to this expression, we find

$$\begin{aligned}&f(x,x',s,\tau)=f_\parallel(x_\parallel,x'_\parallel,s)f_\perp(\mathbf{x}_\perp,\mathbf{x}'_\perp,s-i\tau)\,,\\ &f_\parallel(x_\parallel,x'_\parallel,s)=\exp\left[iq\Lambda_\parallel+\frac{i}{4}\left(x_3-x'_3\right)^2qE\coth\left(qEs\right)\right]f_0(x_0,x'_0,s)\,,\\ &f_0(x_0,x'_0,s)=\frac{iqE}{4\pi\sinh\left(qEs\right)}\exp\left[qEs\gamma^0\gamma^3-\frac{i}{4}\left(x_0-x'_0\right)^2qE\coth\left(qEs\right)\right]\,,\\ &f_\perp(\mathbf{x}_\perp,\mathbf{x}'_\perp,s-i\tau)=\frac{-iqB}{4\pi\sin\left[qB\left(s-i\tau\right)\right]}\exp\left[iq\Lambda_\perp-i\left(s-i\tau\right)\left(M^2-qB\Sigma^3\right)+\frac{iqB\left(\mathbf{x}_\perp-\mathbf{x}'_\perp\right)^2}{4\tan\left[qB\left(s-i\tau\right)\right]}\right]\,,\\ &iq\Lambda_\perp=-i\frac{qB}{2}\left(x_1-x'_1\right)\left(x_2+x'_2\right)\,.\end{aligned} \tag{90}$$

The action of the exponential operator in (89) is determined by the expression

$$\varphi(s,\tau)=\exp\left[-\tau\left(i\partial_3+qET/2\right)^2\right]\exp\left[iq\Lambda_{\parallel}+\frac{i}{4}\left(x_3-x_3'\right)^2qE\coth\left(qEs\right)\right].$$

Using the integral representation

$$\int_{-\infty}^{+\infty}e^{-ip^2a+ibp}dp=\left(e^{-i\pi/2}\pi a^{-1}\right)^{1/2}e^{ib^2/4a},$$

we find

$$\begin{aligned}\varphi(s,\tau)&=\left(\frac{a}{a-i\tau}\right)^{1/2}\exp\left\{iq\Lambda_{\parallel}-\left(x_0+x_0'+T\right)^2\left(\frac{qE}{2}\right)^2\frac{\tau a}{a-i\tau}\right.\\&\left.+i\frac{b^2+i2b\tau qE\left(x_0+x_0'+T\right)}{4\left(a-i\tau\right)}\right\},\end{aligned}\tag{91}$$

where

$$a=a\left(s\right)=\left(qE\right)^{-1}\tanh\left(qEs\right)\,,\;b=x_3-x_3'\,.$$

Therefore,

$$g(x,x',s,\tau)=\varphi(s,\tau)f_0(x_0,x_0',s)f_{\perp}(x_{\perp},x_{\perp}',s-i\tau)\,.\tag{92}$$

Using (84), (86), and (88), the explicit form (92) for $g(x,x',s,\tau)$, and calculating all the derivatives and traces, we find the following expressions for $\mathrm{Re}\langle j_\mu\left(t\right)\rangle_\theta^c$ and $\mathrm{Re}\langle T_{\mu\nu}\left(t\right)\rangle_\theta^c$:

$$\begin{aligned}&\mathrm{Re}<j_\mu\left(t\right)>_\theta^c=\sum_{\zeta=\pm}\sum_{l=1}^{\infty}\left(-1\right)^{l+1}e^{\beta l\mu^{(\zeta)}}\,\mathrm{Re}\,J_\mu^{(l)}\,,\\&J_\mu^{(l)}=\int_0^\infty ds\int_0^\infty du\,h\left(s,u\right)Y\left(s,u\right)j_\mu\left(s,u\right)\,,\;j_\mu\left(s,u\right)=i\delta_\mu^3q^2E\left(t+T/2\right)\frac{\tau}{a-i\tau}\,,\\&h\left(s,u\right)=-\frac{i}{8\pi^2\sqrt{\pi}}\frac{q^2EB\cot\left[qB\left(s-i\tau\right)\right]}{u^{1/2}\sinh\left(qEs\right)}\left(\frac{a}{a-i\tau}\right)^{1/2}e^{-u/4}e^{-M^2\left(is+\tau\right)}\,,\\&Y\left(s,u\right)=\exp\left[-\left(qE\right)^2\left(t+T/2\right)^2\frac{\tau a}{a-i\tau}\right],\end{aligned}\tag{93}$$

and

$$\begin{aligned}&\mathrm{Re}\langle T_{\mu\nu}\left(t\right)\rangle_\theta^c=\sum_{\zeta=\pm}\sum_{l=1}^{\infty}\left(-1\right)^{l+1}e^{\beta l\mu^{(\zeta)}}\,\mathrm{Re}\,T_{\mu\nu}^{(l)}\,,\\&T_{\mu\nu}^{(l)}=\int_0^\infty ds\int_0^\infty du\,h\left(s,u\right)Y\left(s,u\right)t_{\mu\nu}\left(s,u\right)\,,\end{aligned}\tag{94}$$

where it is only the diagonal values $t_{\mu\nu}\left(s,u\right)$ that are nonzero,

$$\begin{aligned}t_{00}\left(s,u\right)&=\sum_{k=1,2,3}t_{kk}\left(s,u\right)+M^2\cosh\left(qEs\right)\,,\;t_{11}\left(s,u\right)=t_{22}\left(s,u\right)=\frac{-iqB}{\sin\left[2qB\left(s-i\tau\right)\right]}\,,\\t_{33}\left(s,u\right)&=\frac{-i}{2\left(a-i\tau\right)\cosh\left(qEs\right)}-\frac{\left(qE\right)^2\left(t+T/2\right)^2}{\cosh\left(qEs\right)}\left(\frac{\tau}{a-i\tau}\right)^2.\end{aligned}\tag{95}$$

Note that the integrals over s and u in the representations (93) and (94) are finite. Such representations hold for any E and temperature when $t\in[t_1,t_2]$ under the restriction (79) on the interval T.

Representations (93) and (94) could be transformed to another forms being more convenient for regions near $s=0$ in the integrands. First of all, we note that at finite temperatures the kernels in (93) and (94) have singularities due to the zeros of $(a-i\tau)$. Let us introduce a new variable s',

$$s'=is+c_0\,,\;c_0=\left(qE\right)^{-1}\arctan\left(qE\tau\right)\,,$$

then

$$\left(a-i\tau\right)^{-1}=\frac{qE\left[1+\tan\left(qEs'\right)qE\tau\right]}{\tan\left(qEs'\right)\left[1+\left(qE\tau\right)^2\right]}\,.\tag{96}$$

The singular points of $(a-i\tau)^{-1}$ are $s'=\pm\pi k$, $k=0,1,2,\ldots$. Let us deform the integration contour over s in integrals (93) and (94) so that it leaves the origin and proceedes downwards along the imaginary axis until the point $-ic_1$, $c_1=(\pi/2-c_0)\,|qE|^{-1}$, without touching it, and then proceedes in parallel to the axis $\mathrm{Re}\, s$ towards the positive infinity. Then, expressions (93) and (94) can be reorganized as follows:

$$
\begin{aligned}
J_\mu^{(l)} &= J_\mu^a + J_\mu^b\,, \\
J_\mu^a &= -i\int_0^{c_1-0} d\tilde{s}\int_0^\infty du\ \left[h(s,u)\,Y(s,u)\,j_\mu(s,u)\right]\big|_{s=-i\tilde{s}}\,, \\
J_\mu^b &= \int_0^\infty d\tilde{s}\int_0^\infty du\ \left[h(s,u)\,Y(s,u)\,j_\mu(s,u)\right]\big|_{s=\tilde{s}-i(c_1-0)}\,,
\end{aligned} \tag{97}
$$

and

$$
\begin{aligned}
T_{\mu\nu}^{(l)} &= T_{\mu\nu}^a + T_{\mu\nu}^b\,, \\
T_{\mu\nu}^a &= -i\int_0^{c_1-0} d\tilde{s}\int_0^\infty du\ \left[h(s,u)\,Y(s,u)\,t_{\mu\nu}(s,u)\right]\big|_{s=-i\tilde{s}}\,, \\
T_{\mu\nu}^b &= \int_0^\infty d\tilde{s}\int_0^\infty du\ \left[h(s,u)\,Y(s,u)\,t_{\mu\nu}(s,u)\right]\big|_{s=\tilde{s}-i(c_1-0)}\,,
\end{aligned} \tag{98}
$$

One can verify that the quantities

$$
\begin{aligned}
\delta J_\mu^a &= -i\int_0^\infty du\int_{e^{i\pi}(c_0-0)}^0 d\tilde{s}\,\left[h(s,u)\,Y(s,u)\,j_\mu(s,u)\right]\big|_{s=-i\tilde{s}}\,, \\
\delta T_{\mu\nu}^a &= -i\int_0^\infty du\int_{e^{i\pi}(c_0-0)}^0 d\tilde{s}\,\left[h(s,u)\,Y(s,u)\,t_{\mu\nu}(s,u)\right]\big|_{s=-i\tilde{s}}\,,
\end{aligned}
$$

are imaginary; therefore the real-valued parts of J_μ^a and $T_{\mu\nu}^a$ can be represented as

$$
\begin{aligned}
&\mathrm{Re}\, J_\mu^a = \mathrm{Re}\, J_\mu'^a\,,\ \ \mathrm{Re}\, T_{\mu\nu}^a = \mathrm{Re}\, T_{\mu\nu}'^a\,, \\
&J_\mu'^a = J_\mu^a + \delta J_\mu^a\,,\ \ T_{\mu\nu}'^a = T_{\mu\nu}^a + \delta T_{\mu\nu}^a\,.
\end{aligned} \tag{99}
$$

We now change the variable $\tilde{s}$ by $s'=\tilde{s}+c_0$ in the integrals for $J_\mu'^a$ and $T_{\mu\nu}'^a$. Then, one can see that the contributions from the integration region $0\le u<u_0$ over u in these integrals are imaginary. Then, changing the variable u by u',

$$
u'=u-u_0\,,\ \ u_0=qE\,(\beta l)^2\cot(qEs')\ ,
$$

and introducing the notation $\mathrm{Re}\, J_\mu'^a=\tilde{J}_\mu^a$, $\mathrm{Re}\, T_{\mu\nu}'^a=\tilde{T}_{\mu\nu}^a$, we represent $\tilde{J}_\mu^a$ and $\tilde{T}_{\mu\nu}^a$ as

$$
\begin{aligned}
\tilde{J}_\mu^a &= \int_0^{\pi/|2qE|-0} ds'\int_0^\infty du'\ \tilde{h}(s',u')\,\tilde{Y}(s',u')\,\tilde{j}_\mu(s',u')\ , \\
\tilde{T}_{\mu\nu}^a &= \int_0^{\pi/|2qE|-0} ds'\int_0^\infty du'\ \tilde{h}(s',u')\,\tilde{Y}(s',u')\,\tilde{t}_{\mu\iota}(s',u')\ ,
\end{aligned} \tag{100}
$$

where

$$
\begin{aligned}
&\tilde{h}(s',u') = -ih\left(i\,(c_0-s')\,,u'+u_0\right) \\
&= \frac{1}{8\pi^2\sqrt{\pi}}\frac{q^2EB\coth\left[qB\,(s'+\delta\tau)\right]}{u'^{1/2}\sin(qEs')}\exp\left[-\frac{u'+u_0}{4}-M^2\,(s'+\delta\tau)\right]\,, \\
&\tilde{Y}(s',u') = Y\left(i\,(c_0-s')\,,u'+u_0\right) = \exp\left[-\frac{(qE)^2\,(t+T/2)^2\,\tau u'}{(u'+u_0)\left[1+(qE\tau)^2\right]}\right]\,, \\
&\delta\tau = \tau-(qE)^{-1}\arctan(qE\tau)\,,\ \ \tau=\frac{(\beta l)^2}{u'+u_0}\,,
\end{aligned} \tag{101}
$$

and

$$
\begin{aligned}
\tilde{j}_\mu(s',u') &= j_\mu(i(c_0-s'),u'+u_0)\,, \quad \tilde{t}_{\mu\nu}(s',u') = t_{\mu\nu}(i(c_0-s'),u'+u_0)\,,\\
\tilde{j}_\mu(s',u') &= -\delta^3_\mu \frac{q(qE)^2(t+T/2)\tau[1+qE\tau\tan(qEs')]}{\tan(qEs')\left[1+(qE\tau)^2\right]}\,,\\
\tilde{t}_{11}(s',u') &= \tilde{t}_{22}(s',u') = \frac{qB}{\sinh[2qB(s'+\delta\tau)]}\,,\\
\tilde{t}_{33}(s',u') &= \frac{qE}{2\sin(qEs')\left[1+(qE\tau)^2\right]^{1/2}} + \frac{(qE)^4(t+T/2)^2\tau^2[1+qE\tau\tan(qEs')]}{\sin(qEs')\tan(qEs')\left[1+(qE\tau)^2\right]^{3/2}}\,,\\
\tilde{t}_{00}(s',u') &= \sum_{k=1,2,3}\tilde{t}_{kk}(s',u') + M^2\cos(qEs')\frac{1+qE\tau\tan(qEs')}{\left[1+(qE\tau)^2\right]^{1/2}}\,.
\end{aligned}
\tag{102}
$$

Note that $\tau \le (qE)^{-1}\tan(qEs')$ in $\mathrm{Re}\,\tilde{J}^a_\mu$ and $\mathrm{Re}\,\tilde{T}^a_{\mu\nu}$.

The result is that $\mathrm{Re}\,J^{(l)}_\mu$ and $\mathrm{Re}\,T^{(l)}_{\mu\nu}$ in (93) and (94) can be presented as

$$
\mathrm{Re}\,J^{(l)}_\mu = \tilde{J}^a_\mu + \mathrm{Re}\,J^b_\mu\,, \quad \mathrm{Re}\,T^{(l)}_{\mu\nu} = \tilde{T}^a_{\mu\nu} + \mathrm{Re}\,T^b_{\mu\nu}\,,
\tag{103}
$$

respectively. In this form, it is convenient to study both low-temperature approximations to a weak electric field and high-temperature approximations to an arbitrary electric field.

In a weak electric field, $|qE|/M^2 \ll 1$, the quantities $\mathrm{Re}\,J^b_\mu$ and $\mathrm{Re}\,T^b_{\mu\nu}$ are exponentially small, so that in order to find the expansions of $\mathrm{Re}\,J^{(l)}_\mu$ and $\mathrm{Re}\,T^{(l)}_{\mu\nu}$ in the powers of $|qE|/M^2$,we use the relations

$$
\mathrm{Re}\,J^{(l)}_\mu = \tilde{J}^a_\mu, \quad \mathrm{Re}\,T^{(l)}_{\mu\nu} = \tilde{T}^a_{\mu\nu},
\tag{104}
$$

where the exponentially small contributions have been neglected.

4.2.2 Zero electric field

When the electric field is zero, the mean current density is zero as well. Considering the limit $E \to 0$, one can see that the integral (100) over s' for $\tilde{T}^a_{\mu\nu}$ is taken from zero to infinity. In addition, the dependence on the variable u' in the kernel of the matrix elements $\tilde{T}^a_{\mu\nu}$ factors out as the multiplier $e^{-u'/4}/\sqrt{u'}$. Calculating the integral over u' by using (80), in case $E=0$, we find

$$
\begin{aligned}
&\tilde{T}^a_{\mu\nu} = T^0_{\mu\nu} = \int_0^\infty ds\rho_0(s)\,t^0_{\mu\iota}(s)\,,\\
&\rho_0(s) = \frac{qB\coth(qBs)}{4\pi^2 s}\exp\left[-M^2 s - \frac{(\beta l)^2}{4s}\right], \quad t^0_{33}(s) = \frac{1}{2s}\,,\\
&t^0_{11}(s) = t^0_{22}(s) = \frac{qB}{\sinh(2qBs)}\,, \quad t^0_{00}(s) = \sum_{k=1,2,3} t^0_{kk}(s) + M^2\,.
\end{aligned}
\tag{105}
$$

The quantities $T^0_{\mu\nu}$ are real-valued. Integrating by parts, one can see that T^0_{00} has the form (105), where

$$
t^0_{00}(s) = -\frac{1}{2s} + \frac{(\beta l)^2}{4s^2}\,.
$$

In the case under consideration, the vacuum is stable, while the renormalized EMT (69) is real-valued, time-independent, and is given only by the conventional terms

$$
\begin{aligned}
\langle T_{\mu\nu}(t)\rangle_{ren}\big|_{E=0} &= \langle T_{\mu\nu}(t)\rangle^c_{ren}\big|_{E=0} + \langle T_{\mu\nu}(t)\rangle^c_\theta\big|_{E=0}\,,\\
\langle T_{\mu\nu}(t)\rangle^c_\theta\big|_{E=0} &= \sum_{\zeta=\pm}\sum_{l=1}^{\infty}(-1)^{l+1}e^{\beta l\mu^{(\zeta)}}T^0_{\mu\nu}\,,
\end{aligned}
\tag{106}
$$

where $\langle T_{\mu\nu}(t)\rangle^c_{ren}\big|_{E=0}$ is determined by (59) at $E=0$.

The system under consideration is a relativistic fermionic gas in an external magnetic field, B, immersed in a sufficiently large quantization volume V. This system is described by the partition function

$$Z = \mathrm{Tr} \exp\left\{\beta\left(\sum_{\zeta=\pm}\mu^{(\zeta)}N^{(\zeta)} - H\right)\right\},$$

following from (15), $H = H(t_1)$, and by the thermodynamic potential $\Omega = -\theta \ln Z$. At the state of equilibrium, the temperature- and field-dependent effective Lagrangian $\mathcal{L}^{\theta}_{ren}$ is related to Ω as follows:

$$\mathcal{L}^{\theta}_{ren} = -\Omega/V. \tag{107}$$

The mean values of energy density $\langle H\rangle/V$ and particle density $n^{(\zeta)}$ in the state (15) can be expressed in terms of $\mathcal{L}^{\theta}_{ren}$ as

$$\begin{aligned} \frac{\langle H\rangle}{V} &= -\mathcal{L}^{\theta}_{ren} - \beta\frac{\partial \mathcal{L}^{\theta}_{ren}}{\partial \beta} + \sum_{\zeta=\pm}\mu^{(\zeta)}n^{(\zeta)}, & (108)\\ n^{(\zeta)} &= \frac{\partial \mathcal{L}^{\theta}_{ren}}{\partial \mu^{(\zeta)}}. & (109)\end{aligned}$$

Note that the energy density has been already calculated as a component of (106), namely,

$$\langle H_0\rangle/V = \langle T_{00}(t)\rangle_{ren}|_{E=0}.$$

Thus, we find that the effective Lagrangian has the form

$$\begin{aligned} \mathcal{L}^{\theta}_{ren} &= \mathcal{L}_{ren}|_{E=0} + \Delta\mathcal{L}_{\theta}, \\ \Delta\mathcal{L}_{\theta} &= \frac{1}{8\pi^2}\sum_{\zeta=\pm}\sum_{l=1}^{\infty}(-1)^{l+1}e^{\beta l\mu^{(\zeta)}}\int_0^{\infty}\frac{ds}{s^2}\exp\left[-M^2 s - \frac{(\beta l)^2}{4s}\right]qB\coth(qBs), \quad (110)\end{aligned}$$

where $\mathcal{L}_{ren}|_{E=0}$ is the renormalized Heisenberg–Euler effective Lagrangian (58) at $E = 0$.

It turns out that the space components of $\langle T_{ik}(t)\rangle_{ren}|_{E=0}$ (106) can be derived from the effective Lagrangian $\mathcal{L}^{\theta}_{ren}$ as follows:

$$\begin{aligned} \langle T_{11}(t)\rangle_{ren}|_{E=0} &= \langle T_{22}(t)\rangle_{ren}|_{E=0} = \mathcal{L}^{\theta}_{ren} - B\frac{\partial \mathcal{L}^{\theta}_{ren}}{\partial B}, \\ \langle T_{33}(t)\rangle_{ren}|_{E=0} &= \mathcal{L}^{\theta}_{ren}. \quad (111)\end{aligned}$$

It should be noted that the finite temperature and density effects are contained only in $\Delta\mathcal{L}_{\theta}$. In particular, the thermodynamic potential Ω_{free} of perfect free Fermi gas at $B = 0$ is related to $\Delta\mathcal{L}_{\theta}|_{B=0}$ in the form $\Omega_{free} = -\Delta\mathcal{L}_{\theta}|_{B=0}V$. Expressing the proper-time integral for $\Delta\mathcal{L}_{\theta}|_{B=0}$ in terms of the modified Bessel function of the third kind (Macdonald's function) $K_{\nu}(z)$ [34], and using formula 3.471.9 in [39],

$$\int_0^{\infty} x^{\nu-1}e^{-a/x-bx}dx = 2\left(\frac{a}{b}\right)^{\nu/2}K_{\nu}\left(2\sqrt{ab}\right), \tag{112}$$

we find

$$\Delta\mathcal{L}_{\theta}|_{B=0} = \left(\frac{M}{\beta\pi}\right)^2\sum_{\zeta=\pm}\sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{l^2}e^{\beta l\mu^{(\zeta)}}K_2(M\beta l). \tag{113}$$

Thus, we obtain for the quantity Ω_{free} the well-known expression from textbooks.

Considering the usual case $\mu^{(+)} = -\mu^{(-)} = \mu$, we can see that (110) coincides with the results obtained earlier in the case $E = 0$: see Eq. (52) in the paper [15], where the imaginary-time formalism was used, and Eq. (5.20) in [16], where the real-time formalism was used (we desregard the factor 1/2 in Eq. (5.20) of [16] as an misprint). In the case $\mu = 0$, Eq. (110) is in agreement with the result obtained in [14].

One ought to say that our analysis deals primarily with the effects of electric field at finite initial temperatures, and therefore we do not examine in detail the particular case of magnetic field (at $E = 0$). Note, however, that the leading magnetic field-dependent terms follow immediately from (110). We present these terms for comparison with a nonzero electric field: in the weak-field limit, when either $|qB|/M^2 \ll 1$ or $|qB|\beta^2 \ll 1$, we have

$$\mathcal{L}_1 = \Delta\mathcal{L}_{\theta} - \Delta\mathcal{L}_{\theta}|_{B=0} = \frac{(qB)^2}{12\pi^2}\sum_{\zeta=\pm}\sum_{l=1}^{\infty}(-1)^{l+1}e^{\beta l\mu^{(\zeta)}}K_0(M\beta l). \tag{114}$$

In the strong-field limit, $|qB| \gg M^2, \beta^{-2}$, we have

$$\mathcal{L}_1 = \frac{|qB|\, M}{2\pi^2 \beta} \sum_{\zeta=\pm} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} e^{\beta l \mu^{(\zeta)}} K_1 (M\beta l) \, .$$

The latter quantity is small in comparison with the corresponding vacuum polarization term, which is given by the following expression

$$\mathcal{L}_{ren}|_{E=0} - \mathcal{L}_{ren}|_{E=0,B=0} = \frac{(qB)^2}{24\pi^2} \ln \frac{|qB|}{M^2} \, .$$

In the weak-field limit, we can see that the leading field-dependent thermal contributions to EMT have the form

$$\begin{aligned} \tau_{11}^{B} &= \tau_{22}^{B} = \mathcal{L}_1 - B\frac{\partial \mathcal{L}_1}{\partial B} = -\mathcal{L}_1 \, , \; \tau_{33}^{B} = \mathcal{L}_1 \, , \\ \tau_{00}^{B} &= -\mathcal{L}_1 - \beta \frac{\partial \mathcal{L}_1}{\partial \beta} + \sum_{\zeta=\pm} \mu^{(\zeta)} \frac{\partial \mathcal{L}_1}{\partial \mu^{(\zeta)}} \, . \end{aligned} \tag{115}$$

Note that Fermi systems with an external magnetic field have been studied intensively and various asymptotic expansions of the effective Lagrangian at finite temperatures have been made: for a review, see [15, 16]; and a review on finite-temperature free relativistic systems can be found, e.g., in [32, 40, 41].

4.2.3 Weak electric field and low temperatures

In a weak electric field, $|qE|/M^2 \ll 1$, the quantities $\operatorname{Re} J_\mu^{(l)}$ and $\operatorname{Re} T_{\mu\nu}^{(l)}$ in (93) and (94) are given by $\tilde{J}_\mu^a$ and $\tilde{T}_{\mu\nu}^a$ (100)–(102); see (104). The dependence of expressions (100)–(102) on the parameters E and β is described by $qE\tau$, which is a function of two variables: s' and u'. Approximations of this function and of the corresponding integral over u' at low temperature and at high temperature are quite different. Therefore, unlike the case $E = 0$, these limiting cases must be studied separately. In this subsection, we shall examine the case of low temperature, $M\beta \gg 1$.

Contributions to integrals (100) given by the region $M^2 s' \gg 1$ of large s' are exponentially small. Therefore, we limit the range of integration over s' from above by the value χ/M^2, so that $1 \ll \chi \ll M^2/|qE|$. At low temperature, the leading contributions to the integral over s' are formed at the region $M^2 s' \sim M\beta l/2 \gg 1$, provided that $M\beta l/2 \ll \chi$. It is interesting to examine an approximation which admits the limiting process $E \to 0$. For every $l \leq l_{\max}$ in a sufficiently weak electric field, there exists such χ that $M\beta l_{\max}/2 \ll \chi$. Under this condition, we obtain the approximation for finite values of l. Note that at low temperatures the terms $e^{\mu^{(\zeta)}\beta l} \operatorname{Re} J_\mu^{(l)}$ and $e^{\mu^{(\zeta)}\beta l} \operatorname{Re} T_{\mu\nu}^{(l)}$ of order $e^{-(M-\mu^{(\zeta)})\beta l} l^{-\alpha}$ in case $\alpha \geq 1/2$ decrease when l increases. Therefore, any accuracy of calculation for the sums in (93) and (94) can be provided by a finite number of summands. Accordingly, at $|\mu^{(\zeta)}| \ll M$, in order to estimate the leading contributions, it is sufficient to take into account only the first summand, at $l = 1$. However, even for large values of chemical potentials, $|\mu^{(\zeta)}| \sim M$, a sufficient accuracy is provided by summation up to a corresponding finite number l_{est}. Accordingly, we suppose that $l_{est} \leq l_{\max}$. Thus, the approximated expressions for $\operatorname{Re}\langle j_\mu(t)\rangle_\theta^c$ and $\operatorname{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c$ at low temperature have the form

$$\begin{aligned} &\operatorname{Re}\langle j_\mu(t)\rangle_\theta^c = \sum_{\zeta=\pm} \sum_{l=1}^{l_{est}} (-1)^{l+1} e^{\beta l \mu^{(\zeta)}} J_\mu^\chi \, , \\ &J_\mu^\chi = \int_0^{\chi/M^2} ds \int_0^\infty du \, \tilde{h}(s,u) \tilde{Y}(s,u) \tilde{j}_\mu(s,u) \; ; \\ &\operatorname{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c = \sum_{\zeta=\pm} \sum_{l=1}^{l_{est}} (-1)^{l+1} e^{\beta l \mu^{(\zeta)}} T_{\mu\nu}^\chi \, , \\ &T_{\mu\nu}^\chi = \int_0^{\chi/M^2} ds \int_0^\infty du \, \tilde{h}(s,u) \tilde{Y}(s,u) \tilde{t}_{\mu\iota}(s,u) \, , \end{aligned} \tag{116}$$

where $\tilde{h}(s,u)$, $\tilde{Y}(s,u)$, $\tilde{j}_\mu(s,u)$, and $\tilde{t}_{\mu\iota}(s,u)$ are defined in (101) and (102).

Let us find the leading contributions depending on the electric field for expressions (116). These quantities depend on two dimensionless parameters that characterize the influence of electric field: $|qE|/M^2$ is a parameter determined by the strength of constant electric field, whereas $qE(t + T/2)/M$ is a parameter depending on time t and determined by the increment in the kinetic momentum of a particle in electric field by the time instant t (i.e., the increment that accumulates during the time interval $t-t_1=t+T/2$). The first parameter $|qE|/M^2$ is small in a weak electric field. We shall expand all the functions in the integrals (116) as series in the powers of this parameter and examine only the leading terms of the expansion. In addition, the value $|qE|(t+T/2)/M$ is not necessarily small for a sufficiently large time interval, $t+T/2$. Thus, the dependence on this value in the following expression is taken into account precisely. For simplicity, we assume that the magnetic field is weak, so we can neglect the biquadratic terms of the order $q^2|EB|/M^4$. Therefore, with accuracy up to the leading term depending on the electric field, we obtain

$$
\begin{aligned}
J_\mu^X &= \int_0^\infty ds \int_0^\infty du\, h^0 Y^W j_\mu^W\,, \\
T_{\mu\nu}^X &= \int_0^\infty ds \int_0^\infty du\, \left(h^0+\delta h\right) Y^W \left(t_{\mu\iota}^0+\delta t_{\mu\iota}\right)\,.
\end{aligned} \tag{117}
$$

Here, we have neglected the exponentially small contribution and extended the range of integration over s to infinity, $+\infty$; besides,

$$
\begin{aligned}
& h^0 = \frac{e^{-u/4}}{2\sqrt{\pi u}}\rho_0,\ \ \delta h = \frac{e^{-u/4}}{2\sqrt{\pi u}}\frac{(qE)^2}{4\pi^2}\exp\left[-M^2 s-\frac{(\beta l)^2}{4s}\right]D\,, \\
& D = \left[\frac{(\beta l)^2}{12s}-\frac{\tau_W^3}{3s^2}\left(\frac{1}{s}+M^2\right)+\frac{1}{6}\right],\ \ Y^W=\exp\left[-\frac{(qE)^2(t+T/2)^2\tau_W u}{u+(\beta l)^2/s}\right], \\
& \tau_W = \frac{(\beta l)^2}{u+(\beta l)^2/s},\ \ j_\mu^W=-\delta_\mu^3 q^2 E\,(t+T/2)\,\tau_W/s\,,\ \ \delta t_{11}=\delta t_{22}=-\frac{(qE)^2\tau_W^3}{6s^2}\,, \\
& \delta t_{33} = \frac{(qE)^2}{4}\left(\frac{s}{3}-\frac{\tau_W^2}{s}\right)+(qE)^2(t+T/2)^2\left(\frac{\tau_W}{s}\right)^2, \\
& \delta t_{00} = \sum_{k=1,2,3}\delta t_{kk}+\delta t_M,\ \ \delta t_M=-\frac{1}{2}M^2(qE)^2(s-\tau_W)^2\,,
\end{aligned} \tag{118}
$$

where $t_{\mu\iota}^0$ and ρ_0 are given by (105). Note that in expressions (118) we have not imposed any conditions on temperature. Let us do it now. Namely, since at low temperature the leading contributions to the integral over s are formed at the region of $s\sim \beta l/(2M)$, the function τ_W can be expanded as a series in the powers of u,

$$
\tau_W = s\left\{1-\frac{su}{(\beta l)^2}+\left[\frac{su}{(\beta l)^2}\right]^2-\dots\right\}. \tag{119}
$$

We consider only the leading contribution to the expansion in inverse powers of βM. To calculate this contribution, it is sufficient to take into account the first three terms of expansion (119); moreover, in the coefficients at time-dependent functions it is sufficient to take into account only the first term of the expansion. Therefore, at low temperature the function Y^W in the leading approximation has the form

$$
Y^W = Y^{WL} = \exp\left[-(qE)^2(t+T/2)^2\frac{s^2u}{(\beta l)^2}\right]. \tag{120}
$$

Using expansion (119) and expression (120), we can calculate the integrals over u in (117). To this end, it is convenient to use the generating function $\Phi_0(\alpha)$, defined by the integral

$$
\Phi_0(\alpha) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{du}{u^{1/2}}e^{-u(1/4+\alpha)}\,. \tag{121}
$$

Calculating this integral according to formula (80), we obtain $\Phi_0(\alpha)=(1+4\alpha)^{-1/2}$. Then, taking derivatives with respect to the parameter α, we obtain a formula we need for calculation:

$$
\Phi_n(\alpha) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{du}{u^{1/2}}e^{-u(1/4+\alpha)}u^n = (-1)^n\frac{\partial^n\Phi_0(\alpha)}{\partial\alpha^n}\,. \tag{122}
$$

In the case under consideration, we have $\alpha=(qE)^2(t+T/2)^2\left(\frac{s}{\beta l}\right)^2$.

Until the T-constant field has switched on, the system in question is at the state of thermal equilibrium. When the T-constant field switches on at the instant t_1 the particles at the initial state begin to accelerate. The corresponding current density and EMT change with time until the moment t_2, when the T-constant field turns off. Nevertheless, such a non-equilibrium state during a certain interval of time until the increment of kinetic momentum is sufficiently small, $|qE|(t+T/2)/M \ll 1$, is not very different from the initial state of thermal equilibrium, and it makes sense to compare these states. It is exactly this condition that guarantees the existence of a limiting process $E \to 0$. Therefore, it is important to examine such a case in more detail. On the other hand, it is of special interest to examine the case of asymptotically large values of time, when $|qE|(t+T/2)/M \gg 1$; in this case, the presence of electric field manifests itself to the maximum. In what follows, we examine two limiting cases.

Small increment of kinetic momentum When the increment of kinetic momentum is small, $|qE|(t+T/2)/M \ll 1$, the kernels in (117) can be expanded in its powers. To this end, we take into account the fact that $\alpha = 0$ implies that the function $\Phi_n(\alpha)$ is expressed in terms of binomial coefficients, $\Phi_0(0) = 1$ and $\Phi_n(0) = 2^n \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$, for $n \geq 1$. Using (117) after integrating over u, and considering only the leading time-dependent contributions, we obtain

$$J_\mu^\chi = j_\mu^0 \int_0^\infty ds \rho_0 \,, \quad T_{\mu\nu}^\chi = T_{\mu\nu}^0 + \Delta T_{\mu\nu} \,, \tag{123}$$

where $T_{\mu\nu}^0$ and ρ_0 are independent of electric field and given by (105); besides

$$j_\mu^0 = -\delta_\mu^3 q^2 E (t + T/2) \,. \tag{124}$$

In the above representation of $T_{\mu\nu}^\chi$, the part that depends on the electric field is separated as $\Delta T_{\mu\nu}$ and has the form

$$\begin{aligned}
\Delta T_{\mu\nu} &= \frac{1}{4\pi^2} \int_0^\infty ds \exp\left[-M^2 s - \frac{(\beta l)^2}{4s}\right] \Delta t_{\mu\nu} \,, \\
\Delta t_{11} &= \Delta t_{22} = \frac{(qE)^2}{2s}\left(D_0 - \frac{1}{3}\right) - \frac{(qE)^2 (t+T/2)^2}{s(\beta l)^2} \,, \\
\Delta t_{33} &= \Delta t_{11} + \frac{(qE)^2}{2(\beta l)^2} + \frac{(qE)^2 (t+T/2)^2}{s^2} \,, \\
\Delta t_{00} &= (qE)^2 M^2 D_0 + \frac{(qE)^2 (t+T/2)^2}{(\beta l)^2}\left[\frac{(\beta l)^2}{s^2} - 2M^2\right] \,, \\
D_0 &= -\frac{1}{6} + \frac{(\beta l)^2}{12s} - \frac{M^2 s}{3} + 2\frac{M^2 s^2}{(\beta l)^2} - 24\frac{M^2 s^3}{(\beta l)^4} \,.
\end{aligned} \tag{125}$$

Integration in (125) leads to the modified Bessel functions of the third kind $K_\nu(z)$, according to formula (112). For simplicity, we assume that the chemical potentials are such that $e^{-(M-\mu^{(\zeta)})\beta} \ll 1$. We then use the asymptotic expansion 7.13.1.(7) [34]:

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left[1 + \left(\nu^2 - \frac{1}{4}\right)\frac{1}{2z} + \left(\nu^2 - \frac{1}{4}\right)\left(\nu^2 - \frac{9}{4}\right)\frac{1}{8z^2} + O\left(|z|^{-3}\right)\right] . \tag{126}$$

As a result of the asymptotic expansion at low temperature, we obtain the leading contributions of the $\Delta T_{\mu\nu}$ in the form

$$\begin{aligned}
\Delta T_{11} &= \Delta T_{22} = -C_l \frac{2}{M\beta l}\left[1 + \frac{M(t+T/2)^2}{\beta l}\right]\left[1 + O\left(\frac{1}{M\beta l}\right)\right] , \\
\Delta T_{33} &= C_l \left[-\frac{3}{2M\beta l} + 4\frac{M(t+T/2)^2}{\beta l}\right]\left[1 + O\left(\frac{1}{M\beta l}\right)\right] , \\
\Delta T_{00} &= C_l \left[-\frac{235}{256} + 2\frac{M(t+T/2)^2}{\beta l}\right]\left[1 + O\left(\frac{1}{M\beta l}\right)\right] , \\
C_l &= \frac{(qE)^2}{4\pi^2}\left(\frac{\pi}{2M\beta l}\right)^{1/2} e^{-M\beta l} .
\end{aligned} \tag{127}$$

This makes it possible to present the expression for EMT $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c$ in (116) as follows:

$$\mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c = \langle T_{\mu\nu}(t)\rangle_\theta^c|_{E=0} + \sum_{\zeta=\pm}\sum_{l=1}^{l_{est}}(-1)^{l+1}e^{\beta l\mu^{(\zeta)}}\Delta T_{\mu\nu}, \tag{128}$$

where we have explicitly separated the part $\langle T_{\mu\nu}(t)\rangle_\theta^c|_{E=0}$, independent of the electric field, which is defined in (106), while the terms $\Delta T_{\mu\nu}$ depending on the electric field are given by (127).

With the help of the expression for J_μ^X in (123), the current density $\mathrm{Re}\langle j_\mu(t)\rangle_\theta^c$ in (116), with the corresponding accuracy, can be expressed in the form

$$\mathrm{Re}\langle j_\mu(t)\rangle_\theta^c = -\delta_\mu^3\frac{q^2E(t+T/2)}{M}\sum_{\zeta=\pm}n^{(\zeta)}, \tag{129}$$

where the initial particle density $n^{(\zeta)}$ is given by (109). Considering only the leading asymptotic contributions, we obtain $n^{(\zeta)}$ as follows:

$$n^{(\zeta)} = 2\left(\frac{M}{2\pi\beta}\right)^{3/2}\sum_{l=1}^{l_{est}}(-1)^{l+1}e^{\beta l(\mu^{(\zeta)}-M)}\frac{1}{l^{3/2}}\frac{qB\beta l}{2M}\coth\frac{qB\beta l}{2M}. \tag{130}$$

If the chemical potential is small, $|\mu^{(\zeta)}| \ll M$, the leading contribution in the sums (128), (130) yields one term, at $l=1$. In this case, the leading magnetic field-dependent thermal contributions (115) in $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c$ are $\tau_{11}^B = \tau_{22}^B = -\tau_{33}^B < 0$, $\tau_{00}^B > 0$, and all time-independent contributions for electric field-dependent terms in (128) are negative.

Large increment of kinetic momentum When the increment of kinetic momentum is large, $|qE|(t+T/2)/M \gg 1$, the parameter α in (121) is large as well. Then, we calculate the integrals over u in (117), by using the asymptotics of $\Phi_0(\alpha)$, $\Phi_0(\alpha) \simeq \alpha^{-1/2}/2$, and obtain

$$\begin{aligned} J_\mu^X &= \frac{j_\mu^0\beta l}{2|qE|(t+T/2)}\int_0^\infty ds\rho_0 s^{-1}, \\ T_{\mu\nu}^X &= \frac{\beta l}{2|qE|(t+T/2)}\int_0^\infty ds\rho_0 s^{-1}\left(t_{\mu\iota}^0 + \delta t_{\mu\iota}|_{\tau_W=s}\right), \end{aligned} \tag{131}$$

where we have used only the first term of expansion (119). At low temperature, the main contribution to the integral over s is formed at the region $s \sim \beta l/(2M)$, and thus the leading contributions in (131) can be found immediately,

$$\begin{aligned} J_\mu^X &= \frac{j_\mu^0 M}{|qE|(t+T/2)}\int_0^\infty ds\rho_0, \\ T_{\mu\nu}^X &= \frac{M}{|qE|(t+T/2)}\int_0^\infty ds\rho_0\left(t_{\mu\iota}^0 + \delta^0 t_{\mu\iota}\right), \end{aligned} \tag{132}$$

where

$$\delta^0 t_{11} = \delta^0 t_{22} = 0, \quad \delta^0 t_{00} = \delta^0 t_{33} = (qE)^2(t+T/2)^2.$$

Consequently, using expressions (132) we obtain from the representation (116) that

$$\begin{aligned} \mathrm{Re}\langle j_\mu(t)\rangle_\theta^c &= -\delta_\mu^3|q|\,\mathrm{sgn}(E)\sum_{\zeta=\pm}n^{(\zeta)}, \\ \mathrm{Re}\langle T_{11}(t)\rangle_\theta^c &= \mathrm{Re}\langle T_{22}(t)\rangle_\theta^c = \frac{M}{|qE|(t+T/2)}\mathrm{Re}\langle T_{11}(t)\rangle_\theta^c|_{E=0}, \\ \mathrm{Re}\langle T_{00}(t)\rangle_\theta^c &= \mathrm{Re}\langle T_{33}(t)\rangle_\theta^c = |qE|(t+T/2)\sum_{\zeta=\pm}n^{(\zeta)}, \end{aligned} \tag{133}$$

where $n^{(\zeta)}$ is the initial particle density, given by (130), and $\mathrm{Re}\langle T_{11}(t)\rangle_\theta^c|_{E=0}$ is the initial value of $\mathrm{Re}\langle T_{11}(t)\rangle_\theta^c$, defined by (106) at $t < t_1$.

4.2.4 High temperatures

Let us find the leading field-dependent contributions for $\mathrm{Re}\langle j_\mu(t)\rangle_\theta^c$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c$ at high temperature. In the presence of external magnetic and electric fields, the conditions of applicability of high-temperature asymptotic include the following conditions for the field strengths:

$$M\beta \ll 1\,, \quad \sqrt{|qB|}\beta \ll 1\,, \quad \sqrt{|qE|}\beta \ll 1\,. \tag{134}$$

In the case of fields that are not very strong, $|qB|/M^2 \leq 1$, $|qE|/M^2 \leq 1$, these subsidiary conditions are not relevant; however, they become necessary in the case of strong fields, $|qB|/M^2 > 1$, $|qE|/M^2 > 1$. There are no other necessary restrictions for the field strengths. Under these restrictions, it is convenient to use the representation (103). At high temperature, the leading contributions into the expressions for $\mathrm{Re}\,J_\mu^{(l)}$ and $\mathrm{Re}\,T_{\mu\nu}^{(l)}$ are given by the terms $\tilde{J}_\mu^a$ and $\tilde{T}_{\mu\nu}^a$, respectively. These terms arise from the integrals (100). In turn, at high temperature, the leading contributions to these integrals arise from small values of s', namely, $s' \sim (\beta l)^2$. Therefore, in calculating the leading contributions to all field-dependent functions one can asume that $|qB|\,s' \ll 1$ and $|qE|\,s' \ll 1$. This allows one to use in the integrals (100) expansions in the powers of $|qB|\,s'$ and $|qE|\,s'$, and then to extend the range of integration over s' to $+\infty$, while neglecting the exponentially small contribution. As a result, one arrives at the conclusion that $\tilde{J}_\mu^a$ and $\tilde{T}_{\mu\nu}^a$, are approximated by the expressions J_μ^χ and $T_{\mu\nu}^\chi$ in (117), respectively. Note that at high temperature the leading contribution into the sums (93) and (94) is formed by a finite number of summands, when l belongs to the range $1 \leq l \leq l_{\max}$. A choice for the number $l_{\max}$ is determined by a given accuracy of calculation, and by the fact that $l_{\max} \ll \left(\sqrt{|qE|}\beta\right)^{-1}$ and $l_{\max} \ll \left(\sqrt{|qB|}\beta\right)^{-1}$. The terms (117) are well-decreasing functions of the number l, and therefore in the high-temperature limit one can assume that $l_{\max} \to \infty$. Thus, the sum is taken over positive integers, as in the case of thermal equilibrium, which happens in the absence of electric field. As a result, we obtain the following asymptotic expressions for $\mathrm{Re}\langle j_\mu(t)\rangle_\theta^c$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c$:

$$\begin{aligned}
\mathrm{Re}\langle j_\mu(t)\rangle_\theta^c &= \sum_{\zeta=\pm}\sum_{l=1}^{\infty}(-1)^{l+1}\,e^{\beta l\mu^{(\zeta)}}J_\mu^\chi\,, \\
\mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c &= \sum_{\zeta=\pm}\sum_{l=1}^{\infty}(-1)^{l+1}\,e^{\beta l\mu^{(\zeta)}}T_{\mu\nu}^\chi\,,
\end{aligned} \tag{135}$$

where J_μ^χ and $T_{\mu\nu}^\chi$ are given by expressions (117) and (118). As usual, in these asymptotics, we suppose $|\mu^{(\zeta)}|\,\beta \ll 1$.

As a whole, we are interested in the leading electric-field-dependent contributions into expressions (135), as well as in the limiting process $E \to 0$, and in the possibility to compare the leading electric-field-dependent term with the leading magnetic-field-dependent term. Therefore, we assume for simplicity that the magnetic field is not stronger than the electric field, $|B/E| \leq 1$; then, within the given accuracy of calculation, we can neglect the contributions that contain the magnetic field at powers higher than the second, as well as the contributions depending on the product B^2E^2.

At high temperature, the dimensionless parameter that depends on time (t) and characterizes the influence of electric field has the form $qE\,(t+T/2)\,\beta$. As in the low-temperature case, this parameter is determined by the relative increment of the kinetic momentum of a particle in electric field reached by the time instant t; however, the scale on which one estimates this increment is now the temperature θ rather than the mass M.

When the T-constant field switches on, the particles found at the initial state begin to accelerate. Nevertheless, such a non-equilibrium state is not very different from the initial state of thermal equilibrium when $|qE|\,(t+T/2)\,\beta \ll 1$, and it makes sense to compare these states. It is exactly this condition that guarantees the existence of the limiting process $E \to 0$. It is clear that at any finite value $|qE|\,(t+T/2)$ the given condition takes place at $\beta \to 0$, i.e., it is exaclty in this case that we deal with an asymptotic expression at high temperature. Therefore, this case is the most interesting one. Let us note, in addition, that this condition implies comparatively weaker restrictions for the value of kinetic momenta, $|qE|\,(t+T/2) \ll 1/\beta$ than the values at low temperature when one requires that $|qE|\,(t+T/2) \ll M$. For sufficiently high temperatures, ultrarelativistic kinetic momenta can be regarded sufficiently small. The opposite case of asymptotically large values of time at large but finite β, when $|qE|\,(t+T/2)\,\beta \gg 1$, is a complicated case, when, according to conditions (134), the temperature is sufficiently high in comparison with M, B, and E; however, at the same time, the temperature is low in comparison with the kinetic momentum $qE\,(t+T/2)$. This case is also interesting, since the presence of electric field manifests itself to the maximum. In what follows, we examine these two limiting cases.

Small increment of kinetic momentum When the increment of kinetic momentum is small, $|qE|(t+T/2)\beta \ll 1$, the function Y^W in (118) can be expanded as

$$Y^W = 1 - \delta Y\,,\ \ \delta Y = (qE)^2 (t+T/2)^2 \left(\tau_W - s^{-1}\tau_W^2\right)\,.$$

Let us the following approximation:

$$\begin{aligned} J_\mu^X &= j_\mu^0 \int_0^\infty \frac{ds}{s}\int_0^\infty du\ h^0\big|_{B=0}\tau_W\,, \\ T_{\mu\nu}^X &= T_{\mu\nu}^0 + \Delta T_{\mu\nu}\,, \end{aligned} \tag{136}$$

where $T_{\mu\nu}^0$ and h^0 are expressions being independent of electric field; $T_{\mu\nu}^0$ is given by (105); h^0 and τ_W are given by (118); j_μ^0 is given by (124); besides

$$\begin{aligned} \Delta T_{\mu\nu} &= \frac{1}{4\pi^2}\int_0^\infty ds \exp\left[-M^2 s - \frac{(\beta l)^2}{4s}\right]\left(\Delta t_{\mu\nu}^1 + \Delta t_{\mu\nu}^2\right)\,, \\ \Delta t_{\mu\nu}^1 &= \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{du}{u^{1/2}} e^{-u/4}\frac{1}{s^2}\,\delta t_{\mu\nu}\big|_{B=0}\,, \\ \Delta t_{\mu\nu}^2 &= \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{du}{u^{1/2}} e^{-u/4}\left[(qE)^2 D - \delta Y\frac{1}{s^2}\right] t_{\mu\nu}^0\big|_{B=0}\,, \end{aligned} \tag{137}$$

where $t_{\mu\nu}^0$ is given by (105), $\delta t_{\mu\nu}$ and D are given by (118), respectively.

To calculate the integrals in (136) and (137), we need the following functions of a real argument p:

$$I_n(p) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{du}{u^{1/2}} e^{-u/4}\frac{1}{(u+p)^n},\ \ n \geq 1.$$

By using formula (3.363.2) in [39], the function $I_1(p)$ can be expressed in terms of the incomplete gamma-function:

$$I_1(p) = \frac{1}{2p^{1/2}} e^{p/4}\Gamma\left(\frac{1}{2},\frac{p}{4}\right)\,. \tag{138}$$

Differentiating this relation with respect to the variable p, one can find the corresponding representation for the remaining functions $I_n(p)$ at $n \geq 2$. In particular, we have

$$I_2(p) = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{2}\right) I_1(p) + \frac{1}{4p}\,. \tag{139}$$

In the case under consideration, $p = (\beta l)^2/s$. Let us express J_μ^X and Δt_{kk} for $k = 1,2,3$ in terms of the functions $I_n(p)$:

$$\begin{aligned} & J_\mu^X = \frac{j_\mu^0 (\beta l)^2}{4\pi^2}\int_0^\infty \frac{ds}{s^3}\exp\left[-M^2 s - \frac{(\beta l)^2}{4s}\right] I_1\left(\frac{(\beta l)^2}{s}\right)\,, \\ & \Delta t_{11}^1 = \Delta t_{22}^1 = -\frac{(qE)^2(\beta l)^6}{6s^4} I_3\left(\frac{(\beta l)^2}{s}\right)\,, \\ & \Delta t_{33}^1 = \frac{(qE)^2}{4}\left[\frac{1}{3s} - \frac{(\beta l)^4}{s^3} I_2\left(\frac{(\beta l)^2}{s}\right)\right] + (qE)^2 (t+T/2)^2 \frac{(\beta l)^4}{s^4} I_2\left(\frac{(\beta l)^2}{s}\right)\,, \\ & \Delta t_{kk}^2 = \frac{(qE)^2}{6}\left[\frac{1}{2s} + \frac{(\beta l)^2}{4s^2} - \frac{(\beta l)^6}{s^4} I_3\left(\frac{(\beta l)^2}{s}\right)\right] \\ & -(qE)^2 (t+T/2)^2 \frac{(\beta l)^2}{2s^3}\left[I_1\left(\frac{(\beta l)^2}{s}\right) - \frac{(\beta l)^2}{s} I_2\left(\frac{(\beta l)^2}{s}\right)\right]\,, \end{aligned} \tag{140}$$

where the relation $\Phi_0(0) = 1$ for the integral (121) has been used, and we have neglected the terms smaller that the remaining ones when $s \sim (\beta l)^2$.

Using the representation 9.1(2) in [34],

$$\Gamma\left(\frac{1}{2},\frac{p}{4}\right) = \frac{p^{1/2}}{2}\int_1^\infty \frac{du}{u^{1/2}} e^{-up/4}\,,$$

and formula (112), one can calculate the following integrals:

$$\chi_n = \int_0^\infty s^{-N} \exp\left[-M^2 s - \frac{(\beta l)^2}{4s}\right] I_n\left(\frac{(\beta l)^2}{s}\right) ds\,, \quad N \geq 2\,.$$

Let us first examine the case $n = 1$. Calculating the integral over s, we obtain

$$\chi_1 = \frac{2^{N-1} M^{2N-3}}{\beta l} \int_{M\beta l}^\infty z^{1-N} K_{N-1}(z)\, dz\,. \tag{141}$$

This representation allows one to extract the leading contribution for $M\beta \to 0$. The function $K_{N-1}(z)$ decreases sufficiently fast at large z, and it is easy to see that the leading contribution is formed at the region of small z, $z \leq \delta_z$, where δ_z is a small value, such that $M\beta l \ll \delta_z \ll 1$. It is sufficient to take into account such l that $M\beta l \ll 1$, since the contributions χ_1 into the sum (135) due to the higher values of l can be neglected.[4] Neglecting the small contribution given by the region of $z > \delta_z$, we now represent χ_1 in the form

$$\chi_1 = \frac{2^{N-1} M^{2N-3}}{\beta l} \int_{M\beta l}^{\delta_z} z^{1-N} K_{N-1}(z)\, dz\,.$$

This integral can be calculated due to expansion 8.446 [39] for $K_{N-1}(z)$ at $z \ll 1$,

$$K_{N-1}(z) = \frac{1}{2}\left(\frac{2}{z}\right)^{N-1} (N-2)!\,. \tag{142}$$

Then, we finally obtain

$$\chi_1 = \frac{2^{2N-1}(N-2)!}{(2N-3)(\beta l)^{2N-2}}\,. \tag{143}$$

in the leading approximation. Expressing $I_2(p)$ through $I_1(p)$ by using relation (139) and following the same procedure as in the case of calculating χ_1, we obtain the leading term for χ_2 at $N \geq 3$

$$\chi_2 = \frac{2^{2N-3}}{(\beta l)^{2N-2}}\left[\frac{(N-3)!}{8} + \frac{1}{2(2N-5)} - \frac{1}{2N-3}\right]. \tag{144}$$

In a similar way, we can estimate the leading contribution for χ_3. For our purposes, it is sufficient to estimate the orders of χ_3: $\chi_3 \sim (\beta l)^{-4}\ln(M\beta l)$ at $N = 3$, and $\chi_3 \sim (\beta l)^{-6}$ at $N = 4$. Using these results for a calculation of the expressions defined by (137) and (140), and leaving only the leading contributions, we find

$$\begin{aligned} J_\mu^\chi &= \frac{8}{3\pi^2}\frac{j_\mu^0}{(\beta l)^2}, \\ \Delta T_{11} = \Delta T_{22} &= \frac{1}{4\pi^2}\left[\frac{(qE)^2}{6} K_0(M\beta l) - \frac{58}{15}\frac{(qE)^2 (t+T/2)^2}{(\beta l)^2}\right], \\ \Delta T_{33} &= \frac{1}{4\pi^2}\left[\frac{(qE)^2}{3} K_0(M\beta l) - \frac{14}{15}\frac{(qE)^2 (t+T/2)^2}{(\beta l)^2}\right]. \end{aligned} \tag{145}$$

Proceeding in the same way, we obtain the leading contribution to ΔT_{00} as follows:

$$\Delta T_{00} = \sum_{k=1,2,3} \Delta T_{kk}\,. \tag{146}$$

Taking the sum of the terms (145), in accordance with the representation (135), we apply the alternating zeta-function

$$\eta(s) = \sum_{l=1}^{\infty} (-1)^{l+1}\frac{1}{l^s}$$

[4] As an alternative, one can sum over l the contribution χ_1 by using the representation (141) and the summation formula with the function $K_n(xl)$ [32]. This approach is more effective when one needs to obtain high-temperature expansions beyond the leading terms.

and the summation formula (see, e.g., Appendix in [32])

$$\sum_{l=1}^{\infty}(-1)^{l+1}K_0(M\beta l) = -\frac{1}{2}\ln(M\beta) + O(1)\,, \tag{147}$$

where we have fixed only the leading term, in accordance with our approximation. Since $\eta(2) = \pi^2/12$, we obtain the final result as follows:

$$\begin{aligned}
&\mathrm{Re}\langle j_\mu(t)\rangle^c_\theta = -\frac{4}{9}\delta^3_\mu q^2 E(t+T/2)/\beta^2\,,\\
&\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^c_\theta = \mathrm{Re}\langle T_{\mu\nu}(t)\rangle^c_\theta|_{E=0} + \tau^\theta_{\mu\nu}(t)\,,\\
&\tau^\theta_{11}(t) = \tau^\theta_{22}(t) = \frac{(qE)^2}{12\pi^2}\left[-\frac{1}{2}\ln(M\beta) + O(1) - \frac{29\pi^2}{15}\frac{(t+T/2)^2}{\beta^2}\right],\\
&\tau^\theta_{33}(t) = \frac{(qE)^2}{12\pi^2}\left[-\ln(M\beta) + O(1) - \frac{7\pi^2}{15}\frac{(t+T/2)^2}{\beta^2}\right],\\
&\tau^\theta_{00}(t) = \frac{(qE)^2}{12\pi^2}\left[-2\ln(M\beta) + O(1) - \frac{13\pi^2}{3}\frac{(t+T/2)^2}{\beta^2}\right].
\end{aligned} \tag{148}$$

Even though we examine the case of a small kinetic-momentum increment, the time-dependent terms in $\tau^\theta_{\mu\nu}(t)$ (148) may be larger that the logarithmic contributions, because they are limited only by the condition that $(t+T/2)^2/\beta^2 \ll 1/\left(qE\beta^2\right)^2$. The derivative of these terms with respect to time t determines the rate of change for the components $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^c_\theta$.

Let us find the relation of the resulting expression for the current density $\mathrm{Re}\langle j_\mu(t)\rangle^c_\theta$ with respect to the initial particle density $n^{(\zeta)}$. The leading terms of the effective Lagrangian $\Delta\mathcal{L}_\theta|_{B=0}$ (113) can be expressed as

$$\Delta\mathcal{L}_\theta|_{B=0} = \frac{2}{\pi^2\beta^4}\sum_{\zeta=\pm}\left[\eta(4) + \eta(3)\beta\mu^{(\zeta)}\right],$$

where formula (142) has been used. Then, using (109) we obtain

$$n^{(+)} = n^{(-)} = \frac{2\eta(3)}{\pi^2\beta^3}\,,$$

where $\eta(3) = 0,9015426774$, and we have

$$\mathrm{Re}\langle j_\mu(t)\rangle^c_\theta = -\frac{\pi^2}{9\eta(3)}\delta^3_\mu q^2 E(t+T/2)\beta\sum_{\zeta=\pm}n^{(\zeta)}\,. \tag{149}$$

Let us compare the time-independent terms in (148) with the magnetic field-dependent terms in $\langle T_{\mu\nu}(t)\rangle^c_\theta|_{E=0}$ (115). Using formula (147), we find the leading field-dependent term of the effective Lagrangian $\Delta\mathcal{L}_\theta$ (114) at high temperature:

$$\mathcal{L}_1 = -\frac{(qB)^2}{12\pi^2}\left[\ln(M\beta) + O(1)\right],$$

which coincides with previously known results (see, e.g., [15]). Then, using formula (115), we find that the magnetic field-dependent terms of EMT are given by

$$\tau^B_{00} = \tau^B_{11} = \tau^B_{22} = -\tau^B_{33} = -\mathcal{L}_1\,.$$

We can see that the equation of state for magnetic field in a medium at high temperature is identical with the equation of state for this field in vacuum. The increment of the energy density of magnetic field caused by vacuum polarization, $\langle T_{00}(t)\rangle^c_{ren}|_{E=0}$, and the increment τ^B_{00} are both negative, which holds true in the case of a field of any intensity. To the contrary, the increment of the energy density of electric field caused by vacuum polarization, $\langle T_{00}(t)\rangle^c_{ren}$, is positive in a weak field and is negative in a strong one. The time-independent terms of $\tau^\theta_{\mu\nu}(t)$ (148) are positive. Therefore, even at the initial period of the existence of electric field, when the time-dependent terms of $\tau^\theta_{\mu\nu}(t)$ can be neglected, its equation of state is essentially different from the vacuum equation and has different forms in the cases of weak and strong fields.

Large increment of kinetic momentum When the increment of kinetic momentum is large, $|qE|(t+T/2)\beta \gg 1$, the leading contribution to the integral over u in (135) is formed at the region $u \gg 1$. In order to find the asymptotic behavior of expressions (135), it is sufficient to know a rough approximation of the function Y^W, namely,

$$Y^W = Y_0 = \exp\left[-(qE)^2 (t+T/2)^2 (\beta l)^2 /u\right].$$

Then J_μ^χ and $T_{\mu\nu}^\chi$ can be approximated as

$$\begin{aligned} J_\mu^\chi &= \int_0^\infty ds \int_0^\infty du\, h^0 Y_0 j_\mu^W , \\ T_{\mu\nu}^\chi &= \int_0^\infty ds \int_0^\infty du\, \left(h^0 + \delta h\right) Y_0 \left(t_{\mu\iota}^0 + \delta t_{\mu\iota}\right) , \end{aligned} \tag{150}$$

with $t_{\mu\iota}^0$ given by (105), and h^0, δh, $\delta t_{\mu\iota}$ given by (118), where the approximation $\tau_W = (\beta l)^2/u$ has been used. It is convenient to carry out the integration over u in (150) with the help of the generating function $Y_0(\sigma) = \exp\left[-(qE)^2(t+T/2)^2(\beta l)^2\sigma/u\right]$, $\sigma > 0$, which is identical with Y_0 at $\sigma = 1$. Using formula (80), we obtain

$$\frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{du}{u^{1/2}} e^{-u/4} Y_0(\sigma) = \exp\left[-|qE|(t+T/2)\beta l \sigma^{1/2}\right].$$

This formula, along with the first and second derivative with respect to the parameter σ, allows one to carry out the integration over u in (150). As a result, we find, with accuracy with the pre-exponential factor, that J_μ^χ and $T_{\mu\nu}^\chi$ are of the order $\exp\left[-|qE|(t+T/2)\beta l\right]$. Consequently, the leading contribution is given by the terms J_μ^χ and $T_{\mu\nu}^\chi$ for $l = 1$. Therefore, at asymptotically large kinetic momenta, one can see that $\mathrm{Re}\langle j_\mu(t)\rangle_\theta^c$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c$ decrease exponentially:

$$\begin{aligned} \mathrm{Re}\langle j_\mu(t)\rangle_\theta^c &\sim \exp\left[-|qE|(t+T/2)\beta\right], \\ \mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c &\sim \exp\left[-|qE|(t+T/2)\beta\right]. \end{aligned} \tag{151}$$

A calculation of the pre-exponential factor requires a more accurate approximation and some very tedious calculations.

4.2.5 Strong electric field

In the limit of a strong electric field, $|qE| \gg M^2, \beta^{-2}, |qB|$, the leading contributions depending on the electric field for the integrals $\mathrm{Re}\, J_\mu^{(l)}$ and $\mathrm{Re}\, T_{\mu\nu}^{(l)}$ in (93) and (94) arise from large values of $|qE|s$ and $|qE|\tau$, $|qE|\tau \gg |qE|s \gg 1$. For simplicity, we assume that $B = 0$. In addition, we know from Subsec. 3.2 that the leading time-dependent vacuum contributions, $\mathrm{Re}\langle j_\mu(t)\rangle^p$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^p$, do not depend on the details of switching on and off when the time interval $t - t_1 = t + T/2$ is sufficiently large, so that condition (38) holds true. Then, it is instructive to examine thermal contributions for the same large interval. In this case, we can use the following approximation:

$$\begin{aligned} \mathrm{Re}\langle j_\mu(t)\rangle_\theta^c &= \sum_{\zeta=\pm}\sum_{l=1}^{\infty}(-1)^{l+1} e^{\beta l \mu^{(\zeta)}}\, \mathrm{Re}\, J_\mu^{(l)} , \\ J_\mu^{(l)} &= j_\mu^0 \int_0^\infty ds \int_0^\infty du\, h^s Y^s ; \end{aligned} \tag{152}$$

and

$$\begin{aligned} \mathrm{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c &= \sum_{\zeta=\pm}\sum_{l=1}^{\infty}(-1)^{l+1} e^{\beta l \mu^{(\zeta)}}\, \mathrm{Re}\, T_{\mu\nu}^{(l)} , \\ T_{\mu\nu}^{(l)} &= \int_0^\infty ds \int_0^\infty du\, h^s Y^s t_{\mu\nu}^s , \\ t_{00}^s &= \sum_{k=1,2,3} t_{kk}^s + M^2 e^{|qE|s}/2 ,\ \ t_{11}^s = t_{22}^s = \frac{u}{2(\beta l)^2} , \\ t_{33}^s &= 2e^{-|qE|s}\left[\frac{u}{2(\beta l)^2} + (qE)^2 (t+T/2)^2\right] ; \end{aligned} \tag{153}$$

where

$$h(s,u)=\frac{1}{2}D_l\,|qE|\,u\exp\left[\frac{i\pi}{4}-\left(|qE|+iM^2\right)s-\frac{u}{4}-\frac{(M\beta l)^2}{u}\right],$$
$$D_l=\left[2\pi^{5/2}\sqrt{|qE|}\,(\beta l)^3\right]^{-1},$$
$$Y^s=\exp\left[-i\,|qE|\,(t+T/2)^2-(t+T/2)^2\,(\beta l)^{-2}\,u\right], \tag{154}$$

and j_μ^0 is given by expression (124). In this representation, the kernels are products: function of the variable s times function of the variable u; therefore, these double integrals can be represented as a product: an integral over s times an integral over u. Having calculated the integral over s, and using formula (112) for a calculation of the integral over u, we find that

$$\begin{aligned}
\operatorname{Re}J_\mu^{(l)} &= 2j_\mu^0\cos\left[\frac{\pi}{4}-|qE|\,(t+T/2)^2\right]D_lf_1\,,\\
\operatorname{Re}T_{00}^{(l)} &= \operatorname{Re}T_{33}^{(l)}+\sin\left[\frac{\pi}{4}-|qE|\,(t+T/2)^2\right]D_l\,|qE|\,f_1\,,\\
\operatorname{Re}T_{11}^{(l)} &= \operatorname{Re}T_{22}^{(l)}=\cos\left[\frac{\pi}{4}-|qE|\,(t+T/2)^2\right]\frac{D_lf_2}{(\beta l)^2}\,,\\
\operatorname{Re}T_{33}^{(l)} &= \cos\left[\frac{\pi}{4}-|qE|\,(t+T/2)^2\right]D_l\left[\frac{f_2}{(\beta l)^2}+2\,(qE)^2\,(t+T/2)^2f_1\right],
\end{aligned} \tag{155}$$

where

$$f_n=\left(\frac{M\beta l}{\sqrt{1/4+(t+T/2)^2\,(\beta l)^{-2}}}\right)^{n+1}K_{n+1}\left(\sqrt{(M\beta l)^2+4M^2\,(t+T/2)^2}\right).$$

We note that the dimensionless parameter $M\,(t+T/2)$ is equal to unity for the time period taken by light to cross the Compton radius of a particle of mass M. Thus, for a macroscopic period of time, $M\,(t+T/2)\gg 1$, one can use the asymptotic expansion (126), namely,

$$\begin{aligned}
&K_{n+1}\left(\sqrt{(M\beta l)^2+4M^2\,(t+T/2)^2}\right)\\
&=\sqrt{\frac{\pi}{2\sqrt{(M\beta l)^2+4M^2\,(t+T/2)^2}}}\exp\left[-\sqrt{(M\beta l)^2+4M^2\,(t+T/2)^2}\right].
\end{aligned}$$

At low initial temperature, $M\beta\gg 1$, this asymptotic expansion is valid for any period of time. We can see that in the limit of a strong electric field the behavior of the current density $\operatorname{Re}\langle j_\mu(t)\rangle_\theta^c$ and the EMT components $\operatorname{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c$ is described by relaxation oscillations. The amplitudes of these oscillations are small in comparison with the increasing vacuum contributions $\operatorname{Re}\langle j_\mu(t)\rangle^p$ and $\operatorname{Re}\langle T_{\mu\nu}(t)\rangle^p$ in (50), respectively, starting from the instant t when conditions (38) are fulfilled.

5 Discussion and Summary

We have obtained nonperturbative one-loop representations for the mean current density $\langle j_\mu(t)\rangle$ and renormalized EMT $\langle T_{\mu\nu}(t)\rangle_{ren}$ of a Dirac field in constant electic-like background. Two cases of initial states are considered, the vacuum one and a thermal equilibrim state. In the general case, each of the found expressions consist of three characteristic terms:

$$\begin{aligned}
\langle j_\mu(t)\rangle &= \operatorname{Re}\langle j_\mu(t)\rangle_\theta^c+J_\mu^p(t)\,,\\
\langle T_{\mu\nu}(t)\rangle_{ren} &= \operatorname{Re}\langle T_{\mu\nu}(t)\rangle_{ren}^c+\operatorname{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c+\tau_{\mu\nu}^p(t)\,,
\end{aligned}$$

see (69).

The components $\operatorname{Re}\langle T_{\mu\nu}(t)\rangle_{ren}^c$ given by (59) describe the contribution due to the vacuum polarization. The corresponding components of current density are zero, $\langle j_\mu(t)\rangle^c=0$. These components $\operatorname{Re}\langle T_{\mu\nu}(t)\rangle^c$ can be related with the real part of Heisenberg–Euler Lagrangian. They are local, i.e., they depend on t, but does not depend on the history of the process.

The components $\operatorname{Re}\langle j_\mu(t)\rangle_\theta^c$ and $\operatorname{Re}\langle T_{\mu\nu}(t)\rangle_\theta^c$ given by (93), (94), (103) describe the contribution due to the work of the external field on the particles at the initial state. The components $J_\mu^p(t)$ and $\tau_{\mu\nu}^p(t)$

given by (48), (50), (73) describe the contribution due to the creation of real particles from the vacuum. These contributions depend on the time interval $t - t_1$ that passes since the instant the electic field turns on, and therefore they are global quantities.

All these contributions are analyzed in detail in different regimes, limits of weak and strong field and low and high temperatures. For all these limiting cases, we have obtained the leading contributions, which are given as elementary functions of the basis dimensionless parameters.

The action of the electric field manifests itself differently for different components of the EMT $\langle T_{\mu\nu}(t)\rangle_{ren}$. For instance, in a strong electric field or in a field which is not strong but acts for a sufficiently long time interval, there is the following correspondence: a coincidence of the leading terms for energy density, $\mathrm{Re}\langle T_{00}(t)\rangle_\theta^c + \tau_{00}^p(t)$, and the pressure along the direction of the electric field, $\mathrm{Re}\langle T_{33}(t)\rangle_\theta^c + \tau_{33}^p(t)$. Let us note that for vacuum polarization the terms in the mentioned correspondence have the opposite signs: $\mathrm{Re}\langle T_{00}(t)\rangle_{ren}^c = -\mathrm{Re}\langle T_{33}(t)\rangle_{ren}^c$. This happens because in the former relation we deal with the equation of state for the relativistic fermions, while in the latter relation we deal with the equation of state for the electromagnetic field. The dependence of the electric field and its duration for the transversal component $\mathrm{Re}\langle T_{11}(t)\rangle_\theta^c + \tau_{11}^p(t)$ is quite different from the dependence of the longitudinal component $\mathrm{Re}\langle T_{33}(t)\rangle_\theta^c + \tau_{33}^p(t)$. This means that one cannot define a universal functional for the electric field, whose variations should give all the components of the EMT. This holds true not only in case there exist particles at the initial state, but also when the initial state is vacuum. Therefore, we can see that there cannot be any generalization of the Heisenberg–Euler Lagrangian for the problem of calculating the mean values. This fact, in our opinion, takes place for any pair-creating background and is related, in such a case, to the existence of nonlocal contributions to the EMT. This explains the failure of numerous attempts to consider one-loop effects on the basis of different variants [19, 26] of such a generalization.

Making a comparison of the constant temperature-dependent components of the mean energy density and pressue in a weak electric field immediately after it switches on, we can see that in this case any generalization of the Heisenberg–Euler Lagrangian to the case of finite temperature is impossible. In other words, even a seemingly small disturbance of thermal equilibrium, produced by a weak electric field, exceeds the limits of applicability of the approaches based on thermal equilibrium.

We have established the restriction $|qE|\,T^2 \ll \frac{\pi^2}{2q^2}$ (which takes place both for the initial vacuum state and for a low-temperature initial thermal state) for the strength and duration of electric field under which QED with a strong constant electric field remains consistent. Under this restriction, one can neglect the back-reaction of particles created by the electric field. On the other hand, there exists another restriction, $1 \ll |qE|\,T^2$ (see 38) that allows one to disregard the details of switching on and off the electric field. Gathering both restrictions, we obtain the following range of the dimensionless parameter $|qE|\,T^2$ under which QED with a strong constant electric field is consistent:

$$1 \ll |qE|\,T^2 \ll \pi^2/2q^2 \,.$$

This inequality is consistent due to $\pi^2/2q^2 \gg 1$, and one can be certain that QED with a T-constant field does exist.

Similar restrictions can be obtained when the intial thermal equilibrium is taken at sufficiently high temperatures, $\beta\,|qE|\,T \ll 1$. In this case, we have two inequalities, $\beta\,|qE|^2\,T^3 \ll \frac{3\pi^2}{q^2}$ and $1 \ll |qE|\,T^2$, which imply

$$1 \ll |qE|\,T^2 \ll \frac{3\pi^2}{q^2\beta\,|qE|\,T} \,.$$

We can see that the upper restriction for $|qE|\,T^2$ is weaker than in the low-temperature case.

We believe that a similar consideration for an electric-like non-Abelian external field will lead to the same restrictions when the created Fermi particles (partons) can be treated as weakly coupled. In the case of high temperatures, one can neglect the back-reaction of the created particles on the chromoelectric field, in comparison with the contribution from the bosons, which are influenced by temperature in a different way (the case of bosons will be examined in another work).

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